RESEARCHES ON THE CURVATURE OF SURFACES[1]

In order to know the curvature of curved lines, the determination of the radius of the osculating circle offers the proper method, which for each point on the curve provides us with a circle whose curvature is the same. But when one asks for the curvature of a surface, the question is rather equivocal, and not at all subject to a definitive answer, as in the previous case. It is only spherical surfaces for which one can measure the curvature, considering that the curvature of a sphere is the same as that of its great circles, and its radius can be considered as the proper measure of curvature. But for other surfaces, one would not even know how to compare the curvature of the surface with that of a sphere, as one can always compare the curvature of a curved line with that of a circle. The reason for this is evident, since through each point of a surface, there are infinitely many different curves. One only need consider the surface of a cylinder, where along directions parallel to the axis, there is no curvature, while cross sections perpendicular to the axis, which are circles, have the same curvature, and all other sections taken obliquely to the axis yield particular values of the curvature. Similarly for all other surfaces, where it can even happen that in one direction the curvature might be convex and in another concave, as for surfaces which resemble a saddle.

Thus the question about curvature of surfaces is not amenable to a simple answer, but requires at once infinitely many determinations, because as soon as one is able to draw an infinitude of directions through each point, the curvature must be known along each direction before one is able to form an accurate idea about the curvature of the surface. Now through each point of the surface there are infinitely many cross sections, not only with respect to all the directions on the surface, but also with respect to different inclinations of the sections. But for the matter at hand, of all these infinitely many sections, it suffices to consider only those which are perpendicular to the surface, the number of which is still infinite. To this end one only has to draw a line perpendicular to the surface and all sections which pass through this line are also perpendicular to the surface. Then for each of these sections it remains to find the curvature, or the radius of the osculating circle, and the collection of all these radii will give us an accurate measure of the curvature of the surface at a given point. It must be observed that each of these radii falls along the same perpendicular direction to the

[1]Translated from the original French, which was the scholarly language of the Berlin *académie des sciences*.

surface, and that the elementary arcs of all these sections are part of the shortest curves which can be drawn on the surface.

To render this work more general, I will begin by determining the radius of curvature of an arbitrary planar section which cuts the surface. Then I will apply this solution to sections which are perpendicular to the surface at an arbitrary point, and finally I will compare the radii of curvature for these sections[2] with respect to their mutual inclination, which will allow us to establish a good idea for the curvature of surfaces. All this work reduces then to the following problems.

PROBLEM 1

1. A surface whose nature is known is cut by an arbitrary plane. Determine the curvature of the section which is formed.

SOLUTION

FIGURE 1. Plane of the Cross Section

When one regards (see Figure 1) the surface with respect to a fixed plane, and from an arbitrary point Z on the surface drops the perpendicular ZY, and from Y drops the perpendicular YX to an axis AC, then the three coordinates $AX = x$, $XY = y$, and $YZ = z$ are given.[3] Since the nature of the surface is known, the quantity z will be equal to a certain function of the two others x and y. Suppose then that by differentiation one obtains $dz = p\,dx + q\,dy$, where

$$p = \left(\frac{dz}{dx}\right) \quad \text{and} \quad q = \left(\frac{dz}{dy}\right).$$

Let the section which cuts the surface pass through the point Z, and let the intersection of the plane of this section and our fixed plane be

[2] the perpendicular sections

[3] See Figure 2 for a modern sketch of the cross section.

Figure 2. Modern Sketch of the Cross Section

the line EF. Let

$$z = \alpha y - \beta x + \gamma,$$

be the equation which determines the plane of the section, and letting $z = 0$, the equation $y = \frac{\beta x - \gamma}{\alpha}$ will give EF, from which we obtain

$$AE = \frac{\gamma}{\beta} \quad \text{and the tangent of angle } CEF = \frac{\beta}{\alpha}.$$

Thus

$$\text{the sine } = \frac{\beta}{\sqrt{\alpha\alpha + \beta\beta}} \quad \text{and the cosine } = \frac{\alpha}{\sqrt{\alpha\alpha + \beta\beta}}.$$

From this and equating the two values of dz, we will have an equation for the section

$$\alpha\, dy - \beta\, dx = p\, dx + q\, dy$$

or just as well

$$\frac{dy}{dx} = \frac{\beta + p}{\alpha - q}.$$

But to reduce this equation to rectangular coordinates, let us draw from Y the perpendicular YT to EF, and the straight line ZT will also be perpendicular to EF. Now, since $EX = x - \frac{\gamma}{\beta}$, we will have

$$ET = \frac{\alpha x + \beta y}{\sqrt{\alpha\alpha + \beta\beta}} - \frac{\alpha\gamma}{\beta\sqrt{\alpha\alpha + \beta\beta}}$$

and

$$TY = \frac{\alpha y - \beta x}{\sqrt{\alpha\alpha + \beta\beta}} + \frac{\gamma}{\sqrt{\alpha\alpha + \beta\beta}} = \frac{z}{\sqrt{\alpha\alpha + \beta\beta}},$$

and finally

$$TZ = \frac{z\sqrt{1 + \alpha\alpha + \beta\beta}}{\sqrt{\alpha\alpha + \beta\beta}} = \frac{(\alpha y - \beta x + \gamma)\sqrt{1 + \alpha\alpha + \beta\beta}}{\sqrt{\alpha\alpha + \beta\beta}}.$$

Then setting

$$ET = \frac{\alpha x + \beta y}{\sqrt{\alpha\alpha + \beta\beta}} - \frac{\alpha\gamma}{\beta\sqrt{\alpha\alpha + \beta\beta}} = t$$

and

$$TZ = \frac{(\alpha y - \beta x + \gamma)\sqrt{\alpha\alpha + \beta\beta + 1}}{\sqrt{\alpha\alpha + \beta\beta}} = u,$$

we will be able to consider the t and u lines as orthogonal coordinates for the section in question. Thus, if we set $du = s\,dt$, the radius of the osculating circle for the section at the point Z will be

$$= -\frac{dt\,(1 + ss)^{\frac{3}{2}}}{ds}$$

provided that it is turning towards the base EF. Presently it is only a matter of reducing this expression to x and y coordinates. To this end, since

$$dt = \frac{\alpha\,dx + \beta\,dy}{\sqrt{\alpha\alpha + \beta\beta}} \quad \text{and} \quad du = \frac{\alpha\,dy - \beta\,dx}{\sqrt{\alpha\alpha + \beta\beta}}\sqrt{1 + \alpha\alpha + \beta\beta},$$

because of $\frac{dy}{dx} = \frac{\beta + p}{\alpha - q}$, we then obtain

$$s = \frac{du}{dt} = \frac{\alpha p + \beta q}{\alpha\alpha + \beta\beta - \alpha q + \beta p}\sqrt{1 + \alpha\alpha + \beta\beta}.$$

Thus

$$1 + ss = \frac{(\alpha\alpha + \beta\beta)\big(\alpha\alpha + \beta\beta - 2\alpha q + 2\beta p + (\alpha p + \beta q)^2 + pp + qq\big)}{(\alpha\alpha + \beta\beta - \alpha q + \beta p)^2}.$$

Thus, for the differential of s, we will have

$$ds = \frac{(\alpha\alpha + \beta\beta)(\alpha\,dp + \beta\,dq - q\,dp + p\,dq)\sqrt{1 + \alpha\alpha + \beta\beta}}{(\alpha\alpha + \beta\beta - \alpha q + \beta p)^2}.$$

Let us now notice that

$$dp = dx\Big(\frac{dp}{dx}\Big) + dy\Big(\frac{dp}{dy}\Big) \quad \text{and} \quad dq = dx\Big(\frac{dq}{dx}\Big) + dy\Big(\frac{dq}{dy}\Big),$$

from which we conclude

$$\frac{dp}{dt} = \frac{(\alpha - q)\left(\frac{dp}{dx}\right) + (\beta + p)\left(\frac{dp}{dy}\right)}{\alpha\alpha + \beta\beta - \alpha q + \beta p}\sqrt{\alpha\alpha + \beta\beta}$$

and

$$\frac{dq}{dt} = \frac{(\alpha - q)\left(\frac{dq}{dx}\right) + (\beta + p)\left(\frac{dq}{dy}\right)}{\alpha\alpha + \beta\beta - \alpha q + \beta p}\sqrt{\alpha\alpha + \beta\beta},$$

and finally[4]

$$\frac{ds}{dt} = \frac{ABC}{D}$$
$$A = (\alpha\alpha + \beta\beta)^{\frac{3}{2}}, \quad C = \sqrt{1 + \alpha\alpha + \beta\beta}$$
$$B = \left[(\alpha - q)^2\left(\frac{dp}{dx}\right) + (\beta + p)^2\left(\frac{dq}{dy}\right) + 2(\alpha - q)(\beta + p)\left(\frac{dp}{dy}\right)\right]$$
$$D = (\alpha\alpha + \beta\beta - \alpha q + \beta p)^3,$$

since $\left(\frac{dq}{dx}\right) = \left(\frac{dp}{dy}\right)$ as is otherwise known. As a consequence, the osculatory radius for the section at the point Z will be expressed in the form[5]

$$-\frac{\left(\alpha\alpha + \beta\beta - 2\alpha q + 2\beta p + (\alpha p + \beta q)^2 + pp + qq\right)^{\frac{3}{2}}}{BC}.$$

This is then the veritable expression for the osculatory radius of an arbitrary section which cuts the given surface.

PROBLEM 2

5. If (see Figure 3) the plane of the section is perpendicular to the surface at the point Z, determine the osculatory radius of this section at the same point Z.

SOLUTION

To this end one only need draw from the point Z the line ZP which is perpendicular to the surface, and require that the plane of the section pass through this line ZP. Let us consider two other sections through the point Z, both perpendicular to the fixed plane, the intersection of one[6] being the line YM, parallel to the axis AL, the intersection of the other being YN, perpendicular to the first. For the first of these two

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[4] The original formula is split for legibility.
[5] Using the simplifying notation above.
[6] The intersection of the plane of the section with the fixed plane.

FIGURE 3. A Perpendicular Cross Section

sections, the quantity $XY = y$ should be taken to be constant, and the equation $dz = p\,dx$ will give the subnormal

$$YM = \frac{z\,dz}{dx} = pz.$$

Now, for the other section, taking x constant, the equation $dz = q\,dy$ gives the subnormal

$$YN = \frac{z\,dz}{dy} = qz.$$

Now drawing through the points M and N the lines MP and NP parallel to the coordinates XY and AX, which intersect at the point P, the straight line ZP will be perpendicular to each of our two sections[7], and finally ZP will also be perpendicular to the surface at the point[8] Z. Thus the sections under consideration in the problem are required to pass through this line ZP, which will give at the same time the position of the osculatory radius that we seek. Hence we only need pass the line of inersection, EF, through the point P. Let ζ be the angle PEL, which intersects the axis AL so that $\beta = \alpha \tan\zeta$; and since the perpendicular drawn from N to EP will be

$$= NP \sin\zeta = pz \sin\zeta,$$

we conclude from this that the perpendicular

$$YT = z(p \sin\zeta - q \cos\zeta),$$

and finally

$$\text{the tangent of angle } YTZ = \frac{1}{p \sin\zeta - q \cos\zeta},$$

[7]The segment ZP is perpendicular to the two curves on the surface formed by the intersection of the surface with the planes ZYM and ZYN.

[8]A derivation of the normal vector ZP can be found in exercise 9

which will be the value of $\tan\vartheta$. From this we have

$$\alpha = \frac{\cos\zeta}{p\,\sin\zeta - q\,\cos\zeta} \quad \text{and} \quad \beta = \frac{\sin\zeta}{p\,\sin\zeta - q\,\cos\zeta};$$

thus, since $\beta : \alpha = \sin\zeta : \cos\zeta$, both of these give

$$\beta p - \alpha q = 1.$$

Now, substituting these values for α and β in the expression found for the osculatory radius, the numerator will become

$$\frac{(1+pp+qq)^{\frac{3}{2}}\left(1+(p\,\sin\zeta - q\,\cos\zeta)^2\right)^{\frac{3}{2}}}{(p\,\sin\zeta - q\,\cos\zeta)^3},$$

and for the denominator

$$\sqrt{1+\alpha\alpha+\beta\beta} = \frac{\sqrt{1+(p\,\sin\zeta - q\cos\zeta)^2}}{p\sin\zeta - q\cos\zeta}$$

and for the other factor[9]:

$$\frac{A+B+C}{(p\,\sin\zeta - q\cos\zeta)^2}$$

$$A = \left((1+qq)\cos\zeta - pq\,\sin\zeta\right)^2\left(\frac{dp}{dx}\right)$$

$$B = \left((1+pp)\sin\zeta - pq\,\cos\zeta\right)^2\left(\frac{dq}{dy}\right)$$

$$C = 2\left((1+qq)\cos\zeta - pq\,\sin\zeta\right)\left((1+pp)\sin\zeta - pq\,\cos\zeta\right)\left(\frac{dp}{dy}\right)$$

and finally the osculatory radius at the point Z will be[10]

$$\frac{-\left(1+(p\,\sin\zeta - q\,\cos\zeta)^2\right)(1+pp+qq)^{\frac{3}{2}}}{A+B+C}.$$

CONCLUSION

23. After these reported examples, to clarify the preceding work, one can draw the following conclusion to judge the curvature of any surface in general. Let us consider the plane which touches the surface at a point where the curvature is sought. Let the plane of reference be this plane (see Figure 4), which touches the surface at the point Z and all sections for which I have just defined the osculatory radii will then be perpendicular to this plane and cut the plane in some straight line EF or MN, which in turn passes through the point Z. Let all possible sections be represented by some straight line drawn through the point

[9]The original equation is split for legibility.

[10]Using the symbols introduced above.

FIGURE 4. The Touching Plane

Z on the touching plane[11]. Let EF be the section above which I called the principal one, and consider another arbitrary section MN which forms an angle of $EZM = \varphi$ with the principal. Since the osculatory radius of this section MN was found in paragraph 10

$$-\frac{u^3(pp+qq)}{A+B+C}$$

$$A = (p\cos\varphi - qu\sin\varphi)^2\Big(\frac{dp}{dx}\Big)$$

$$B = (q\cos\varphi + pu\sin\varphi)^2\Big(\frac{dq}{dy}\Big)$$

$$C = 2(p\cos\varphi - qu\sin\varphi)(q\cos\varphi + pu\sin\varphi)\Big(\frac{dp}{dy}\Big),$$

the denominator of this expression can be expressed in the form:

$$+\cos\varphi^2\cdot\Big(pp\Big(\frac{dp}{dx}\Big)+qq\Big(\frac{dq}{dy}\Big)+2pq\Big(\frac{dp}{dy}\Big)\Big)$$
$$+2u\sin\varphi\cos\varphi\cdot\Big(-pq\Big(\frac{dp}{dx}\Big)+pq\Big(\frac{dq}{dy}\Big)+(pp-qq)\Big(\frac{dp}{dy}\Big)\Big)$$
$$+uu\sin\varphi^2\cdot\Big(qq\Big(\frac{dp}{dx}\Big)+pp\Big(\frac{dq}{dy}\Big)-2pq\Big(\frac{dp}{dy}\Big)\Big),$$

where it should be noted that the quantities u, p, q, along with the formulae $(\frac{dp}{dx})$, $(\frac{dq}{dy})$ and $(\frac{dp}{dy})$ depend only on the choice of the point Z, and consequently are common to all sections, the variability of which is comprised by the single angle φ. Thus in general the expression for any osculatory radius of any surface whatever should always have the

[11] A clear reference to the tangent plane.

form
$$\frac{V}{P\,\cos\varphi^2 + Q\,\sin\varphi^2 + 2R\,\sin\varphi\,\cos\varphi},$$
which, since
$$\cos\varphi^2 = \tfrac{1}{2}+\tfrac{1}{2}\,\cos 2\varphi,\;\; \sin\varphi^2 = \tfrac{1}{2}-\tfrac{1}{2}\,\cos 2\varphi \;\text{ and }\; \sin\varphi\,\cos\varphi = \tfrac{1}{2}\sin 2\varphi,$$
reduces to this
$$\frac{1}{L + M\,\cos 2\varphi + N\,\sin 2\varphi},$$
and furnishes me the following reflections.

III. Reflection

26. From our general formula we can find the sections which correspond to the largest and the smallest osculatory radius. The method of the largest and smallest provides us with the equation
$$-2M\,\sin 2\varphi + 2N\,\cos 2\varphi = 0,$$
from which we have $\tan 2\varphi = \frac{N}{M}$. Thus, if ζ is the angle whose tangent is $= \frac{N}{M}$, the angle $180^\circ + \zeta$ works just as well, and from this we find the two values of the angle φ:
$$\text{I. }\; \varphi = \tfrac{1}{2}\zeta \quad\text{and}\quad \text{II. }\; \varphi = 90^\circ + \tfrac{1}{2}\zeta,$$
of which one corresponds to the largest osculatory radius and the other the smallest. From this one has the important consequence that whatever the curvature of an element of the surface, these two sections, one having the largest curvature and the other the smallest, are always perpendicular to each other.

VI. Reflection

29. Let the largest osculatory radius $= f$, which corresponds to the section EF, the smallest $= g$, for the section GH, perpendicular to the preceding. This established, for any other section MN inclined to the first EF at an angle of $EZM = \varphi$, the osculatory radius $= r$, will be solely determined by two previous ones and the angle φ in the following manner. The general formula
$$r = \frac{1}{L + M\,\cos 2\varphi},$$
gives, setting $\varphi = 0$,
$$L + M = \frac{1}{f},$$
and setting $\varphi = 90^\circ$, there results
$$L - M = \frac{1}{g},$$

from which one concludes

$$L = \frac{f+g}{2fg} \quad \text{and} \quad M = -\,\frac{f-g}{2fg}\,,$$

and finally we will have:

$$r = \frac{2fg}{f+g-(f-g)\cos 2\varphi}.$$