An observation on the sums of divisors*

Leonhard Euler†

1. For any number \( n \), I define the expression \( \int n \) to denote the sum of all the divisors of \( n \). Thus for unity, which aside from itself has no other divisors, it will be \( \int 1 = 1 \). Since a prime number has two divisors, unity and itself, if \( n \) is a prime number, then \( \int n = 1 + n \). Next, for a perfect number, which is equal to the sum of its aliquot parts, where the aliquot parts are the divisors of the number aside from the number itself, it is clear that the sum of its divisors are twice as large as it; so if \( n \) is a perfect number, then \( \int n = 2n \). Furthermore, since it is usual for a number to be called excessive when the sum of its aliquot parts is larger than it, if \( n \) is an excessive number, then \( \int n > 2n \), and if \( n \) is a deficient number, such that the sum of its aliquot parts is less than it, then \( \int n < 2n \).

2. Thus in this way the characters of numbers, at least as far as those that can be expressed by the sum of their aliquot parts, or divisors, is easily related with a symbol. Indeed, if \( \int n = 1 + n \), then \( n \) is a prime number, if \( \int n = 2n \) then \( n \) is a perfect number, and if \( \int n > 2n \) or \( \int n < 2n \), then \( n \) is respectively excessive or deficient. It is also possible for another question to be considered, about numbers which are called amicable, where the sum of the aliquot parts of the one is equal to the other. Namely, if \( m \) and \( n \) are amicable numbers, where for \( m \) the sum of its aliquot parts is equal to

---


†Date of translation: December 5, 2004. Translated from the Latin by Jordan Bell, 2nd year undergraduate in Honours Mathematics, School of Mathematics and Statistics, Carleton University, Ottawa, Ontario, Canada. Email: jbell3@connect.carleton.ca. Part of this translation was written during an NSERC USRA supervised by Dr. B. Stevens.
\[ m - m \text{ and for the number } n \text{ it is equal to } n - n, \text{ it will be had that } \int m = \int n = m + n. \text{ Therefore two amicable numbers have the same sum of divisors, which is furthermore equal to the sum of both the numbers.} \]

3. The sum of the divisors of any given number can be found without much difficulty, and it is the easiest when the number is made of two factors which between themselves are prime. For if \( p \) and \( q \) are prime numbers between themselves, that is if they have no common divisors aside from unity, then the sum of the divisors of their product \( pq \) is equal to the product of the sums of the divisors of both; that is, it will be \( \int pq = \int p \cdot \int q \). This method that has been devised for finding the sums of small numbers of divisors can easily be extended to finding the sums of larger numbers of divisors.

4. If \( a, b, c, d, \) etc. are prime numbers, every number, no matter how large it is, is always able to be reduced to the form \( a^\alpha b^\beta c^\gamma d^\delta \) etc., and with this form that has been devised, the sum of the divisors of any number can be found:

\[
\int a^\alpha b^\beta c^\gamma d^\delta \text{ etc.} = \int a^\alpha \int b^\beta \int c^\gamma \int d^\delta \text{ etc.}
\]

Thereupon, because \( a, b, c, d, \) etc. are prime numbers, it will be

\[ \int a^\alpha = 1 + a + a^2 + \text{etc.} + a^\alpha = \frac{a^{\alpha+1} - 1}{a - 1}, \]

and so

\[
\int a^\alpha b^\beta c^\gamma d^\delta \text{ etc.} = \frac{a^{\alpha+1} - 1}{a - 1} \cdot \frac{b^{\beta+1} - 1}{b - 1} \cdot \frac{c^{\gamma+1} - 1}{c - 1} \cdot \frac{d^{\delta+1} - 1}{d - 1} \cdot \text{etc.}
\]

Thus it is sufficient to find the multiplicities of the prime factors of a number in order to find the sum of the divisors of that number.

5. I achieve nothing beyond a search here, but I give this to help with what I will soon discuss. I set forth the natural numbers in the second column coming after the sums of their divisors, which were found by inspection:
6. If we consider this sequence of numbers 1, 3, 4, 7, 6, 12, 8, 15, 13, 18, 12, 28, etc., which constitutes the progression corresponding to the sums of the divisors of the natural numbers in order, there is no clear rule for the progression, and the order of these numbers seems so confused that no rule at all is seen to have been found. Clearly an arrangement of prime numbers is interwoven within this series, where for each term \( n \) such that \( \int n = n + 1 \), \( n \) is a prime number; however, it is understood that no fixed rule for the progression of the prime numbers is able to be given. So with our sequence not only of prime numbers but also all the other numbers, to give a rule for how far each one is from a prime is seen to be even harder than discovering a rule for the progression of the prime numbers themselves.
7. Thus I see that I am not equal to moving forward the science of numbers. However, I have found a certain fixed rule, according to which the terms of the given sequence 1, 3, 4, 7, 6, etc. come forward, where I have found that from this rule, any term of the sequence can be defined by the preceding terms. It is seen to be most wonderful that I have come upon this progression, which is a type of progression usually called recurrences; by the nature of this it has been found that any term is determined with this rule according to the preceding terms by a certain fixed relationship. However, this sequence will be very disturbed, and there seems to be no level rule for recurrence in these sequences. Nevertheless, is it possible to give a ladder of relation for these sequences to be formed by?

8. For a term \( n \), the series corresponding to it gives the sum of the divisors of this number \( n \), and is equal to \( \int n \). For it, the pre-existing terms are arranged in reverse order, \( \int (n-1) \), \( \int (n-2) \), \( \int (n-3) \), \( \int (n-4) \), \( \int (n-5) \), etc. For any term this series for \( \int n \) is composed of some number of its pre-existing aliquot parts, such that it is:

\[
\int n = \int (n-1) + \int (n-2) - \int (n-5) - \int (n-7) + \int (n-12) + \int (n-15) - \\
\int (n-22) - \int (n-26) + \int (n-35) + \int (n-40) - \int (n-51) - \int (n-57) + \\
\int (n-70) + \int (n-77) - \int (n-92) - \int (n-100) + \int (n-117) - \int (n-126) - \text{ etc.}
\]

As the + and - signs change alternatingly by pairs of terms, for convenience this series is broken into two, in this way:

\[
\int n = \int (n-1) - \int (n-5) + \int (n-12) - \int (n-22) + \int (n-35) - \int (n-51) + \text{ etc.}
\]

\[
+\int (n-2) - \int (n-7) + \int (n-15) - \int (n-26) + \int (n-40) - \int (n-57) + \text{ etc.}
\]

9. From the form of the above arrangement of numbers, which on both sides are sequences of successive numbers taken away from the number \( n \), it is easily seen for the series on each side that their second differences hold
constant. For instance, the numbers in the first sequence, with their first
differences and then their seconds, are:

<p>| | | | | | | | | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4</td>
<td>7</td>
<td>12</td>
<td>22</td>
<td>35</td>
<td>51</td>
<td>70</td>
<td>92</td>
<td>117</td>
<td></td>
<td>etc.</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td></td>
<td>etc.</td>
</tr>
</tbody>
</table>

from which the general term of this sequence is equal to \( \frac{3x^2-x}{2} \), which
forms precisely all the pentagonal numbers. The other sequence is:

<p>| | | | | | | | | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>7</td>
<td>15</td>
<td>26</td>
<td>40</td>
<td>57</td>
<td>77</td>
<td>100</td>
<td>126</td>
<td></td>
<td></td>
<td>etc.</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td></td>
<td></td>
<td>etc.</td>
</tr>
</tbody>
</table>

according to which the general term is thus \( \frac{3x^2+x}{2} \), and in fact it contains a
sequence of numbers consecutively ahead each of the pentagonals.

10. Thus, it is observed that the special sequence of pentagonal numbers
and those ahead of them can be applied in a fundamental way to the series
of sums of divisors, and this link between the pentagonal numbers and the
sums of divisors one would certainly not suspect. For, if the sequence of
pentagonal numbers are joined in the way shown here with the sequence of
the ones consecutively ahead them,

etc. , 77, 57, 40, 26, 15, 7, 2, 0, 1, 5, 12, 12, 22, 35, 51, 70, 92, etc.

our formula for the sums of the divisors, with the alternating signs for the
series taken into account, will then be able to be produced in this way:

\[ \text{etc. } - (n-15) + (n-7) - (n-2) + (n-0) + (n-5) - (n-12) + (n-22) - \text{ etc.} \]

where this series goes off to infinity on both sides, although the determination
of the number of terms to take is according to our choice.

11. In fact, by the use of this formula, our original one can be produced:

\[
\int_1^n = \\
\int (n-1) + \int (n-2) - \int (n-5) - \int (n-7) + \int (n-12) + \int (n-15) \\
- \int (n-22) - \int (n-26) + \int (n-35) + \int (n-40) - \int (n-51) - \int (n-57) \\
+ \int (n-70) + \int (n-77) - \int (n-92) - \int (n-100) + \text{ etc.}
\]
We want to find the sum of the divisors of the number \(n\) from the known sums of the divisors of smaller numbers. It is not possible to tell how many terms of this formula to take, until, that is, the sums of divisors of negative numbers are reached. All the terms which after the \(\int\) symbol contain negative numbers are ignored; from which if \(n\) is a small number, a small number of terms are sufficient, whereas if \(n\) is a larger number, in general many terms are taken by our formula.

12. Therefore the sum of the divisors of some number \(n\) is given by the sums of the aliquot divisors of smaller numbers, which I assume to be known; this is stopped at the sum of negative numbers. There is an exception to this however; it is not possible for the sum of the divisors of negative numbers to be taken, but along with them ought to be mentioned the case that with our formula gives the term \(\int(n - n)\), that is \(\int 0\). Indeed, any number with zero as a divisor is seen to be either infinity or indeterminate. Whenever this happens, as long as \(n\) is a number from the sequence of pentagonal numbers themselves or the sequence of those consecutively ahead of them, it is then always held that for the terms \(\int(n - n)\), that is \(\int 0\), the number \(n\) itself is written; thus with this rule, the terms \((n - n)\) are replaced.

13. For this rule that has been related, the use of our formula ought to be demonstrated, and so I give an example for small numbers, which are easily found by our formula and which are also seen be correct.
\[ \begin{align*}
\int 1 &= \int 0 \\
\int 1 &= 1 = 1 \\
\int 2 &= \int 1 + \int 0 \\
\int 2 &= 1 + 2 = 3 \\
\int 3 &= \int 2 + \int 1 \\
\int 3 &= 3 + 1 = 4 \\
\int 4 &= \int 3 + \int 2 \\
\int 4 &= 4 + 3 = 7 \\
\int 5 &= \int 4 + \int 3 - \int 0 \\
\int 5 &= 7 + 4 - 5 = 6 \\
\int 6 &= \int 5 + \int 4 - \int 1 \\
\int 6 &= 6 + 7 - 1 = 12 \\
\int 7 &= \int 6 + \int 5 - \int 2 - \int 0 \\
\int 7 &= 12 + 6 - 3 - 7 = 8 \\
\int 8 &= \int 7 + \int 6 - \int 3 - \int 1 \\
\int 8 &= 8 + 12 - 4 - 1 = 15 \\
\int 9 &= \int 8 + \int 7 - \int 4 - \int 2 \\
\int 9 &= 15 + 8 - 7 - 3 = 13 \\
\int 10 &= \int 9 + \int 8 - \int 5 - \int 3 \\
\int 10 &= 13 + 15 - 6 - 4 = 18 \\
\int 11 &= \int 10 + \int 9 - \int 6 - \int 4 \\
\int 11 &= 18 + 13 - 12 - 7 = 12 \\
\int 12 &= \int 11 + \int 10 - \int 7 - \int 5 + \int 0 \\
\int 12 &= 12 + 18 - 8 - 6 + 12 = 28 \\
\end{align*} \]

14. Even larger numbers can proceed by considering this example, which is marvelous; in this way, the expectation is always met of giving the sum of the divisors of any given number. It is easily seen that none of the numbers I have given have sums of divisors larger than 100, from which point the correctness of our formula for larger numbers will be investigated. Whenever a prime number is given, we will find with delight that with our formula, the sum of the divisors of the number will be determined to be one larger than it. We follow this example with the given number \( n = 101 \) to be investigated, as if ignorant of whether, is it prime? This will be clear from the following:

\[ \begin{align*}
\int 101 &= \int 100 + \int 99 - \int 96 - \int 94 + \int 89 + \int 86 - \int 79 - \int 75 \\
&\quad + \int 66 + \int 61 - \int 50 - \int 44 + \int 31 + \int 24 - \int 9 - \int 1 \\
&= 217 + 156 - 252 - 144 + 90 + 132 - 80 - 124 \\
&\quad + 144 + 62 - 93 - 84 + 32 + 60 - 13 - 1
\]
By collecting together every two terms, it will therefore be

\[ \int 101 = +373 -396 = +222 -204 = +206 -177 = +92 -14 \]

that is, \[ \int 101 = +893 -791 = 102 \]

It is found that the sum of the divisors of the number 101 is one larger than it, i.e. 102, from which, even if were not already known, it would therefore be clear that the number 101 is prime. While this is rightly seen to be marvelous, no method is established by which all the divisors can be given; moreover, a fixed rule for finding the sum of the divisors remains unknown, although sometimes from contemplation the sums are able to be found.

15. These interesting properties that the sums of divisors are gifted with are nothing less than remarkable, even if they are hard to clearly prove. However, even if a proof is not found, with these properties of numbers hidden, this rule that has been discovered for the progression is seen to be quite valuable, and indeed a search for the truth is very praiseworthy, but this is quite concealed. To be sure, I am exhausted, with no proof of the truth of this being able to be given by me, and have just about given up. I do not know whether or not this should be thought of as verified, whose proof cannot be found. So although many examples confirming the truth of this have been seen, it is not possible for a proof of this to be given by me.

16. Therefore we have excellent examples of this proposition, which we are not able to doubt in any way even if we cannot prevail with a proof of it. Indeed, it is seen that a great deal of the amazing and common propositions in mathematics do not admit proof, although the truth of these principles is free from doubt. In the mean time however, investigation of the truth of this should not be approached just by chance; it comes to mind that for the sums of divisors, is there a recurrence series that has a connection with the pentagonal numbers? I do not consider this difficult matter in and of itself, rather as a consequence; I will make this clear shortly, though in a roundabout way.

17. I have been led to this observation by considering the infinite formula:

\[ s = (1 - x)(1 - x^2)(1 - x^3)(1 - x^4)(1 - x^5)(1 - x^6)(1 - x^7)(1 - x^8) \text{ etc.} \]
as whose value, if every factor is multiplied in sequence, then the powers of 
\(x\) are put together, I have found to be converted into the following series:

\[
s = 1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + x^{26} - x^{35} - x^{40} + x^{51} + x^{57} - \text{etc.}
\]

where the numbers occurring as the exponents of \(x\) are the same as those de-
scribed earlier, that is, consecutively the pentagonal numbers and the num-
bers ahead of them. From this arrangement, it is easy to see that it is possible
for a series to be made which goes off into infinity on both sides:

\[
s = \text{etc.} + x^{26} - x^{15} + x^7 - x^2 + x^0 - x^1 + x^5 - x^{12} + x^{22} - x^{35} + x^{51} - \text{etc.}
\]

18. I am not able to confirm that these two formulas giving \(s\) are equal
with a rigorous proof; however, it is not hard to see that if the first formula
\(s = (1 - x)(1 - x^2)(1 - x^3)(1 - x^4)(1 - x^5)\) etc. is expanded, with each of
its factors successively multiplied by the others, the first terms of the other
series, \(s = 1 - x - x^2 + x^5 + x^7 + \text{etc.}\) will be found; as well, the two signs
+ and - come in turns by pairs, and the exponents for \(x\) follow the very
same rule that I have already given. Moreover, by conceding that these two
infinite formulas are equal, the properties of the sums of divisors, which I
noted earlier, are able to be rigorously proved; in turn, if these properties
were seen as true, from this the truth of the agreement of these two formulas
would proceed.

19. If we assume for this demonstration that it is set \(s = (1-x)(1-x^2)(1-
-x^3)(1-x^5)\) etc., and it to be \(s = 1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + x^{26}-
\text{etc.}\), then these logarithms will be obtained:

\[
\ln s = \ln(1-x) + \ln(1-x^2) + \ln(1-x^3) + \ln(1-x^4) + \ln(1-x^5) + \text{etc.}
\]

and

\[
\ln s = \ln(1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + x^{26} - \text{etc.})
\]

Then if the differential of each side of these formulas is taken, it will be:

\[
\frac{ds}{s} = \frac{-dx}{1-x} - \frac{2xdx}{1-x^2} - \frac{3x^2dx}{1-x^3} - \frac{4x^3dx}{1-x^4} - \frac{5x^4}{1-x^5} - \text{etc.}
\]

and

\[
\frac{ds}{s} = \frac{-dx - 2x + 5x^4 + 7x^6 - 12x^{11} - 15x^{14} + \text{etc.}}{1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + x^{26} - \text{etc.}}
\]
Then both would be multiplied through by $\frac{x}{dx}$, so that it is had that:

\[-\frac{x ds}{sdx} = \frac{x}{1-x} + \frac{2x^2}{1-x^2} + \frac{3x^3}{1-x^3} + \frac{4x^4}{1-x^4} + \frac{5x^5}{1-x^5} + \text{etc.} \tag{1}\]

and

\[-\frac{x ds}{sdx} = \frac{x + 2x^2 - 5x^5 - 7x^7 + 12x^{12} + 15x^{15} - 22x^{22} - 26x^{26} + \text{etc.}}{1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + x^{26} - \text{etc.}} \tag{2}\]

20. For showing that these expressions are equal between themselves, the first of them is considered, and we convert every term into geometric progressions; from this will come forth infinitely many of these geometric progressions, for each power of $x$ in the arrangement:

\[-\frac{x ds}{sdx} = \frac{x^n}{1-x} + \frac{x^{n+1}}{2} + \frac{x^{n+2}}{3} + \frac{x^{n+3}}{4} + \frac{x^{n+4}}{5} + \frac{x^{n+5}}{6} + \frac{x^{n+6}}{7} + \text{etc.}\]

where it is clear that for each power of $x$ all the numbers which are its coefficients are given as the divisors of that power. So of course for the power $x^n$, its coefficients will be the sum of the divisors of the number $n$, which is then given in the earlier way with the expression $\int n$. Then indeed this series will be produced as equal to $-\frac{x ds}{sdx}$, such that:

\[-\frac{x ds}{sdx} = x \int 1 + x^2 \int 2 + x^3 \int 3 + x^4 \int 4 + x^5 \int 5 + x^6 \int 6 + x^7 \int 7 + \text{etc.}\]

and by setting $x = 1$, the progression for the sums of divisors is produced, with all numbers following their natural progressions.
22. We denote this series with \( t \), so that it would be

\[
t = x^1 \int 1 + x^2 \int 2 + x^3 \int 3 + x^4 \int 4 + x^5 \int 5 + x^6 \int 6 + x^7 \int 7 + \text{ etc.}
\]

and because \( t = -\frac{xdx}{sdx} \), it will then be

\[
t = x^1 + 2x^2 - 5x^5 - 7x^7 + 12x^{12} + 15x^{15} - 22x^{22} - 26x^{26} + \text{ etc.}
\]

Therefore it is necessary, from the expansion of this fraction, that a series for \( t \) is obtained that is equal to the first form. From this it is clear that the series for \( t \) that is found is recurrent, each term of which is determined by the preceding ones, by a stepping relation, all of which is indicated by the denominator \( 1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + x^{26} - \text{ etc.} \)

23. Now the nature of this recurrence series can now be seen easily, if we assume that these two values for \( t \) are equal between themselves, with both sides multiplied by the denominator \( 1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + x^{26} - \text{ etc.} \) of the fraction; with this having been done, the following terms are seen to be given in place of the powers of \( x \):

\[
\begin{align*}
x^1f_1 + x^2f_2 + x^3f_3 + x^4f_4 + x^5f_5 + x^6f_6 + x^7f_7 + x^8f_8 + x^9f_9 + x^{10}f_{10} + x^{11}f_{11} + x^{12}f_{12} \\
- f_1 - f_2 - f_3 - f_4 - f_5 - f_6 - f_7 - f_8 - f_9 - f_{10} - f_{11} - f_{12} \\
+ f_1 + f_2 + f_3 + f_4 + f_5 + f_6 + f_7 \\
= x^1 = 2x^2 = 0 = 0 = -5x^5 = 0 = -7x^7 = 0 = 0 = 0 = 0 = 12x^{12}
\end{align*}
\]

24. Now with each power of \( x \), its coefficients are bound to mutually cancel each other out, from which we extract the following equalities:
\[ \int 1 = 1 \]
\[ \int 2 = \int 1 + 2 \]
\[ \int 3 = \int 2 + \int 1 \]
\[ \int 4 = \int 3 + \int 2 \]
\[ \int 5 = \int 4 + \int 1 - 5 \]
\[ \int 6 = \int 5 + \int 4 - \int 1 \]
\[ \int 7 = \int 6 + \int 5 - \int 2 - 7 \]
\[ \int 8 = \int 7 + \int 6 - \int 3 - \int 1 \]
\[ \int 9 = \int 8 + \int 7 - \int 4 - \int 2 \]
\[ \int 10 = \int 9 + \int 8 - \int 5 - \int 3 \]
\[ \int 11 = \int 10 + \int 9 - \int 6 - \int 4 \]
\[ \int 12 = \int 11 + \int 10 - \int 7 - \int 5 + 12 \]

etc.

which clearly can be reduced to:

\[ \int 1 = 1 \]
\[ \int 2 = \int (2 - 1) + 2 \]
\[ \int 3 = \int (5 - 1) + \int (3 - 2) \]
\[ \int 4 = \int (4 - 1) + \int (4 - 2) \]
\[ \int 5 = \int (5 - 1) + \int (5 - 2) - 5 \]
\[ \int 6 = \int (6 - 1) + \int (6 - 2) - \int (6 - 5) \]
\[ \int 7 = \int (7 - 1) + \int (7 - 2) - \int (7 - 5) - 7 \]
\[ \int 8 = \int (8 - 1) + \int (8 - 2) - \int (8 - 5) - \int (8 - 7) \]
\[ \int 9 = \int (9 - 1) + \int (9 - 2) - \int (9 - 5) - \int (9 - 7) \]
\[ \int 10 = \int (10 - 1) + \int (10 - 2) - \int (10 - 5) - \int (10 - 7) \]
\[ \int 11 = \int (11 - 1) + \int (11 - 2) - \int (11 - 5) - \int (11 - 7) \]
\[ \int 12 = \int (12 - 1) + \int (12 - 2) - \int (12 - 5) - \int (12 - 7) + 12 \]

25. It is clear that the numbers that evolve for a given number, to be subtracted for finding the sum of its divisors, is the series of numbers 1, 2, 5, 7, 12, 15, 22, 26, etc. itself, as long as they do not exceed the given number: thus this is a proof that the rule described before holds. Therefore for any given \( n \), it is clear to be:

\[ \int n = \int (n - 1) + \int (n - 2) - \int (n - 5) - \int (n - 7) + \int (n - 12) + \int (n - 15) - \text{ etc.} \]

where these terms continue as long as the numbers prefixed with the \( \int \).
symbol are not negative. Similarly, the original recurrence series given by this rule does not proceed beyond that point either.

26. By this rule that has been made, for any number considered before the end, it is clear that the value of the numerator of the fraction is expressed by the $t$ that was constructed in §22, and so this continues uninterrupted for the series for the number $n$, 1, 2, 5, 7, 12, 15, 22, 26, etc., and indeed nothing upsets this rule. In the case that the number being taken away is equal to the given number, if we consider the rule described before, we see that this number corresponds to the term $\int(n - n)$, from which the rule is clear that whenever in the application of the formula

$$\int n = \int (n - 1) + \int (n - 2) - \int (n - 5) - \int (n - 7) + \int (n - 12) + \text{etc.}$$

whenever the term $\int(n - n)$ is found, it is not ignored but rather the value of $n$ itself is written for it. Thus this confirms all the parts of the method described earlier.