ON THE INTEGRATION OF THE DIFFERENTIAL EQUATION

$$\frac{m\,dx}{\sqrt{(1-x^4)}} = \frac{n\,dy}{\sqrt{(1-y^4)}}$$

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[*Opera Omnia*, I_{20}, 58–79]

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SUMMARY

In this essay and several following, which treat similar topics, an entirely new field, as it were, is opened in Analysis, in which it becomes possible to compare with one another the integrals of various formulas which individually resist all techniques of integration. For example, by means of the known relationships among angles, the relation between the two variables x and y satisfying the differential equation

$$\frac{m\,dx}{\sqrt{(1-xx)}} = \frac{n\,dy}{\sqrt{(1-yy)}}$$

can be expressed algebraically, although neither formula can be integrated algebraically by itself, but rather expresses an angle or an arc of a circle. Now this relation can be seen to come simply from the fact that the sines of angles which have a given rational ratio to one another satisfy an algebraic relation. But it would seem that such a relation could not exist unless both formulas could be integrated, either in terms of angles or in terms of logarithms. To be sure, whenever the solution of any problem can be reduced to a differential equation of the form $X\,dx = Y\,dy$, where X is a function of x and Y a function of y, then, since the variables are separated from one another, the problem is generally considered to have been essentially solved, inasmuch as the solution could be constructed by means of the quadratures of two curves, of which the area of the one is expressed by $\int X\,dx$, the other by $\int Y\,dy$. Nevertheless, if, for any given value of x, it is required to determine the corresponding value of y, it appears that both quadratures would be required, without which there would be no way to express the relation between x and y. How much more astonishing, therefore, will it appear, that, even though the integral of the formula $\frac{dz}{\sqrt{(1-z^4)}}$ cannot be expressed either by means of angles or of logarithms, the only transcendental quantities thought suitable for such expressions, nevertheless for the proposed differential equation the relation between x and y can be exhibited algebraically, so that the curved line whose indefinite arc is expressed by the integral formula $\frac{dz}{\sqrt{(1-z^4)}}$ enjoys a property similar to that of the circle, namely that all its arcs can be compared with one another; or, if any of its arcs be given, it is possible to determine geometrically any other arc which has a given ratio with the first. Indeed, what amount to the same thing, the equation for the integral of the given differential equation, which gives the true relation between x and y, not only does not involve such an integral, but will in fact be algebraic.

And furthermore, this holds not just in a particular case, but, what is more, the complete integral, which involves an arbitrary constant quantity, will be algebraic. Nor indeed does this admirable result hold only for that particular differential equation, but in an entirely similar way the celebrated Author shows that this much more general differential equation

$$\frac{m\,dx}{\sqrt{(A+Bx^2+Cx^4)}} = \frac{n\,dy}{\sqrt{(A+By^2+Cy^4)}}$$

can be completely integrated by means of an algebraic equation, provided that the numbers m and n are rational; and indeed he extends the same method of integration to the even more general equation

$$\frac{m\,dx}{\sqrt{(A+Bx+Cx^2+Dx^3+Ex^4)}} = \frac{n\,dy}{\sqrt{(A+By+Cy^2+Dy^3+Ey^4)}},$$

where all the powers of x and y up to the fourth occur in the radicals in the denominators. From this, one might conjecture that the same algebraic integration would continue to hold, even if the expressions were continued to higher powers; but apart from the fact that the Author's method is limited to that fourth power, it is easy to show that, at least for the sixth power, an algebraic integration is in general excluded. For if the coefficients are taken in such a way that it is possible to extract the square root, it is clear just from the one case $\frac{m\,dx}{1+x^3} = \frac{n\,dy}{1+y^3}$ that there is no way to express the relation between x and y algebraically, inasmuch as the integrals of both formulas would involve angles as well as logarithms; but clearly angles and logarithms do not admit algebraic comparisons with one another. Meanwhile, on the other hand, by means of a particular method, the integration of the equation

$$\frac{m\,dx}{\sqrt{(A+Bx^2+Cx^4+Dx^6)}} = \frac{n\,dy}{\sqrt{(A+By^2+Cy^4+Dy^6)}}$$

will be shown algebraically, whence it is clear that this essay contains many more investigations that its title seems to indicate.

1. When, prompted by the discoveries of the illustrious Count FAGNANO, I first considered this equation, I found indeed an algebraic relation between the variables x and y which satisfied the equation;[1)] but this relation could not have been the equation of the complete integral, as it did not involve an arbitrary constant quantity, such as is always brought into the calculation by integration. Hence, as is well known, it is customary to distinguish between incomplete[2)] and particular integrals, of which the former exhaust the whole force of the differential equation, whereas the latter satisfy it, to be sure, though there are other expressions besides which satisfy it equally well. Moreover, the criterion for a complete integral consists in this, that it must involve a constant quantity which does not appear in the differential equation.

2. In order to make this clearer, it will suffice to consider the simplest differential equation $dx = dy$, which is certainly satisfied by the integral $x = y$; in fact, however, this integral is less general than the differential $dx = dy$, since that is satisfied also by the integral $x = y \mp a$, which is much more general, a being taken to be an arbitrary constant quantity; now this last integral can be seen to exhaust the whole force of the differential equation $dx = dy$, and thus we call it the equation of the complete integral, since it contains the constant quantity a which did not occur in the differential equation. But if in place of that indefinite constant a determinate values are substituted, we will obtain from the complete integral particular integrals, which for that very reason are less general than the given differential equation.

3. Now it is frequently possible to give an algebraic particular integral for a differential equation, even though the complete integral is transcendental; this comes about, evidently, if the transcendental part is multiplied by the arbitrary constant, so that if that constant is set equal to zero the transcendental part will vanish, and an algebraic particular integral will remain. Thus the equation $dy = dx + (y-x)\,dx$ is clearly satisfied by the value $y = x$, which, however, gives only a particular integral, since the complete integral is $y = x + ae^x$, where e denotes the number whose logarithm is $= 1$. Thus, unless the arbitrary constant a is made to vanish, the integral will always be transcendental.

4. Since therefore it can occur that a differential equation admits an algebraic particular integral, even though the complete integral is transcendental, there is also reason to suspect that the complete integral of the proposed differential equation

$$\frac{m\,dx}{\sqrt{(1-x^4)}} = \frac{n\,dy}{\sqrt{(1-y^4)}}$$

would involve transcendental quantities, although we can exhibit an algebraic particular integral for it. For the complete integral is

$$m\int\frac{dx}{\sqrt{(1-x^4)}} = n\int\frac{dy}{\sqrt{(1-y^4)}} + C;$$

1) See "Observationes de comparatione arcuum curvarum irrectificabilium" (E 252), Theorema 4 (*Opera Omnia*, I_{20}, p. 92). *Tr.*

2) The text has *incompleta*, but EULER seems to intend "complete". *Tr.*

but there is no way to determine these integrals, whether by means of the quadrature of the circle or of the hyperbola, so that it seems very unlikely that these formulas, whose nature is so transcendental, could be developed into an algebraic relation between x and y, as long as the constant C remains indeterminate.

5. It is known, however, that the complete integral of the differential equation

$$\frac{m\,dx}{\sqrt{(1-xx)}} = \frac{n\,dy}{\sqrt{(1-yy)}}$$

can always be given algebraically, so long as the ratio of the coefficients m and n is rational. But since the integral of each formula produces an arc of a circle, the complete integral will be $m\,\mathrm{A}\sin.x = n\,\mathrm{A}\sin.y + C$; and since the sines of arcs whose ratio to one another is rational have a mutual relation which can be expressed algebraically, it is not surprising that in these cases the complete integral can also be exhibited algebraically. Since however there is no similar way to work out the transcendental formulas $\int \frac{dx}{\sqrt{(1-x^4)}}$ and $\int \frac{dx}{\sqrt{(1-x^4)}}$, or at any rate none is known, the reduction of the integral to algebraic quantities cannot be expected.

6. Nonetheless, I discovered that, for the differential equation

$$\frac{m\,dx}{\sqrt{(1-x^4)}} = \frac{n\,dy}{\sqrt{(1-y^4)}},$$

once again the complete integral, which we know has to contain an arbitrary constant quantity, can always be expressed algebraically whenever the ratio $m:n$ is rational; which to me indeed seems the more noteworthy, in that I was not led to this integral by any definite method, but rather found it by guesswork, or by trial and error. On this account there is no doubt but that a direct method for evaluating this integral would enlarge the boundaries of Analysis not a little; therefore it seems appropriate to recommend to Analysts that it be investigated with all diligence.

7. Now I was able to get the complete integral of that differential equation, whenever the ratio of the coefficients m and n is rational, from the complete integral of the equation

$$\frac{dx}{\sqrt{(1-x^4)}} = \frac{dy}{\sqrt{(1-y^4)}};$$

for, this being granted, I will present a definite method by which the complete integral of the more general equation

$$\frac{m\,dx}{\sqrt{(1-x^4)}} = \frac{n\,dy}{\sqrt{(1-y^4)}}$$

can be obtained from it. More generally still, this method can also be applied to determine the integrals of equations of the form $mX\,dx = nY\,dy$, given the complete integral of $X\,dx = Y\,dy$, where Y is a function of y and X a function of x.

8. Let me therefore begin with the equation

$$\frac{dx}{\sqrt{(1-x^4)}} = \frac{dy}{\sqrt{(1-y^4)}}.$$

It is clear, indeed, at first glance, that this equation is satisfied by the equation $x = y$, which consequently is a particular integral of it. In addition, however, this same equation is also satisfied by the algebraic value

$$x = -\sqrt{\frac{1-yy}{1+yy}};$$

for since

$$dx = +\frac{2y\,dy}{(1+yy)\sqrt{(1-yy)(1+yy)}} \quad \text{and} \quad \sqrt{(1-x^4)} = \frac{2y}{1+yy},$$

it follows that

$$\frac{dx}{\sqrt{(1-x^4)}} = \frac{dy}{\sqrt{(1-y^4)}}.$$

Hence that value also, in other words the equation $xxyy + xx + yy - 1 = 0$, is a particular integral of the given differential equation. It follows that the complete integral, which has to contain an arbitrary constant, must be such that it produces

$$x = y$$

when that constant is assigned a certain value, but produces

$$x = -\sqrt{\frac{1-yy}{1+yy}} \quad \text{or} \quad xxyy + xx + yy - 1 = 0$$

when that same constant is assigned some other value.

THEOREM

9. *I say therefore that the complete integral of the differential equation*

$$\frac{dx}{\sqrt{(1-x^4)}} = \frac{dy}{\sqrt{(1-y^4)}}$$

is

$$xx + yy + ccxxyy = cc + 2xy\sqrt{(1-c^4)}.$$

PROOF

For by taking the differential of this equation, we will get

$$x\,dx + y\,dy + ccxy(x\,dy + y\,dx) = (x\,dy + y\,dx)\sqrt{(1-c^4)},$$

whence

$$dx\big(x + ccxyy - y\sqrt{(1-c^4)}\big) + dy\big(y + ccxxy - x\sqrt{(1-c^4)}\big) = 0.$$

But solving the same equation, we obtain

$$y = \frac{x\sqrt{(1-c^4)} + c\sqrt{(1-x^4)}}{1+ccxx} \quad \text{and} \quad x = \frac{y\sqrt{(1-c^4)} - c\sqrt{(1-y^4)}}{1+ccyy}.$$

For if the radical $\sqrt{(1-x^4)}$ is given the sign $+$, then the radical $\sqrt{(1-y^4)}$ must be given the sign $-$, so that setting $x = 0$ in both formulas the same value $y = c$ will be produced. Thus we will have

$$\begin{aligned} x + ccxyy - y\sqrt{(1-c^4)} &= -c\sqrt{(1-y^4)}, \\ y + ccxxy - x\sqrt{(1-c^4)} &= c\sqrt{(1-x^4)}, \end{aligned}$$

and these values being substituted into the differential equation, the result is

$$-c\,dx\sqrt{(1-y^4)} + c\,dy\sqrt{(1-x^4)} = 0$$

or

$$\frac{dx}{\sqrt{(1-x^4)}} = \frac{dy}{\sqrt{(1-y^4)}}.$$

Hence an integral of this differential equation is

$$xx + yy + ccxxyy = cc + 2xy\sqrt{(1-c^4)},$$

and since it contains the constant c which depends on our arbitrary choice, it will also be the complete integral. Q. E. D.

10. Therefore, for the equation $\frac{dx}{\surd(1-x^4)} = \frac{dy}{\surd(1-y^4)}$, the value of x coming from the complete integral is

$$x = \frac{y\surd(1-c^4) \pm c\surd(1-y^4)}{1+ccyy},$$

whence, if the arbitrary constant c vanishes, we get $x = y$; but if we put $c = 1$, we have $x = \pm\frac{\surd(1-y^4)}{1+yy} = \sqrt{\frac{1-yy}{1+yy}}$, which are the two particular values already mentioned above. From this formula can be obtained other particular values which are especially simple, though they may become imaginary. Thus by setting $c = \infty$ we get

$$x = \frac{\surd{-1}}{y}$$

and by setting $cc = -1$, we get

$$x = \sqrt{\frac{yy+1}{yy-1}},$$

which likewise satisfy the given equation.

11. But in order to make the nature of this integral more clearly evident, consider the curve AM (Fig. 1), having the following property: setting the abscissa $AP = u$, let the corresponding arc be $AM = \int \frac{du}{\surd(1-u^4)}$.

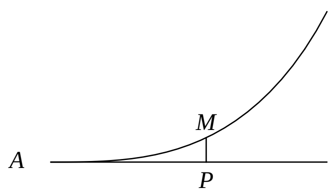

Fig. 1

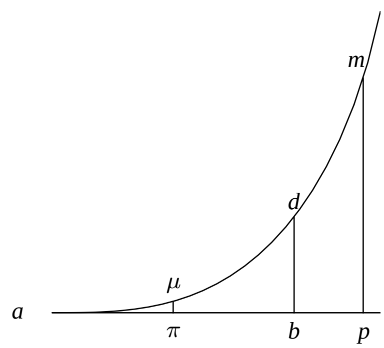

Fig. 2

Then, describing the same curve once again (Fig. 2), take the abscissa $ap = x$; so that arc $am = \int \frac{dx}{\surd(1-x^4)}$. Therefore, if we let

$$x = \frac{u\surd(1-c^4) \pm c\surd(1-u^4)}{1+ccuu},$$

it will follow that $\frac{dx}{\surd(1-x^4)} = \frac{du}{\surd(1-u^4)}$, and hence arc. am = arc. AM + Const. Determining the constant by setting $u = 0$, in which case the arc AM vanishes, produces $x = c$. Thus if the abscissa $ab = c$ is taken, to which the arc ad corresponds, it will follow that arc dm = arc AM.

12. By means, therefore, of the complete integration of the equation $\frac{dx}{\surd(1-x^4)} = \frac{du}{\surd(1-u^4)}$, it will be possible to cut off in the given curve an arc dm, beginning at a given point d, which is equal to any given arc AM, corresponding to the abscissa $AP = u$. For taking the abscissa corresponding to the given point d to be $ab = c$, if we take the abscissa

$$ap = x = \frac{c\surd(1-u^4) + u\surd(1-c^4)}{1+ccuu},$$

then the arc dm will be equal to the arc AM. But in the same way, inasmuch as $\surd(1-c^4)$ could be taken to be negative, if we take the abscissa

$$a\pi = \frac{c\surd(1-u^4) - u\surd(1-c^4)}{1+ccuu},$$

the arc $d\mu$ will likewise be equal to the arc AM, so that in this curve, from any given point d, it is possible to cut off arcs dm and $d\mu$ in both directions, which are equal to the arc AM.

13. From this it is therefore clear that if the arc ad is taken to be equal to the arc AM, or $c = u$, then the arc am will be double the arc AM. Hence if we let $ap = x = \frac{2u\sqrt{(1-u^4)}}{1+u^4}$, then arc $am = 2\,\text{arc.}\,AM$. Similarly, if we take arc $ad = 2AM$ or $c = \frac{2u\sqrt{(1-u^4)}}{1+u^4}$, and let $x = \frac{c\sqrt{(1-u^4)}+u\sqrt{(1-c^4)}}{1+ccuu}$, we will get arc $am = 3\,\text{arc.}\,AM$. And if that value of x is once again substituted for c, so that $ad = 3AM$, and we again let $x = \frac{c\sqrt{(1-u^4)}+u\sqrt{(1-c^4)}}{1+ccuu}$, the resulting arc am will be quadruple arc AM; and thus in succession any desired multiples of the arc AM can be determined geometrically.

14. Let the arc $ad = n \cdot AM$ and $ab = z$, so that

$$\int \frac{dz}{\sqrt{(1-z^4)}} = n \int \frac{du}{\sqrt{(1-u^4)}};$$

and from this it is clear, taking

$$x = \frac{z\sqrt{(1-u^4)} + u\sqrt{(1-z^4)}}{1+uuzz},$$

that

$$\int \frac{dx}{\sqrt{(1-x^4)}} = (n+1) \int \frac{du}{\sqrt{(1-u^4)}};$$

on the other hand, if we took

$$x = \frac{z\sqrt{(1-u^4)} - u\sqrt{(1-z^4)}}{1+uuzz},$$

then we would get

$$\int \frac{dx}{\sqrt{(1-x^4)}} = (n-1) \int \frac{du}{\sqrt{(1-u^4)}}.$$

If therefore the equation $\frac{dz}{\sqrt{(1-z^4)}} = \frac{n\,du}{\sqrt{(1-u^4)}}$ is integrated to get the required value of z, then it will also be possible to integrate the equation $\frac{dx}{\sqrt{(1-x^4)}} = \frac{(n\pm1)\,du}{\sqrt{(1-u^4)}}$, and indeed the integral will be $x = \frac{z\sqrt{(1-u^4)}\pm u\sqrt{(1-z^4)}}{1+uuzz}$. Thus, assuming that we have the complete value of z, which, as we know, must involve an arbitrary constant, then we can also get the complete value of x.

15. From this it is therefore clear how the complete integral belonging to the differential equation $\frac{dx}{\sqrt{(1-x^4)}} = \frac{n\,du}{\sqrt{(1-u^4)}}$ should be found, so long as n is a whole number. But in a similar way it will be possible to determine y so that $\frac{dy}{\sqrt{(1-y^4)}} = \frac{m\,du}{\sqrt{(1-u^4)}}$; whence, if an equation between x and y is obtained by eliminating u, it will be an integral of the equation $\frac{m\,dx}{\sqrt{(1-x^4)}} = \frac{n\,dy}{\sqrt{(1-y^4)}}$, no matter which rational numbers are substituted for m and n; and in order that this be the complete integral, it will suffice on the other hand just to have determined the complete values of x and y in terms of u, since in this way a new arbitrary constant will be brought into the calculation.

16. Although the method which I have used in the proof of this theorem is not derived from the nature of the problem, but rather leads indirectly to the desired result, it is nevertheless of much broader applicability; for in a similar way we can determine that the complete integral of the differential equation

$$\frac{dx}{\sqrt{(1+mxx+nx^4)}} = \frac{dy}{\sqrt{(1+myy+ny^4)}}$$

is

$$0 = cc - xx - yy + nccxxyy + 2xy\sqrt{(1+mcc+nc^4)}.$$

And in turn, by applying the previous reasoning, we can also get the complete integral of the equation

$$\frac{\mu\,dx}{\sqrt{(1+mxx+nx^4)}} = \frac{\nu\,dy}{\sqrt{(1+myy+ny^4)}},$$

where the letters μ and ν designate whole numbers.

17. Now the study of this integration can be carried out as follows: Let us first set up arbitrarily a relation between the variables x and y given by the equation

$$(1) \qquad \alpha xx + \alpha yy = 2\beta xy + \gamma xxyy + \delta,$$

which by differentiation gives

$$\alpha x\,dx + \alpha y\,dy = \beta x\,dy + \beta y\,dx + \gamma xyy\,dx + \gamma xxy\,dy,$$

whence, collecting terms,

$$(2) \qquad dx(\alpha x - \beta y - \gamma xyy) + dy(\alpha y - \beta x - \gamma xxy) = 0.$$

Then, in equation (1), solve for the values of each of the variables

$$x = \frac{\beta y + \sqrt{(\alpha\delta + (\beta\beta - \alpha\alpha - \gamma\delta)yy + \alpha\gamma y^4)}}{\alpha - \gamma yy},$$
$$y = \frac{\beta x - \sqrt{(\alpha\delta + (\beta\beta - \alpha\alpha - \gamma\delta)xx + \alpha\gamma x^4)}}{\alpha - \gamma xx}.$$

Thus we get

$$(3) \qquad \alpha x - \beta y - \gamma xyy = \sqrt{(\alpha\delta + (\beta\beta - \alpha\alpha - \gamma\delta)yy + \alpha\gamma y^4)},$$
$$(4) \qquad \alpha y - \beta x - \gamma xxy = -\sqrt{(\alpha\delta + (\beta\beta - \alpha\alpha - \gamma\delta)xx + \alpha\gamma x^4)}.$$

Substituting in equation (2), this gives

$$(5) \qquad \frac{dx}{\sqrt{(\alpha\delta + (\beta\beta - \alpha\alpha - \gamma\delta)xx + \alpha\gamma x^4)}} = \frac{dy}{\sqrt{(\alpha\delta + (\beta\beta - \alpha\alpha - \gamma\delta)yy + \alpha\gamma y^4)}},$$

so that the integral of this equation is equation (1).

18. In order to put this into a simpler form, let us take

$$\alpha\delta = A, \quad \beta\beta - \alpha\alpha - \gamma\delta = C, \quad \alpha\gamma = E,$$

whence we will have

$$\delta = \frac{A}{\alpha}, \quad \gamma = \frac{E}{\alpha}, \quad \text{and} \quad \beta = \sqrt{\left(C + \alpha\alpha + \frac{AE}{\alpha\alpha}\right)}.$$

It follows that the equation for the integral of the differential equation

$$(6) \qquad \frac{dx}{\sqrt{(A + Cxx + Ex^4)}} = \frac{dy}{\sqrt{(A + Cyy + Ey^4)}}$$

is

$$(7) \qquad \alpha(xx + yy) = \frac{A}{\alpha} + \frac{E}{\alpha}xxyy + 2xy\sqrt{\left(C + \alpha\alpha + \frac{AE}{\alpha\alpha}\right)},$$

which is also the complete integral.

19. Now let us take

$$A = f\alpha\alpha, \quad C = g\alpha\alpha \quad \text{and} \quad E = h\alpha\alpha,$$

to get the differential equation

$$\frac{dx}{\sqrt{(f+gxx+hx^4)}} = \frac{dy}{\sqrt{(f+gyy+hy^4)}},$$

of which consequently the equation of the complete integral will be

$$xx+yy = f + hxxyy + 2xy\sqrt{(1+g+fh)};$$

which, although it does not appear to involve a new constant, is nevertheless complete, since in the differential it is only the ratio of the quantities f, g, and h that matters, so that it is permissible to write fcc, gcc and hcc, whence we get the obviously complete integral

$$xx+yy = fcc + hccxxyy + 2xy\sqrt{(1+gcc+fhc^4)}$$

or

$$f(xx+yy) = fee + heexxyy + 2xy\sqrt{f(f+gee+he^4)},$$

where $cc = \frac{ee}{f}$.

20. So if the differential equation

$$\frac{dx}{\sqrt{(f+gxx+hx^4)}} = \frac{dy}{\sqrt{(f+gyy+hy^4)}}$$

is given, the value of y can be expressed as an algebraic function of x, namely[1)]

$$y = \frac{x\sqrt{(1+gcc+fhc^4)} \pm c\sqrt{(1+gxx+fhx^4)}}{1-hccxx}$$

or

$$y = \frac{x\sqrt{f(f+gee+he^4)} \pm e\sqrt{f(f+gxx+hx^4)}}{f-heexx}.$$

And in particular if we set $g=0$, getting the differential equation

$$\frac{dx}{\sqrt{(f+hx^4)}} = \frac{dy}{\sqrt{(f+hy^4)}},$$

the value of y from the complete integral will be

$$y = \frac{x\sqrt{f(f+he^4)} \pm e\sqrt{f(f+hx^4)}}{f-heexx},$$

whence, fixing the constant e as we like, innumerable particular values for y can be found.

21. Furthermore, by the method I used above, it will also be possible, if m and n are rational numbers, to exhibit the complete integral of the equation

$$\frac{m\,dx}{\sqrt{(f+gxx+hx^4)}} = \frac{n\,dy}{\sqrt{(f+gyy+hy^4)}},$$

and in fact that complete integral is algebraic.

22. Although, in the equation which was considered above, the variables x and y were set up symmetrically, so that the two formulas came out similar to one another, by putting aside that restriction we will come to the study of formulas involving unlike differentials. Let us therefore take

$$(1) \qquad \alpha xx + \beta yy = 2\gamma xy + \delta xxyy + \varepsilon,$$

1) The second term in the numerator of the first equation should be $\pm c\sqrt{(f+gxx+hx^4)}$. *Tr.*

whence we have

$$x = \frac{\gamma y + \sqrt{\left(\alpha\varepsilon + (\gamma\gamma - \delta\varepsilon - \alpha\beta)yy + \beta\delta y^4\right)}}{\alpha - \delta yy}$$

and

$$y = \frac{\gamma x - \sqrt{\left(\beta\varepsilon + (\gamma\gamma - \delta\varepsilon - \alpha\beta)xx + \alpha\delta y^4\right)}}{\beta - \delta xx}$$

and hence

$$(2) \qquad \alpha x - \gamma y - \delta xyy = \sqrt{\left(\alpha\varepsilon + (\gamma\gamma - \delta\varepsilon - \alpha\beta)yy + \beta\delta y^4\right)},$$

$$(3) \qquad \beta y - \gamma x - \delta xxy = -\sqrt{\left(\beta\varepsilon + (\gamma\gamma - \delta\varepsilon - \alpha\beta)xx + \alpha\delta x^4\right)};$$

now differentiating equation (1) gives

$$dx(\alpha x - \gamma y - \delta xyy) + dy(\beta y - \gamma x - \delta xxy) = 0,$$

whence we obtain the differential equation

$$\frac{dx}{\sqrt{\left(\beta\varepsilon + (\gamma\gamma - \delta\varepsilon - \alpha\beta)xx + \alpha\delta x^4\right)}} = \frac{dy}{\sqrt{\left(\alpha\varepsilon + (\gamma\gamma - \delta\varepsilon - \alpha\beta)yy + \beta\delta y^4\right)}},$$

whose integral therefore is the given equation.

23. Indeed, the disparity here is easily removed by substituting $z\sqrt{\frac{\alpha}{\beta}}$ in place of y, the reason for which could have been seen immediately from the equations we began with. But there is another way to produce dissimilar formulas, of which the following example should be sufficient. Let the equation

$$x^4 + 2axxyy + 2bxx = c$$

be taken, whose differential is

$$dx(x^3 + axyy + bx) + axxy\,dy = 0$$

or

$$\frac{dx}{xy} = \frac{-a\,dy}{xx + ayy + b}.$$

Now first let xy be found in terms of x from the original equation, so that

$$xy = \sqrt{\frac{c - 2bxx - x^4}{2a}},$$

then $xx + ayy + b$ in terms of y; but from $(xx + ayy + b)^2 = c + (ayy + b)^2$ we will get

$$xx + ayy + b = \sqrt{\left(c + (ayy + b)^2\right)}.$$

From this we will have the differential equation

$$\frac{dx\sqrt{2a}}{\sqrt{(c - 2bxx - x^4)}} = \frac{-a\,dy}{\sqrt{(c + bb + 2abyy + aay^4)}},$$

of which therefore the integral is the given equation, or $y = \frac{\sqrt{(c-2bxx-x^4)}}{x\sqrt{2a}}$.

24. And although this integral is not complete, nevertheless from the above formulas the complete integral can easily be obtained. For suppose we take

$$\frac{a\,dy}{\sqrt{(c + bb + 2abyy + aay^4)}} = \frac{a\,dz}{\sqrt{(c + bb + 2abzz + aaz^4)}};$$

from $f = c + bb$, $g = 2ab$, $h = aa$, we will get

$$y = \frac{z\sqrt{(c+bb)(c+bb+2abee+aae^4)} \pm e\sqrt{(c+bb)(c+bb+2abzz+aaz^4)}}{c+bb-aaeezz};$$

therefore let this value be set equal to $\frac{\sqrt{(c-2bxx-x^4)}}{x\sqrt{2a}}$, and the resulting equation between x and z will be the complete integral of the differential equation

$$\frac{dx\sqrt{2a}}{\sqrt{(c-2bxx-x^4)}} = \frac{-a\,dz}{\sqrt{(c+bb+2abzz+aaz^4)}}.$$

Furthermore, from what has been done it will clear how the complete integral should be found, if the two sides are also multiplied by any rational numbers whatever.

25. Now, putting aside unlike members, let us consider the formation of similar members in a more general way; let us therefore take

$$(1) \qquad 0 = \alpha + 2\beta(x+y) + \gamma(xx+yy) + 2\delta xy + 2\varepsilon xy(x+y) + \zeta xxyy,$$

whence by differentiation we will get

$$dx(\beta + \gamma x + \delta y + 2\varepsilon xy + \varepsilon yy + \zeta xyy) + dy(\beta + \gamma y + \delta x + 2\varepsilon xy + \varepsilon xx + \zeta xxy) = 0$$

and therefore

$$(2) \qquad \frac{dy}{\beta + \gamma x + \delta y + 2\varepsilon xy + \varepsilon yy + \zeta xyy} = \frac{-dx}{\beta + \gamma y + \delta x + 2\varepsilon xy + \varepsilon xx + \zeta xxy}.$$

But solving the original equation, we get

$$y = \frac{-\beta-\delta x-\varepsilon xx \pm \sqrt{\left(\beta\beta-\alpha\gamma+2(\beta\delta-\alpha\varepsilon-\beta\gamma)x+(\delta\delta-\gamma\gamma-\alpha\zeta-2\beta\varepsilon)xx+2(\delta\varepsilon-\beta\zeta-\gamma\varepsilon)x^3+(\varepsilon\varepsilon-\gamma\zeta)x^4\right)}}{\gamma+2\varepsilon x+\zeta xx}.$$

Putting, for the sake of brevity,

$$\begin{matrix} \beta\beta - \alpha\gamma = A, & \beta\delta - \alpha\varepsilon - \beta\gamma = B, \\ \varepsilon\varepsilon - \gamma\zeta = E, & \delta\varepsilon - \beta\zeta - \gamma\varepsilon = D, \end{matrix} \quad \delta\delta - \gamma\gamma - \alpha\zeta - 2\beta\varepsilon = C$$

we will have

$$\beta + \delta x + \varepsilon xx + \gamma y + 2\varepsilon xy + \zeta xxy = \pm\sqrt{(A + 2Bx + Cxx + 2Dx^3 + Ex^4)},$$
$$\beta + \delta y + \varepsilon yy + \gamma x + 2\varepsilon xy + \zeta xyy = \mp\sqrt{(A + 2By + Cyy + 2Dy^3 + Ey^4)}.$$

26. We conclude therefore that for the differential equation

$$\frac{dx}{\sqrt{(A + 2Bx + Cxx + 2Dx^3 + Ex^4)}} = \frac{dy}{\sqrt{(A + 2By + Cyy + 2Dy^3 + Ey^4)}},$$

the equation of the complete integral is

$$0 = \alpha + 2\beta(x+y) + \gamma(xx+yy) + 2\delta xy + 2\varepsilon xy(x+y) + \zeta xxyy,$$

where of course the coefficients must be determined as above. To do this, first let β or ε be defined by the equation

$$\frac{BB(\varepsilon\varepsilon - E) - DD(\beta\beta - A)}{A\varepsilon\varepsilon - E\beta\beta} + \frac{2AD\varepsilon - 2BE\beta}{B\varepsilon - D\beta} = C;$$

then we will have

$$\gamma = \frac{A\varepsilon\varepsilon - E\beta\beta}{B\varepsilon - D\beta}, \quad \alpha = \frac{\beta\beta - A}{\gamma}, \quad \zeta = \frac{\varepsilon\varepsilon - E}{\gamma}$$

and

$$\delta = \frac{B\beta(\varepsilon\varepsilon - E) - D\varepsilon(\beta\beta - A)}{A\varepsilon\varepsilon - E\beta\beta} + \gamma \quad \text{or} \quad \delta = \gamma + \frac{B + \alpha\varepsilon}{\beta}.$$

27. Hence therefore it is clear that the differential equation

$$\frac{dx}{\sqrt{(A + 2Dx^3)}} = \frac{dy}{\sqrt{(A + 2Dy^3)}}$$

can also be integrated; for from $B = 0$, $C = 0$ and $E = 0$ we will get

$$\frac{-DD(\beta\beta - A)}{A\varepsilon\varepsilon} - \frac{2A\varepsilon}{\beta} = 0 \quad \text{or} \quad \varepsilon = \sqrt[3]{\frac{DD}{2AA}\beta(A - \beta\beta)},$$

but the values we get in this way are too complicated. An easier computation will result from solving the equations for the vanishing letters B, C and E; for

$$E = 0 \quad \text{gives} \quad \zeta = \frac{\varepsilon\varepsilon}{\gamma}; \quad \text{then} \quad B = 0 \quad \text{gives} \quad \delta = \gamma + \frac{\alpha\varepsilon}{\beta}$$

and

$$C = 0 \quad \text{gives} \quad \delta\delta - \gamma\gamma = \alpha\zeta + 2\beta\varepsilon = \frac{\alpha\varepsilon\varepsilon}{\gamma} + 2\beta\varepsilon = \frac{\alpha^2\varepsilon\varepsilon}{\beta\beta} + \frac{2\alpha\gamma\varepsilon}{\beta},$$

whose factors are $\beta\beta = \alpha\gamma$ and $\alpha\varepsilon\varepsilon + 2\beta\gamma\varepsilon = 0$. But if $\beta\beta = \alpha\gamma$, then $A = 0$; on the other hand if $\varepsilon = 0$, then $\zeta = 0$ and $D = 0$, contrary to our intention. We must therefore have $\alpha\varepsilon = -2\beta\gamma$; whence

$$\alpha = -\frac{2\beta\gamma}{\varepsilon}, \quad \delta = -\gamma \quad \text{and} \quad \zeta = \frac{\varepsilon\varepsilon}{\gamma}.$$

Finally,

$$\beta\beta + \frac{2\beta\gamma\gamma}{\varepsilon} = A \quad \text{and} \quad -2\gamma\varepsilon - \frac{\beta\varepsilon\varepsilon}{\gamma} = D.$$

From this we have $\varepsilon = \frac{2\beta\gamma\gamma}{A - \beta\beta}$, and from $\frac{\gamma D}{\varepsilon} = -(2\gamma\gamma + \beta\varepsilon)$ and $2\gamma\gamma + \beta\varepsilon = \frac{A\varepsilon}{\beta}$ we will get $\frac{\gamma D}{\varepsilon} = -\frac{A\varepsilon}{\beta}$, and hence $\varepsilon\varepsilon = -\frac{\beta\gamma D}{A}$. Therefore

$$\frac{4\beta\gamma^3}{(A - \beta\beta)^2} + \frac{D}{A} = 0.$$

28. Now although only the ratio of the letters A and D affects the outcome, this last equation will enable us to find the actual value of A itself, which, however, it is not necessary to know. The letters γ and β will therefore remain indeterminate. Therefore let

$$\gamma = -Ac \quad \text{and} \quad \beta = Dc;$$

then $\varepsilon\varepsilon = DDcc$ or

$$\varepsilon = Dc \quad \text{so that} \quad \delta = Ac, \quad \zeta = -\frac{DDc}{A} \quad \text{and} \quad \alpha = 2Ac.$$

Thus an integral of the differential equation

$$\frac{dx}{\sqrt{(A + 2Dx^3)}} = \frac{dy}{\sqrt{(A + 2Dy^3)}}$$

is

$$0 = 2A + 2D(x+y) - A(xx+yy) + 2Axy + 2Dxy(x+y) - \frac{DD}{A}xxyy.$$

Again, this integral is not complete, but can be made so by taking $\gamma = -A$ and $\beta = Dcc$, whence $\varepsilon\varepsilon = DDcc$ and $\varepsilon = Dc$; next we will have $\delta = A$, $\zeta = -\frac{DDcc}{A}$, $\alpha = 2Ac$, so that the complete integral is

$$0 = 2Ac + 2Dcc(x+y) - A(xx+yy) + 2Axy + 2Dcxy(x+y) - \frac{DDcc}{A}xxyy,$$

where c is a constant depending on our arbitrary choice; whence

$$y = \frac{Dcc + Ax + Dcxx \pm \sqrt{c\left(2A + \frac{DD}{A}c^3\right)\left(A + 2Dx^3\right)}}{A - 2Dcx + \frac{DDcc}{A}xx}.$$

29. The case where $A = 1$ and $D = \frac{1}{2}$ deserves to be mentioned, in which case we will have the differential equation

$$\frac{dx}{\sqrt{(1+x^3)}} = \frac{dy}{\sqrt{(1+y^3)}},$$

where, in order to remove fractions, write $2c$ in place of c, and the complete integral will be

$$0 = 4c + 4cc(x+y) - xx - yy + 2xy + 2cxy(x+y) - ccxxyy$$

or

$$y = \frac{2cc + x + cxx \pm 2\sqrt{c(1+c^3)(1+x^3)}}{1 - 2cx + ccxx}.$$

Therefore some particular integrals will be

I. if $c = 0$, $\quad y = x$;

II. if $c = \infty$, $\quad y = \frac{2 \pm 2\sqrt{(1+x^3)}}{xx}$;

III. if $c = -1$, $\quad y = \frac{2 + x - xx}{1 + 2x + xx} = \frac{2-x}{1+x}$.

30. By the same token, if in §26 the letters A, B, C, D, E are each multiplied by any quantity p, the differential equation will still be

$$\frac{dx}{\sqrt{(A + 2Bx + Cxx + 2Dx^3 + Ex^4)}} = \frac{dy}{\sqrt{(A + 2By + Cyy + 2Dy^3 + Ey^4)}}$$

and it will be found that

$$p = \frac{BB\varepsilon\varepsilon - DD\beta\beta}{BBE - ADD} + 2\frac{(AD\varepsilon - BE\beta)(A\varepsilon\varepsilon - E\beta\beta)}{(B\varepsilon - D\beta)(BBE - ADD)} - \frac{C(A\varepsilon\varepsilon - E\beta\beta)}{BBE - ADD};$$

then

$$\gamma = \frac{A\varepsilon\varepsilon - E\beta\beta}{B\varepsilon - D\beta}, \quad \alpha = \frac{\beta\beta - Ap}{\gamma}, \quad \zeta = \frac{\varepsilon\varepsilon - Ep}{\gamma}, \quad \text{and} \quad \delta = \gamma + \frac{\alpha\varepsilon + Bp}{\beta},$$

so, keeping the letters β and ε indeterminate, the equation of the integral will become

$$0 = \alpha + 2\beta(x+y) + \gamma(xx+yy) + 2\delta xy + 2\varepsilon xy(x+y) + \zeta xxyy,$$

whence

$$y = \frac{-\beta - \delta x - \varepsilon xx \pm \sqrt{p(A + 2Bx + Cxx + 2Dx^3 + Ex^4)}}{\gamma + 2\varepsilon x + \zeta xx}.$$

31. Finally it should be noted that not only the differential equation whose complete integral I have just presented, but also this much more general one

$$\frac{m\,dx}{\sqrt{(A+2Bx+Cxx+2Dx^3+Ex^4)}}=\frac{n\,dy}{\sqrt{(A+2By+Cyy+2Dy^3+Ey^4)}}$$

can always be integrated algebraically and nevertheless completely, as long as the ratio of the coefficients m and n is rational; for this integration is carried out in a way similar to that which I used earlier to integrate the equation I originally started with. Indeed this method, of which I have given examples here, seems to me to be of such a nature, that through diligent cultivation it would be apt to produce significant applications, whence benefits by no means to be despised would redound to Analysis.

32. Here, however, I note that by generalizing the formula given in §26, it will be possible to compare differentials which are unlike, and indeed the example of dissimilarity already given (§22) could have been obtained in this way, so that everything which has been presented so far will be contained in the following general investigation. In particular, take the following equation representing an integral

$$(1)\qquad \alpha xxyy+2\beta xxy+2\gamma xyy+\delta xx+\varepsilon yy+2\zeta xy+2\eta x+2\theta y+\varkappa=0,$$

from which we have

$$(2)\qquad y=\frac{-\beta xx-\zeta x-\theta+\sqrt{\big((\beta xx+\zeta x+\theta)^2-(\alpha xx+2\gamma x+\varepsilon)(\delta xx+2\eta x+\varkappa)\big)}}{\alpha xx+2\gamma x+\varepsilon},$$

$$(3)\qquad x=\frac{-\gamma yy-\zeta y-\eta-\sqrt{\big((\gamma yy+\zeta y+\eta)^2-(\alpha yy+2\beta y+\delta)(\varepsilon yy+2\theta y+\varkappa)\big)}}{\alpha yy+2\beta y+\delta}.$$

Next for the sake of brevity put

$$\begin{array}{rl|rl}
App &= \beta\beta-\alpha\delta & \mathfrak{A}qq &= \gamma\gamma-\alpha\varepsilon\\
2Bpp &= 2\beta\zeta-2\alpha\eta-2\gamma\delta & 2\mathfrak{B}qq &= 2\gamma\zeta-2\alpha\theta-2\beta\varepsilon\\
Cpp &= \zeta\zeta+2\beta\theta-\alpha\varkappa-\delta\varepsilon-4\gamma\eta & \mathfrak{C}qq &= \zeta\zeta+2\gamma\eta-\alpha\varkappa-\delta\varepsilon-4\beta\theta\\
2Dpp &= 2\zeta\theta-2\gamma\varkappa-2\varepsilon\eta & 2\mathfrak{D}qq &= 2\zeta\eta-2\beta\varkappa-2\delta\theta\\
Epp &= \theta\theta-\varepsilon\varkappa & \mathfrak{E}qq &= \eta\eta-\delta\varkappa
\end{array}$$

and we will get

$$(4)\qquad p\sqrt{(Ax^4+2Bx^3+Cxx+2Dx+E)}=\alpha xxy+2\gamma xy+\varepsilon y+\beta xx+\zeta x+\theta,$$

$$(5)\qquad -q\sqrt{(\mathfrak{A}y^4+2\mathfrak{B}y^3+\mathfrak{C}yy+2\mathfrak{D}y+\mathfrak{E})}=\alpha xyy+2\beta xy+\delta x+\gamma yy+\zeta y+\eta.$$

33. But if the given equation for the integral is differentiated, the result is

$$(6)\qquad \begin{aligned} &dx(\alpha xyy+2\beta xy+\gamma yy+\delta x+\zeta y+\eta)\\ &+dy(\alpha xxy+\beta xx+2\gamma xy+\varepsilon y+\zeta x+\theta)=0,\end{aligned}$$

whence, if the values for the factors found in (4) and (5) are substituted, the result will be the following differential equation

$$(7)\qquad \frac{q\,dx}{\sqrt{(Ax^4+2Bx^3+Cxx+2Dx+E)}}=\frac{p\,dy}{\sqrt{(\mathfrak{A}y^4+2\mathfrak{B}y^3+\mathfrak{C}yy+2\mathfrak{D}y+\mathfrak{E})}},$$

of which the integral is therefore the given equation (1).

But since we have 10 equations above, while there are 9 coefficients α, β, γ, δ etc., of which one can be taken arbitrarily, there will remain eight letters to be determined. Next, however, the two letters p and q must also be defined, so that there are now ten unknown quantities, from which it seems that the coefficients

A, B, C, D, E and $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, $\mathfrak{E}$ of the two formulas could be taken arbitrarily. It is clear, however, that if the one set is taken arbitrarily, the others could not be completely arbitrary, for otherwise any formula could be reduced to an algebraic one.

34. At this point, however, it is possible to obtain other, not inelegant, transformations of the given formula, if in place of y we substitute other values. For example, if we put $\mathfrak{E} = 0$, or $\eta\eta = \delta\varkappa$, and set $y = zz$, the result will be the following differential equation

$$\text{(8)} \qquad \frac{q\,dx}{\sqrt{(Ax^4 + 2Bx^3 + Cxx + 2Dx + E)}} = \frac{2p\,dz}{\sqrt{(\mathfrak{A}z^6 + 2\mathfrak{B}z^4 + \mathfrak{C}z^2 + 2\mathfrak{D})}},$$

of which, therefore, the integral is the given equation, if we take $y = zz$ and set $\eta\eta = \delta\varkappa$, and the remaining letters are determined accordingly. Furthermore, the complete integral can be found without difficulty; for even though as it happens the integral we have just found does not involve a new constant, let

$$\frac{q\,dx}{\sqrt{(Ax^4 + 2Bx^3 + Cxx + 2Dx + E)}} = \frac{q\,du}{\sqrt{(Au^4 + 2Bu^3 + Cuu + 2Du + E)}}$$

and the complete integral of this equation can be determined from the foregoing, so that we can construct the complete integral, including constant, of the equation having unlike sides.

35. Just as the complete integral of this equation

$$\frac{dx}{\sqrt{(f + gx)}} = \frac{dy}{\sqrt{(f + gy)}},$$

to begin with the simplest, is

$$gg(xx + yy) - 2ggxy - 2ccg(x + y) + c^4 - 4ccf = 0,$$

next, the complete integral of the differential equation

$$\frac{dx}{\sqrt{(f + gxx)}} = \frac{dy}{\sqrt{(f + gyy)}}$$

is

$$xx + yy - 2xy\sqrt{(1 + fgcc)} - ccff = 0,$$

third, the complete integral of the differential equation

$$\frac{dx}{\sqrt{(f + gx^3)}} = \frac{dy}{\sqrt{(f + gy^3)}}$$

is

$$f(xx + yy) + \frac{ggcc}{4f}xxyy - gcxy(x + y) - 2fxy - gcc(x + y) - 2fc = 0,$$

fourth, the complete integral found for the differential equation

$$\frac{dx}{\sqrt{(f + gx^4)}} = \frac{dy}{\sqrt{(f + gy^4)}}$$

is

$$f(xx + yy) - fcc - gccxxyy - 2xy\sqrt{f(f + gc^4)} = 0,$$

and in addition the complete integral of the equation

$$\frac{dx}{\sqrt{(f + gx^6)}} = \frac{dy}{\sqrt{(f + gy^6)}}$$

can also be found.

36. Let the values be determined, as in §33, which would produce the equation

$$\frac{dx}{\sqrt{(fx+gx^4)}} = \frac{dy}{\sqrt{(fy+gy^4)}},$$

of which the complete integral turns out to be

$$gg(xx+yy) - 4ggcxxyy - 4fgccxy(x+y) - 2ggxy - 2fgc(x+y) + ffcc = 0.$$

Now set $x = tt$ and $y = uu$, to get the differential equation

$$\frac{dt}{\sqrt{(f+gt^6)}} = \frac{du}{\sqrt{(f+gu^6)}},$$

the complete integral of which will therefore be

$$gg(t^4+u^4) - 4ggct^4u^4 - 4fgccttuu(tt+uu) - 2ggttuu - 2fgc(tt+uu) + ffcc = 0;$$

here the case resulting from the hypothesis $c = \infty$ should be mentioned, which gives

$$4gttuu(tt+uu) = f.$$