TRANSLATOR'S NOTE

In preparing this translation, my assumption has been that the interested audience likely consists of both mathematicians and historians of mathematics. To satisfy the latter, I have attempted, as near as possible, to mimic Euler's phrasing, and especially his mathematical notation, with a few exceptions in cases where Euler's notation might be confusing to a modern reader. These include:

- Use of "arcsin, arccos" rather than "A sin, A cos" for the inverse trigonometric functions
- Use of "$\log$ expression" rather than "l expression" for the base 10 logarithm
- Use of, for example, "$\cos^2 u$" rather than "$\cos u^2$" to denote the square of the cosine;
- Clarifying the argument of the square root operator

In most other cases, I have attempted to copy Euler's original notation.

My parenthetical notes are enclosed in brackets; material found in parentheses is found so in the original texts.

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Elements of Spheroidal Trigonometry Drawn from the Method of the Maxima and Minima

Leonhard Euler
(G. Heine, tr.)

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1. Having established the elements of Spherical Trigonometry on the principle of the maxima and minima, my principal goal was to fix a general principle from which one could draw the resolution of triangles, not only on a spherical surface, but more generally upon any surface whatever. The sides of a spherical triangle are arcs of great circles, which are the shortest path from which can be drawn on the surface of a sphere from one point to another; in the same way, I envisage the sides of a triangle described on any surface whatever, so that these sides are the shortest routes which lead from one angle to another on this surface. So, conceiving three points upon any surface whatever, the shortest lines are drawn from each to the others, forming a triangle, for which the question is to show the resolution.

2. I limit myself here to spheroidal surfaces, which are formed by the revolution of an ellipse about one of its axes; in particular, I shall consider the triangles formed on the surface of the earth by their sides, which are the shortest possible between their endpoints. For, whether the sides are formed by ropes tightly stretched from one point to the other, or are drawn in following the direction of rays of light, so that the plane which contains any two contiguous elements is everywhere perpendicular to the surface of the earth, they will represent the shortest path from one end to the other. Indeed, this is also the method followed in practice, if it is required to draw the shortest line from one point to another on the surface of the earth; and when

one speaks in Geography of the distance between two places, this is always understood to mean the shortest route which goes from one to the other. It is thus necessary to distinguish this shortest route from the loxodromethat one follows in navigation, which requires specialized investigations.

3. Let, therefore, AEB be the half ellipse, which produces the spheroid of the Earth by revolution about the axis ACB, and we set

$$\text{the semiaxis } CA = CB = a, \text{ and the semidiameter of the equator } CE = e.$$

Now, the half ellipse AEB will represent an arbitrary meridian, and whatever point that can be conceived on the surface of the earth, in order to know its situation, one must consider the meridian which passes through that point, let it be M, and then one will have three things to determine.

1° The latitude, or the elevation of the pole observed in that place.

2° Its distance to the plane of the equator, measured by the perpendicular MP, equal to CQ, and

3° The latitude, or the elevation of the pole observed in that place.

One sees clearly that, knowing one of these three things, it is easy to determine the other two by the properties of the ellipse. Next it will be convenient to seek the radius of curvature of the meridian at the point M, together with the measure of the arc of the meridian MA, by which the point is separated from the pole A.

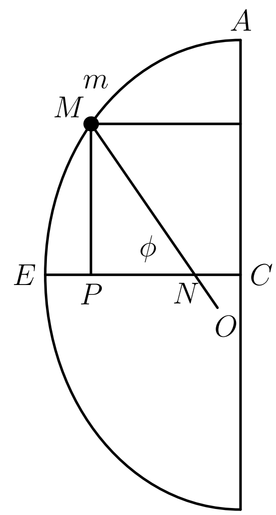

Figure 1

4. Let $CP = MQ = x$, and $PM = CQ = y$. Then one will have

$$y = \frac{a}{e}\sqrt{(ee - xx)}, \quad \text{and thus} \quad dy = -\frac{ax\,dx}{e\sqrt{(ee - xx)}}.$$

Now let the line MN be drawn perpendicular to the meridian, marking the direction of gravity, and the angle ENM will measure the latitude, or the elevation of the pole at the place M. Thus, we set this angle $ENM = \varphi$; ordinarily it is the first element that is known, and having

$$\text{the sub-normal} \qquad PN = -\frac{y\,dy}{dx} = \frac{aa}{ee}\,x\,,$$

we infer that

$$\operatorname{tang}\varphi = \frac{PM}{PN} = \frac{e\sqrt{(ee-xx)}}{ax},$$

and furthermore, that

$$CP = x = \frac{ee\cos\varphi}{\sqrt{(aa\sin^2\varphi + ee\cos^2\varphi)}}, \quad \text{and} \quad PM = y = \frac{aa\sin\varphi}{\sqrt{(aa\sin^2\varphi + ee\cos^2\varphi)}}.$$

From these formulae, knowing the latitude of a place M, the distance to the axis of the Earth, as well as to the plane of the equator, will easily be determined. From this one can also find the distance from the point M to the center of the Earth C, in other words, the length of the line

$$CM = \sqrt{\frac{a^4\sin^2\varphi + e^4\cos^2\varphi}{aa\sin^2\varphi + ee\cos^2\varphi}},$$

and the angle CMN, which this line makes with the direction of gravity MN, for one finds that

$$\operatorname{tang} CMN = \frac{(ee-aa)\sin\varphi\cos\varphi}{aa\sin^2\varphi + ee\cos^2\varphi}, \quad \text{and} \quad \sin CMN = \frac{(ee-aa)\sin\varphi\cos\varphi}{\sqrt{(a^4\sin^2\varphi + e^4\cos^2\varphi)}}.$$

5. We search also the radius of curvature MO, for which, setting $\frac{dy}{dx} = p$, the expression is

$$MO = -\frac{dx(1+pp)^{\frac{3}{2}}}{dp}.$$

Now, having $\frac{dy}{dx} = \frac{aax}{eey}$, one will have also $p = -\frac{\cos\varphi}{\sin\varphi}$, $dp = \frac{d\varphi}{sin^2\varphi}$, therefore $\sqrt{1+pp} = \frac{1}{\sin^2\varphi}$, so

$$(1+pp)^{\frac{3}{2}} = \frac{1}{\sin^3\varphi}, \quad \text{and therefore} \quad \frac{(1+pp)^{\frac{3}{2}}}{dp} = \frac{1}{d\varphi\,\sin\varphi}.$$

But since

$$x = \frac{ee\cos\varphi}{\sqrt{aa\sin^2\varphi + ee\cos^2\varphi}},$$

we shall have

$$dx = -\frac{aa\,ee\,d\varphi\,\sin\varphi}{(aa\sin^2\varphi + ee\cos^2\varphi)^{\frac{3}{2}}}.$$

Consequently, the radius of curvature will be

$$MO = -\frac{aa\,ee}{(aa\sin^2\varphi + ee\cos^2\varphi)^{\frac{3}{2}}}.$$

Thus, if we take on the same meridian as M an infinitely close point m, with latitude $= \varphi + d\varphi$, the element Mm will be the arc of a described circle of radius MO, with length

$$Mm = \frac{aa\,ee\,d\varphi}{(aa\sin^2\varphi + ee\cos^2\varphi)^{\frac{3}{2}}}\ .$$

6. The integral of this formula will give the length of the elliptic arc EM, and in order to find the approximate value, one only need set

$$\sin^2\varphi = \tfrac{1}{2} - \tfrac{1}{2}\cos 2\varphi \quad \text{and} \quad \cos^2\varphi = \tfrac{1}{2} + \tfrac{1}{2}\cos 2\varphi,$$

to obtain

$$Mm = \frac{aa\,ee\,d\varphi}{\left(\frac{1}{2}(aa+ee) + \frac{1}{2}(ee-aa)\cos 2\varphi\right)^{\frac{3}{2}}}\quad .$$

For, since $ee - aa$ is extremely small in comparison to $aa + ee$, in setting $\frac{ee-aa}{ee+aa} = \delta$, our formula is changed to

$$Mm = \frac{2\,aa\,ee\,d\varphi\sqrt{2}}{(aa+ee)^{\frac{3}{2}}}\ (1 + \delta\cos 2\varphi)^{-\frac{3}{2}}\,,$$

whose integral, since

$$(1 + \delta\cos 2\varphi)^{-\frac{3}{2}} = 1 + \tfrac{15}{16}\,\delta\delta - \tfrac{3}{2}\,\delta\cos 2\varphi + \tfrac{15}{16}\,\delta\delta\cos 4\varphi,$$

will be

$$EM = \frac{2\,aa\,ee\sqrt{2}}{(aa+ee)^{\frac{3}{2}}}\left((1 + \tfrac{15}{16}\,\delta\delta)\varphi - \tfrac{3}{4}\,\delta\sin 2\varphi + \tfrac{15}{64}\,\delta\delta\sin 4\varphi\right),$$

which is a very close approximation; one could even discard the terms in the square of δ.

7. One can also use the differential formula to determine the size of each degree of the meridian; it suffices to give to $d\varphi$ the value of one degree, or the 180th part of 3,14159265, which is the length of an arc of 180° on a circle of radius 1. Therefore one sets $d\varphi = 0,017453292$, and φ will designate the latitude at the middle of the degree. Then the size of this degree will be

$$= \frac{2aa\,ee\,d\varphi\,\sqrt{2}}{(aa+ee)^{\frac{3}{2}}}\left(1 - \tfrac{3}{2}\,\delta\,\cos 2\varphi\right),$$

neglecting terms in the square of δ; and this formula is adequate to determine at latitude φ the size of one degree $d\varphi$ of the meridian. Inversely, from this formula it will also be possible to determine the size of the two semi-diameters of the earth by taking the actual measurement of several degrees, supposing the figure of the earth to be an elliptic spheroid. Two measured degrees would suffice for this result, if the measure was exact to the last point, but, since an error of one second produces one of about 16 toises[1] in the size of a degree, it would be better to make use of several degrees, allowing to each a small error of at least 32 toises, in order to then reconcile the conclusions.

8. As an abbreviation, let us set

$$\frac{2aa\,ee\,d\varphi\sqrt{2}}{(aa\,ee)^{\frac{3}{2}}} = A,$$

since this quantity is the same for every latitude. The measures of a degree conducted in Peru, at the Cape of Good Hope, in France, and in Lapland[2] furnish us with these four equations:

$$\begin{aligned}
A(1 - \tfrac{3}{2}\delta\cos\ 1^\circ\ ) &= 56753 + p \text{ Toises}\\
A(1 - \tfrac{3}{2}\delta\cos\ 66^\circ\ 36'\ ) &= 57037 + q \text{ Toises}\\
A(1 - \tfrac{3}{2}\delta\cos\ 98^\circ\ 46'\ ) &= 57074 + r \text{ Toises}\\
A(1 - \tfrac{3}{2}\delta\cos\ 132^\circ\ 40'\ ) &= 57438 + s \text{ Toises}
\end{aligned}$$

[1] One "Paris toise" was equal to approximately 1.949 meters.[tr]

[2] The survey in "Peru"(now Ecuador) was conducted by Bouguer, Godin, and LaCondamine about 1740; at the Cape of Good Hope in 1752 or 1753 by Nicholas Louis de LaCaille; in Lapland (near the present boundary between Sweden and Finland) in 1735–6 by Maupertuis, Clairaut, Celsius, and others. There were numerous surveys in France throughout the 18th century. Todhunter(1873), in *A History of the Mathematical Theories of Abstraction and the Figure of the Earth* cites a survey directed by Cassini de Thury in 1739–40, with exactly the result Euler uses here. [tr]

marking by p, q, r, s th errors which might have slipped into the measurements, and could be positive or negative. These are assumed to be as small as possible, because so much care was taken in the measurements, that the errors would not exceed a few seconds of arc, except in the third case, where the error r might be larger than 32 toises.

9. If we now substitue the values of these cosines, we shall have the following four equations:

$$\begin{aligned} \text{I.} \quad & A(1 - 1,4997715\,\delta) = 56753 + p; \\ \text{II.} \quad & A(1 - 0,5957219\,\delta) = 57037 + q; \\ \text{III.} \quad & A(1 + 0,2286163\,\delta) = 57074 + r; \\ \text{IV.} \quad & A(1 + 1,0165980\,\delta) = 57438 + s; \end{aligned}$$

We subtract the first from each of the others to obtain the three equations:

$$\begin{aligned} 0,9040496\,\delta\,A &= 284 + q - p \\ 1,7283878\,\delta\,A &= 321 + r - p \\ 2,5163696\,\delta\,A &= 685 + s - p \end{aligned}$$

and, dividing the two others by the first of these, we obtain

$$\frac{321 + r - p}{284 + q - p} = \frac{65}{34} \qquad \text{and} \qquad \frac{685 + s - p}{284 + q - p} = \frac{437}{157},$$

from which it follows that

$$31\,p - 65\,q + 34\,r = 7546 \quad \text{and} \qquad 280\,p - 437\,q + 157\,s = 16563.$$

10. If we eliminate p from these two equations, the result is

$$-150\,q + 307\,r - 157\,s = 51594,$$

from which we see that the measurement errors at the Cape and Lapland should be assumed negative, while that from France is positive. If one were willing to assume these three errors equal, each would become 84 toises, which would be too exorbitant to reconcile with the extreme exactness by which the second and fourth of these operations were carried out. But it is not to be doubted that a rather considerable error might have slipped into the determination of the degree in France, and that it could well have amounted to 100 toises or more; and if we wanted to assume the measures at the Cape and at Lapland to be entirely correct, or that $q = 0$ and $s = 0$, we would find the error of a degree in France to be $r = 168$ toises; so that an error of $10''$ would have been made in the astronomical observations. Now, if we assumed $r = 100$ toises and $s = q$, it would be found that $q = s = -68$ toises; and one would not know how to admit so large an error. Let us therefore set $r = 120$ and one will have $q = s = -48$ toises, which would be just barely acceptable; but setting $r = 125$ one will obtain $q = s = -43$ toises.

11. Since it is absolutely necessary to identify the errors in these several measurements of degrees and the largest in that of the degree of France, which is assumed to be not smaller than 125 toises, let us set $r = 125$. We will then have

$$-150\,q - 157\,s = 13219, \quad \text{thus, approximately} \quad q + s = -86 \text{ toises.}$$

Before deciding separately between one and the other of the errors q and s, let us consider the result for p, from the following equalities:

$$31\,p - 65\,q = 3296 \quad \text{or} \quad p = 106\tfrac{1}{2} + 2\tfrac{1}{10}\,q;$$

If one assumes $p = 0$, then one finds $q = -51$ and therefore $s = -35$, but if one assumes $p = 15$, it is found that $q = -43\frac{1}{2}$ and therefore $s = 42\frac{1}{2}$. From this it is seen that if we wish to assume the error of a degree at Peru to be larger, we will be obliged to attribute a larger error to that of Lapland. So, unless the figure of the earth differs considerably from an elliptic spheroid, it appears that one should assume the following errors:

$$p = 15 \text{ toises}, \quad q = -43 \text{ toises}, \quad r = +125 \text{ toises} \quad \text{and} \quad s = -43 \text{ toises.}$$

12. This being fixed, the true sizes of the four degrees will be:

		Central Latitude
Peru	= 56768 Toises	$\varphi = 0^\circ\,30'$
Cape	= 56994 Toises	$\varphi = 33^\circ\,18'$
France	= 57199 Toises	$\varphi = 49^\circ\,23'$
Lapland	= 57395 Toises	$\varphi = 66^\circ\,20'$

and having made these corrections, the figure of the earth will be reducible to an elliptic spheroid that can be determined by any two of these four measured degrees. We choose the first and the last, which give

$$A(1 - 1,4997715\,\delta) = 56768 \text{ Toises}$$
$$A(1 + 1,0165980\,\delta) = 57395 \text{ Toises,}$$

from which is obtained

$$\frac{1 + 1,0165980\,\delta}{1 - 1,4997715\,\delta} = \frac{57395}{56768};$$

consequently $143789\,\delta = 627$, so that

$$\delta = 0,004360055 = \frac{ee - aa}{ee + aa}.$$

Therefore

$$\frac{ee}{aa} = \frac{1+\delta}{1-\delta} = 1 + 2\delta + 2\delta\delta = 1,0087593 \quad \text{and} \quad \frac{e}{a} = 1,00437.$$

Thus, the diameter of the equator will be to the axis of the earth as 230 to 229, which is precisely the ratio that NEWTON asserted, from which it can be concluded that the hypotheses which the great Geometer made concerning the structure and the attraction of the earth, are in agreement with reality.

13. Having found the value of δ, we immediately deduce that

$$A = 57142 = \frac{2\,aa\,ee\sqrt{2}}{(aa + ee)^{\frac{3}{2}}} = 0,01745329,$$

thus

$$\frac{aa\,ee}{(aa + ee)^{\frac{3}{2}}} = 1157526 \text{ Toises.}$$

Now we set $\frac{e}{a} = \operatorname{tang}\omega$, so that $\omega = 45^\circ\, 7'\, 30''$, and we shall have

$$a\,\sin\omega^2\cos\omega = 1157526,$$

from which we conclude:

the semiaxis of the earth	$a = 3266892$	Toises
the semidiameter of the equator	$e = 3281168$	Toises

Now NEWTON, although he had established the same ratio between the axis and diameter of the equator, gives 3262168 toises to the semiaxis and 3276433 to the semidiameter of the equator. The reason for this difference is that I have assumed here a degree measured in France larger than did NEWTON. Now, having discovered the true size of the earth's axis and diameter, it is possible to determine at each latitude the size of a degree on the meridian. For, setting φ for the latitude at the middle of the degree, the size of this degree will be

$$57142(1 - 0,00654082\ \cos 2\varphi).$$

14. Here I observe again, that if the degree of France had been entirely omitted, the three others would agree very well among themselves; one would only need to assume for each an error of 19 toises, so the degrees of Peru and Lapland would need to be augmented, and that of the Cape diminished. From this would result a greater difference between the axis and the diameter of the equator, just as was already noted before the measures at the Cape were known. But then, supposing $p = 19$, $q = -19$, and $s = 19$, one would find $r = 169$ toises, by which the degree of France would need to be increased; in this case the needed correction to the degree of Lapland would be positive, instead of negative as I have supposed it above; this is a very sure sign of the correctness of this measure. Now, whether one rejects the degree of France or not, it is always necessary to assume q negative, from which one must conclude that the degree measured at the Cape is too large as stated. One sees also that the measure made at Quito is very exact; one would not know how to suppose an error greater than 20 toises, in such a way that the four measures would be in agreement. For the degree measured in Lapland, it must also be noted that there the refraction of stars near the zenith, taken into accont in the other measures, was neglected. Now if one furnishes this minor correction, one finds that the degree of Lapland is reduced from 57438 to 57422 toises, which approaches even more closely the preceding correction, where I assumed this degree to be 57395 toises, and the error would only be 27 toises, instead of 43.

15. However, I have not determined anything specific about the figure of the earth, since there is still room to doubt whether it can be regarded as a perfect elliptic spheroid in which the two halves on each side of the equator be equal and similar: although, whatever other hypothesis that one makes, one is obliged to take note of some small errors in the observations, and especially in that of France. My inquiries will turn upon the surface of a general and perfect elliptic spheroid, whose semi-axis is $= a$ and the semi-diameter of the equator $= e$; I will suppose the difference between these to be very small. Now, to abbreviate the formulae found below, I shall set

$$\frac{ee-aa}{ee+aa}=\delta \qquad \text{and} \qquad \frac{2\,aa\,ee\sqrt{2}}{(aa+ee)^{\frac{3}{2}}}=c,$$

where it is sufficient to remark that if one wishes to apply these results to the earth, the following values of these two letters will be reasonably exact:

$$\delta = 0,00436055 \qquad \text{and} \qquad c = 3\,273\,980 \text{ toises.}$$

Thus, that which I have found is reduced to:

$$Mm = \frac{c\,d\varphi}{(1+\delta\cos 2\varphi)^{\frac{3}{2}}}$$

and in integrating, by approximation

$$EM = c\left((1+\tfrac{15}{16}\,\delta\delta)\varphi - \tfrac{3}{4}\,\delta\sin 2\varphi + \tfrac{15}{64}\,\delta\delta\sin 4\varphi\right).$$

16. Here I take as known the latitude of the point M, or the angle ENM, which I name $= \varphi$, and from this the length of the arc EM is easily found; from this, it is seen that setting $\varphi = 90^\circ$ or $\varphi = \frac{1}{2}\pi$, setting for π the number 3,14159265 etc., the quarter of the ellipse will be

$$EMA = \tfrac{1}{2}\pi\left(1+\tfrac{15}{16}\delta\delta\right)c.$$

Then, using these abbreviations, one will have the radius of curvature of the meridian at the point M, or

$$MO = \frac{c}{(1+\delta\cos 2\varphi)^{\frac{3}{2}}}\,.$$

For the same latitude at the point $M = \varphi$, its distance $MQ = CP$ to the axis will also be known immediately, with

$$\frac{CP^3}{MO} = \frac{e^4}{aa}\cos^3\varphi\,;$$

and therefore,

$$CP = \frac{\cos\varphi}{\sqrt{(1+\delta\cos 2\varphi)}}\sqrt[3]{\frac{c\,e^4}{aa}} = \frac{ee\sqrt{2}}{\sqrt{(aa+ee)}}\cdot\frac{\cos\varphi}{\sqrt{(1+\delta\cos 2\varphi)}}\,,$$

and in the same way,

$$PM = CQ = \frac{aa\sqrt{2}}{\sqrt{(aa+ee)}}\cdot\frac{sin\varphi}{\sqrt{(1+\delta\cos 2\varphi)}}\,.$$

Now for the Earth, we have just found

$$\tfrac{e}{a} = \tfrac{230}{229} \qquad \text{and} \qquad a = 3\,266\,892 \text{ toises} \qquad \text{and} \qquad e = 3\,281\,168 \text{ toises.}$$

Or, using only the letters δ and c,

$$a = \frac{c}{(1+\delta)\sqrt{(1-\delta)}}\,, \qquad\qquad e = \frac{c}{(1-\delta)\sqrt{(1+\delta)}}\,;$$

so that

$$CP = MQ = \frac{c}{1-\delta}\cdot\frac{\cos\varphi}{\sqrt{(1+\delta\cos 2\varphi)}} \quad \text{and} \quad PM = CQ = \frac{c}{1+\delta}\cdot\frac{\sin\varphi}{\sqrt{(1+\delta\cos 2\varphi)}}\,.$$

PROBLEM 1

17. *Having observed the elevation of the pole at two places M and M' on the same meridian, to determine the length of the meridian arc MM' between these two places.*

SOLUTION

Let φ be the latitude at M, and ψ at M', and since these two places are on the same meridian, it is evident that the meridian arc MM' between them is the shortest path which leads from one place to the other. So, introducing the terms c and δ, which determine the type and size of the the elliptic spheroid, the size of arc MM' will be expressed in the manner

$$MM' = c\left(\left(1+\tfrac{15}{16}\delta\delta\right)(\psi-\varphi) - \tfrac{3}{4}\delta(\sin 2\psi - \sin 2\varphi) + \tfrac{15}{64}\delta\delta(\sin 4\psi - \sin 4\varphi)\right),$$

neglecting terms in the cube and higher powers of δ. But in any case it would be easy to extend the approximation further, even to infinity. One can also determine the situation of each place relative to the axis and the equator, and do so exactly, without approximation, for one will have

$$CQ = \frac{c\sin\varphi}{(1+\delta)\sqrt{(1+\delta\cos 2\varphi)}}, \qquad QM = \frac{c\cos\varphi}{(1-\delta)\sqrt{(1+\delta\cos 2\varphi)}},$$
$$CQ' = \frac{c\sin\psi}{(1+\delta)\sqrt{(1+\delta\cos 2\psi)}}, \qquad Q'M' = \frac{c\cos\psi}{(1-\delta)\sqrt{(1+\delta\cos 2\psi)}};$$

furthermore, the radii of curvature will be

$$\text{at } M = \frac{c}{(1+\delta\cos 2\varphi)^{\frac{3}{2}}}, \qquad \text{and at } M' = \frac{c}{(1+\delta\cos 2\psi)^{\frac{3}{2}}},$$

and these determinations encompass all that can be asked with respect to these two places.

SCHOLIE

18. My intention is to consider any two points on the surface of the earth, finding the distance between them, as well as their location with respect to the meridian. Now I have begun my investigation with the simplest case,

when the two points are on the same meridian, since it is obvious that the meridian arc between these two points is also the shortest path leading from one to the other. But, when the two points are not on the same meridian, one must use the method of maxima and minima to find the shortest path between them, and I take up this investigation in the following problem.

PROBLEM 2

19. *Knowing (Fig. 2) the latitude of two locations L and M, together with their difference in Longitude, to find the shortest path LM on the surface of the earth, which leads from one to the other.*

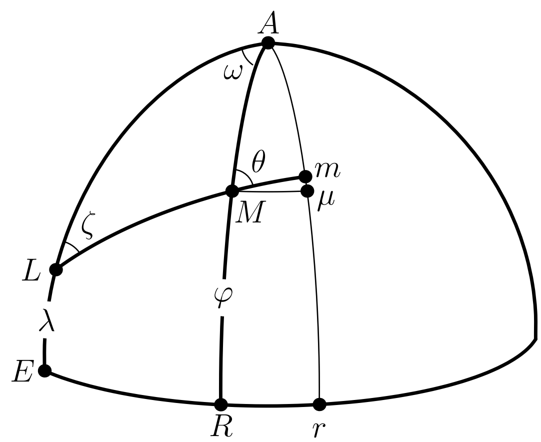

Figure 2: Original figure, with angles ζ, θ, λ, φ, and ω added.

SOLUTION

Through the points L and M let the meridians ALE and AMR be drawn; then the angle formed at the pole by these two meridians will measure the difference in longitude; given this, set

$$\begin{aligned}\text{the difference in Longitude, or the angle } LAM &= \omega,\\ \text{the Latitude of the Place } L &= \lambda,\\ \text{and the Latitude of the Place } M &= \varphi,\end{aligned}$$

which are the three quantities given, besides the size and shape of the earth. Now let LM be the shortest path between the points L and M, and let it be lengthed by an infinitely small amount beyond M to the point m; let the infinitely close meridian through m be Amr, upon which one takes $A\mu = AM$, so that the latitude in μ is the same as in M, namely $= \varphi$, and the latitude in m will be $\varphi + d\,\varphi$; thus, the meridian element between m and μ will be

$$m\mu = \frac{c\,d\varphi}{(1+\delta\cos 2\varphi)^{\frac{3}{2}}}.$$

Furthermore, one will have the angle $MA\mu = d\omega$, which is equal to the angle formed at the axis of the earth by perpendicular lines drawn from M and μ. Now these perpendicular lines, represented in Figure 1 by MQ, are

$$= \frac{c\cos\varphi}{(1-\delta)\sqrt{(1+\delta\cos 2\varphi)}},$$

from which is extracted the element

$$M\mu = \frac{c\,d\omega\cos\varphi}{(1-\delta)\sqrt{(1+\delta\cos 2\varphi)}};$$

and therefore the element of the path LM will be

$$Mm = c\sqrt{\left(\frac{d\varphi^2}{(1+\delta\cos 2\varphi)^3} + \frac{d\omega^2\cos\varphi^2}{(1-\delta)^2(1+\delta\cos 2\varphi)}\right)},$$

for which the integral must be minimized. Let us set $d\varphi = p\,d\omega$ in order to find a *minimum* for the integral formula

$$\int d\omega\sqrt{\left(\frac{pp}{(1+\delta\cos 2\varphi)^3} + \frac{\cos^2\varphi}{(1-\delta)^2(1+\delta\cos 2\varphi)}\right)}.$$

We set

$$\sqrt{\left(\frac{pp}{(1+\delta\cos 2\varphi)^3} + \frac{\cos^2\varphi}{(1-\delta)^2(1+\delta\cos 2\varphi)}\right)} = V,$$

and I have shown[3] that, if the differential of V is expressed by

$$dV = M\,d\omega + N\,d\varphi + P\,dp,$$

then the equation which contains the *minimum* is expressed in the form

$$0 = N - \frac{dP}{d\omega},$$

or, since in this case $M = 0$, this equation is reduced to this form

$$V - Pp = \text{Const.}$$

[3]See, for example, Proposition III, Chapter II, from E65(1743), *Methodus inveniendi lineas curvas maximi minimive proprietate gaudentes, sive solutio problematis isoperimetrici lattissimo sensu accepti.* [tr]

Now, differentiating V gives us

$$P = \frac{p}{(1+\delta\cos 2\varphi)^3\, V}\ ,$$

from which we conclude

$$VV - \frac{pp}{(1+\delta\cos 2\varphi)^3} = \frac{\cos^2\varphi}{(1-\delta)^2(1+\delta\cos 2\varphi)} = \frac{V}{\alpha}$$

and then taking the squares, our equation becomes

$$\frac{\alpha\alpha\ \cos^4\varphi}{(1-\delta)^4(1+\delta\cos 2\varphi)^2} = \frac{pp}{(1+\delta\cos 2\varphi)^3} + \frac{cos^2\varphi}{(1-\delta)^2(1+\delta\cos 2\varphi)} \quad .$$

Thus, the element of the shortest path

$$Mm = \frac{\alpha\, c\, d\omega\cos^2\varphi}{(1-\delta)^2(1+\delta\cos 2\varphi)}.$$

Now, taking for p its value $\dfrac{d\varphi}{d\omega}$, we shall have

$$\frac{d\omega^2\ \cos^2\varphi\,(\alpha\alpha\cos^2\varphi - (1-\delta)^2(1+\delta\cos 2\varphi))}{(1-\delta)^4(1+\delta\cos 2\varphi)^2} = \frac{d\varphi^2}{(1+\delta\cos 2\varphi)^3},$$

from which we obtain

$$d\omega = \frac{(1-\delta)^2\, d\varphi}{(1+\delta\cos 2\varphi)^{\frac{1}{2}}\cos\varphi\sqrt{(\alpha\alpha\cos^2\varphi - (1-\delta)^2(1+\delta\cos 2\varphi))}}$$

and

$$Mm = \frac{\alpha c\, d\varphi\cos\varphi}{(1+\delta\cos 2\varphi)^{\frac{3}{2}}\sqrt{(\alpha\alpha\cos^2\varphi - (1-\delta)^2(1+\delta\cos 2\varphi))}}.$$

But, before integrating these formulae, it is already possible to determine the angle AMm, which the arc LM makes with the meridian AM, for one will have

$$\sin AMm = \frac{M\mu}{Mm} = \frac{(1-\delta)\sqrt{(1+\delta\cos 2\varphi)}}{\alpha\cos\varphi};$$

whence, putting $\varphi = \lambda$, one will have the angle ALM, so that

$$\sin ALM = \frac{(1-\delta)\sqrt{(1+\delta\cos 2\lambda)}}{\alpha\cos\lambda}.$$

Putting this angle $ALM = \zeta$, in order to introduce it into the calculation in place of the constant α, we shall have

$$\alpha = \frac{(1-\delta)\sqrt{(1+\delta\cos 2\lambda)}}{\sin\zeta\cos\lambda}.$$

This value, being substituted, yields

$$\sin AMm = \frac{\sin\zeta\cos\lambda\sqrt{(1+\delta\cos 2\varphi)}}{\cos\varphi\sqrt{(1+\delta\cos 2\lambda)}}$$

and

$$d\omega = \frac{(1-\delta)\,d\varphi\sin\zeta\cos\lambda}{(1+\delta\cos 2\varphi)^{\frac{1}{2}}\cos\varphi\sqrt{(\cos^2\varphi(1+\delta\cos 2\lambda)-\sin^2\zeta\cos^2\lambda(1+\delta\cos 2\varphi))}}$$

$$Mm = \frac{c\,d\varphi\cos\varphi\sqrt{(1+\delta\cos 2\lambda)}}{(1+\delta\cos 2\varphi)^{\frac{3}{2}}\sqrt{(\cos^2\varphi(1+\delta\cos 2\lambda)-\sin^2\zeta\cos^2\lambda(1+\delta\cos 2\varphi))}}.$$

In order to integrate these formulae, it is necessary to separate the parts which depend on the small fraction δ, and neglecting terms with a square or higher power, one will have

$$d\omega = \frac{d\varphi\sin\zeta\cos\lambda}{\cos\varphi\sqrt{(\cos^2\varphi-\sin^2\zeta\cos^2\lambda)}} - \frac{\delta\,d\varphi\sin\zeta\cos\lambda\cos\varphi}{\sqrt{(\cos^2\varphi-\sin^2\zeta\cos^2\lambda)}} - \frac{\delta\,d\varphi\sin\zeta\cos^2\zeta\cos^3\lambda\cos\varphi}{(\cos^2\varphi-\sin^2\zeta\cos^2\lambda)^{\frac{3}{2}}},$$

of which the integral is found to be as follows:

$$\omega = \arcsin\left(\frac{\sin\zeta\cos\lambda\sin\varphi}{\cos\varphi\sqrt{(1-\sin^2\zeta\cos^2\lambda)}}\right) - \delta\sin\zeta\cos\lambda\arcsin\left(\frac{\sin\varphi}{\sqrt{(1-\sin^2\zeta\cos^2\lambda)}}\right)$$
$$- \frac{\delta\sin\zeta\cos^2\zeta\cos^3\lambda\sin\varphi}{(1-\sin^2\zeta\cos^2\lambda)\sqrt{(\cos^2\varphi-\sin^2\zeta\cos^2\lambda)}} + \text{Const.},$$

where the constant is to be determined so that in setting $\varphi = \lambda$, the angle ω, or the difference in longitude, vanishes.

Consequently one will have:

$$\omega = \arcsin\left(\frac{\sin\zeta\cos\lambda\sin\varphi}{\cos\varphi\sqrt{(1-\sin^2\zeta\cos^2\lambda)}}\right) - \arcsin\left(\frac{\sin\zeta\sin\lambda}{\sqrt{(1-\sin^2\zeta\cos^2\lambda)}}\right)$$
$$-\delta\sin\zeta\cos\lambda\arcsin\left(\frac{\sin\varphi}{\sqrt{(1-\sin^2\zeta\cos^2\lambda)}}\right) + \delta\sin\zeta\cos\lambda\arcsin\left(\frac{\sin\lambda}{\sqrt{(1-\sin^2\zeta\cos^2\lambda)}}\right)$$
$$-\frac{\delta\sin\zeta\cos^2\zeta\cos^3\lambda\sin\varphi}{(1-\sin^2\zeta\cos^2\lambda)\sqrt{(\cos^2\varphi-\sin^2\zeta\cos^2\lambda)}} + \frac{\delta\sin\zeta\cos\zeta\sin\lambda\cos^2\lambda}{1-\sin^2\zeta\cos^2\lambda}\,.$$

We proceed in the same manner to find the length of the path LM, and since we shall have

$$Mm = \frac{c\,d\varphi\cos\varphi}{\sqrt{\cos^2\varphi-\sin^2\zeta\cos^2\lambda}}\left(1+\tfrac{3}{2}\delta+\delta\sin^2\zeta\cos^2\lambda-3\delta\cos^2\varphi-\frac{\delta\sin^2\zeta\cos^2\zeta\cos^4\lambda}{1-\sin^2\zeta\cos^2\lambda}\right)\,,$$

the integral with the proper constant shall be found:

$$LM = c\left(1-\tfrac{1}{2}\delta\sin^2\zeta\cos^2\lambda\right)\arcsin\left(\frac{\sin\varphi}{\sqrt{(1-\sin^2\zeta\cos^2\lambda)}}\right)$$
$$-c\left(1-\tfrac{1}{2}\delta\sin^2\zeta\cos^2\lambda\right)\arcsin\left(\frac{\sin\lambda}{\sqrt{(1-\sin^2\zeta\cos^2\lambda)}}\right)$$
$$-\tfrac{3}{2}\,\delta\,c\sin\varphi\sqrt{(\cos^2\varphi-\sin^2\zeta\cos^2\lambda)}+\tfrac{3}{2}\,\delta\,c\cos\zeta\sin\lambda\cos\lambda$$
$$-\frac{\delta c\sin^2\zeta\cos^2\zeta\cos^3\lambda\sin\varphi}{(1-\sin^2\zeta\cos^2\lambda)\sqrt{(\cos\varphi-\sin^2\zeta\cos^2\lambda)}}+\frac{\delta c\sin^2\zeta\cos\zeta\sin\lambda\cos^3\lambda}{1-\sin^2\zeta\cos^2\lambda}\,.$$

Thus, knowing the two elevations of the pole λ and φ, at L and M, together with the angle $ALM = \zeta$ which the path LM makes with the meridian at L, one can determine the angle AMm which the path makes with the meridian at M , as well as the difference in longitude, i.e., the angle LAM, and the length of the path LM itself.

COROLLARY 1

20. If we introduce also the angle $AMm = \theta$, we shall have

$$\sin\theta = \frac{\sin\zeta\cos\lambda\sqrt{1+\delta\cos 2\varphi}}{\cos\varphi\sqrt{1+\delta\cos 2\lambda}}$$

and this value holds in general, because no approximation is yet employed. But, if one wants to make use of it, one will have either

$$\sin\theta = \frac{\sin\zeta\cos\lambda}{\cos\varphi}\left(1 - \tfrac{1}{2}\delta\cos 2\lambda + \tfrac{1}{2}\delta\cos 2\varphi\right)$$

or else

$$\sin\theta = \frac{\sin\zeta\cos\lambda}{\cos\varphi}\left(1 - \delta\cos^2\lambda + \delta\cos^2\varphi\right).$$

COROLLARY 2

21. To shorten the calculation of approximate values of ω and LM, one can seek an angle α which is

$$\alpha = \arcsin\left(\frac{\sin\zeta\cos\lambda\sin\varphi}{\cos\varphi\sqrt{1-\sin^2\zeta\cos^2\lambda}}\right) - \arcsin\left(\frac{\sin\zeta\sin\lambda}{\sqrt{1-\sin^2\zeta\cos^2\lambda}}\right)$$

and then one will obtain

$$\omega = \alpha - \delta\sin\zeta\cos\lambda\arcsin\left(\frac{\sin\alpha\cos\varphi}{\sin\zeta}\right) - \frac{\delta\sin\alpha\cos\zeta\cos^2\lambda\cos\varphi}{\sqrt{\cos^2\varphi - \sin^2\zeta\cos^2\lambda}},$$

and

$$LM = c\left(1 - \tfrac{1}{2}\delta\sin^2\zeta\cos^2\lambda\right)\arcsin\left(\frac{\sin\alpha\cos\varphi}{\sin\zeta}\right) - \frac{\delta\, c\,\sin\alpha\sin\zeta\cos\zeta\cos^2\lambda\cos\varphi}{\sqrt{\cos^2\varphi - \sin^2\zeta\cos^2\lambda}}$$
$$- \tfrac{3}{2}\,\delta\, c\,\sin\varphi\sqrt{(\cos^2\varphi - \sin^2\zeta\cos^2\lambda)} + \tfrac{3}{2}\,\delta\, c\cos\zeta\sin\lambda\cos\lambda.$$

22. If $\delta = 0$, one would derive from these formulae all the known rules of Spherical Trigonometry; but having already treated this subject amply,[4] I go no further here. However it is to be remarked that given the three elements ζ, λ, and φ, one can determine the fourth, θ, without integration or approximation; while the two final elements ω ad LM are not known without additional information.

SCHOLIE

23. So there they are, the formulae which contain in general the principles of Spheroidal Trigonometry, having found the solution of triangle LAM. For, although I assume one of its angles is at the pole A, this is a limitation which the nature of the problem seems absolutely to require, and if none of the three angles falls at one of the poles, it will be necessary to draw through each its meridian, and to reduce by the case of two or three such triangles as I have just considered. Since a spheroidal surface is not similar everywhere, in order to fix the situation of an arbitrary point, it is necessary to know its place with respect to the poles of the spheroid, which is determined most conveniently by the latitude or the elevation of the pole: and it is in view of this, that for the points L and M I have introduced the angles λ and φ, which mark the latitudes, into the calculation. Now from this one is in a position to fix the meridian arcs AL and AM themselves, which give the absolute distances of these points from the pole A. However, these distances have no immediate influence on the solution of the problem, and it suffices to know the latitude of these points. And then for the sides, or the shortest lines that can be drawn from one point to any other, the principal determination upon which it is necessary to reflect is the angle that such a line makes with the meridians. For these reasons, having two arbitrary points L and M on a spheroidal surface, in order to seek the shortest path which leads from one to the other, it is first necessary to pay regard to the latitude of each of these two points, as in the preceding computation the angle λ marks that of point L and φ that of point M. Next, after having drawn for the points L and M their meridians AL and AM, it is a matter of knowing first the angle $LAM = \omega$, which marks the difference in longitude of the two

[4] E214(1755), *Principes de la Trigonométrie Sphérique, Tirés de la Mèthode des Plus Grands et Plus Petits*[tr]

proposed locations L and M, and then the angles $ALM = \zeta$ and $AMm = \theta$, that the shortest route makes with the meridians at L and M. And finally one will have to determine the length itself of the shortest path LM, so that we must in all consider six elements $\lambda, \varphi, \zeta, \theta, \omega$ and LM, which have, as in ordinary Trigonometry, such a relation among them that knowing three, one can determine the three others. In the solution which I have just presented, I considered the three elements $\lambda, \varphi,$ and ζ as given, from which the fourth, θ, is easily determined by the equation

$$\sin\theta = \frac{\sin\zeta\cos\lambda\sqrt{1+\delta\cos 2\varphi}}{\cos\varphi\sqrt{1+\delta\cos 2\lambda}},$$

so that the solution will always be the same, some other three of these four elements $\lambda, \varphi, \zeta,$ and θ being given, since the fourth of them will already be known. But the situation is not the same, if one of the two others ω and LM are found among the known, since because the formulae which express the values of ω and LM are so complicated and only valid when the fraction δ is extremely small, one will not know how to eliminate the unknown elements. However, in the case where δ is extremely small, one will only have to regard the problem as in ordinary trigonometry, and then look for the corrections which result from the aberrations of the spherical figure, by the ordinary method of approximations. But it seems that the evolution of such a case will almost never be necessary, seeing as it can be assumed that the latitudes of points L and M, as well as the angles ζ and θ, formed by the line LM and the meridians through L and M, are all known, being determinable by the most simple operations. Now the greatest advantage which one could draw from this solution, is without doubt a new method that it furnishes for discovery of the true ellipticity of the figure of the earth, or the ratio between its axis and the diameter of its equator; and this can be carried out, without need of an actual measurement of any line drawn on the surface of the earth, as I shall explain more fully in the following problem.

PROBLEM 3

24. *To determine the ratio between the diameter of the equator and the axis of the earth without the aid of actual measures of several degrees of the meridian, by a method which can be carried out in a single country on the earth.*

SOLUTION

It has been believed until now, that the only means to know the figure of the earth was to measure the size of a degree along the meridian in widely different latitudes; so that from their difference one could conclude that which is found between the axis of the earth and the diameter of its equator. But the method which I shall propose here requires no more than operations which can be completed within a rather bounded country, or a nation of moderate extent, and this without needing to measure geometrically any line drawn on the surface of the earth. Thus, we suppose that the point L is found at one end of a large plane; first it is necessary that the height of the pole be observed, and that the line of the meridian be drawn with the utmost precision. It is sufficient that the absolute elevation of the pole be known to within one minute, but it is necessary that the distance of several fixed stars from the zenith at the time of their culmination be observed very exactly. So let λ be the height of the pole at the point L. Then, from the point L, let one depart along a route which makes an oblique angle with the meridian, but let this angle, which the beginning of the route makes with the meridian through L, be measured as precisely as possible. Next let the same route be followed, periodically setting down vertical pikes, in such a way that they all appear to be arranged in a straight line; let this apparently straight line be continued so long as the terrain will permit, keeping always in view that that path so described agrees with that which would be formed by a rope stretched on the ground. It would be well that this operation could be extended for a length of several German miles. In this way one will be sure of having traced the shortest line on the earth's surface, and it will not be necessary to measure its length. So let ζ be the angle which this route makes with the meridian at L, and having followed this route very far to the point M, let the distances to the zenith of the same fixed stars from M at their passage through the meridian be observed with the greatest care, so that the difference of latitudes between L and M may be exactly determined, which can be done to within several seconds; so let φ be the elevation of the pole at M, and although this measure may perhaps be not exact to the last degree, at least the difference between λ and φ should be as exact as possible. Finally, the meridian at M should also be traced, and the angle AMm which it makes with the continuation of the same route LM be measured; one will name this angle$= \theta$; these are all the operations which must be done in order to determine the ratio between the semi-axis$= a$ and the semi-diameter$= e$. For, setting $\frac{ee - aa}{ee + aa} = \delta$, so that

$$\frac{aa}{ee} = \frac{1-\delta}{1+\delta} \quad \text{and} \quad \frac{a}{e} = \frac{2-\delta}{2+\delta}$$

approximately, or $\frac{a}{e} = 1 + \delta$, we need only solve this equation:

$$\sin\theta\cos\varphi\sqrt{1+\delta\cos 2\lambda} = \sin\zeta\cos\lambda\sqrt{1+\delta\cos 2\varphi}\,,$$

from which, being given the four angles λ, φ, ζ, and θ, is extracted

$$\delta = \frac{\sin^2\zeta\cos^2\lambda - \sin^2\theta\cos^2\varphi}{\sin^2\theta\cos^2\varphi\cos 2\lambda - \sin^2\zeta\cos^2\lambda\cos 2\varphi}\,,$$

and therefore

$$\frac{ee}{aa} = \frac{\sin^2\zeta\cos^2\lambda\sin^2\varphi - \sin^2\theta\cos^2\varphi\sin^2\lambda}{\sin^2\theta\cos^2\varphi\cos^2\lambda - \sin^2\zeta\cos^2\lambda\cos^2\varphi}\,,$$

or

$$\frac{ee}{aa} = 1 + \frac{\sin^2\zeta\cos^2\lambda - \sin^2\theta\cos^2\varphi}{\cos^2\lambda\cos^2\varphi(\sin^2\theta - \sin^2\zeta)}\,.$$

Thus, having exactly observed and measured the angles λ, φ, ζ, and θ, it will be possible to determine the ratio betwen the axis of the earth and the diameter of its equator, by means of this formula which I have just discovered; concerning which, it must be remarked, that it is exact, and requires absolutely no approximation, as must be resorted to in using the ordinary method. But, to make this conclusion all the more certain, it is required to choose such a country on the earth and such a direction for the path which is traced, that small errors committed in the measurement of the angles influence the conclusion as little as possible. Now it is evident that, the larger is the denominator

$$\cos^2\lambda \cdot \cos^2\varphi \cdot (\sin^2\theta - \sin^2\zeta),$$

the larger must be the numerator, or the difference between

$$\sin^2\zeta\cos^2\lambda \quad \text{and} \quad \sin^2\theta\cos^2\varphi,$$

and this is without doubt the most favorable case, since then a small error commtted in the measure of the angles influences less the value of the numerator, upon which depends the justice of the conclusion.

REMARK

25. After having observed the two heights of the pole at L and M, together with the angle $ALM = \zeta$, if the earth were spherical, the angle AMm would be such that

$$\sin\theta = \frac{\sin\zeta\cos\lambda}{\cos\varphi};$$

so the ellipticity of the earth could not be inferred unless this angle was found to be smaller or larger. Now, on account of the ellipticity, we have

$$\sin\theta = \frac{\sin\zeta\cos\lambda}{\cos\varphi}\sqrt{\frac{1+\delta\cos 2\varphi}{1+\delta\cos 2\lambda}},$$

or indeed, since δ is very small,

$$\sin\theta = \frac{\sin\zeta\cos\lambda}{\cos\varphi}(1+\delta\cos^2\varphi - \delta\cos^2\lambda).$$

Here it is evident that the difference between the two latitudes must be perceptible; for if it were too small, the slightest error committed in their observation would produce a very considerable error in the conclusion. So the route LM must not be perpendicular to the meridians, since in following such a route, the latitude does not change perceptibly. Here one should principally take into consideration the length of the route LM, although it is not necessary to measure it; for it is advantageous that can reach a reasonably certain conclusion, without being obliged to follow this route too far. So let s be the length of the route LM, or rather let s be the angle to which corresponds an arc equal to this path on a spherical surface equal to that of the earth, and since this angle is quite small, one will have approximately $\varphi = \lambda + s \cos\zeta$, from which it is seen that $\cos\zeta$ should not be too small, since then the difference between the angles ζ and θ would become too small, which would render the conclusion equally uncertain. For, having

$$\cos\varphi = \cos\lambda - s\cos\zeta\sin\lambda,$$

since s is very small, one would have

$$\sin\theta = \frac{\sin\zeta}{1 - s\cos\zeta\,\mathrm{tang}\,\lambda}(1 - 2\delta s\cos\zeta\sin\lambda\cos\lambda)\,,$$

from which it is seen, that for the difference $2\delta s\cos\zeta\sin\lambda\cos\lambda$ to become perceptible, the country LM should not be too close, either to the pole, or to the equator.

EXAMPLE 1

26. Suppose that the place L is found at the latitude of 48°, so that $\lambda = 48^\circ$, and that the route LM makes an angle of $\zeta = 30^\circ$ with the meridian. Let this route be continued until it arrives at M with a latitude of $48^\circ\,52'$, which will occur after the route is extended by a distance of approximately 15 German miles. Having then $\varphi = 48^\circ\,52'$, if the earth were spherical, one would find the angle $AMm = \theta = 30^\circ\,34'\,15''$. But because of the ellipticity of the earth, if we assume $\delta = \frac{1}{229}$, the angle θ is found to be smaller by $8''$ and we shall have $\theta = 30^\circ\,34'\,7''$.

But if from the same place L, or $\lambda = 48^\circ$, one has set out at the angle

$$ALM = \zeta = 60^\circ\ ,$$

until one has arrived at the latitude $\varphi = 48^\circ\ 30'$, which would also happen after having gone a distance of about 15 German, miles, under the hypothesis of a spherical earth one would find $AMm = \theta = 60^\circ\,59'\,23''$. But under the hypothesis $\delta = \frac{1}{229}$, this angle would be $\theta = 60^\circ\,59'\,9''$ and therefore $14''$ less.

Thus, this case will be preferable to the preceding for knowing from it the ellipticity of the earth.

Suppose that in setting out from the same place L or $\lambda = 48^\circ$, the route LM is such that the angle $ALM = \zeta = 80^\circ$, and that after having made a course of about 15 German miles, one arrives at M with latitude $\varphi = 48^\circ\, 10'$. Then under the hypothesis of a spherical earth the angle AMm would be $\theta = 81^\circ\, 6'\, 59''$; but under the hypothesis of ellipticity $\delta = \frac{1}{229}$ this angle would be $\theta = 81^\circ\, 6'\, 42''$ and therefore $17''$ less.

EXAMPLE 2

27. Let us assume the country large enough that the route LM can be continued to a distance of about 30 miles and the latitude of place L be the same as previously, that is, $\lambda = 48^\circ$. First we suppose that the angle of the route $ALM = \zeta = 30^\circ$ and the latitude at M will be $\varphi = 49^\circ\, 44'$. So, if the earth were spherical, the angle AMm would be $\theta = 31^\circ\, 10'\, 20''$; but under the hypothesis of ellipticity $\delta = \frac{1}{229}$ this angle will be $\theta = 31^\circ\, 10'\, 4''$ and therefore about $16''$ smaller.

But in departing from the same place L or $\lambda = 48^\circ$, at the angle $ALM = \zeta = 60^\circ$, for a distance of about 30 miles, until one arrives at the latitude $\varphi = 49^\circ$, the hypothesis of a spherical earth would give the angle $AMm = \theta = 62^\circ\, 2'\, 27''$; but the hypothesis of ellipticity $\delta = \frac{1}{229}$ would produce $\theta = 62^\circ\, 1'\, 58''$, the difference being $29''$.

Now suppose that the route traced, LM, makes with the meridian at L the angle $\zeta = 80^\circ$ and that after having continued for about 30 miles, one has arrived at M with latitude $\varphi = 48^\circ\, 21'$. Then under the spherical hypothesis the angle AMm would be $\theta = 82^\circ\, 32'\, 53''$; but under the hypothesis of ellipticity $\delta = \frac{1}{229}$ this angle will be $\theta = 82^\circ\, 32'\, 11''$, about $42''$ less.

Since in this case the difference in latitude is still quite perceptible, one could bring the angle ζ closer to 90°; so, let the angle $ALM = \zeta = 85^\circ$, and say that after a path of about 30 miles one arrives at M, where the latitude is $\varphi = 48^\circ\, 10'$. Then under the spherical hypothesis the angle AMm would be $\theta = 88^\circ\, 3'\, 43''$, but under the hypothesis of ellipticiy $\delta = \frac{1}{229}$ this angle will be $\theta = 88^\circ\, 2'\, 27''$, and therefore about $76''$ less.

COROLLARY

28. It is seen by this, unless the country LM be very close to the equator, it is always advantageous to take the angle ALM to be approximately right. It could even be made entirely right, but then the calculation will become a little different than that which I have used until now, since in following the route LM, one will get closer and closer to the equator. But it must well be noted that one sets aside here all error that could be found in the height of the pole. So it will be worth the trouble to develop particularly this case.

PROBLEM 4

29. *If, in departing (*Fig. 3*) from the place L, the route LM is traced so that it is perpendicular to the meridian ALE, and the route be continued to M, where the height of the pole is observed: to find the angle AML, which this route will make with the meridian drawn through M, under the hypothesis of a spherical earth, as well as under the hypothesis of ellipticity expressed by the fraction δ.*

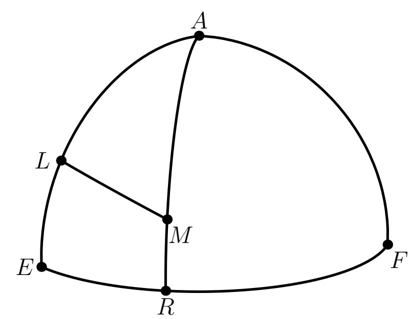

Figure 3

SOLUTION

Let λ be the height of the pole observed at L, and tracing the route LM perpendicularly to the meridian drawn at L, the height of the pole will be found there to be smaller and smaller. For, if under the hypothesis of a spherical earth the arc LM be continued until it attains the angle s, and if the height of the pole at M be named $= \varphi$, one will have by spherical trigonometry that $\sin\varphi = \sin\lambda\cos s$ and therefore $\varphi < \lambda$. So in general, let φ be the height of the pole observed at M, and since $\varphi < \lambda$, we set $\varphi = \lambda - \omega$, so that ω is an angle very small with respect to λ, because we suppose that the place L is quite distant from the equator and the line LM very short.

This being set, if the earth is spherical and we call the angle $AML = \theta$,[5] we shall have, since $\zeta = 90°$,

$$\sin\theta = \frac{\cos\lambda}{\cos\varphi} = \frac{\cos\lambda}{\cos\lambda\cos\omega + \sin\lambda\sin\omega} ,$$

and since ω is very small,

$$\cos\theta = \frac{\sqrt{2\omega\sin\lambda\cos\lambda + \omega^2\sin^2\lambda - \omega^2\cos^2\lambda}}{\cos\lambda + \omega\sin\lambda}$$

or

$$\cos\theta = \left(\frac{1}{\cos\lambda} - \frac{\omega\sin\lambda}{\cos^2\lambda}\right)\left(\sqrt{2\omega\sin\lambda\cos\lambda} - \frac{\omega^2(\cos^2\lambda - \sin^2\lambda)}{2\sqrt{2\omega\sin\lambda\cos\lambda}}\right) ,$$

from which we conclude:

$$\cos\theta = \frac{\sqrt{\omega\sin 2\lambda}}{\cos\lambda} - \frac{\omega^2(2-\cos 2\lambda)}{2\cos\lambda\sqrt{\omega\sin 2\lambda}} .$$

Thus, since the angle θ is nearly right, if we set $\theta = 90° - \mu$, we shall have, because $\cos\theta = \sin\mu = \mu - \frac{1}{6}\mu^3$,

$$\mu = \frac{\sqrt{\omega\sin 2\lambda}}{\cos\lambda} - \frac{\omega^2(4-\cos 2\lambda)}{6\cos\lambda\sqrt{\omega\sin 2\lambda}} .$$

But we now consider the earth as an ellipsoid, and having

$$\sin\theta = \frac{\cos\lambda}{\cos\varphi}(1 + \delta\cos^2\varphi - \delta\cos^2\lambda)$$

or

$$\sin\theta = (1 - \omega\,\mathrm{tang}\,\lambda)(1 + 2\delta\omega\sin\lambda\cos\lambda) ,$$

since the angle θ is again nearly right, we set for this case $\theta = 90° - \mu$ and we shall find

$$\mu = \frac{\sqrt{\omega\sin 2\lambda}}{\cos\lambda} - \frac{\omega^2(4-\cos 2\lambda)}{6\cos\lambda\sqrt{\omega\sin 2\lambda}} - \frac{2\delta\omega\sin\lambda\cos^2\lambda}{\sqrt{\omega\sin 2\lambda}} .$$

[5] $ALM = \theta$ in both *Memoires de l'academie des sciences de Berlin* and *Opera Omnia*[tr]

From this it is seen that the difference between the spherical and elliptical figure of the earth produces in the angle $AML = \theta$ a difference which amounts to

$$\delta \cos \lambda \sqrt{\omega \sin 2\lambda}\,,$$

by which the angle AML is larger under the hypothesis of the ellipsoid than under the spherical hypothesis.

But, if the route ML is measured on the spherical surface by the angle s, having

$$\sin \varphi = \sin \lambda \cos s = \sin \lambda \cos \omega - \cos \lambda \sin \omega\,;$$

one will have $\omega = \frac{1}{2} ss \operatorname{tang} \lambda$ and therefore the said distance will be

$$= \delta s \sin \lambda \cos \lambda\,.$$

COROLLARY 1

30. From this one sees that this difference becomes the largest under the latitude of $45°$, the length of the route s remaining the same. Now, if we set $s = 2°$, $\lambda = 45°$, and $\delta = \frac{1}{229}$, this difference amounts to only $\frac{1}{229} \cdot \frac{1}{2}\, s = 15\,\frac{2}{3}''$, which is much less than the last case of the second Example (paragraph 27), where for an equal route having taken the angle $\zeta = 85°$, the difference amounts to $76''$.

COROLLARY 2

31. So it is not advantageous to make the angle $ALM = \zeta$ right, even though the difference becomes quite perceptible, if this angle is brought so near to $90°$ that it no longer differs perceptibly; since in the case of $\zeta = 85°$, the difference is found to be $76''$, while in making the angle ζ right, it only amounts to about $15''$.

COROLLARY 3

32. It is thus very important to determine the angle ζ, so that when the route LM follows it and is produced up to a certain distance, the difference between the values of the angle θ which correspond to the spherical and elliptical figures of the earth, becomes the largest.

PROBLEM 5

33. *Find* (Fig. 2) *the direction of the route LM which must be chosen so that, after having arrived at M and there having observed the height of the pole, the angle of the route with the meridian at M differs by the greatest possible amount under the hypotheses of sphericity and ellipticity of the earth.*

SOLUTION

Let the height of the pole at L be $= \lambda$, the angle of the route LM, which is sought, be $ALM = \zeta$, and the height of the pole at M be $= \varphi$. Now the principal quantity which must be considered here is the length of the path LM, so that in taking it about the same, the difference between sphericity and ellipticity of the earth becomes the most marked in the angle AMm. Let the angle s denote the length of the route LM, if the earth be spherical, and then one will have by spherical trigonometry

$$\sin\varphi = \cos\zeta\cos\lambda\sin s + \sin\lambda\cos s\,.$$

So this angle φ is the height of the pole at M, and it must certainly be noted, that when the earth is not spherical, the angle s no longer corresponds to the length of the route LM, where it only approximates the size.

Now consider the earth as spherical and set the angle $AMm = \theta$, and we have seen

$$\sin\theta = \frac{\sin\zeta\cos\lambda}{\cos\varphi}\,.$$

But giving to the earth an ellipticity expressed by δ, this angle AMm will be a little smaller; we set therefore this angle $AMm = \theta - \omega$, and we shall have

$$\sin(\theta - \omega) = \frac{\sin\zeta\cos\lambda}{\cos\varphi}\bigl(1 - \delta(\cos^2\lambda - \cos^2\varphi)\bigr)\,;$$

now this equation reduces to

$$\cos\omega - \operatorname{cotg}\theta\sin\omega = 1 - \delta(\cos^2\lambda - \cos^2\varphi),$$

from which it is seen that, the angle ω being very small, the case will not be known to be more favorable than when $\theta = 90^\circ$, because then, since

$$\cos\omega = 1 - \tfrac{1}{2}\omega\omega \quad \text{and} \quad \omega^2 = 2\delta(\cos^2\lambda - \cos^2\varphi)\,,$$

the difference ω is determined by $\sqrt{\delta}$, and is therefore much more considerable than if it were proportional to δ.

So, set $\theta = 90^\circ$ and it is required that

$$\sin\zeta = \frac{\cos\varphi}{\cos\lambda}, \quad \text{or} \quad \cos\varphi = \sin\zeta\cos\lambda\,,$$

thus $\sin\varphi = \sqrt{(1 - \sin^2\zeta\cos^2\lambda)}$. This value being substituted gives

$$1 - \sin^2\zeta\cos^2\lambda = \cos^2\zeta\cos^2\lambda\sin^2 s + 2\cos\zeta\sin\lambda\cos\lambda\sin s\cos s + \sin^2\lambda\cos^2 s\,,$$

or

$$0 = (\cos\zeta\cos\lambda\cos s - \sin\lambda\sin s)^2\,;$$

thus

$$\cos\zeta = \operatorname{tang}\lambda\operatorname{tang}s$$

and from this we conclude

$$\sin\varphi = \frac{\sin\lambda\sin^2 s}{\cos s} + \sin\lambda\cos s = \frac{\sin\lambda}{\cos s}\,.$$

Having thus established the length of the route more or less according to the nature of the country, so that fifteen German miles are counted as one degree: first one will have the angle $ALM = \zeta$, which the route must make with the meridian at L, by the formula

$$\cos\zeta = \text{tang}\,\lambda\,\text{tang}\,s$$

and on this route one will get to a place M, where the elevation of the pole will be φ, so that

$$\sin\phi = \frac{\sin\lambda}{\cos s}\,.$$

Now having arrived at this height of the pole on the route marked LM, it is certain, that if the earth were spherical, the route would be found to be perpendicular to the meridian at M; that is, the angle θ would be right. But, under the hypothesis of the ellipticity of the earth, the angle AMm will be found to be less than right; thus, suppose that this angle is $90^\circ - \omega$ and we shall have, because $\theta = 90^\circ$,

$$\cos\omega = 1 - \delta(\cos^2\lambda - \cos^2\varphi) = 1 - \tfrac{1}{2}\omega\omega\,,$$

thus

$$\omega = \sqrt{2\delta(\cos^2\lambda - \cos^2\varphi)}\,;$$

now

$$\cos^2\varphi = 1 - \frac{\sin^2\lambda}{\cos^2 s}$$

and therefore

$$\cos^2\lambda - \cos^2\varphi = -\sin^2\lambda + \frac{\sin^2\lambda}{\cos^2 s} = \sin^2\lambda\,\text{tang}^2\,s\,.$$

Consequently we shall have

$$\omega = \sin\lambda\,\text{tang}\,s\sqrt{2\delta}\,,$$

or, since the arc s is always so small that it can be identified with the tangent, the difference of the angles AMm corresponding to the sphericity or to the ellipticity of the earth, will be

$$\omega = s\sin\lambda\cdot\sqrt{2\delta}\,.$$

And reciprocally, having carefully observed the angle $AMm = 90^\circ - \omega$, from this is deduced the elliptic figure of the earth

$$\delta = \frac{\omega^2}{2ss\sin^2\lambda}\,,$$

or, because $s = \dfrac{\cos\zeta}{\operatorname{tang}\lambda}$, one will have

$$\delta = \frac{\omega\omega}{2\cos^2\zeta\cos^2\lambda}\,.$$

COROLLARY 1

34. So this method seems quite advantageous in all regions of the earth which are not too close to the equator, since, when $\sin\lambda$ is very small, the difference ω will become imperceptible. But the further the country is from the equator, so much more will this operation be practiced with success. However, it is evident that when too close to the poles, this method loses its usefulness in other ways, since under the poles themselves there are no meridian lines at all.

COROLLARY 2

35. The further one continues along the route LM the more the difference ω becomes large in the same ratio. But it is not the same for the ellipticity δ, which follows an under-doubled ratio, so that if the value of δ becomes four times as large, the angle ω will only be doubled. Now the size of this angle ω will amply compensate for this shortcoming.

COROLLARY 3

36. Suppose that the ellipticity $\delta = \frac{1}{229}$ and that the length of the route LM is about 15 German miles. Furthermore let the height of the pole at L be 45° and, because $\sin\lambda = \dfrac{1}{\sqrt{2}}$ and $s = 1^\circ$, approximately we shall have $\omega = \frac{1}{15}$ degree $= 4'$; and this difference is sizeable enough to discover the true ellipticity of the earth.

EXAMPLE

37. Suppose that the place L is found at a latitude of $52^\circ\,31'$ and that it is convenient to trace a line towards the west, or approximately, for a distance of about 15 miles and it is a matter of finding the most advantageous direction to trace the line LM. Since $\lambda = 52^\circ\,31'$ and $s = 1^\circ$, one will have

$$\begin{array}{lrl} & \log\text{tang}\lambda = & 10{,}1152811 \\ + & \log\text{tang}\, s = & 8{,}2419215 \\ \hline & \log\cos\zeta = & 8{,}3572026 \\ \text{thus} & \zeta = & 88^\circ\,41'\,45'' \end{array} \qquad \text{and} \qquad \begin{array}{rrl} & \log\sin\lambda = & 9{,}8995636 \\ - & \log\cos s = & 9{,}9999338 \\ \hline & \log\sin\varphi = & \text{- } 9{,}8996298 \\ & \varphi = & 52^\circ\,31'\,41'' \end{array} .$$

Now, since it would be impossible to observe these measures exactly and it suffices to get them about right, let us suppose that a line LM was traced so that it makes with the meridian drawn through L towards the north an angle of $88^\circ\,41'\,30''$ and that this line was extended to M, where the height of the pole was observed to be $40''$ higher than at L, so that

$$\lambda = 52^\circ\,31'\,, \quad \zeta = 88^\circ\,41'\,30''\,, \quad \text{and} \quad \varphi = 52^\circ\,31'\,40'' \ .$$

This being set, let us see at what angle this line LM will be inclined to the meridian drawn through M, if the earth be spherical or an elliptic spheroid under the hypothesis $\delta = \frac{1}{229}$. First, if the earth were spherical and the angle[6] $AMm = \theta$, then having

$$\sin\theta = \frac{\sin\zeta\cos\lambda}{\cos\varphi} \ ,$$

[6] $AMm = \theta$ in *Memoires de l'academie des sciences de Berlin*, but $AMm = 0$ in *Opera Omnia*[tr]

we would have this calculation to make:

$$\begin{array}{rrr} & \log\ \sin\zeta = & 9,9998868 \\ & \log\cos\lambda = & \underline{9,7842824} \\ & & 9,7841692 \\ & \log\cos\varphi = & \underline{9,7841726} \\ \text{thus} & \log\ \sin\theta = & 9,9999966 \\ \text{and therefore} & \theta = & 89^\circ\, 46'\, 24'' \quad . \end{array}$$

But, if the earth were spheroidal according to the value $\delta = \frac{1}{229}$, and θ once again denoted the angle AMm, having

$$\sin\theta = \frac{\sin\zeta\cos\lambda}{\cos\varphi}\left(1 - \delta(\cos^2\lambda - \cos^2\varphi)\right) \;,$$

the following calculation would be required:

$$\begin{array}{lcrl} \log\cos^2\lambda = 9,5685648 & \ldots\ldots\ldots & \cos^2\lambda & = 0,3703094 \\ \log\cos^2\varphi = 9,5683452 & \ldots\ldots\ldots & \cos^2\varphi & = 0,3701223 \\ & & \cos^2\lambda - \cos^2\varphi & = 0,0001871 \\ & & \delta(\cos^2\lambda - \cos^2\varphi) & = 0,0000008 \end{array}$$

and therefore

$$\sin\theta = 0,9999992 \cdot \frac{\sin\zeta\cos\lambda}{\cos\varphi}$$

now

$$\begin{aligned} \log \tfrac{\sin\zeta\cos\lambda}{\cos\varphi} &= 9,9999966 \\ \log 0,9999992 &= 9,9999996 \\ \log\sin\theta &= 9,9999962 \end{aligned}$$

and therefore

$$\theta = 89^\circ\, 45'\, 36'' \;.$$

So the difference is $48''$, but it could well amount to $4'\,30''$, if the determinations found by the calculation had been followed exactly.

38. But in this example and other similar ones, it must be remarked that the difference of the latitudes λ and φ is so small, that the least error committed in the observations would have too much influence on the conclusion. For, since after having observed the four angles λ, φ and ζ, θ, one has for the ellipticity of the earth

$$\delta = \left(1 - \frac{\sin\theta\cos\varphi}{\sin\zeta\cos\lambda}\right) : \left(\cos^2\lambda - \cos^2\varphi\right) \ ;$$

in order that the conclusion be sure, the denominator must not come out too small. Now, to show how much an error committed in the measure of these angles may influence the conclusion, let us consider the last case of the 2^e example (paragraph 27), where, supposing $\delta = \frac{1}{229}$, the four angles would be

$$\lambda = 48^\circ\,, \quad \varphi = 48^\circ\,10'\,, \quad \zeta = 85^\circ\,, \text{ and } \quad \theta = 88^\circ\,2'\,27''$$

and let us suppose that in the difference of the latitudes λ and φ and in that of the angles ζ and θ there be error of $5''$, so that from the actual observations is drawn the result

$$\lambda = 48^\circ\,, \quad \varphi = 48^\circ\,10'\,5''\,, \quad \zeta = 85^\circ\,, \text{ and } \quad \theta = 88^\circ\,2'\,22''$$

and let us see what will be the ellipticity found from this:

$\log\sin\theta$	=	9,9997457	$\log\cos^2\lambda$	= 9,6510218
$\log\cos\varphi$	=	9,8240919	$\log\cos^2\varphi$	= 9,6481838
		9,8238376	$\cos^2\lambda$	= 0.4477358
$\log\sin\zeta$	=	9,9983442	$\cos^2\varphi$	= 0.4448195
$\log\cos\lambda$		9,8255109	Denominator	= 0,0029163
		9,8238551		
$\log\frac{\sin\theta\cos\varphi}{\sin\zeta\cos\lambda}$	=	9,9999825	thus δ	$= \frac{403}{29163} = \frac{1}{72}$
number	=	0,9999597		
Numerator	=	0.0000403		

Thus, one would find the ellipticity much larger than it is in reality and this large difference results principally from the error of the angle φ in the numerator; for the denominator does not suffer greatly. For, if there is an error in the angle φ of $d\varphi$, the value of δ thereby becomes false by

$$\frac{d\varphi \sin\theta \sin\varphi}{\sin\zeta \cos\lambda(\cos^2\lambda - \cos^2\varphi)} \,,$$

that is, this error will be to the quantity δ itself, as $d\varphi \sin\theta \sin\varphi$ to $\sin\zeta\cos\lambda - \sin\theta\cos\varphi$. Thus, in order that the error not be too large, $\sin\zeta\cos\lambda - \sin\theta\cos\varphi$ and therefore also the denominator

$$\cos^2\lambda - \cos^2\varphi$$

must not become too small.

REMARK 2

39. So it will be better to make the angle $ALM = \zeta$ smaller, even though the difference in the angle θ for the spherical and elliptical hypotheses becomes smaller; for the advantages noted above, when the angle ζ is taken almost right, suppose absolutely that the slightest error is not committed in the observation of latitudes and as soon as some error should be suspected, this route must be abandoned and others, where the angle ζ is taken much smaller, be preferred to it. So for example having found

$$\delta = \tfrac{1}{229}, \quad \lambda = 48^\circ\,, \quad \varphi = 49^\circ\,, \quad \zeta = 60^\circ\,, \text{ the angle } \quad \theta = 62^\circ\, 1'\, 58''\,,$$

we suppose that one had found by actual operations that

$$\lambda = 48^\circ\,, \quad \varphi = 49^\circ\, 0'\, 5''\,, \quad \zeta = 60^\circ\,, \text{ and } \quad \theta = 62^\circ\, 1'\, 53''$$

and we look for the ellipticity δ :

$\log\sin\theta$	=	9,9460614	$\log\cos^2\lambda$	= 9,6510218
$\log\cos\varphi$	=	9,8169308	$\log\cos^2\varphi$	= 9,6338616
		9,7629922		
$\log\sin\zeta$	=	9,9375306	$\cos^2\lambda$	= 0,4477358
$\log\cos\lambda$		9,8255109	$\cos^2\varphi$	= 0,4303894
		9,7630415	Denominator	= 0,0173464
$\log\frac{\sin\theta\cos\varphi}{\sin\zeta\cos\lambda}$	=	9,9999507	thus δ	$= \frac{1135}{173464} = \frac{1}{153}$
number	=	0,9998865		
Numerator	=	0,0001135		

So in this case the error of the observations has much less influence on the conclusion. However, it will be found all the same, that it is not advantageous to take the angle ζ too small, for if we took it to be 30°, and we set φ to be $5''$ too large and θ to be $5''$ too small, we would find $\delta = \frac{1}{229}$; whence it can be concluded that the best course is always to trace the line LM so that it makes an angle with the meridian of less than 60° and greater than 30°. There are other reasons which suggest taking this angle ζ at $54^\circ\,44'$, so that $\sin^2\zeta\cos\zeta$ becomes a *maximum*; but, since it is not certain if the errors affect the angles λ and φ, and what relationship these double errors have between themselves, one would not at all know how to determine the above precisely, and it suffices to have fixed the limits 30° and 60°, between which the angle ζ should be chosen. Now the most advantageous is to continue the line LM as far as possible; for the further it can be extended, the more sure will one be of the conclusion that will be drawn. However, I must confess that this method would never be known to executed in practice, for not only would one encounter insurmountable difficulties in tracing the shortest line, but also the meridian line would never be known to be traced so exactly, as the success of this method requires, an error of $20''$ being almost inevitable.