E258: "Principia Motus Fluidorum". Presented to Berlin Academy 1752, published 1761 in *Novi commentarii academiae scientiarum Petropolitanae. Tom. VI ad Annum MDCCLVI et MDCCLVII*, pp. 271-311, +1 page of figures. See also E225-227.

Principles of the Motion of Fluids

Part One

1. Fluid bodies differ from solids principally in that in general their particles are not bound to each other, so these different particles can be subject to very different motions. A motion which is imparted to one fluid particle is not so determined by the motion of other particles that it could not proceed in its own way. For solid bodies it is quite a different situation; if they were inflexible, their figures would undergo no change, and the individual parts would keep at a constant distance from each other; so that the motion of all parts would be known, once that of two or three points were given. Even then, the motions of these two or three points are not completely arbitrary, since they must keep the same distance from each other.

2. If the solid bodies were flexible, however, the motion of individual particles is less fixed; because of flexure, the distance or the relative location

of diverse particles is subject to change. Even then, the manner of bending obeys a certain rule, which diverse particles of bodies of this type must follow in their motion, to wit that the parts that are subject to the bending will not tear apart, or pierce each other; which indeed will be ruled out for all such bodies by a common character of impenetrability.

3. [Fluids have an infinitely large number of conceivable flows.] In fluid bodies, however, whose particles are not joined to each other by any bond, the motion of diverse particles are much less restricted, and from the motion of a number of particles the motion of the others cannot be determined. For if the motion of even a hundred particles were known, it is clear that the motion which the remaining particles could take is infinitely variable. From this it can be concluded that the motion of each particle of the fluid clearly does not depend on the the motion of others, unless it were bound with them in such a way that it must follow with them.

4. At the same time, it cannot be that the motion of all the particles of the fluid is bound in *no* way by any law; nor can *any* conceivable motion of a single particle be allowed. For since the particles are impenetrable, it is clear that no motion can take place where some particles go through others, or that they penetrate each other. An infinite number of such motions should be excluded, and only the remaining are to be considered, and clearly the task is to determine by which property these remaining possibilities can be

distinguished from the others.

5. [Circumscribe the types of kinematic flows to be considered, from which one can be picked out by dynamics.] Before we can fix on the appropriate motion when a fluid is acted on by a force, we must delimit those motions which could take place in this fluid. I shall call them *possible* motions, to distinguish them from those *impossible* motions which could not take place. To this end we must decide the character appropriate to the possible motions, separating them from the impossible ones; when this is done we need to determine in any situation which one of the possible motions actually should be chosen. At that point we must look at the *forces* to which the fluid is subjected, and then the motion compatible with these forces can be determined by the principles of mechanics.

6. [Restriction to incompressible fluids.] I have decided therefore to look at the character of motions that are possible for a fluid that cannot be penetrated. I shall posit moreover that the fluid cannot be compressed into a smaller space, and its continuity cannot be interrupted. I stipulate without qualification that, in the course of the motion within the fluid, no empty space is left by the fluid, but it always maintains continuity in this motion. After we have theory suitable for fluids of this nature, it will not be difficult to extend it further to fluids whose density is variable, and which do not necessarily require continuity.

7. If we consider any portion of a fluid of this type, the motion by which its individual particles are moved should be so constructed that at each time they fill the same amount of space. If this happens for individual particles, the portion as a whole is prevented from expanding into a greater amount of space, or being compressed into a smaller space; and it is just motion of this type, in which the fluid is considered incapable of either expansion or compression, that we shall take as *possible motions*. What we have said here about an arbitrary portion of fluid, is to be understood as applying to each element of the fluid, so that the volume of each element of the fluid ought to remain unchanged.

8. With this condition satisfied, we are to consider what the motion will be at the individual points of the fluid. For an arbitrary element of the fluid, we have to find out the instantaneous translation of its bounding surfaces, so as to determine the new portion of space in which it will be contained after a very small time period. The new portion of space must be equal in size to the old portion which the element had occupied. This equating of size will fully characterize what can be said about the motion. For if the individual elements occupy equal spaces at each time, no compression or expansion will arise in the fluid; so the motion will be compatible with our condition, and we must allow it as a possible motion.

9. [Resolution into two or three directions.] When we take into account not only the speed but also the direction of the motion at each point of the fluid, it becomes useful to resolve that motion into fixed directions. This can be done into two or three directions, the first if the motion of individual points remain planar; otherwise the motion should be resolved along three fixed axes. Since this latter case is more difficult than the former, it is convenient to start with the possible motions in the first case, and when that is worked through we can more easily solve the latter case.

10. [Two-dimensional flow.] Therefore I shall attribute to the fluid flow two such directions, so that the individual particles and their motions lie in their plane.

Fig: 1.
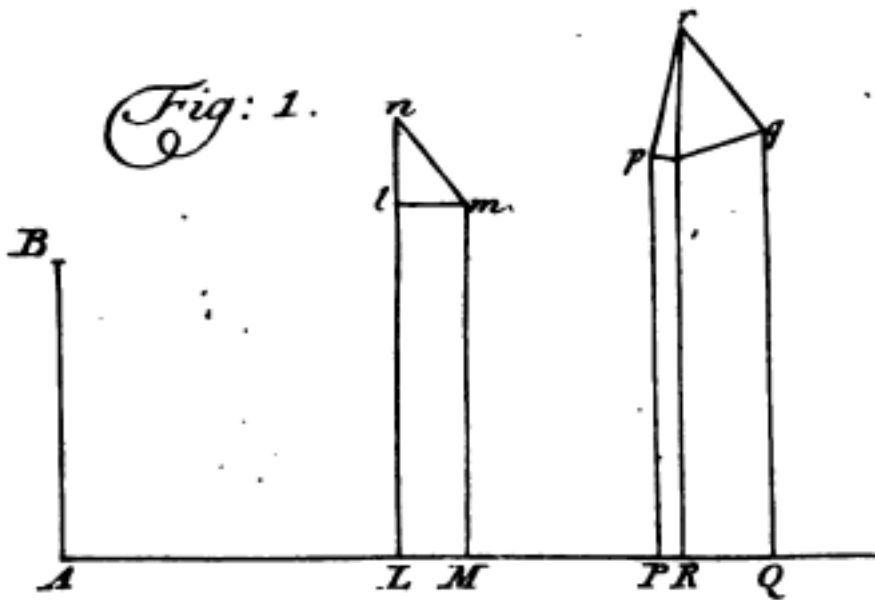
Let this plane be represented by the plane of Figure 1[1], and consider any point l of the fluid, whose location is referred to the orthogonal coordinates $AL = x$ and $Ll = y$. Its motion when resolved along the same two directions displays a velocity along AL, namely $lm = u$, and along the other axis AB, namely $ln = v$: thus the actual speed of this point is $= \sqrt{uu + vv}$ [$= \sqrt{u^2 + v^2}$], and its direction will be at an angle inclined to the axis AL, whose tangent is $= v/u$.

11. Since we are proposing to develop the state of the motion that applies to each individual point, the velocities u and v depend only on the location of the point l, and they are to viewed as functions of the coordinates x and y. We can therefore write a differential relation

$$
\begin{aligned}
du &= Ldx + ldy, \\
dv &= Mdx + mdy,
\end{aligned}
$$

and since these are to be complete differentials we must have[2] $dL/dy = dl/dx$ and $dM/dy = dm/dx$. It is to be noted in an expression like dL/dy that the differential dL of L is to be taken only from the variability of y, and in a similar way in the expression dl/dx the differential dl is such that would arise if only x were to vary.

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[1]The Appendix gives the complete set of graphics.
[2]Euler does not use the partial derivative notation $\partial/\partial x$.

12. [Meaning of differentials.] Care should be taken that in "fractions" of the type dL/dy, dl/dx, dM/dy, dm/dx the numerators dL, dl, dM, dm not be thought to denote complete differentials of the functions L, l, M, m; rather they always denote how much of those differentials will arise from variability of just that coordinate (and that only) that appears in the denominator, so that the expressions always represent finite and determinate quantities. A similar meaning is to be understood for $L = du/dx$, $l = du/dy$, $M = dv/dx$, $m = dv/dy$; this notation was first used by the illustrious FONTAINE who has furnished us with such a worthy compendium of calculus, and I shall adhere to it also.

13. Thus, since we have $du = Ldx + ldy$ and $dv = Mdx + mdy$, we may also infer the two velocities at any other point an infinitely small distance from the point l; for if such a point is at a distance from l along the axis $AL = dx$, and along the axis $AB = dy$, then the velocity of this point along the axis AL will be $= u + Ldx + ldy$; and the velocity along the other axis AB will be $= v + Mdx + mdy$. Therefore in an infinitely small time interval dt this point is moved in the direction of axis AL by the amount $= dt(u + Ldx + ldy)$ and in the direction of the other axis AB by the amount $= dt(v + mdx + mdy)$.

14. Having noted this, let us consider a triangular element of water lmn, and we seek the location to which it is transferred, by the motion intrinsic to that element. Let the side lm of this triangular element be parallel to the axis

AL, the side ln parallel to the axis AB; and take $lm = dx$ and $ln = dy$; so that the point m has coordinates $x + dx$ and y, and point n has coordinates x and $y + dy$. It is clear that the differentials dx and dy could be either positive or negative, since we have not fixed them; and also that the whole mass of the fluid can be mentally divided up into elements like this, so that what we prescribed for one will apply equally well to all.

15. To make clear how the element lmn is transferred in the small time interval dt by its intrinsic motion, we seek the points p, q and r, into which its angles [vertices] l, m and n are transferred in the time dt. Since we shall have velocities

point:	l	m	n
along AL	u	$u + Ldx$	$u + ldy$
along AB	v	$v + Mdx$	$v + mdy$

point l will come to p, that is:

$$
\begin{aligned}
AP - AL &= udt, \\
Pp - Ll &= vdt.
\end{aligned}
$$

Point m will come to q, that is:

$$
\begin{aligned}
AQ - AM &= (u + Ldx)dt, \\
Qq - Mm &= (v + Mdx)dt.
\end{aligned}
$$

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But point p will be brought to r, that is:

$$AR - AL = (u + ldy)dt,$$
$$Rr - Ln = (v + mdy)dt.$$

16. Since points l, m and n are brought to points p, q and r in the small time interval dt, the triangle lmn is to be thought as going to the location indicated by triangle pqr, joined by the line segments pq, pr and qr. Since the triangle lmn was set to be infinitely small, after the translation over the little time dt it will still retain a triangular figure pqr, that is rectilinear. Since the element lmn ought not to be extended into a greater area, nor to be compressed into a smaller one, its motion must be so composed that the area of triangle pqr equals the area of triangle lmn.

17. [Paragraphs **17-20** will establish that $\triangledown \cdot \mathbf{u} = 0$, without the benefit of the divergence theorem.] But the triangle lmn, if it is a right angle at l, has an area $= \frac{1}{2}dxdy$, and the area of triangle pqr must also be equal to this. To find that area, we must consider the coordinates of the points p, q, r, which are:

	p	q	r
x	$AP=x+udt$	$AQ=x+dx+(u+Ldx)dt$	$AR=x+(u+ldy)dt$
y	$Pp=y+vdt$	$Qq=y+(v+Mdx)dt$	$Rr=y+dy+(v+mdy)dt$

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Then the area of the triangle pqr is found from the areas of the following trapezoids, thus[3]:

$$\triangle pqr = PprR + RrqQ - PpqQ.$$

Since however these trapezoids have two sides parallel and perpendicular to the base AQ, their areas are easily determined.

18. For we have, as in geometry,

$$
\begin{aligned}
PprR &= \frac{1}{2}PR(Pp + Rr), \\
RrqQ &= \frac{1}{2}RQ(Rr + Qq), \\
PpqQ &= \frac{1}{2}PQ(Pp + Qq).
\end{aligned}
$$

Collecting these together, we find:

$$\triangle pqr = \frac{1}{2}PQ \cdot Rr - \frac{1}{2}RQ \cdot Pp - \frac{1}{2}PR \cdot Qq.$$

$\left[\text{Euler defines new quantities } Q,\ R,\ q,\ r.\right]$

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[3]See translator's comment to this paragraph at the end of this document.

For the sake of brevity, put

$$
\begin{aligned}
AQ &= AP + Q \\
AR &= AP + R \\
Qq &= Pp + q \\
Rr &= Pp + r
\end{aligned}
$$

so that

$$PQ = Q, \ PR = R, \ RQ = Q - R.$$

Then $\triangle pqr = \frac{1}{2}Q(Pp+r) - \frac{1}{2}(Q-R)Pp - \frac{1}{2}R(Pp+q)$ or $\triangle pqr = \frac{1}{2}Qr - \frac{1}{2}Rq$.

19.

But from the coordinate values shown above [paragraph **17**]

$$
\begin{aligned}
Q &= dx + Ldxdt; & q &= 0 + Mdxdt; \\
R &= 0 + ldydt; & r &= dy + mdydt.
\end{aligned}
$$

After substitution, the area of the triangle becomes

$$\triangle pqr = \frac{1}{2}dxdy(1 + Ldt)(1 + mdt) - \frac{1}{2}Mldxdydt^2,$$

or

$$\triangle pqr = \frac{1}{2}dxdy(1 + Ldt + mdt + Lmdt^2 - Mldt^2)$$

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and since this should be equal to the area of triangle lmn, which is $= \frac{1}{2}dxdy$, there results this equation:

$$Ldt + mdt \quad +Lmdt^2 - Mldt^2 \quad = 0,$$
$$L + m \quad\quad +Lmdt - Mldt \quad = 0.$$

20. Since the terms $Lmdt$ and $Mldt$ are vanishingly small compared to finite L and m, we shall have the equation $L + m = 0$. For this reason, if we are dealing with a *possible* motion, the velocities u and v of any point l must be such that in their differentials

$$du \quad = \quad Ldx + ldy,$$
$$dv \quad = \quad Mdx + mdy$$

we shall have $L + m = 0$. Since $L = du/dx$ and $m = dv/dy$, the velocities u and v, which are conceived as those in point l in the directions of AL and AB, should be thought of as functions of the coordinates x and y such that $du/dx + dv/dy = 0$, and the criterion of possible motions consists in the condition $du/dx + dv/dy = 0$. Iif this condition does not hold, the fluid motion cannot take place.

21. [Three-dimensional flows.] We must proceed in the same way when the fluid motion does *not* resolve into a plane. To investigate the question taken

in its widest sense, we shall take the individual particles of the fluid affecting each other in any sort of motion, with the only proviso being that neither compression nor expansion occur in any part. We seek to determine from this what sort of velocities can occur and give a possible motion; or, what comes to the same thing, we want to exclude from the list of possible motions those which do not observe these conditions, so that the criterion for possible motions can be determined.

22. So we shall consider any point λ of the fluid, whose location we shall represent using three orthogonal axes $AL, \ AB, \ AC$. [Figure 2 below.] Let the three coordinates of the point λ parallel to these axes be $AL = x$, $Ll = y$ and $l\lambda = z$; which will be gotten if from the point λ a perpendicular λl is dropped to the plane determined by the two axes AL and AB. From the point l we then take the perpendicular lL to the axis AL. In this way the location of the point λ can generally be expressed by three coordinates. This will apply at all points of the fluid.
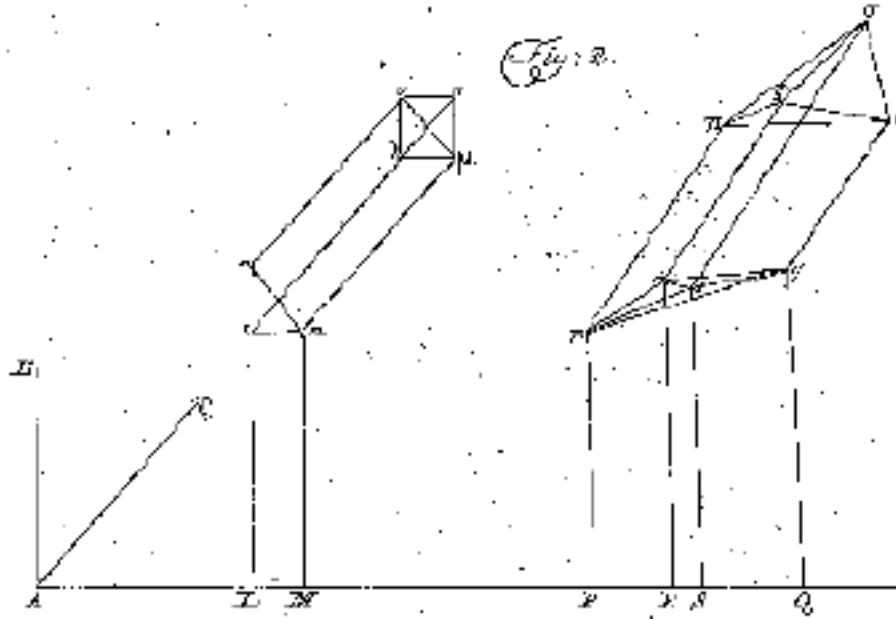
23. The motion of the point λ can be resolved into three directions $\lambda\mu$. $\lambda\nu$ and λo parallel to the axes AL, AB and AC. So let the three directions of the velocity of the point λ be $\lambda\mu = u$, $\lambda\nu = v$, $\lambda o = w$; and since these velocities can vary with the point λ, they can be considered as functions of the three coordinates x, y and z. Taking differentials, we get the forms:

$$
\begin{aligned}
du &= Ldx + ldy + \lambda dz \\
dv &= Mdx + mdy + \mu dz \\
dw &= Ndx + ndy + \nu dz
\end{aligned}
$$

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and the coefficients L, l, λ, M, m, μ, N, n, ν will be functions of the coordinates x, y and z.

24. As these differential forms are complete, it follows, in the same way as the above, that

$$dL/dy = dl/dx; \quad dL/dz = d\lambda/dx; \quad dl/dz = d\lambda/dy$$
$$dM/dy = dm/dx; \quad dM/dz = d\mu/dx; \quad dm/dz = d\mu/dy$$
$$dN/dy = dn/dx; \quad dN/dz = d\nu/dx; \quad dn/dz = d\nu/dy$$

each fraction showing how much the variable in the numerator changes for a given change in the coordinate in the denominator.

25. In an infinitesimal time dt, the point λ can move in all three directions: by the amounts udt in the direction of AL, vdt in the direction of AB, wdt in the direction of AC. Since however the speed of the point λ, which we may call V, arises from the composition of the motions in the three directions, which are orthogonal, we shall have $V = \sqrt{(uu + vv + ww)}$, and the distance traveled in the time dt will be $= Vdt$.

26. Let us now consider any volume element of the fluid, to see where it may advance to in an infinitesimal time dt. Since it does not matter what figure we attribute to it, as long as the whole fluid mass can be divided into such figures, for ease of calculation let the figure be a rectangular triangular

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pyramid, ending at the four solid angles[4] λ, μ, ν, o so that the the coordinates are given by the scheme:

along:	λ	μ	ν	o
AL	x	$x+dx$	x	x
AB	y	y	$y+dy$	y
AC	z	z	z	$z+dz$

and since the base of this pyramid is $\lambda\mu\nu = lmn = \frac{1}{2}dxdy$, and the altitude is $\lambda o = dz$, the volume will be $= \frac{1}{6}dxdydz$.

27. We shall now investigate where these individual vertices λ, μ, ν, o will be carried in the infinitesimal time dt. For each of these, we must consider the three velocities along the three coordinate axes, for these will differ from the three original velocities u, v, w according to the following scheme.[5]

Parallel to	λ	μ	ν	o
AL	u	$u+Ldx$	$u+ldy$	$u+\lambda dz$
AB	v	$v+Mdx$	$v+mdy$	$v+\mu dz$
AC	w	$w+Ndx$	$w+ndy$	$w+\nu dz$

28. If the points λ, μ, ν, o are carried in the infinitesimal time dt to points π, ϕ, ρ, and σ, whose coordinates are given parallel to the three axes, the

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[4]This is a different set of meanings for these symbols.

[5]Remember that Euler uses λ, μ, ν with two sets of meanings.

instantaneous translations along these axes will be: [for $\lambda \to \pi$,]

$$
\begin{aligned}
AP - AL &= udt, \\
Pp - Ll &= vdt, \\
p\pi - l\lambda &= wdt;
\end{aligned}
$$

[for $\mu \to \phi$]

$$
\begin{aligned}
AQ - AM &= (u + Ldx)dt, \\
Qq - Mm &= (v + Mdx)dt, \\
q\phi - m\mu &= (w + Ndx)dt,
\end{aligned}
$$

[for $\nu \to \rho$]

$$
\begin{aligned}
AR - AL &= (u + ldy)dt, \\
Rr - Ln &= (v + mdy)dt, \\
r\rho - n\nu &= (w + ndy)dt,
\end{aligned}
$$

[for $o \to \sigma$]

$$
\begin{aligned}
AS - AL &= (u + \lambda dz)dt, \\
Ss - Ll &= (v + \mu dz)dt, \\
s\sigma - lo &= (w + \nu dz)dt.
\end{aligned}
$$

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Thus we shall have as coordinates for the four points[6]: [for π]

$$AP = x + udt,$$

$$Pp = y + vdt,$$

$$p\pi = z + wdt;$$

[for ϕ]

$$AQ = x + dx + (u + Ldx)dt,$$

$$Qq = y + (v + Mdy)dt,$$

$$q\phi = z + (w + Ndx)dt;$$

[for ρ]

$$AR = x + (u + ldy)dt,$$

$$Rr = y + dy + (v + mdy)dt,$$

$$r\rho = z + (w + ndy)dt;$$

---

[6][The original had a misprint.]

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[for σ]

$$
\begin{aligned}
AS &= x + (u + \lambda dz)dt, \\
Ss &= y + (v + \mu dz)dt, \\
s\sigma &= z + dz + (w + \nu dz)dt.
\end{aligned}
$$

29. When therefore the vertices λ, μ, ν, o of the pyramid are translated into points π, ϕ, ρ, σ in the infinitesimal time dt, these new points are to determine a triangular pyramid such that the volume of both be equal, namely $= \frac{1}{6}dxdydz$. So the task comes down to determining the volume of the pyramid $\pi\phi\rho\sigma$.[7]

It is clear, however, that this pyramid is what we have left if from the volume element $pqr\pi\phi\rho\sigma$ we take away the element $pqr\pi\phi\rho$, for the latter element is a prism sitting perpendicularly[8] on the triangular base pqr, with the upper oblique section $\pi\phi\rho$ cut off.

30. In any truncated prism of this type, the element $pqr\pi\phi\rho$ can be resolved into three other volumes, which are:

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[7]The following paragraphs up through **36** are devoted to this task. If we allow use of the determinantal formula for a parallelpiped, then the equation at the end of **35** follows immediately, which implies that $\nabla \cdot \mathbf{u} = 0$.

[8]The triangle $\pi\phi\rho$ differs from pqr only in the $z-$coordinates.

I.	$pqs\pi\phi\sigma$
II.	$prs\pi\rho\sigma$
III.	$qrs\phi\rho\sigma$

in such a way that we must have

$$\frac{1}{6}dxdydz = pqrs\pi\phi\sigma + prs\pi\rho\sigma + qrs\phi\rho\sigma - pqr\pi\phi\rho.$$

When however a prism of this sort sits perpendicularly on its lower base, with three different altitudes, then its volume is found if the base is multiplied by the sum of the three altitudes, divided by three.

31. Therefore the volume of these truncated prisms will be:

$$
\begin{aligned}
pqs\pi\phi\sigma &= \frac{1}{3}pqs(p\pi + q\phi + s\sigma), \\
prs\pi\rho\sigma &= \frac{1}{3}prs(p\pi + r\rho + s\sigma), \\
qrs\phi\rho\sigma &= \frac{1}{3}qrs(q\phi + r\rho + s\sigma), \\
pqr\pi\phi\rho &= \frac{1}{3}pqr(p\pi + q\phi + r\rho).
\end{aligned}
$$

Since however $pqr = pqs + prs + qrs$, the sum of the first three volumes, minus the last, will be

$$\frac{1}{6}dxdydz = -\frac{1}{3}p\pi.qrs - \frac{1}{3}q\phi.prs - \frac{1}{3}r\rho.pqs + \frac{1}{3}s\sigma.pqr;$$

21

or

$$dxdydz = 2pqr.s\sigma - 2pqs.r\rho - 2prs.q\phi - 2qrs.p\pi.$$

32. It remains to ascertain the bases of these prisms. Before we do this, to reduce calculations we put[9]

$$AQ = AP + Q; \quad Qq = Pp + q; \quad q\phi = p\pi + \phi,$$

$$AR = AP + R; \quad Rr = Pp + r; \quad r\rho = p\pi + \rho,$$

$$AS = AP + S; \quad Ss = Pp + s; \quad s\sigma = p\pi + \sigma$$

and with these substitutions, the terms containing $p\pi$ cancel each other, and we shall have

$$dxdydz = 2pqr.\sigma - 2pqs.\rho - 2prs.\phi$$

and the number of bases to be investigated is reduced by one.

33. Now the triangle pqr will be found, if the trapezoid $PpqQ$ is cut out from the figure $PprqQ$, or from the combined trapezoids $PprR + RrqQ$. Hence

$$\triangle pqr = \frac{1}{2}PR(Pp + Rr) + \frac{1}{2}RQ(Rr + Qq) - \frac{1}{2}PQ(Pp + Qq);$$

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[9]This orrespondd to the abbreviations in paragraph **18**.

but since $PR = R$, $RQ = Q - R$, and $PQ = Q$, we shall have

$$\triangle pqr = \frac{1}{2}R(Pp - Qq) + \frac{1}{2}Q(Rr - Pp) = \frac{1}{2}Qr - \frac{1}{2}Rq.$$

In a similar way we shall have:

$$
\begin{aligned}
\triangle pqs &= \frac{1}{2}PS(Pp + Ss) + \frac{1}{2}SQ(Ss + Qq) - \frac{1}{2}PQ(Pp + Qq), \\
\triangle pqs &= \frac{1}{2}S(Pp + Ss) + \frac{1}{2}(Q - S)(Ss + Qq) - \frac{1}{2}Q(Pp + Qq)
\end{aligned}
$$

whence
$$\triangle pqs = \frac{1}{2}S(Pp - Qq) + Q(Ss - Pp) = \frac{1}{2}Qs - \frac{1}{2}Sq.$$

Next,

$$
\begin{aligned}
\triangle prs &= \frac{1}{2}PR(Pp + Rr) + \frac{1}{2}RS(Rr + Ss) - \frac{1}{2}PS(Pp + Ss), \\
\triangle prs &= \frac{1}{2}R(Pp + Rr) + \frac{1}{2}(S - R)(Rr + Ss) - \frac{1}{2}S(Pp + Ss)
\end{aligned}
$$

whence

$$\triangle prs = \frac{1}{2}R(Pp - Ss) + \frac{1}{2}S(Rr - Pp) = \frac{1}{2}Sr - \frac{1}{2}Rs.$$

34. Substituting in these values, we shall obtain

$$dxdydz = (Qr - Rq)\sigma + (Sq - Qs)\rho + (Rs - Sr)\phi,$$

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so the volume of the pyramid $\pi\phi\rho\sigma$ will be

$$\frac{1}{6}(Qr - Rq)\sigma + \frac{1}{6}(Sq - Qs)\rho + \frac{1}{6}(Rs - Sr)\phi.$$

From the values recorded in paragraph **28** above,

$$Q = dx + Ldxdt \qquad q = Mdxdt \qquad \phi = Ndxdt$$

$$R = ldydt \quad r = dt + mdydt \quad \rho = ndydt$$

$$S = \lambda dzdt \qquad s = \mu dzdt \qquad \sigma = dz + \nu dzdt.$$

35. Since it follows that

$$
\begin{aligned}
Qr - Rq &= dxdy(1 + Ldt + mdt + LMdt^2 - MLdt^2) \\
Sq - Qs &= dxdz(-\mu dt - L\mu dt^2 + M\lambda dt^2) \\
Rs - Sr &= dydz(-\lambda dt - m\lambda dt^2 + l\mu dt^2),
\end{aligned}
$$

therefore we find that the volume of the pyramid $\pi\phi\rho\sigma$ is expressed as

$$\frac{1}{6}dxdydz\left\{1 + \begin{array}{ll} +Ldt & +Lmdt^2 \quad +Lm\nu dt^3 \\ +mdt & -Mldt^2 \quad -Ml\nu dt^3 \\ +\nu dt & +L\nu dt^2 \quad -Ln\mu dt^3 \\ & +m\nu dt^2 \quad +Mn\lambda dt^3 \\ & -n\mu dt^2 \quad -Nm\lambda dt^3 \\ & -N\lambda dt^2 \quad +Nl\mu dt^3 \end{array}\right\}$$

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which should be held equal to the volume of the pyramid $\lambda\mu\nu o = \frac{1}{6}dxdydz$. After dividing by dt, there results the equation

$$
\begin{aligned}
0 = \quad & L + m + \nu \quad + dt(Lm + L\nu + m\nu - Ml - N\lambda - n\mu) \\
& + dt^2(Lm\nu + Mn\lambda + Nl\mu - Ln\mu - Ml\nu - Nl\mu).
\end{aligned}
$$

36. Disregarding the infinitely small terms, we have the equation $L + m + \nu = 0$, in which we have ascertained the condition on the velocities u, v, w to admit the fluid motion as possible. Since $L = du/dx$, $m = dv/dy$ and $\nu = dw/dz$, the condition for a possible motion, when any point with coordinate values x, y, z has corresponding velocities u, v, w, will therefore be:

$$
\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0.
$$

By this condition[10] no part of the fluid will pass into a greater or smaller space, and the continuity of the fluid (and also the density) will be maintained without interruption.

37. This property of the fluid, however, is to be interpreted to hold for all parts of the fluid at each moment in time: that is, at each moment the three velocities u, v, w for all points ought to be such functions of the three coordinates x, y and z, that $du/dx + dv/dy + dw/dz = 0$ will hold, and this character of those functions limits any proposed motion of the individual

---

[10]Euler has actually proved that this is a necessary condition.

points of the fluid. At any other time, however, the motion of those points could be quite different, restricted only by the requirement that the above property still take place. Of course, the same behavior up to the present is assumed.

38. If however we wish to think of time also as variable, so that the motion of a point after an elapsed time t is to be defined, when the position λ is given by the coordinates $AL = x$, $Ll = y$, and $l\lambda = $ [11], it is clear that the three velocities u, v, w depend not only on the coordinates x, y and z, but also on the time t, so that they are functions of these four quantities x, y, z and t, so that[12]

$$du = Ldx + ldy + \lambda dz + \mathcal{L}dt,$$

$$dv = Mdx + mdy + \mu dz + \mathcal{M}dt,$$

$$dw = Ndx + ndy + \nu dz + \mathcal{N}dt.$$

Meanwhile however, we shall always have $L + m + \nu = 0$, because at any instant the time t is to be taken as constant, so that $dt = 0$. Thus it is necessary¸ however the functions u¸ v¸ w may change with time¸ that at each

---

[11]A, L, l, λ represent points is space, while the L, l and λbelow represent differential coefficients.

[12]I have substituted calligraphic for German letters in this translation. It is too difficult for non-Germans to distinguish between the German N and R.

moment there holds the condition

$$\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0.$$

Since this condition assures that any portion of the fluid will be carried in the infinitesimal time dt into an equal volume, and also likewise under the same condition in the following element of time, it follows that this must happen in all following elements of time.

Part Two

39. [Dynamics.] From those possible motions¸ which have satisfied the above condition¸ we shall now investigate the nature of that motion which actually can be sustained in the fluid. That is¸ besides the continuity of the fluid¸ and its density being constant, account must be taken here of the forces affecting the movement of individual elements of the fluid. For whatever the motion of each element¸ if it is not uniform or not pointed in a [common] direction¸ the change of motion ought to conform to the forces applied to this element. No matter how the change of motion is determined by these given forces¸ the above formulas must still be a constraint on this change of motion¸ so new conditions must be found, by which any hitherto possible motion is restricted to the actual motion.

40. [Planar motion.] Let us also set up this investigation in two parts; and¸ first¸ we shall think of all the motion of the fluid as taking place in the same plane˙ Therefore let the position coordinates of any point l be defined as before as $AL = x$, $Ll = y$; and now in the elapsed time t let the two velocities of l in directions parallel to the axes AL and AB be u and v; because we now have to take account the change in time¸ u and v will be functions of x¸ y and t, so that[13]

$$du = Ldx + ldy + \mathcal{L}dt,$$
$$dv = Mdx + mdy + \mathcal{M}dt;$$

and on account of the condition that we found above we must have $L + m = 0$.

41. In the infinitesimal elapsed time dt¸ let therefore the point l be brought to p¸ with a displacement $= udt$ in the direction of the axis AL and with a displacement $= vdt$ in the direction of the other axis AB; to obtain the increase in the velocities u and v of the point l which occur in the infinitesimal time dt, the distance dx ought to be written udt and the distance dy as vdt, so that

$$du = Ludt + lvdt + \mathcal{L}dt,$$
$$dv = Mudt + mvdt + \mathcal{M}dt,$$

---

[13]In modern notation, $L = \partial u \partial t$, and $M = \partial v/\partial t$.

from which the accelerative forces[14] needed to produce these accelerations in the corresponding directions will be:

$$\text{along } AL : \quad 2(Lu + lv + \mathcal{L})$$
$$\text{along } AB : \quad 2(Mu + mv + \mathcal{M})$$

and the forces acting on the particle of water l should be equal to these expressions.

42. Among the forces that act on the particles of water, we have to give first consideration to gravity. Its effect will be null, if the plane of motion is horizontal. If however the plane of motion is at an incline, in the direction of the axis AL, the horizontal axis being AB, the accelerative force due to gravity will take a constant value α in the direction of AL. Moreover, we ought not to neglect friction, because the motion is often there-by appreciably impeded. Although the laws governing friction are not yet satisfactorily established, never-the-less we shall perhaps not err too much from the mark if , in analogy with the friction of solid bodies, we set the friction to be proportional to the pressure of the particles of water acting on each other.

---

[14]Euler defines the measure of force as the acceleration needed to move a unit mass a unit distance in the direction of the force, in a unit time. Since our notion of acceleration will give one-half of a unit distance, Euler's measure of force will require *twice* our acceleration.

43. As the first step then, we must compute the pressure with which the particles of water are acting on each other. For a particle pressed all around by adjacent particles, to the extent that the pressure in some direction is not balanced, just so much will the motion of the particle be affected. [15] That is to say, the water at each point is subject to a certain state of compression, which will be like what occurs in still water at a certain depth. It is convenient to use this depth, at which still water is found to be in the same state of compression, as an expression for the pressure at any point l of the water. Therefore, if p is that still-water depth that expresses the pressure at point l, then p will be a certain function of the coordinates x and y, and if the the pressure at l also varies with time, p will also be a function of time t.

44. Therefore we shall put $dp = Rdx + rdy + \mathcal{R}dt$, and we shall consider a rectangular element of water $lmno$ [Fig. 3], whose sides are $lm = no = dx$ and $ln = mo = dy$; the area being $= dxdy$.

---

[15]Euler is considering a material particle extending in each direction.

Fig: 3.
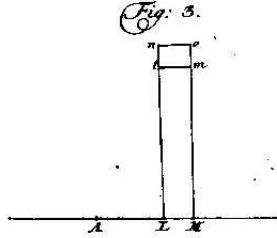
If now the pressure at l is $= p$, the pressure at m will be $= p + Rdx$, at n $= p + rdy$ and at o will be $= p + Rdx + rdy$. Then the side lm is pressed by a force $= dx(p + \frac{1}{2}Rdx)$. while the opposite side no will be pressed by a force $= dx(p + \frac{1}{2}Rdx + rdy)$. From these two forces, the element $lmno$ will be have a resulting force in the direction of ln that is $= -rdxdy$. In the same way, from the forces $dy(p + \frac{1}{2}rdy)$ and $dy(p + Rdx + \frac{1}{2}rdy)$ which act on the sides ln and mo, the resulting force acting on the element in the direction lm will be $= -Rdxdy$.

45. Hence there arises an accelerative force in the direction of lm that is $= -R$, and an accelerative force in the direction of ln that is $= -r$. The first of these along with the force due to gravity will then be $\alpha - R$. Absent friction, we shall have the equations $\alpha - R = 2Lu + 2lv + 2\mathcal{L}$ or

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$$R = \alpha - 2Lu - 2lv - 2\mathcal{L},$$

and $-r = 2Mu + 2mv + 2\mathcal{M}$ or

$$r = -2Mu - 2mv - 2\mathcal{M}$$

which together give us

$$dp = \alpha dx - 2(Lu + lv + \mathcal{L})dx - 2(Mu + mv + \mathcal{M})dy + \mathcal{R}dt.$$

This differential should be complete, that is, integrable.

46. Since the term αdx is already integrable, and as of yet we know nothing of $\mathcal{R}dt$, by the very nature of complete differentials it is necessary for the above expression that

$$\frac{d(Lu + lv + \mathcal{L})}{dy} = \frac{d(Mu + mv + \mathcal{M})}{dx}$$

and thence, because $du/dx = L$, $du/dy = l$, $dv/dx = M$, and $dv/dy = m$,

$$Ll + \frac{udL}{dy} + lm + \frac{vdl}{dy} + \frac{d\mathcal{L}}{dy} = ML + \frac{udM}{dx} + mM + \frac{vdm}{dx} + \frac{d\mathcal{M}}{dx}$$

which reduces to this formula:

$$(L+m)(l-M) + u\left(\frac{dL}{dy} - \frac{dM}{dx}\right) + v\left(\frac{dl}{dy} - \frac{dm}{dx}\right) + \frac{d\mathcal{L}}{dy} - \frac{d\mathcal{M}}{dx} = 0.$$

47. Since $Ldx+ldy+\mathcal{L}dt$ and $Mdx+mdy+\mathcal{M}dt$ are complete differentials[16], we know that

$$\frac{dL}{dy} = \frac{dl}{dx}; \quad \frac{dm}{dx} = \frac{dM}{dy}; \quad \frac{d\mathcal{L}}{dy} = \frac{dl}{dt} \quad \text{and} \quad \frac{d\mathcal{M}}{dx} = \frac{dM}{dt},$$

and when these expressions are substituted, we shall have the equation:[17]

$$(L+m)(l-M) + u\left(\frac{dl - dM}{dx}\right) + v\left(\frac{dl - dM}{dy}\right) + \frac{dl - dM}{dt} = 0$$

which clearly will be satisfied by $l = M$: that is, by $du/dy = dv/dx$. This latter condition requires in turn that $udx + vdy$ be a complete differential, and this latter is the desired condition describing those motions that are to be allowed.[18]

48. This criterion is independent of the previous one of continuity and uniform density. For this property that $udx + vdy$ be a complete differential would still apply, even if the moving fluid were to change its density, as in the motion of elastic fluids, or as happens in air. That is, the velocities u

---

[16]$= du$ and dv respectively.

[17]I have corrected some obvious misprints on this page.

[18]See translator's comment to this paragraph.

and v would be such functions of the coordinates x and y that at any fixed time t the expression $udx + vdy$ would be completely integrable.

49. We are now in a position to define the pressure p, which we need to determine completely the motion of the fluid. Since we have found that $M = l$, we shall have[19]

$$dp = \alpha dx - 2u(Ldx + ldy) - 2v(ldx + mdy) - 2\mathcal{L}dx - 2\mathcal{M}dy + \mathcal{R}dt.$$

From $Ldx + ldy = du - \mathcal{L}dt$ and $ldx + mdy = dv - \mathcal{M}dt$,

$$dp = \alpha dx - 2udu - 2vdv + 2\mathcal{L}udt + 2\mathcal{M}vdt - 2\mathcal{L}dx - 2\mathcal{M}dy + \mathcal{R}dt.$$

If we wish to define the pressure at each location for a given fixed time[20], the equation to be considered is:

$$dp = \alpha dx - 2udu - 2vdv - 2\mathcal{L}dx - 2\mathcal{M}dy$$

and, on writing $\mathcal{L} = du/dt$ and $\mathcal{M} = dv/dt$, we then get

$$dp = \alpha dx - 2udu - 2vdv - 2\frac{du}{dt}dx - 2\frac{dv}{dt}dy.$$

---

[19]Actually, the argument in the next two lines does not depend on $M = l$.
[20]That is, to get the spatial gradient of pressure.

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In integrating this equation, the time t is to be held constant.

50. Given the hypotheses, [we shall see that] this equation is integrable, if we take into account the criterion that $udx + vdy$ be a complete differential, keeping the time t constant. Let S be its integral, which is a function of x, y and t, which gives $dS = udx + vdy$ when $dt = 0$. If we further allow t to be variable, this becomes

$$dS = udx + vdy + Udt.$$

Then we shall have $du/dt = dU/dx$ and $dv/dt = dU/dy$. Then $U = dS/dt$.

51. Introducing these expressions gives us:

$$\frac{du}{dt}dx + \frac{dv}{dt}dy = \frac{dU}{dx}dx + \frac{dU}{dy}dy$$

whose integral for a fixed time t is clearly $= U$. To make this more clearly apparent, let us put $dU = Kdx + kdy$, so $dU/dx = K$ and $dU/dy = k$. Then

$$\frac{dU}{dx}dx + \frac{dU}{dy}dy = Kdx + kdy = dU.$$

The integral of this equation being $= U = dS/dt$, then

$$dp = \alpha dx - 2udu - 2vdv - 2dU$$

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which upon integration yields:

$$p = \text{Const.} + \alpha x - uu - vv - 2\frac{dS}{dt},$$

S being a function of x, y and t. For $dt = 0$, its differential is $udx + vdy$.

52. To understand better the nature of this formula, we shall consider the speed at a point l, which will be $= V = \sqrt{(uu + vv)}$. The pressure moreover will be: $p = \text{Const.} + \alpha x - uu - vv - 2dS/dt$. The S in the last term dS denotes $S = \int(udx + vdy)$, where we view the time t as variable.

53. Suppose we now wished to include a friction term that is proportional to the pressure p in effect while the point l traverses an infinitesimal distance ds. The retarding force arising from the friction would then be $= p/f$. Putting $dS/dt = U$, our differential equation at a definite point t in time becomes:

$$dp = \alpha dx - \frac{p}{f}ds - 2VdV - 2dU.$$

Let e be the number whose hyperbolic logarithm is $= 1$. Integration then gives[21]

$$p = e^{-s/f} \int e^{s/f}(\alpha dx - 2VdV - 2dU)$$

---

[21]The orginal has dV instead of dU.

or

$$p = \alpha x - VV - 2U - \frac{1}{f}e^{-s/f}\int e^{s/f}(\alpha x - VV - 2U)ds.$$

54.   For the fluid motion that is actually to be sustained, the criterion is that $udx + vdy$ be a complete differential at any fixed time t[22]. The continuity condition requires that the density stay constant and uniform so that $du/dx + dv/dy$ will be $= 0$, whence it follows that $-udy + vdx$ is a complete differential[23]. Thus the velocity components u and v together must be functions of x, y and t such that both the expressions $udx + vdy$ and $-udy + vdx$ are complete differentials.

55.   [Flows in three dimensions.] Let us now start to investigate the case where the three velocity components u, v, w of the point λ, directed along the axes AL, AB, AC, are functions of the coordinates x, y, z, and of the time t, such that

$$\begin{aligned} du &= Ldx + ldy + \lambda dz + \mathcal{L}dt \\ dv &= Mdx + mdy + \mu dz + \mathcal{M}dt \\ dw &= Ndx + ndy + \nu dz + \mathcal{N}dt; \end{aligned}$$

and, according the the condition given earlier, we must have $L + m + \nu = 0$ even if the time t is allowed to vary. This is the same as

$$\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0.$$

---

[22]See the translator's comment for paragraph **47**.
[23]This expression lacks the minus sign both times it appears in this paragraph.

This condition will not be used in the present part of our analysis.[24]

56. After an infinitesimal time dt, the point λ is brought to position π. It traverses a distance $= udt$ in the direction of the axis AL, a distance $= vdt$ in the direction of AB, and a distance $= wdt$ in the direction of AC. The three velocity components for the point λ at postion π will be:

$$
\begin{aligned}
\text{along } AL &= u + Ludt + lvdt + \lambda wdt + \mathcal{L}dt, \\
\text{along } AB &= v + Mudt + mvdt + \mu wdt + \mathcal{M}dt, \\
\text{along } AC &= w + Nudt + nvdt + \nu wdt + \mathcal{N}dt.
\end{aligned}
$$

Then the accelerations along these same directions will be:

$$
\begin{aligned}
\text{along } AL &= 2(Lu + lv + \lambda w + \mathcal{L}), \\
\text{along } AB &= 2(Mu + mv + \mu w + \mathcal{M}), \\
\text{along } AC &= 2(Nu + nv + \nu w + \mathcal{N}).
\end{aligned}
$$

57. Let us take the axis AC in the vertical direction, so that the other two axes AL and AB are horizontal. Along the axis AC, there shall be an accelerative force $= -1$ due to gravity. The pressure p of point λ will have a differential

$$dp = Rdx + rdy + \rho dz,$$

_____
[24]The condition will be re-introduced in paragraph **64**.

if time is held constant. There arise from this the three accelerations

$$AL : -R, \quad AB : -r, \quad AC : -\rho.$$

These various equations are combined in the same way that we used in paragraphs **44** and **45**, and we need not repeat the argument here. The result will be the equations:

$$
\begin{aligned}
R &= & -2(Lu + lv + \lambda w + \mathcal{L}), \\
r &= & -2(Mu + mv + \mu w + \mathcal{M}), \\
\rho &= -1 & -2(Nu + nv + \nu w + \mathcal{N}).
\end{aligned}
$$

58. Since however the form $dp = Rdx + rdy + \rho dz$ must represent a complete differential, we shall have

$$dR/dy = dr/dx; \quad dR/dz = d\rho/dx; \quad dr/dz = d\rho/dy.$$

After differentiation and dividing by -2, we shall obtain the three equations[25]

$$
\text{I} \quad
\begin{cases}
uL_y + vl_y + w\lambda_y + \mathcal{L}_y & +Ll + lm + \lambda n & = \\
uM_x + vm_x + w\mu_x + \mathcal{M}_x & +ML + mM + \mu N,
\end{cases}
$$

---

[25]To increase legibility in these two paragraphs, I follow the more modern practice indicating partial derivatives by subscripts. In modern terms, Euler has taken the curl to eliminate the pressure gradient. The dependent variables in the resulting differential equations in paragraph **59** become the vorticity components.

$$
\text{II} \quad
\begin{cases}
uL_z + vl_z + w\lambda_z + \mathcal{L}_z \quad\; +L\lambda + l\mu + \lambda\nu \quad\; = \\[2mm]
uN_x + vn_x + w\nu_x + \mathcal{N}_x \quad +NL + nM + \nu N,
\end{cases}
$$

$$
\text{III} \quad
\begin{cases}
uM_z + vm_z + w\mu_z + \mathcal{M}_z \quad +M\lambda + m\mu + \mu\nu \;\; = \\[2mm]
uN_y + vn_y + w\nu_y + \mathcal{N}_y \quad +Nl + nm + \nu n.
\end{cases}
$$

59.   By the property of complete differentials,[26]

$$
L_y = l_x; \quad m_x = M_y; \quad \lambda_y = l_z; \quad \mu_x = M_z; \quad \mathcal{L}_y = l_t; \quad \mathcal{M}_x = M_t,
$$

$$
L_z = \lambda_x; \quad l_z = \lambda_y; \quad n_x = N_y; \quad \nu_x = N_z; \quad \mathcal{L}_z = \lambda_t; \quad \mathcal{N}_x = N_t,
$$

$$
M_z = \mu_x; \quad N_y = n_x; \quad m_z = \mu_y; \quad \nu_y = n_z; \quad \mathcal{M}_z = \mu_t; \quad \mathcal{N}_y = n_t.
$$

Substitute these in the above three equations, which then become

$$
\left(\frac{dl - dM}{dt}\right) + u\left(\frac{dl - dM}{dx}\right) + v\left(\frac{dl - dM}{dy}\right) + w\left(\frac{dl - dM}{dz}\right) + (l - M)(L + m) + \lambda n - \mu N = 0,
$$

$$
\left(\frac{d\lambda - dN}{dt}\right) + u\left(\frac{d\lambda - dN}{dx}\right) + v\left(\frac{d\lambda - dN}{dy}\right) + w\left(\frac{d\lambda - dN}{dz}\right) + (\lambda - N)(L + \nu) + l\mu - nM = 0,
$$

$$
\left(\frac{d\mu - dn}{dt}\right) + u\left(\frac{d\mu - dn}{dx}\right) + v\left(\frac{d\mu - dn}{dy}\right) + w\left(\frac{d\mu - dn}{dz}\right) + (\mu - n)(m + \nu) + M\lambda - Nl = 0.
$$

---

[26]Each line of equalities is used in the corresponding equation of the previous paragraph.

60. It is clear that these three equations are satisfied on setting:[27]

$$l = M; \quad \lambda = N; \quad \mu = N;$$

and therein lies the condition as derived from our analysis of applied forces.[28]
These can be expressed in our usual notation as

$$\frac{du}{dy} = \frac{dv}{dx}; \quad \frac{du}{dz} = \frac{dw}{dx}; \quad \frac{dv}{dz} = \frac{dw}{dy}.$$

These, however, are the very conditions required for the form $udx + vdy + wdz$ to be a complete differential. This condition then states that the three velocity components $u, \ v, \ w$ be functions of $x, \ y, \ z$ along with t, such that at any fixed time the form $udx + vdy + wdz$ admit an integral.

61. For a fixed moment in time (i.e., $dt = 0$), then, we have

$$du = Ldx + Mdy + Ndz$$
$$dv = Mdx + mdy + ndz$$
$$dw = Ndx + ndy + \nu dz$$

---

[27]See translator's comment for this paragraph.
[28]*quibus continetur criterium, quod consideratio sollicitationum suppeditat.*

and the values for $R, \ r, \ \rho$ will be:

$$
\begin{aligned}
R &= & -2(Lu + Mv + Nw + \mathcal{L}), \\
r &= & -2(Mu + mv + nw + \mathcal{M}), \\
\rho &= -1 & -2(Nu + nv + \nu w + \mathcal{N}).
\end{aligned}
$$

We shall have this equation for the pressure:

$$
\begin{aligned}
dp = \quad -dz \quad &-2u(Ldx + Mdy + Ndz) \\
&-2v(Mdx + mdy + ndz) \\
&-2w(Ndx + ndy + \nu dz) \\
&-2\mathcal{L}dx - 2\mathcal{M}dy - 2\mathcal{N}dz
\end{aligned}
$$

$$
= dz - 2udu - 2vdv - 2wdw - 2\mathcal{L}dx - 2\mathcal{M}dy - 2\mathcal{N}dz.
$$

62. Since $\mathcal{L} = du/dt; \ \mathcal{M} = dv/dt; \ \mathcal{N} = dw/dt,$ integration gives

$$
p = C - z - uu - vv - ww - 2\int \left(\frac{du}{dt}dx + \frac{dv}{dt}dy + \frac{dw}{dt}dz \right).
$$

By the condition found above, $udx + vdy + wdz$ will be integrable, and we can take this integral to be $= S$. A variable time t can now be allowed, and we can take

$$
dS = udx + vdy + wdz + Udt,
$$

with $du/dt = dU/dx; \quad dv/dt = dU/dy; \quad dw/dt = dU/dz.$ [29] Consequently, for that point in time assumed in the above integral,

$$\frac{dU}{dx}dx + \frac{dU}{dy}dy + \frac{dU}{dz}dz = dU$$

and we shall have

$$p = C - z - uu - vv - ww - 2U,$$

or[30]

$$p = C - z - uu - vv - ww - 2\frac{dS}{dt}.$$

63. The form $uu + vv + ww$ is seen to express the square of the speed V at the point λ, so the equation for the pressure becomes

$$p = C - z - VV - 2\frac{dS}{dt}.$$

To evaluate this, first we must seek the integral S of the form $udx+vdy+wdz$. Its differential with only the time t being variable will be divided by dt, which will give the value of dS/dt, which then goes into the expression we found for the pressure.

---

[29]Also, $U = \partial S/\partial t.$
[30]The original has $dS/ds.$

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64.   If we now conjoin the earlier criterion which constrains any possible motion, then the three velocity components u, v, w should be such functions of the coordinates x, y and z with [fixed?] time t, that first $u\,dx + v\,dy + w\,dz$ be a complete differential, and also that $du/dx + dv/dy + dw/dz = 0$. Any motion of the fluid must be subject to these conditions, if the density is taken not to vary. Moreover, if the form $u\,dx + v\,dy + w\,dz + U\,dt$ is a complete differential with variable time t, then the state of the pressure at any point λ is expressed in terms of a depth p with

$$p = C - z - uu - vv - ww - 2U.$$

This holds when the fluid experiences gravity in the z-direction, the plane BAL being horizontal.

65.   [More general case.] Suppose we assign a different direction for gravity, or we allow any variable forces to act on individual particles of the fluid, so that a difference in the value of the pressure p would now enter. There would still be no change in the law governing the velocity components of each point of the fluid. The three velocity components must always be so constituted such that $u\,dx + v\,dy + w\,dz$ be a complete differential, and such that $du/dx + dv/dy + dw/dz = 0$. The three velocity components u, v, w can be set up in an infinite number of ways to satisfy these two conditions ; and then the fluid pressure can be assigned at each point.

66.   It will be much more difficult, however, to determine the motion of the fluid at each point when variable applied forces and pressure are assigned. For in these cases we need to find various equations of the form $p = C - z - uu - vv - ww - 2U$. The functions u, v, w have to be defined to satisfy not only such equations, but also the previously specified laws. This will require the utmost analytical skill. The sensible approach is to inquire into the nature of suitable functions which would conform to each criterion.

67.   The best place to start is with the integral whose differential is $udx + vdy + wdz$ when time is held fixed. Let S be this integral, which will be a function of x, y, and z, for fixed but arbitrary time t. If the quantity S is differentiated, the coefficients of the differentials dx, dy, dz will provide the velocity components u, v, w which obtain at the current time at the fluid point whose coordinates are x, y and z. The question now comes down to this: to define what functions of x, y and z may be allowed for S so that we also have $du/dx + dv/dy + dw/dz = 0$. Since $u = dS/dx$, $v = dS/dy$, and $w = dS/dz$, this means that

$$\frac{ddS}{dx^2} + \frac{ddS}{dy^2} + \frac{ddS}{dz^2} = 0.$$

68. [Special solutions.] Since it is not obvious how in general this can be made to happen, I shall consider certain classes of possibilities. Let then

$$S = (Ax + By + Cz)^n.$$

Then $dS/dx = nA(Ax + By + Cz)^{n-1}$ and $ddS/dx^2 = n(n-1)AA(Ax + By + Cz)^{n-2}$, and similar forms will hold for ddS/dy^2 and ddS/dz^2. From this it must be that

$$n(n-1)(Ax + By + Cz)^{n-2}(AA + BB + CC) = 0.$$

This will be satisfied in the first case when either $n = 0$ or $n = 1$. From these will obtain two suitable solutions, namely $S = $ constant and $S = Ax + By + Cz$, where the constants A, B, C and the time can be chosen arbitrarily.

69. If however n is neither 0 nor 1, then we necessarily have[31] $AA + BB + CC = 0$. A suitable solution for S will now be

$$S = (Ax + By + Cz)^n$$

in which the order n can be any number, but the time t can enter into this order n. It is clear that any combination of such forms can be taken for the

---

[31]Evidently we are now considering complex numbers.

solution S, so that:

$$S = \alpha + \beta x + \gamma y + \delta z + \epsilon(Ax + By + Cz)^{n'} + \zeta(A'x + B'y + C'z)^{n''}$$
$$+ \eta(A''x + B''y + C''z)^{n'''} + \theta(A'''x + B'''y + C'''z)^{n''''} \text{ etc.}$$

as long as

$$AA + BB + CC = 0;$$

$$A'A' + B'B' + C'C' = 0;$$

$$A''A'' + B''B'' + C''C'' = 0 \text{ etc.}$$

70. Suitable formulas for S for the smaller orders, where the coordinates x, y, z are raised to the first, second, third or fourth powers, will be the following:[32]

I. $\quad S = \qquad\qquad\qquad\qquad A,$

II. $\quad S = \qquad\qquad\qquad Ax + By + Cx,$

III. $\quad S = \qquad\quad Axx + Byy + Czz + 2Dxy + 2Exz + 2Fyz$
$$(A + B + C = 0),$$

IV. $\quad S = \quad Ax^3 + By^3 + Cz^3 + 3Dxxy + 3Fxxz + 3Hyyz + 6Kxyz$
$$A + E + G = 0; \quad B + D + I = 0; \quad C + F + H = 0;$$

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[32]See comment on this paragraph at the end of the document.

$$\text{V.} \quad S = \begin{array}{l} +Ax^4 + 6Dxxyy + 4Gx^3y + 4Hxy^3 + 12Nxxyz \\ +By^4 + 6Exxzz + 4Ix^3z + 4Kxz^3 + 12Oxyyz \\ +By^4 + 6Exxzz + 4Ix^3z + 4Kxz^3 + 12Oxyyz \end{array}$$

$$\text{where} \qquad \begin{array}{ll} A + D + E = 0 & G + H + P = 0, \\ B + D + F = 0 & I + K + O = 0, \\ C + E + F = 0 & L + M + N = 0. \end{array}$$

71. We can now see how to get the like formulas for any order. First, the same numerical coefficients are given in the individual terms as occur in the law of permutation of quantities – that is, which arise if the trinomial $x+y+z$ is raised to that order power. Next, indefinite literals A, B, C, etc. are multiplied into these numerical coefficients. Then, without regard to these literals, check wherever there occur three terms of the type $LZxx + MZyy + NZzz$, which have the same common factor arising from the variables.[33] As often as this occurs, specify that the sum $L + M + N$ of the three literals be set to zero. For example, for the fifth power there will be had

$$S = \begin{array}{l} +Ax^5 + 5Dx^4y + 5\mathcal{D}x^4z + 10Gx^3yy + 10\mathcal{G}x^3zz + 20Kx^3yz + 30Nxyyzz \\ +By^5 + 5Exy^4 + 5\mathcal{E}y^4z + 10Hx^2y^3 + 10\mathcal{H}y^3zz + 20Lxy^3z + 30Oxxyzz \\ +Cz^5 + 5Fxz^4 + 5\mathcal{F}yz^4 + 10Ixxz^3 + 10\mathcal{I}yyz^3 + 20Mxyz^3 + 30Pxxyyz, \end{array}$$

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[33]That is, arising from the permutations of x, y, z.

with the following conditions for the literals:

$$A + G + \mathcal{G} = 0; \quad D + H + O = 0; \quad \mathcal{D} + I + P = 0$$
$$B + H + \mathcal{H} = 0; \quad E + G + N = 0; \quad \mathcal{E} + \mathcal{I} + P = 0; \quad K + L + M = 0$$
$$C + I + \mathcal{I} = 0; \quad F + \mathcal{G} + N = 0; \quad \mathcal{F} + \mathcal{H} + O = 0.$$

In a similar way, there will be 15 conditions of this type for the sixth order, 21 for the seventh, 28 for the eighth, and so on.

72. $[n{=}0.]$ Now in the first formula $S = A$, the three velocity components will be zero, since the coordinates x, y and z do not appear at all. This describes a fluid at rest. The pressure at any point, however, can be variable with the time. For A is an arbitrary function of time, and so the pressure at a point λ and time t will be $p = C - 2\frac{dA}{dt} - z$. This formula indicates the state of a fluid subject to any forces whatsoever at any point in time, so long as the forces balance each other so that no motion of the fluid will arise. This will happen for example if the fluid is contained in a vase without a means of egress, yet subject to any sort of forces.

73. $[n{=}1.]$ For the formula $S = Ax + By + Cz$, differentiation at the point λ will give three velocity components:

$$u = A; \quad v = B \quad \& \, w = C.$$

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Thus, at a given time, all points of the fluid will be carried with the same motion, in the same direction. Then the fluid as a whole will move like a solid body, carried by a common but changing motion. At a different point in time, as the applied external forces change, the motion will differ accordingly in both speed and direction. If, for point λ, the functions of time are A, B and C, then the pressure will be $p = C - z - AA - BB - CC - 2x\frac{dA}{dt} - 2y\frac{dB}{dt} - 2z\frac{dC}{dt}$.[34]

74. [n=2.] The third formula $S = Axx + Byy + Czz + 2Dxy + 2Exz + 2Fyz$ with $A + B + C = 0$ will yield three velocity components at the point λ : [35]$u = 3Ax + 2Dy + 2Ez$; $v = 2By + 2Dx + 2Fz$; $w = 2Cz + 2Ex + 2Fy$, or $w = 2Ex + 2Fy - 2(A + B)z$. In this case, at any moment in time different points in the fluid will be carried in different motions. At the next moment in time, moreover, the motion of each point can be variable in any way, because the functions for A, B, C, D, E, F can be of any sort. Even more variety can occur if composite functions are used for S.

75. [Investigating the possibility of a common rotation.] In the second case, the motion of the fluid will coincide with the uniform motion of a solid body, so at any moment in time the different parts of the fluid will be carried in an equal and parallel motion. We might suspect that the motion of the fluid in other cases can also coincide with the motion of a solid body, whether rotational or of some other sort. For this to happen, the pyramid $\pi\phi\rho\sigma$ must

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[34]The original reads: $p = C - z - AA - CC - 2x\frac{dA}{dt} - 2y\frac{dB}{dt} - z\frac{dC}{dt}$.
[35]The orginal has α instead of u.

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necessarily be equal and similar to the pyramid $\lambda\mu\nu o$; that is, taking over the values shown in paragraph 32,[36]

$$\pi\Phi = \lambda\mu = dx \quad = \sqrt{QQ + qq + \Phi\Phi}$$
$$\pi\rho = \lambda\nu = dy \quad = \sqrt{RR + rr + \rho\rho}$$
$$\pi\sigma = \lambda o = dz \quad = \sqrt{SS + ss + \sigma\sigma}$$

$$\Phi\rho = \mu\nu \quad = \sqrt{dx^2 + dy^2} \quad = \sqrt{(Q-R)^2 + (q-r)^2 + (\Phi-\rho)^2}$$
$$\Phi\sigma = \mu o \quad = \sqrt{dx^2 + dz^2} \quad = \sqrt{(Q-S)^2 + (q-s)^2 + (\Phi-\sigma)^2}$$
$$\rho\sigma = ro \quad = \sqrt{dy^2 + dz^2} \quad = \sqrt{(R-S)^2 + (r-s)^2 + (\rho-\sigma)^2}.$$

76. On comparison with the three first equations, the three last equations reduce to these:

$$QR + qr + \Phi\rho = 0;$$
$$QS + qs + \Phi\sigma = 0;$$
$$RS + rs + \rho\sigma = 0.$$

If however we substitute for Q, R, S, q, r, s, Φ, ρ, σ the values assigned in

[36]Again, with the same letters doubly used as unrelated labels and distances. See also the comments to paragraphs **75-77** at the end of this document.

paragraph **34**, then the first three equations [of the previous paragraph] will give:

$$1 = 1 + 2Ldt; \quad l + M = 0$$
$$1 = 1 + 2mdt; \quad \lambda + N = 0$$
$$1 = 1 + 2\nu dt; \quad \mu + n = 0$$

from which we would conclude that $L = 0$, $m = 0$, and $\nu = 0$, $M = -l$, $N = -\lambda$ and $n = -\mu$.

77. The three velocity components at each point λ would therefore be so constituted that[37]

$$du = + ldy + \lambda dz$$
$$dv = - ldx + \mu dz$$
$$dw = - \lambda dx - \mu dy.$$

Now the second condition on the fluid motion demands that $l = M$, $\lambda = N$ and $n = \mu$. Then these all vanish, and the velocity components u, v and w will be the same in all parts of the fluid at any given time. It is therefore clear that the fluid motion cannot coincide with the motion of a rigid body in this case.

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[37]Some minus signs were missing in the original.

78. To determine the contribution of the forces which act externally on the fluid, we ought first to find the force needed to produce any specified fluid motion. We found in paragraph **56** that these are equal to the three accelerative forces recorded there. If we consider a fluid element whose volume or mass is $= dxdydz$, the motive forces required are therefore:[38]

$$
\begin{aligned}
AL: &\quad = 2dxdydz(Lu + lv + \lambda w + \mathcal{L}) &\quad = 2dxdydz\,(uu_x + vu_y + wu_z + u_t) \\
AB: &\quad = 2dxdydz\,(Mu + mv + \mu w + \mathcal{M}) &\quad = 2dxdydz\,(uv_x + vv_y + wv_z + v_t) \\
AC: &\quad = 2dxdydz\,(Nu + nv + \nu w + \mathcal{N}) &\quad = 2dxdydz\,(uw_x + vw_y + ww_z + w_t)
\end{aligned}
$$

and triple integration gives the total forces that ought to be applied on the whole mass of fluid in each direction.

79. According to the second condition the form $udx + vdy + wdz$ is to be a complete differential, whose integral is $= S$. With time also variable, we are to set $dS = udx + vdy + wdz + Udt$. Then, from $du/dy = dv/dx$; $du/dz = dw/dx$; $du/dt = dU/dx$ etc., those three motive forces become:

$$
AL := 2dxdydz\frac{udu + vdv + wdw + dU}{dx},
$$

---

[38]For the sake of legibility, I have resorted to subscripts here, writing for example u_x in place of $\frac{du}{dx}$.

$$AB := 2dxdydz\frac{udu + vdv + wdw + dU}{dy},$$

$$AC := 2dxdydz\frac{udu + vdv + wdw + dU}{dz}.$$

80. Let now $uu+vv+ww+2U = T$, T being a function of the coordinates x, y, z.[39] For a fixed point in time, we can write[40]

$$dT = Kdx + kdy + \kappa dz$$

and the three motive forces of the element $dxdydz$ are

$$AL := Kdxdydz$$

$$AB := kdxdydz$$

$$AC := \kappa dxdydz.$$

Upon triple integration, these formulae extend to the whole fluid mass. From these, we obtain equivalent[41] expressions for forces, and their average directions, that may be used everywhere. But this involves a truly higher level of difficulty, and I shall not dwell further on this topic.

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[39]See comments for this and the following paragraph, at the end of the document.

[40]The typesetter misread κ, replacing it with u here and with k further down.

[41]Equivalent to the expressions written out in paragraph **78**.

81. This quantity $T = uu + vv + ww + 2U$ introduced here yields a simpler formula for the equivalent depth p that gives the pressure; it is $p = C - z - T$, as long as each fluid particle is acted upon only by gravity. If however any particle λ is subjected to accelerative forces whose components along the directions AF, AB, and AC are respectively Q, q and Φ, then a similar calculation gives the pressure as

$$ p = C + \int (Qdx + qdy + \Phi dz) - T. $$

It is clear then that $Qdx + qdy + \Phi dz$ must be a complete differential, to be compatible with a state of equilibrium. The celebrated Monsieur Clairaut has indeed shown with great clarity that such a condition must be imposed on the force components Q, q and Φ.

82. [Application to hydrostatics and hydraulics.] At first glance the principles of the general theory of fluid motion did not seem very fruitful, yet almost everything that is known about hydraulics and hydrostatics is contained in them, so it must be allowed that these principles have a very broad reach. To see this more clearly, it will be worth while to show exactly how the known precepts of hydrostatics and hydraulics follow in a clear and straight-forward manner from the principles developed so far.

83.   Let us therefore consider first a fluid at rest, so that $u = 0$, $v = 0$ and $w = 0$. Since then $T = 2U$, the pressure at any fluid particle λ will be

$$p = C + \int (Qdx + qdy + \Phi dz) - 2U.$$

Since U is a function of time t, which we take fixed, we can fold this quantity U into the constant C, so that

$$p = C + \int (Qdx + qdy + \Phi dz),$$

where Q, q and Φ are the forces acting on the fluid particle λ in the direction of the axes AL, AB and AC respectively.

84.   Since now[42] the pressure p is a function of the position of λ, that is, of the coordinates x, y and z, then the form $\int (Qdx + qdy + \Phi dz)$ must be a definite integral function of these coordinates. Then it is clear, using the same sort of argument as above, that the fluid could not be in equilibrium unless the forces acting on the individual fluid elements are so constituted that $Qdx + qdy + \Phi dz$ be a complete differential. If we set its integral $= P$, then the pressure at the point λ will be $p = C + P$. If the only force is gravity acting in the direction of CA, then $p = C - z$. If the pressure at one point λ is specified, thus giving the constant C, then, for that one moment in time, the pressure at all other points will be completely determined.

---

[42]For a fixed moment in time.

85.   With the passage of time, the pressure at each position can change. This will surely happen if the external forces acting on the fluid are variable, as these are not restricted except that they remain in equilibrium and do not produce fluid motion. On the other hand, if these forces do not suffer any change, then the literal C actually does signify a true constant independent of time, and in that position λ the same pressure $p = C + P$ will always be found.

86.   [Free surface.] In a permanent state, the fluid's boundary can be determined if the boundary is not subjected to any force. In a vessel, on the free surface[43] where the fluid is not confined by walls of the vessel, the pressure must neccesarily be zero. Then the equation will be $P =$const., and the shape of the free surface is thereby expressed as a relation among the three coordinates x, y and z. At the free surface we may set $P = E$, $C = -E$; and for any internal point λ the pressure will be $p = P - E$. If the fluid elements are subject only to gravity, so that $p = C - z$, then at the free surface we shall have $z = C$, from which we may conclude that the free surface is horizontal.

87.   [Flow through narrow tubes.] Finally, concerning flows through tubes, everything that has been teased out by various means are easily deduced from these principles. The tubes are usually taken to be very narrow, or

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[43]In this paragraph, Euler uses the terms "extrema figura", "extremitas", "extrema superficies", and other combinations of these words, not using the same phrase twice. He lands finally at "extrema superficies libera". I have translated all these terms with the modern term "free surface".

else the flow is assumed to be uniform across any normal cross-section of the tube. From these assumptions arose the rule that the speed of the fluid at any position in the tube is inversely proportional to the area of the cross-section. So let the shape of the tube be expressed by two equations among the three coordinates x, y and z; so that for any value of the abscissa x, the other two coordinates y and z can be defined. Let also λ be any point of the tube.[44]

Tab. IV

Fig. 2.

88. Let moreover the area of the cross-section at λ be $= rr$, and at another fixed position of the tube let the area be $= ff$, while the speed is $= \breve{o}$,[45] which after an infinitesimal time dt becomes $\breve{o} + d\breve{o}$. Thus $\breve{o}$ will be a function of time t, as will be $d\breve{o}/dt$. The velocity of the fluid at point λ at the current time will be $V = ff\breve{o}/rr$. From the shape of the tube, y and z are given in terms of x, so that $dy = \eta dx$ and $dz = \theta dx$; whence the three velocity components at λ in the directions of AL, AB and AC are respectively[46]:

$$u = \frac{ff\breve{o}}{rr}\frac{1}{\sqrt{1+\eta\eta+\theta\theta}};$$
$$v = \frac{ff\breve{o}}{rr}\frac{\eta}{\sqrt{1+\eta\eta+\theta\theta}};$$
$$w = \frac{ff\breve{o}}{rr}\frac{\theta}{\sqrt{1+\eta\eta+\theta\theta}};$$

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[44]The marginal note refers to Figure 2, whose re-use is a bit of a stretch. Or did Euler lift this passage from another manuscript which contained additional figures?

[45]Euler used the astronomical symbol whose teX code is "\taurus". One is bound to have strong feelings about using such a symbol. Printing difficulties have forced me to substitute the makeshift $\breve{o}$. The combination $ff\breve{o}$ is the mass flux.

[46]These are the speed V times the direction cosines.

so that $uu + vv + ww = VV = f^4 \breve{o}\breve{o}/r^4$. The term rr is a function of x as well as the dependent variables y and z.

89. Since $udx + vdy + wdz$ is a complete differential and we can take its integral to be $= S$, there results:

$$dS = \frac{ff\breve{o}}{rr}\frac{dx\,(1 + \eta\eta + \theta\theta)}{\sqrt{1 + \eta\eta + \theta\theta}} = \frac{ff\breve{o}}{rr}dx\sqrt{1 + \eta\eta + \theta\theta}.$$

But $dx\sqrt{1 + \eta\eta + \theta\theta}$ represents the element[47] of the tube. If we write this as $= ds$, then $dS = \frac{ff\breve{o}ds}{rr}$. For a fixed time t, this is a function of $\breve{o}$. Since however s and rr do not depend on the time t, but only on the shape of the tube, it follows that $S = \breve{o}\int \frac{ffds}{rr}$.

90. To find the pressure p which obtains at the point λ in the tube, we must consider the quantity U arising from differentiating S when only the time t is variable – that is, such that $U = dS/dt$. Since the integral form $\int \frac{ffds}{rr}$ does not involve the time t, the differential will be $dS/dt = U = \frac{d\breve{o}}{dt}\int \frac{ffds}{rr}$; and then, from paragraph 80,

$$T = \frac{f^4\breve{o}\breve{o}}{r^4} + \frac{2d\breve{o}}{dt}\int \frac{ffds}{rr}.$$

For any posited forces Q, q and ϕ acting on the fluid, the corresponding

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[47]arc-length.

pressure at the point λ will be:

$$p = C + \int (Qdx + qdy + \Phi dz) - \frac{f^4 \breve{o}\breve{o}}{r^4} - \frac{2d\breve{o}}{dt} \int \frac{ffds}{rr}.$$

This is the formula that was to be extracted for fluid motion through a tube. Since we allowed any sort of forces acting on the fluid [in this derivation], it will hold all the more so when the only force is gravity. It is well to recall that the forces Q, q and Φ had to be so constituted that the form $Qdx + qdy + \Phi dz$ be a complete differential, i.e., that it be integrable.

Explicit dissertatio de principiis motus fluidorum auctore Leon. Eulero.

Appendix: Table IV

Fig: 1.

Fig: 3.

Fig: 2.
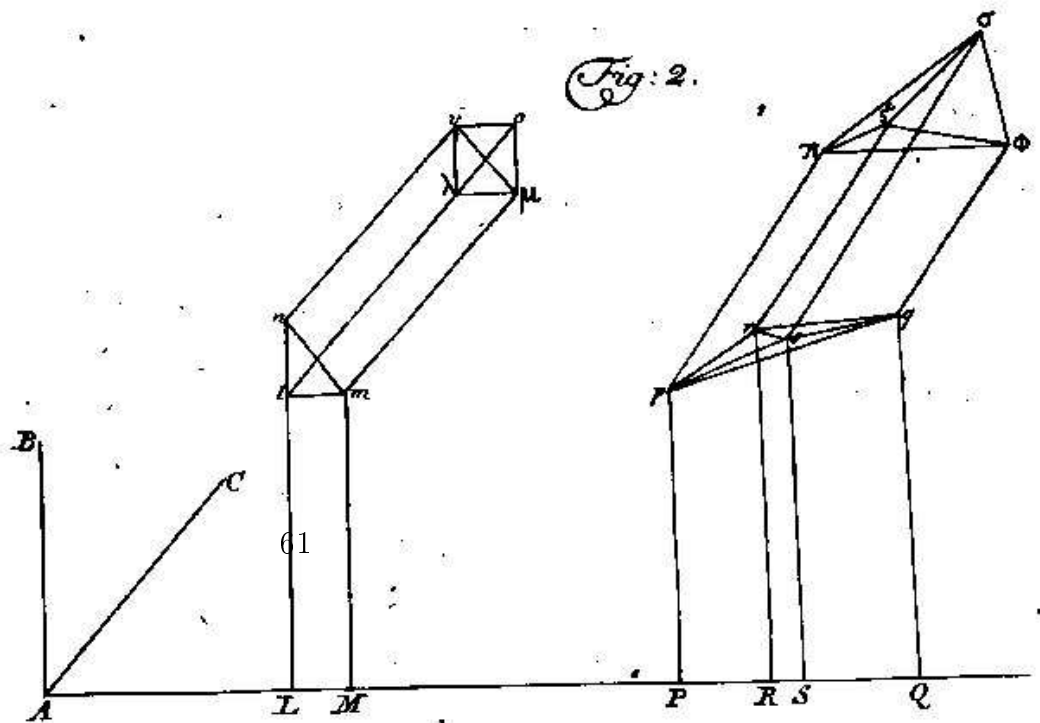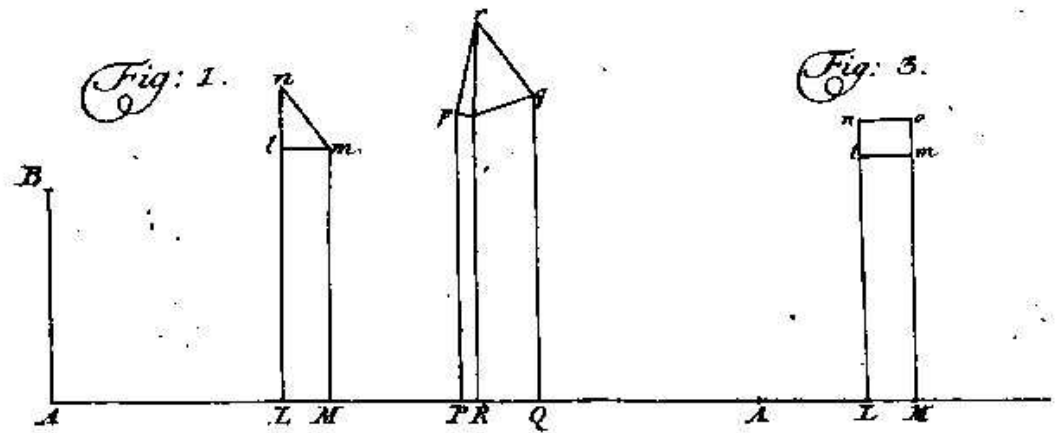
Translator's comments

I find that Euler's Latin is in general very precise. Almost all obscurities in his meaning or ambiguities in his equations can be resolved after careful parsing of his language. The following comments on individual paragraphs are meant to help the reader relate Euler's discoveries to modern treatments of inviscid incompressible fluids. This requires us to take advantage of mathematical concepts developed after his time.

Truesdell gives further discussion, extensive and illuminating, in his massive commentaries for the various parts of *Euleri Opera Omnia* XII.

Paragraph **17**: At this stage, using the determinant formula for the area of a triangle as it appears in high-school texts would immediately give the area as

$$
\frac{1}{2} \cdot
\begin{vmatrix}
0 & 0 & 1 \\
dx + Ldxdt & Mdxdt & 1 \\
ldydt & dy + mdydt & 1
\end{vmatrix}
$$

which $= \frac{1}{2}dxdy + \frac{1}{2}(L+m)dxdydt + \frac{1}{2}(Lm - lM)dxdydtdt$ and we can then conclude that $0 = L + m = \partial u/\partial x + \partial v/\partial y$. Euler is showing how to get this determinantal formula.

Paragraph **47**: The solution given by Euler says that the curl of velocity is zero: the flow is irrotational. The condition that $udx + vdy$ be a complete differential $= d\Phi$, for some potential function Φ, combined with the continuity equation, implies that the potential function Φ is harmonic.

How did Euler sneak in the assumption of irrotational flow? Note that the continuity equation gives $L + m = 0$. The rest of the partial differential equation says that the material derivative of $l - M$ vanishes:

$$\frac{d}{dt}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) = \frac{d}{dt}\xi = 0,$$

ξ being the vorticity. Euler then adopts $\xi = 0$ as the solution, thus singling out irrotational flow. Of course there are other solutions. What Euler has done is to show rigorously that irrotaional flow is a valid solution. The whole argument glows with the excitement of discovery of a *method*.

Paragraph **60**: The quantities $l - M$, $\lambda - N$, $\mu - n$ are the components of vorticity $\xi = \nabla \times \mathbf{U}$, except for order and sign. In modern vector notation, we would write the equations as

$$\frac{D\xi}{Dt} - \xi \cdot \nabla\mathbf{U} + [\nabla \cdot \mathbf{U}]\xi = 0,$$

where D/Dt is the material derivative. Euler's solution says that the components of the vorticity are all zero, i.e., the flow is irrotational. There are of course other solutions, just as there were in the planar case of paragraph **47**. Euler did come to understand that the irrotational case was only a special solution of the general problem. He had given it up by the time of his didactic treatise [E396] "Sectio Secunda de Principiis Motus Fluidorum" [*Novi commentarii academiae scientiarum Petropolitanae* 14 (1769, published 1770)].

Consider now an incompressible fluid. If we let $\mathbf{S}$ be the symmetric part of $\nabla\mathbf{U}$, i.e.

$$\mathbf{S} = \frac{1}{2}\left\{[\nabla\mathbf{U}] + [\nabla\mathbf{U}]^T\right\},$$

then $\xi\cdot\nabla\mathbf{U} = \mathbf{S}\xi$. Furthermore, the trace of $\mathbf{S}$ is 0. Thus, for an incompressible fluid, the vorticity equation becomes

$$\frac{\mathrm{D}\xi}{\mathrm{D}t} = \mathbf{S}\xi,$$

where the symmetric matrix $\mathbf{S}$ has at least one positive eigenvalue. This raises issues about the stability of the solution. These issues are ameliorated somewhat by the fact that in *planar* motion the eigenvector corresponding to the most negative eigenvalue is parallel to vorticity. In many other cases the largest positive eigenvalue can still be expected to have little effect. It may be that the supreme virtuoso of analytic manipulation had gone far enough to satisfy himself that the obvious solution $l - M = \lambda - N = \mu - n = 0$ was at least reasonable.

Paragraph **70**: In the condition in rule III: $A + B + C = 0$, the three components are the AA, BB, CC of paragraph **69**. The A, B, C... of this current paragraph are now real numbers.

For the general solution of Laplace's equation following Euler's line of reasoning, see section 18.3 in Whittaker and Watson's *Course of Modern Analysis*.

Paragraphs **75**-**77**: As Truesdell remarks, Euler fails to account here for all

second order terms and so misleads himself into "proving" that there can be no rotational fluid motion coinciding with the motion of a solid body. Indeed, there is a certain sloppiness in the passage. But mistakes by the residents of Olympus help us mortals understand how they think.

Paragraph **80**: The aim of this paragraph is to establish an accleration potential T whose spatial derivatives are the "motive forces" written out in paragraph **78**.

Paragraph **81**: For a well-defined pressure, the expression

$$p = C + \int (Qdx + qdy + \Phi dz) - T$$

needs to be a function of position independent of the path of the integral, and so $Qdx + qdy + \Phi dz$ must be a complete differential. Euler has thus established the necessity, at least locally. He finishes the sentence however with "*alioquin status aequilibrii, vel saltem possibilis, non daretur*" – "otherwise, a state of equilibrium, or at least a possible one, would not be given." He seems to say that any force field under consideration must be compatible with some possible equilibrium state. He may have d'Alembert's principle in mind. That principle, however, applies locally, and Euler is seeking a global condition. Truesdell points out that there are winding number issues that need a more careful analysis for resolution.

Paragraphs **88-90**: this final section is breath-taking.