ON LIFE ANNUITIES

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1. Having established the right principle on which it is necessary to base the calculation of life annuities, I believe that the development of this calculation will not fail to be very interesting, both to those who will want to undertake such an establishment as well as to those who will want to benefit from them. I sketched this matter in my *General Investigations on the Mortality and the Multiplication of Man*, where I showed the fair method for determining, by this calculation, how much a man of a certain age must pay, to enjoy during his entire life a given annual income. But since the calculation seemed to me then very troublesome, I was not able to bring myself to carry it out. Well, a certain occasion compelled me finally to undertake this work, and by means of some techniques to shorten the calculation, I have happily reached the end.

2. There are two things upon which the determination of life annuities must be based. One is a good list of mortality which shows us, for each age, how many will probably die during the course of one or several years. The other is the way in which the entrepreneur is able to profit from the money that he will have received from the annuitants, or at what interest he is able to invest it. These two items combine very essentially to determine the payments to which the entrepreneur will be able to commit himself, both in relation to the sum which was paid to him initially as well as in relation to the age of the annuitant. For it is clear that the more the entrepreneur is able to draw in profit from the capital that he has in hand, the more he will also be in a position to pay better annuities.

3. For the list of mortality, the entrepreneur would no doubt risk much if he wished to go by the mortality of people in general, which are inferred from observations made in a large town or from an entire country, where equal account is taken of all people, both strong and weak. Now, when it is a question of getting life annuities, it is very natural that we must exclude all those whose constitution does not seem to promise a long life. Thus, we have

reason to regard the annuitants as a more robust kind. It is in this view that I have chosen in my cited work the list of Mr. Kersseboom, which was drawn from observations made solely of people who have enjoyed life annuities, and so also this same list will serve as my basis in the following calculations.

4. If the entrepreneur were not in a position to adequately invest the capital that was paid to him by the annuitants, he could only offer annuities so mediocre that nobody would want to purchase them. Formerly, the town of Amsterdam paid ten percent annuities to each person under the age of twenty years, or for 1000 florins they paid them 100 each year, which is such a rich payment that the town would have suffered a very considerable loss if it had not earned nearly 10 percent each year from the funds that this enterprise provided. Thus, if one can count on only 5 percent interest, the annuities must become considerably less. However it is on that basis that it seems necessary at present to adjust life annuities, seeing that those who will have occasion to make a larger profit will hardly involve themselves with such an enterprise, which can only draw to a close after a large number of years.

5. In order to determine the price of these annuities, we fix for each age an average length of life, so that it is as probable to survive as to die before having attained it. Or else this length is taken so that out of a large number of men of the same age, as many will die before as after. Then, we suppose that all the men of this age live precisely this long, and then they all die at once. From this, we imagine being able to confidently set the annuity prices, since it is a matter of finding the present value of an annual income payable during a certain number of consecutive years. And we guess that the profit the entrepreneur gains due to those who die early would be precisely offset by his loss caused by those who live longer. But we will easily understand that this reasoning goes astray, since we do not take into account the decrease in present value of an income that will be paid only after a number of years. Because of this situation, it will be necessary to base the calculation on the right principles, which I have described in my above-mentioned article, without using any reasoning that could appear to be suspect.

6. For this purpose, let us consider 1000 infants born at the same time, and by the characters

$$(1), \ (2), \ (3), \ (4), \ \text{etc.}$$

indicate the number of those who will still be alive at the end of

$$1, \ 2, \ 3, \ 4, \ \text{etc.}$$

years, so that in general (m) represents the number of those who will reach the age of m years. Now let r be the annual income that a man aged m years would want to acquire, and x the price that he must pay for it to the entrepreneur, which must be a fair equivalent of the expense which the entrepreneur undertakes by this bargain. To determine this price x, it is necessary to consider a number of men of the same age m, who enter under the same circumstances. Let (m) be the number of these men, and the sum that they pay at present to the entrepreneur will be $(m)x$, which must be sufficient to furnish all the payments that he will have to make thereafter.

7. Now, of these (m) men, there will be alive: after one year $(m+1)$, after two years $(m+2)$, after three years $(m+3)$, and so on. Therefore, the entrepreneur will have to pay: after one year $(m+1)r$, after two years $(m+2)r$, after three years $(m+3)r$, etc., until all these annuitants have passed away. We have only, then, to reduce each of these payments to the present time, at the rate of 5 percent, and equate the sum to $(m)x$, to find the fair value of x. Now, to make the calculation more general, in place of 105/100 or 21/20, we write the letter λ; and the sum of all the payments that the entrepreneur must subsequently pay will be worth at present

$$\frac{(m+1)r}{\lambda} + \frac{(m+2)r}{\lambda^2} + \frac{(m+3)r}{\lambda^3} + \frac{(m+4)r}{\lambda^4} + \text{ etc.}$$

which being equal to $(m)x$ gives

$$x = \frac{r}{(m)}\left[\frac{(m+1)}{\lambda} + \frac{(m+2)}{\lambda^2} + \frac{(m+3)}{\lambda^3} + \frac{(m+4)}{\lambda^4} + \text{ etc.}\right]$$

8. This then is the fair price that a man aged m years must pay in order to enjoy the right to an annual income r during his entire life, and which, being initially invested at 5 percent, sets the entrepreneur precisely in a position to subsequently make the payments, provided that the number of annuitants is large enough. We understand well that having thus invested at the beginning all the capital that the entrepreneur has received, the next year the interest will not be sufficient to make the payments, but that it will be necessary to use a part of the capital; therefore the capital will suffer each year a diminution. However, it will be entirely used up only when all the annuitants have died. For this reason, the entrepreneur will be well obliged to raise the price of the annuities that I just found, according to the circumstances and the particular expenses that such an establishment requires.

9. We see clearly that the determination of the price named x demands a long and boring calculation, especially for the lower ages, where the number of terms to add together is quite considerable. But it is not hard to see that having already done this calculation for a certain age, we can easily derive that which corresponds to one year more or less. To explain more clearly this technique, I will use this notation

$$\overline{m}r$$

to indicate the price that a man aged m years must pay for the life annuity r, so that

$$\overline{m} = \frac{1}{(m)}\left[\frac{(m+1)}{\lambda} + \frac{(m+2)}{\lambda^2} + \frac{(m+3)}{\lambda^3} + \frac{(m+4)}{\lambda^4} + \text{ etc.}\right]$$

and so, for men aged $m+1$ years, we have

$$\overline{m+1} = \frac{1}{(m+1)}\left[\frac{(m+2)}{\lambda} + \frac{(m+3)}{\lambda^2} + \frac{(m+4)}{\lambda^3} + \frac{(m+5)}{\lambda^4} + \text{ etc.}\right]$$

from which we conclude

$$\lambda(m)\overline{m} = (m+1) + (m+1)\overline{m+1}$$

and therefore

$$\overline{m} = \frac{1}{\lambda}\cdot\frac{(m+1)}{(m)}\left[1 + \overline{m+1}\right]$$

so that having found the value of $\overline{m+1}$, we will calculate from it easily enough that of $\overline{m}$.

10. By the aid of this technique, after having started with the age of 90 years, I calculated the price of the income r successively for all the lesser ages, down to the newborn infants. From this I obtained the following table, fixing the income r at 100 écus and the interest at 5 percent.

TABLE: PRICE OF A LIFE ANNUITY OF 100 ÉCUS
FOR AGES 0 TO 50 YEARS

age in years	number of living	price of annuity	age in years	number of living	price of annuity
0	1000	1155.50	25	552	1403.60
1	804	1409.04	26	544	1395.45
2	768	1448.84	27	535	1389.87
3	736	1487.43	28	525	1387.16
4	709	1521.27	29	516	1382.54
5	690	1541.32	30	507	1376.82
6	676	1551.90	31	499	1368.84
7	664	1558.94	32	490	1363.68
8	654	1561.92	33	482	1355.63
9	646	1560.33	34	475	1344.38
10	639	1556.29	35	468	1332.71
11	633	1549.59	36	461	1320.60
12	627	1542.64	37	454	1308.01
13	621	1535.42	38	446	1298.04
14	616	1525.28	39	439	1284.67
15	611	1514.65	40	432	1270.76
16	606	1503.50	41	426	1253.09
17	601	1491.81	42	420	1234.54
18	596	1479.54	43	413	1218.24
19	590	1469.31	44	406	1201.21
20	584	1458.63	45	400	1180.19
21	577	1450.18	46	393	1161.27
22	571	1438.68	47	386	1141.44
23	565	1426.66	48	378	1123.88
24	559	1414.07	49	370	1105.59
25	552	1403.60	50	362	1086.52
m	(m)	$\overline{m}$	m	(m)	$\overline{m}$

TABLE: PRICE OF A LIFE ANNUITY OF 100 ÉCUS
FOR AGES 50 TO 90 YEARS

age in years	number of living	price of annuity	age in years	number of living	price of annuity
50	362	1086.52	70	175	638.30
51	354	1066.62	71	165	610.83
52	345	1049.17	72	155	582.75
53	336	1031.14	73	145	554.09
54	327	1012.49	74	135	524.89
55	319	989.78	75	125	495.22
56	310	969.44	76	114	470.16
57	301	948.35	77	104	441.13
58	291	929.98	78	93	417.98
59	282	907.64	79	82	397.75
60	273	884.44	80	72	375.64
61	264	860.32	81	63	350.77
62	254	838.90	82	54	329.69
63	245	813.21	83	46	306.38
64	235	790.20	84	39	279.44
65	225	766.59	85	32	257.60
66	215	742.30	86	26	232.90
67	205	717.43	87	20	217.91
68	195	691.93	88	15	205.07
69	185	665.14	89	11	193.62
70	175	638.30	90	8	179.54
m	(m)	$\overline{m}$	m	(m)	$\overline{m}$

11. Mr. Kerseboom continued his mortality table only to 95 years, and for this reason I didn't judge it appropriate to continue this one beyond 90 years, since probably nobody at this age will still have eyes for life annuities. In any event, in nearly all these plans these elderly are placed in the same class as that of the 60 year-olds or the 70 year-olds, notwithstanding that it would be very unfair, if one wanted to ask of a nonagenarian more than a third of the price that a septuagenarian would have to pay, and more than a quarter of that of a sexagenarian. However, if one is curious to see the continuation of my table, here it is:

m	90	91	92	93	94
(m)	8	6	4	3	2
$\overline{m}$	179.54	151.35	138.38	93.73	47.62

But I would not want to advise an entrepreneur to get involved with such elderly, at least unless their number were sufficiently large. This is a general rule for all establishments based on probabilities.

12. From the above we will easily conclude how much the entrepreneur must pay in interest to each age, for an arbitrary sum which we have placed at the beginning in his hands. It is not necessary to enter here into the same detail, but it will suffice to indicate for every fifth year the annual percentage yield that the annuitants would be able to demand.

age	yield	age	yield	age	yield
0	$8\,^2/_3$	30	$7\,^1/_4$	60	$11\,^1/_3$
5	$6\,^1/_2$	35	$7\,^1/_2$	65	13
10	$6\,^1/_3$	40	8	70	$15\,^2/_3$
15	$6\,^1/_2$	45	$8\,^1/_2$	75	20
20	$6\,^3/_4$	50	9	80	$25\,^2/_3$
25	7	55	10	85	$38\,^1/_2$
30	$7\,^1/_4$	60	$11\,^1/_3$	90	$55\,^1/_2$

On this basis, the entrepreneur would not have any profit, at least if he were not able to invest his money at more than 5 percent.

13. So, if a State needed money, and it could find at five percent interest as much as required, it would certainly be bad for it to establish life annuities as I have just determined, based on this same five percent, because considering the trouble such an establishment necessarily entails, it would always be better to borrow the required sum at five percent, which could then be repaid as circumstances allow, whereas the life annuities would remain a liability for a very long time. Indeed, it would be necessary to raise the prices of the annuities beyond what I just set them at, in order to gain some benefit. But then there would be much to fear of not finding any customers, unless they were the elderly beyond 60 years, whom the interest rates of 10 percent and more might dazzle.

14. But to wish to establish life annuities more attractive to the annuitants would not be a very suitable way to relieve a State, since this would amount to the same thing as taking on debts at six percent, and more, when one

could borrow at five percent, without being subject to the bother which the life annuities demand. Indeed, if a State wanted to establish the life annuities put forward here and calculated on the basis of five percent, it could only regard this undertaking as a borrowing taken at six percent, due to the many arrangements that would be required. So, I see hardly a case where the establishment of life annuities would seem to be advantageous to a State, as long as one could borrow money at five percent and maybe less.

But we can imagine another kind of annuity, which will be perhaps more appealing, even though it is also based on five percent. I want to talk about annuities which must start paying only after 10, or even 20, years; and we can easily understand that the price of such an annuity will be very modest, and therefore capable of attracting the public.

15. So let us think of this question also in general, and try to find how much a man aged m years must pay at present, to acquire for himself an annual income r, which starts to be payable to him only after n years, so that from then on he may enjoy it regularly until his death. Let x be the current price of this income, and we will find, as above

$$x = \frac{r}{(m)}\left[\frac{(m+n)}{\lambda^n} + \frac{(m+n+1)}{\lambda^{n+1}} + \frac{(m+n+2)}{\lambda^{n+2}} + \text{etc.}\right]$$

Now, by the calculation of ordinary annuities explained previously, we have

$$\overline{m+n-1} = \frac{1}{(m+n-1)}\left[\frac{(m+n)}{\lambda} + \frac{(m+n+1)}{\lambda^2} + \frac{(m+n+2)}{\lambda^3} + \text{ etc.}\right]$$

from which we conclude

$$x = \frac{r}{(m)} \cdot \frac{(m+n-1)}{\lambda^{n-1}} \overline{m+n-1} = \frac{r}{\lambda^{n-1}} \cdot \frac{(m+n-1)}{(m)} \overline{m+n-1}$$

where $\overline{m+n-1}\, r$ expresses the current price of the ordinary annuity for a man aged $m+n-1$ years.

16. So, if we ask the current price of an annual income of 100 écus, which will start to be payable only at the end of ten years, for a man aged m years, we take from the table developed in §10 the price of the ordinary annuity corresponding to age $m+9$ years, and multiply that by $\left[\frac{20}{21}\right]^9 \frac{(m+9)}{(m)}$, to get the value of x that we seek. From this, I have calculated the following tables for every fifth year.

TABLE: PRICE OF A LIFE ANNUITY OF 100 ÉCUS
NOT PAYABLE UNTIL TEN YEARS PASS

age in years	price of annuity	age in years	price of annuity
0	649.75	40	610.40
5	877.77	45	533.55
10	874.50	50	455.78
15	833.95	55	375.25
20	787.43	60	290.55
25	745.72	65	203.11
30	717.05	70	120.14
35	671.73	75	56.20
40	610.40	80	19.07

TABLE: PRICE OF A LIFE ANNUITY OF 100 ÉCUS
NOT PAYABLE UNTIL TWENTY YEARS PASS

age in years	price of annuity	age in years	price of annuity
0	343.06	35	272.96
5	453.36	40	234.47
10	441.81	45	183.72
15	413.60	50	134.52
20	382.17	55	87.91
25	349.63	60	47.28
30	319.30	65	19.17
35	272.96	70	4.82

17. It could be that such a project of life annuities would succeed well, notwithstanding that they would be fixed at the rate of five percent. It seems it would always be advantageous to secure for a newborn infant, for a price of 343 or even 350 écus, a fixed income of 100 écus per year, even though it starts to be payable only when the child attains the age of 20 years. And if one wanted to employ the sum of 3500 écus for this, it would be a fine establishment to enjoy, starting at the age of 20 years, a fixed pension of 1000 écus. However, it is still doubtful that you will find many parents who would want to make such a big sacrifice for the welfare of their children. Perhaps you will find more men aged 60 years who would not hesitate to pay 3000 écus in advance to be assured of the right to a fixed pension of 1000 écus per year as soon as they have passed their 70th year.