TRANSLATOR'S NOTE

In preparing this translation, my assumption has been that the interested audience likely consists of both mathematicians and historians of mathematics. To satisfy the latter, I have attempted, as near as possible, to mimic Euler's phrasing, and especially his mathematical notation, with a few exceptions in cases where Euler's notation might be confusing to a modern reader. These include:

- Use of "arcsin, arccos" rather than "A sin, A cos" for the inverse trigonometric functions
- Use of, for example, "$\cos^2 u$" rather than "$\cos u^2$" to denote the square of the cosine;
- Clarifying the argument of the square root operator in paragraphs 42 and 51.

In most other cases, I have attempted to copy Euler's original notation.

My parenthetical notes are enclosed in brackets; material found in parentheses is found so in the original texts.

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Principles of Spherical Trigonometry Drawn from the Method of the Maxima and Minima

Leonhard Euler
(G. Heine, tr.)

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Since one knows that the arcs of great circles, drawn on the surface of a sphere, represent the shortest path from one point to another, a spherical triangle could be defined as follows: given three points on the surface of a sphere, let a spherical triangle be the space enclosed between these three points. Thus, since the sides of a spherical triangle are the shortest lines which can be drawn from one angle to another, the method of maxima and minima could be used to determine the sides of a spherical triangle and from this could be found the relations which exist between the angles and sides, which is exactly the content of spherical Trigonometry. For the three points, where are found the angles, will determine the three sides as well as the three angles, and these six things will always have such a relationship among them, that any three being known, one can from this determine the three others.

This, therefore, is a property which the spherical triangles have in common with plane triangles, which are the subject of elementary Trigonometry. For, just as a plane triangle is the space enclosed between three points marked on a plane, when the shortest path is drawn from one to another, which is, on the plane, a straight line, so a spherical triangle is the space enclosed between three points marked on the surface of a sphere, when these three points are joined by the shortest lines that can be drawn on the same surface. Now, it is clear that a spherical triangle changes into a planar triangle, when the radius of a sphere becomes infinitely large, so that a planar surface can be regarded as the surface of an infinitely large sphere.

No doubt, one will object that it goes against the rules of the method, to want to use the calculus of the maxima and minima to establish the foundations of spherical Trigonometry; it seems useless, moreover, to derive these again from other priniciples, since those which have been used until now, are founded on elementary Geometry, whose rigor serves as a rule for all the other parts of Mathematics. But I remark, first, that the method of maxima and minima thereby acquires somewhat of a new luster, when I show that by itself it leads to the resolution of spherical triangles; moreover, it is always useful to arrive by different routes at the same truth, since our spirits will not fail thereby to arrive at new understandings.

But I also may argue that the method of maxima and minima is much more general than the ordinary method. For the latter is limited to triangles formed on plane or spherical surfaces, while the former extends to any surface whatever. Thus, if one asks the nature of triangles formed on spheroidal or conic surfaces, whose sides are the shortest lines that can be drawn from one angle to the other, the ordinary method would not be suitable to such research; it would be absolutely necessary to resort to the method of maxima and minima, without which one would not even be in a state to know the shortest lines, which form the sides of these triangles.

From this, it is understood that this research could well become of great importance; for the surface of the Earth is not spherical, but spheroidal; a triangle formed on the surface of the Earth belongs to the sort of which I just mentioned. To see this, one only needs to imagine three points on the surface of the Earth which are joined by the shortest path which leads from one to the other, or formed by a cord stretched from one to the other; for it is thus that those triangles must be represented, which are used in the operations for the measure of the Earth. Certainly, it is true that such triangles are ordinarily regarded as plane and rectilinear, and it is quite accurate, when one calculates on the basis of spherical triangles. But if one succeeds in making these triangles much larger, and one wants to calculate with the greatest possible precision, it would without doubt be necessary to investigate the true nature of such triangles, which cannot be fully known without using the method of the maxima and minima.

Having, therefore, seen the importance of this method in the subject which it concerns, it will do no harm to apply this method to the resolution of spherical triangles, since on one hand, this inquiry will serve as a basis and a model for the resolution of triangles formed on a arbitrary spheroidal surface; on the other hand it will furnish us with considerable explanations as much for

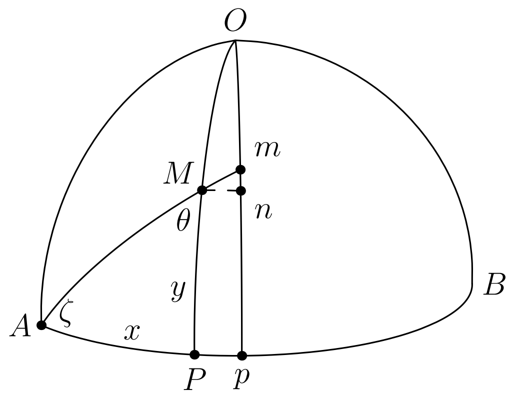

Figure 1: Euler's original figure, labels for ζ and θ added

spherical Trigonometry itself as for the method of the maxima and minima, from which one will know more and better its extent and great usage. For, once one has shown that most mechanical and physical problems are resolved very quickly by means of this method, it will only be very pleasant to see that the same method brings such a great help to the resolution of problems in pure Geometry.

To begin this inquiry in such a manner that it applies equally to the sphere and to any spheroid whatever, I first consider two opposite points on the sphere as its poles, and the great circle equally distant from them will represent the equator, and the shortest lines drawn from a pole to each point on the equator will represent the meridians which are perpendicular to the equator. Now on the sphere, any side of a spherical triangle can be regarded as part of the equator, and when it is a right triangle, one of the sides which forms the right angle can always be supposed to be a portion of the meridian, since the two poles can be chosen freely. But it will not be the same when the surface is not spherical, but spheroidal. However, I will only speak here of spherical surfaces, reserving spheroidal surfaces for another Memoire.

Problem 1

1. *Given* (Fig. 1) *the arc AP on the equator AB, and the arc PM on a meridian OP, find, on the spherical surface, the shortest line AM, which can be drawn from point A to point M.*

Solution

Setting the half-diameter of the sphere $= 1$, let

the arc of the equator $AP = x$, and the arc of the meridian $PM = y$.

In addition, let

$$\text{the sought-after arc } AM = s;$$

and let it be prolonged an infinitely small distance to m, so that $Mm = ds$, and let the meridian Omp be drawn through m, so that the element Mn is perpendicular to Omp. From this, one will have $Pp = dx$ and $mn = dy$, and since Pp is to Mn as the sine of the total meridian OP=1 to the sine of the arc OM, or to the cosine of $PM = y$, one will have $Mn = dx \cos y$, and the triangle Mnm to n, being right, gives

$$Mm = ds = \sqrt{dy^2 + dx^2 \cos^2 y}$$

and therefore

$$AM = s = \int \sqrt{dy^2 + dx^2 \cos^2 y}\ .$$

Thus, the problem is to find such a relation between x and y, that if known values such as AP and PM are given, the value of the integral

$$\int \sqrt{dy^2 + dx^2 \cos^2 y}$$

becomes as small as possible. To this effect, set $dy = p\,dx$, in order to reduce this integral to the form

$$\int dx \sqrt{pp + \cos^2 y}\ ;$$

and since I have demonstrated[1] that when the integral formula $\int Z\,dx$, where Z is a function of x, y, and p, with $dZ = M\,dx + N\,dy + P\,dp$, must become as small or large as possible, this happens by the equation

$$N\,dx - dP = 0.$$

[1]See, for example, Proposition III, Chapter II, from E65(1743) – *Methodus inveniendi lineas curvas maximi minimive proprietate gaudentes, sive solutio problematis isoperimetrici lattissimo sensu accepti.* [*tr*]

Thus, making application to our case, we shall have

$$Z = \sqrt{pp + \cos^2 y};$$

thus

$$dZ = -\frac{dy\ \sin y \cos y}{\sqrt{pp + \cos^2 y}} + \frac{p\,dp}{\sqrt{pp + \cos^2 y}}\ ;$$

and consequently

$$M = 0, \qquad N = -\frac{\sin y \cos y}{\sqrt{pp + \cos^2 y}}, \qquad \text{and} \quad P = \frac{p}{\sqrt{pp + \cos^2 y}}.$$

Now, since $M = 0$ and therefore $dZ = N\,dy + P\,dp$, we multiply the equation $N\,dx - dP = 0$ by p, which, since $dy = p\,dx$, will become $N\,dy - p\,dP = 0$, or $N\,dy = p\,dP$, and, this value being substituted for $N\,dy$ will give

$$dZ = p\,dP + P\,dp\ ,$$

whose integral is $Z = Pp + C$, or

$$\sqrt{pp + \cos^2 y} = \frac{pp}{\sqrt{pp + \cos^2 y}} + C;$$

which reduces to

$$\cos^2 y = C\sqrt{pp + \cos^2 y};$$

from which is elicited

$$CC\,pp = \cos^2 y(\cos^2 y - CC),$$

or

$$p = \frac{dy}{dx} = \frac{\cos y\sqrt{\cos^2 y - CC}}{C}\ .$$

Thus the relation between x and y is expressed in the separated differential equation[2]

$$dx = \frac{C\,dy}{\cos y\sqrt{\cos^2 y - CC}}$$

[2] "équation différentielle séparée" [*tr*]

and from this is obtained

$$ds = dx\sqrt{pp + \cos^2 y} = \frac{dx\cos^2 y}{C};$$

thus

$$ds = \frac{dy\ \cos y}{\sqrt{\cos^2 y - CC}}\ ,$$

and the arc itself

$$s = \int \frac{dy\ \cos y}{\sqrt{\cos^2 y - CC}}\ .$$

Corollary 1

2. Thus, the equation

$$dx = \frac{C\,dy}{\cos y\sqrt{\cos^2 y - CC}}$$

expresses the nature of the line AM, which has the property, that taking any portion whatever, this will be the shortest line that can be drawn between its endpoints on the surface of the sphere. Now, I have shown elsewhere that this line is also a great circle of the sphere; here it does not matter to our purpose, what relation this line has with the sphere, provided that we know it is the shortest between its endpoints.

Corollary 2

3. Having found

$$dx = \frac{C\,dy}{\cos y\sqrt{\cos^2 y - CC}}$$

we shall have

$$Mn = dx\cos y = \frac{C\,dy}{\sqrt{\cos^2 y - CC}}\ .$$

Now $\dfrac{Mn}{mn}$ expresses the tangent of the angle AMP and consequently we have:

$$\operatorname{tang} AMP = \frac{C}{\sqrt{\cos^2 y - CC}}\ .$$

Furthermore, having

$$Mm = ds = \frac{dy\cos y}{\sqrt{\cos^2 y - CC}}\ ,$$

the fraction $\dfrac{Mn}{Mm}$ expresses the sine of the angle AMP, so that

$$\sin AMP = \frac{C}{\cos y} \quad \text{and} \quad \cos AMP = \frac{\sqrt{\cos^2 y - CC}}{\cos y}\ .$$

Corollary 3

4. Moreover, setting $y = 0$, the point M will arrive at A and then the fraction $\dfrac{dy}{dx}$ will express the tangent of the angle PAM, and $\dfrac{dy}{ds}$ its sine and $\dfrac{dx}{ds}$ its cosine. Now having $\cos y = 1$,

$$dx = \frac{C\,dy}{\sqrt{1 - CC}} \quad \text{and} \quad ds = \frac{dy}{\sqrt{1 - CC}}\ ,$$

from which we conclude

$$\operatorname{tang} PAM = \frac{\sqrt{1-CC}}{C}, \quad \sin PAM = \sqrt{1-CC}, \quad \text{and} \quad \cos PAM = C.$$

Corollary 4

5. Thus, if we introduce this angle PAM in place of the constant C, and set $PAM = \zeta$, we shall have, since $C = \cos\zeta$, the two following equations:

$$dx = \frac{dy\cos\zeta}{\cos y\sqrt{\cos^2 y - \cos^2\zeta}} \quad \text{and} \quad ds = \frac{dy\cos y}{\sqrt{\cos^2 y - \cos^2\zeta}}.$$

Moreover, if we name the angle $AMP = \theta$, we shall have

$$\operatorname{tang}\theta = \frac{\cos\zeta}{\sqrt{\cos^2 y - \cos^2\zeta}}, \quad \sin\theta = \frac{\cos\zeta}{\cos y}, \quad \text{and} \quad \cos\theta = \frac{\sqrt{\cos^2 y - \cos^2\zeta}}{\cos y}.$$

Corollary 5

6. It yet remains to integrate the two differential equations which express the values of dx and ds. By integration, it is found that

$$x = \arcsin\frac{C\sin y}{\cos y\sqrt{1-CC}}, \quad \text{or} \quad \sin x = \frac{C\sin y}{\cos y\sqrt{1-CC}} = \frac{\cos\zeta\sin y}{\sin\zeta\cos y}$$

and

$$s = \arccos\frac{\sqrt{\cos^2 y - CC}}{\sqrt{1-CC}}, \quad \text{or} \quad \cos s = \frac{\sqrt{\cos^2 y - CC}}{\sqrt{1-CC}} = \frac{\sqrt{\cos^2 y - \cos^2\zeta}}{\sin\zeta}$$

Corollary 6

7. Here, then, are the quantities ζ and y, and the other quantities x, s, θ so

determined:

$$\sin x = \frac{\cos\zeta \sin y}{\sin\zeta\cos y}, \quad \cos x = \frac{\sqrt{\cos^2 y - \cos^2\zeta}}{\sin\zeta\cos y}, \quad \text{and} \quad \operatorname{tang} x = \frac{\cos\zeta\sin y}{\sqrt{\cos^2 y - \cos^2\zeta}}$$

$$\sin s = \frac{\sin y}{\sin\zeta}, \quad \cos s = \frac{\sqrt{\cos^2 y - \cos^2\zeta}}{\sin\zeta}, \quad \text{and} \quad \operatorname{tang} s = \frac{\cos\zeta}{\sqrt{\cos^2 y - \cos^2\zeta}}$$

$$\sin\theta = \frac{\cos\zeta}{\cos y}, \quad \cos\theta = \frac{\sqrt{\cos^2 y - \cos^2\zeta}}{\cos y}, \quad \text{and} \quad \operatorname{tang}\theta = \frac{\cos\zeta}{\sqrt{\cos^2 y - \cos^2\zeta}}$$

Corollary 7

8. There is only one irrational formula,

$$\sqrt{\cos^2 y - \cos^2\zeta},$$

in the equations we have found; by eliminating it, we shall obtain:

$$\frac{\cos s}{\cos x} = \cos y, \qquad \frac{\cos\theta}{\cos x} = \sin\zeta, \qquad \frac{\cos\theta}{\cos s} = \frac{\sin\zeta}{\cos y},$$

$$\frac{\operatorname{tang} x}{\operatorname{tang} s} = \cos\zeta, \qquad \frac{\operatorname{tang} x}{\operatorname{tang}\theta} = \sin y, \qquad \frac{\operatorname{tang} s}{\operatorname{tang}\theta} = \frac{\sin y}{\cos\zeta},$$

$$\sin x = \frac{\cos\zeta\sin y}{\sin\zeta\cos y}, \quad \cos x \operatorname{tang} s = \frac{\sin y}{\sin\zeta\cos y}, \quad \cos x \operatorname{tang}\theta = \frac{\cos\zeta}{\sin\zeta\cos y},$$

$$\cos s \operatorname{tang} x = \frac{\cos\zeta\sin y}{\sin\zeta}, \qquad \sin s = \frac{\sin y}{\sin\zeta}, \qquad \cos s\operatorname{tang}\theta = \frac{\cos\zeta}{\sin\zeta},$$

$$\cos\theta\operatorname{tang} x = \frac{\cos\zeta\sin y}{\cos y}, \quad \cos\theta\operatorname{tang} s = \frac{\sin y}{\cos y}, \qquad \sin\theta = \frac{\cos\zeta}{\cos y}.$$

Corollary 8

9. Having here five quantities, x, y, s, ζ, and θ, which form part of the right spherical triangle APM, we take from the equalities just found those which

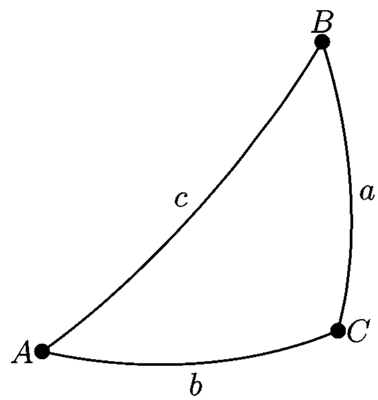

Figure 2

contain three of the quantities, and reduce them to a simpler form:

I. $\cos s = \cos x \cos y$,	VI. $\sin y = \sin\zeta \sin s$,
II. $\cos\theta = \sin\zeta \cos x$,	VII. $\cos s \,\mathrm{tang}\,\zeta \,\mathrm{tang}\,\theta = 1$,
III. $\mathrm{tang}\, x = \cos\zeta \,\mathrm{tang}\, s$,	VIII. $\mathrm{tang}\, y = \cos\theta \,\mathrm{tang}\, s$,
IV. $\mathrm{tang}\, x = \sin y \,\mathrm{tang}\,\theta$,	IX. $\cos\zeta = \sin\theta \cos y$,
V. $\mathrm{tang}\, y = \sin x \,\mathrm{tang}\,\zeta$;	

whence, given any two quantities, one can find from them the three others, without the need to extract any roots, provided that one adds here this tenth one:

$$\text{X.}\quad \sin x = \sin\theta \sin s \quad ,$$

which follows immediately from the three first formulae on the left in paragraph 7.

Problem 2

10. *Exhibit the rules for the resolution of all cases of right spherical triangles.*

Solution

Let the angles (Fig. 2) be marked by A, B, C, with C being the right angle, and the sides by the lower-case letters a,b,c, corresponding to their opposite angles, so that c is the hypotenuse and a and b the supports. Then comparing this triangle with the previous figure, we shall have

$$s = c, \quad x = b, \quad y = a, \quad \zeta = A, \quad \text{and} \quad \theta = B.$$

Now everything turns on the fact that, given two of these five quantities, the three others are determined from them; then the formulae reported will yield the following resolutions for all possible cases:

The two given quantities		Determination of the three others		
I.	a, b	$\cos c = \cos a \cdot \cos b,$	$\operatorname{tang} A = \dfrac{\operatorname{tang} a}{\sin b},$	$\operatorname{tang} B = \dfrac{\operatorname{tang} b}{\sin a}$
II.	a, c	$\cos b = \dfrac{\cos c}{\cos a},$	$\sin A = \dfrac{\sin a}{\sin c},$	$\cos B = \dfrac{\operatorname{tang} a}{\operatorname{tang} c}$
III.	b, c	$\cos a = \dfrac{\cos c}{\cos b},$	$\cos A = \dfrac{\operatorname{tang} b}{\operatorname{tang} c},$	$\sin B = \dfrac{\sin b}{\sin c}$
IV.	a, A	$\sin b = \dfrac{\operatorname{tang} a}{\operatorname{tang} A},$	$\sin c = \dfrac{\sin a}{\sin A},$	$\sin B = \dfrac{\cos A}{\cos a},$
V.	a, B	$\operatorname{tang} b = \sin a \operatorname{tang} B,$	$\operatorname{tang} c = \dfrac{\operatorname{tang} a}{\cos B},$	$\cos A = \cos a \sin B$
VI.	b, A	$\operatorname{tang} a = \sin b \operatorname{tang} A,$	$\operatorname{tang} c = \dfrac{\operatorname{tang} b}{\cos A},$	$\cos B = \cos b \sin A$
VII.	b, B	$\sin a = \dfrac{\operatorname{tang} b}{\operatorname{tang} B},$	$\sin c = \dfrac{\sin b}{\sin B},$	$\sin A = \dfrac{\cos B}{\cos b}$
VIII.	c, A	$\sin a = \sin c \sin A,$	$\operatorname{tang} b = \operatorname{tang} c \cos A,$	$\operatorname{tang} B = \dfrac{1}{\cos c \operatorname{tang} A}$
IX.	c, B	$\sin b = \sin c \sin B,$	$\operatorname{tang} a = \operatorname{tang} c \cos B,$	$\operatorname{tang} A = \dfrac{1}{\cos c \operatorname{tang} B}$
X.	A, B	$\cos a = \dfrac{\cos A}{\sin B},$	$\cos b = \dfrac{\cos B}{\sin A},$	$\cos c = \dfrac{1}{\operatorname{tang} A \operatorname{tang} B}$

Corollary 1

11. From the above it is evident that side a with its opposite angle A enters into these formulae, exactly as does side b with its opposite angle B, so that it makes no difference which of the two sides a and b one wishes to take for the base, exactly as the nature of the subject requires.

Corollary 2

12. The large number of formulae which express the relationship between the various parts of the right triangle, are reduced to the following formulae, whose number is smaller; it suffices to learn these by heart.

$$\text{I.}\quad \sin c = \frac{\sin a}{\sin A} \qquad \text{or} \qquad \sin c = \frac{\sin b}{\sin B}$$

$$\text{II.}\quad \cos c = \cos a \cos b$$

$$\text{III.}\quad \cos c = \cot A \cot B$$

$$\text{IV.}\quad \cos A = \frac{\operatorname{tang} b}{\operatorname{tang} c} \qquad \text{or} \qquad \cos B = \frac{\operatorname{tang} a}{\operatorname{tang} c}$$

$$\text{V.}\quad \sin A = \frac{\cos B}{\cos b} \qquad \text{or} \qquad \sin B = \frac{\cos A}{\cos a}$$

$$\text{VI.}\quad \sin a = \frac{\operatorname{tang} b}{\operatorname{tang} B} \qquad \text{or} \qquad \sin b = \frac{\operatorname{tang} a}{\operatorname{tang} A}$$

Corollary 3

13. It is only necessary to note these six formulae, which contain many of the properties of right spherical triangles, and it will be possible to solve every imaginable case of such triangles.

Problem 3

14. *Find the area of a right spherical triangle.*

Solution

In the right triangle APM (Fig. 1) let the base $AP = x$ and the side $PM = y$, and having drawn the infinitely close meridian Omp, one will have $Pp = dx$ and $mn = dy$. Moreover, having $Mn = dx \cos y$, the element of the area $PMmp$ will be $= dx\, dy \cos y$, taking dx for a constant. Thus, the area itself will

$$PMmp = dx \ \sin y \,,$$

which, being the differential of the area of the triangle APM, this will be

$$= \int dx \ \sin y \,.$$

Now we have found that

$$dx = \frac{dy \cos\zeta}{\cos y \sqrt{\cos^2 y - \cos^2\zeta}} \ ,$$

where ζ marks the angle PAM; consequently, the area of the triangle will be

$$= \int \frac{dy \sin y \cos\zeta}{\cos y \sqrt{\cos^2 y - \cos^2\zeta}} \ .$$

In place of y we introduce the angle $AMP = \theta$; and because

$$\sin\theta = \frac{\cos\zeta}{\cos y} \quad \text{and} \quad \cos\theta = \frac{\sqrt{\cos^2 y - \cos^2\zeta}}{\cos y} \ ,$$

we shall have

$$d\theta \cos\theta = \frac{dy \cos\zeta \sin y}{\cos^2 y} \ ;$$

thus

$$d\theta = \frac{dy \cos\zeta \sin y}{\cos y \sqrt{\cos^2 y - \cos^2\zeta}} \ ,$$

so that the sought-after area of the triangle becomes

$$= \int d\theta = \theta + \text{Const} \ .$$

In order to assign this constant its proper value, one must consider that the area should vanish when the point M collapses into A, in which case the

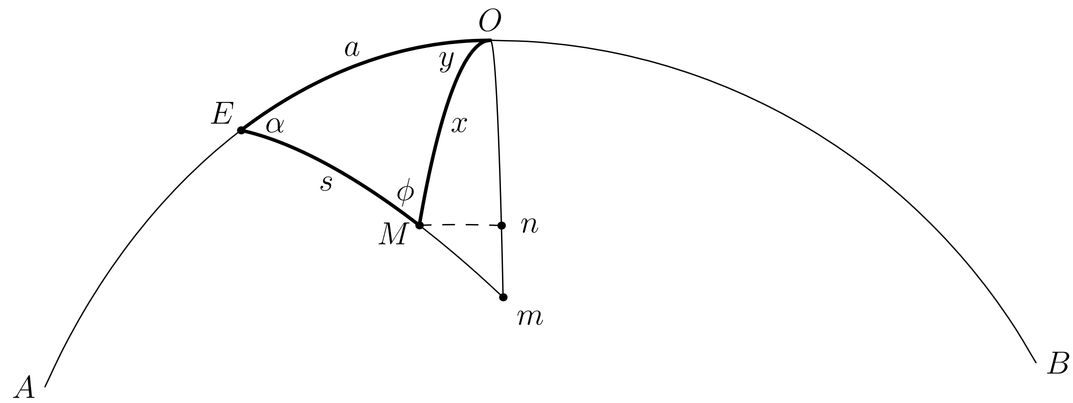

Figure 3: Copy of Euler's figure; labels for a,x,y,s,α, and ϕ added

angle θ becomes $90° - \zeta$; thus it must be that $90° - \zeta + \text{Const} = 0$, so that $\text{Const} = \zeta - 90°$. Consequently, the sought-after area of the triangle will be

$$= \zeta + \theta - 90°\ ;$$

in other words, the excess of the sum of the two angles ζ and θ over a right angle shall express the area of the triangle APM.

Corollary 1

15. Thus, the sum of the angles PAM and AMP is always greater than a right angle, and the excess becomes larger as the area of the triangle increases. And the product of the great-circle arc measuring this excess, multipied by the radius of the sphere, will give the area of the spherical triangle.

Corollary 2

16. From this, the area of an arbitrary spherical triangle is easily deduced: for, since such a triangle can be resolved into two right triangles, one will find its area, when one multiplies the excess over 180° of the sum of the three angles, by the radius of the sphere.

Problem 4

17. *Given two arbitrary points E and M on the surface of a sphere*(Fig. 3), *find the shortest line EM between them.*

Solution

From one of the poles O, let meridians OE and OM be drawn, where the location of the latter is regarded as variable. We denote

$$\text{the meridian } OE = a, \qquad OM = x, \quad \text{and} \quad \text{the angle } EOM = y.$$

Morever, let the desired quantities be

$$\text{the arc } EM = s, \qquad \text{the angle } OEM = \alpha, \quad \text{and} \quad \text{the angle } OME = \phi,$$

the last being variable with the quantities x, y, and s, while a and α remain constant. Let the infinitely close meridian Om be drawn, to which is drawn from M the perpendicular Mn. One will have

$$mn = dx, \qquad \text{the angle } MOm = dy, \quad \text{and} \quad Mn = dy \sin x,$$

where we take the radius of the sphere to be one. Then we shall have

$$\operatorname{tang} \phi = \frac{Mn}{mn} = \frac{dy \sin x}{dx} \ ,$$

or

$$\sin \phi = \frac{dy \sin x}{ds} \quad \text{and} \quad \cos \phi = \frac{dx}{dy} \ .$$

Now having $ds = \sqrt{dx^2 + dy^2 \sin^2 x}$, it is required that the formula

$$\int \sqrt{dx^2 + dy^2 \sin^2 x}$$

be a *minimum.* To this end, we set $dy = p\,dx$; the formula to be minimized becomes

$$\int dx \sqrt{1 + pp \sin^2 x} = \int Z\,dx,$$

so that

$$Z = \sqrt{1 + pp \sin^2 x}.$$

Now in general, if one has $dZ = M\,dx + N\,dy + P\,dp$, the equation to minimize[3] is $N\,dx - dP = 0$; applied to the present case, we have

$$N = 0 \qquad \text{and} \qquad P = \frac{p \sin^2 x}{\sqrt{1 + pp \sin^2 x}}\ .$$

But since $N = 0$, our equation will be

$$dP = 0 \qquad \text{and thus} \qquad P = \text{Const.},$$

Consequently, we shall have

$$\frac{p \sin^2 x}{\sqrt{1 + pp \sin^2 x}} = C, \qquad \text{or} \qquad \frac{dy \sin^2 x}{\sqrt{dx^2 + dy^2 \sin^2 x}} = C\ ;$$

that is,

$$\frac{dy \sin^2 x}{ds} = \sin x \sin \phi = C\ .$$

To evaluate the constant C, we must consider that, when making the angle $EOM = y$ vanish, x becomes $= a$ and $\phi = 180^\circ - \alpha$, or $\sin \phi = \sin \alpha$, so in this case we have $\sin a \cdot \sin \alpha = C$. Consequently, the nature of the minimum yields the equation

$$\frac{dy \sin^2 x}{\sqrt{dx^2 + dy^2 \sin^2 x}} = \sin a \sin \alpha.$$

But it is still necessary to integrate the differential equation; writing C again in place of $\sin a \sin \alpha$, we have

$$dy = \frac{C\,dx}{\sin x \sqrt{\sin^2 x - CC}},$$

and since $ds = \dfrac{dy \sin^2 x}{C}$, we have

$$ds = \frac{dx \sin x}{\sqrt{\sin^2 x - CC}}.$$

[3] Another reference to Euler's work on the calculation of variations. See the footnote to paragraph 1[*tr*].

Now one finds through the rules of integration that

$$y = -\arcsin\frac{C\cos x}{\sin x\sqrt{1-CC}} + \arcsin\frac{C\cos a}{\sin a\sqrt{1-CC}}$$
$$= -\arccos\frac{\sqrt{\sin^2 x - CC}}{\sin x\sqrt{1-CC}} + \arccos\frac{\sqrt{\sin^2 a - CC}}{\sin a\sqrt{1-CC}},$$
$$s = -\arccos\frac{\sqrt{\sin^2 x - CC}}{\sqrt{1-CC}} + \arccos\frac{\sqrt{\sin^2 a - CC}}{\sqrt{1-CC}}$$
$$= -\arcsin\frac{\cos x}{\sqrt{1-CC}} + \arcsin\frac{\cos a}{\sqrt{1-CC}}\ ,$$

where the added constants are such that, making $y = 0$ and $s = 0$, x becomes $= a$. But the two arcs of circles being reduced to one will give

$$y = \arcsin\frac{C\cos a\sqrt{\sin^2 x - CC} - C\cos x\sqrt{\sin^2 a - CC}}{(1-CC)\sin a\sin x},$$
$$s = \arcsin\frac{\cos a\sqrt{\sin^2 x - CC} - \cos x\sqrt{\sin^2 a - CC}}{1-CC},$$

from which we derive the following two equations:

$$(1-CC)\sin a\sin x\sin y = C\cos a\sqrt{\sin^2 x - CC} - C\cos x\sqrt{\sin^2 a - CC},$$
$$(1-CC)\sin s = \cos a\sqrt{\sin^2 x - CC} - \cos x\sqrt{\sin^2 a - CC}\ .$$

But, taking the cosines of the angles y and s, we shall have:

$$(1-CC)\sin a\sin x\cos y = \sqrt{(\sin^2 a - CC)(\sin^2 x - CC)} + CC\cos a\cos x$$
$$(1-CC)\cos s = \sqrt{(\sin^2 a - CC)(\sin^2 x - CC)} + \cos a\cos x.$$

And replacing C by its value $\sin a\sin\alpha$, since

$$\sqrt{\sin^2 a - CC} = -\sin a\cos\alpha,$$

for we regard here the angle α as obtuse, so that the angle ϕ is acute from the point E; because, setting $y = 0$, the angle ϕ becomes $180° - \alpha$, thus its cosine is $-\cos\alpha$. We shall have:

$$\begin{aligned}
(1-\sin^2 a\sin^2\alpha)\sin x\sin y &= \quad \sin\alpha\cos a\sqrt{\sin^2 x-\sin^2 a\sin^2\alpha}+\sin\alpha\cos\alpha\sin a\cos x,\\
(1-\sin^2 a\sin^2\alpha)\sin x\cos y &= -\cos\alpha\sqrt{\sin^2 x-\sin^2 a\sin^2\alpha}+\sin a\cos a\sin^2\alpha\cos x,\\
(1-\sin^2 a\sin^2\alpha)\sin s &= \quad \cos a\sqrt{\sin^2 x-\sin^2 a\sin^2\alpha}+\sin a\cos\alpha\cos x,\\
(1-\sin^2 a\sin^2\alpha)\cos s &= -\sin a\cos\alpha\sqrt{\sin^2 x-\sin^2 a\sin^2\alpha}+\cos a\cos x,
\end{aligned}$$

to which must be added

$$\sin x\sin\phi=\sin a\sin\alpha\,.$$

Corollary 1

18. Since $\sin a\sin\alpha=\sin x\sin\phi$, one will have:

$$\sqrt{\sin^2 x-\sin^2 a\sin\alpha}=+\sin x\cos\phi.$$

Thus, our four formulae become:

$$\begin{aligned}
&\text{(I)} & (1-CC)\sin y &= \quad \sin\alpha\cos a\cos\phi+\cos\alpha\cos x\sin\phi\\
&\text{(II)} & (1-CC)\cos y &= -\cos\alpha\cos\phi+\sin\alpha\cos a\cos x\sin\phi\\
&\text{(III)} & (1-CC)\sin s &= \quad \cos a\sin x\cos\phi+\sin a\cos\alpha\cos x\\
&\text{(IV)} & (1-CC)\cos s &= -\sin a\cos\alpha\sin x\cos\phi+\cos a\cos x,
\end{aligned}$$

setting, to abbreviate, CC in place of $\sin^2 a\sin^2\alpha$ or $\sin^2 x\sin^2\phi$.

Corollary 2

19. These four formulae can be combined in several different ways, from which simpler formulae can be deduced. First, let us take

$$\text{(I)}\cdot\cos\alpha+\text{(II)}\cdot\sin\alpha\cos a,$$

and we shall have

$$(1-CC)(\cos\alpha\sin y+\sin\alpha\cos a\cos y)=(cos^2\alpha+\sin^2\alpha\cos^2 a)\cos x\sin\phi.$$

Now

$$\cos^2\alpha + \sin^2\alpha\cos^2 a = 1 - \sin^2\alpha\sin^2 a = 1 - CC,$$

from which we derive

$$\cos\alpha\sin y + \sin\alpha\cos a\cos y = \cos x\sin\phi == \frac{\sin a\sin\alpha}{\operatorname{tang} x},$$

or

$$\operatorname{tang} x\sin y + \operatorname{tang}\alpha\operatorname{tang} x\cos a\cos y = \operatorname{tang}\alpha\sin a.$$

Corollary 3

20. Let make this combination

$$(\mathrm{I})\cdot\sin\alpha\cos a - (\mathrm{II})\cdot\cos\alpha,$$

resulting in

$$\begin{aligned}(1-CC)(\sin\alpha\cos a\sin y - \cos\alpha\cos y) &= (\sin^2\alpha\cos^2 a + \cos^2\alpha)\cos\phi \\ &= (1-CC)\cos\phi,\end{aligned}$$

from which, dividing by $1 - CC$, we get

$$\sin\alpha\cos a\sin y - \cos\alpha\cos y = \cos\phi.$$

Corollary 4

21. The combination

$$(\mathrm{I})\cdot\sin x - (\mathrm{III})\cdot\sin\alpha$$

gives

$$(1-CC)(\sin x\sin y - \sin\alpha\sin s) = 0\,,$$

or

$$\sin x\sin y = \sin\alpha\sin s,$$

whence, since $\sin x \sin \phi = \sin a \sin \alpha$,

$$\sin a \sin y = \sin \phi \sin s\,,$$

in other words, the proportion

$$\sin a : \sin \phi = \sin x : \sin \alpha = \sin s : \sin y$$

Corollary 5

22. This combination:

$$(\mathrm{I}) \cdot \sin a \cos \alpha \sin x + (\mathrm{IV}) \cdot \sin \alpha \cos a$$

gives

$$\begin{aligned}(1 - CC)&(\sin a \cos \alpha \sin x \sin y + \sin \alpha \cos a \cos s)\\ &= \cos x(\sin a \cos^2 \alpha \sin x \sin \phi + \sin \alpha \cos^2 a).\end{aligned}$$

Now, since $\sin x \sin \phi = \sin a \sin \alpha$, the value of the formula becomes

$$\sin \alpha \cos x(\sin^2 a \cos^2 \alpha + \cos^2 a) = (1 - \sin^2 a \sin^2 \alpha) \sin \alpha \cos x = (1 - CC) \sin \alpha \cos x,$$

so that, dividing by $1 - CC$, one will obtain

$$\sin a \cos \alpha \sin x \sin y + \sin \alpha \cos a \cos s = \sin \alpha \cos x,$$

which becomes, since $\sin y = \dfrac{\sin \alpha \sin s}{\sin x}$,

$$\sin a \cos \alpha \sin s + \cos a \cos s = \cos x$$

Corollary 6

23. Now this combination

$$(\mathrm{I}) \cdot \cos a - (\mathrm{IV}) \cdot \cos \alpha \sin \phi$$

gives

$$(1-CC)(\cos a\sin y-\cos\alpha\sin\phi\cos s)=\cos\phi(\sin\alpha\cos^2 a+\sin a\cos^2\alpha\sin\alpha\sin x\sin\phi)\,,$$

whose value is, since $\sin x\sin\phi=\sin a\sin\alpha$,

$$\sin\alpha\cos\phi(\cos^2 a+\sin^2 a\cos^2\alpha)=(1-CC)\sin\alpha\cos\phi\,;$$

thus, dividing by $1-CC$, one has

$$\cos a\sin y-\cos\alpha\sin\phi\cos s=\sin\alpha\cos\phi\,,$$

which, since $\sin y=\dfrac{\sin\phi\sin s}{\sin a}$, changes into

$$\cos a\sin\phi\sin s-\sin a\cos\alpha\sin\phi\cos s=\sin a\sin\alpha\cos\phi\ ,$$

or

$$\operatorname{tang}\phi\sin s-\cos\alpha\operatorname{tang}a\operatorname{tang}\phi\cos s=\sin\alpha\operatorname{tang}a\ .$$

Corollary 7

24. We consider the combination

$$\text{(II)}\cdot\cos a\sin x-\text{(III)}\cdot\cos\alpha\,.$$

which gives

$$(1-CC)(\cos a\sin x\cos y+\cos\alpha\sin s)=\cos x(\sin\alpha\cos^2 a\sin x\sin\phi+\sin a\cos^2\alpha)\,,$$

whose value, since $\sin x\sin\phi=\sin a\sin\alpha$, will be

$$\sin a\cos x(\sin^2\alpha\cos^2 a+\cos^2\alpha)=(1-CC)\sin a\cos x\,.$$

Then dividing through by $1-CC$, one will have

$$\cos a\sin x\cos y+\cos\alpha\sin s=\sin a\cos x\,,$$

and since $\sin s = \dfrac{\sin x \sin y}{\sin \alpha}$, one will obtain

$$\sin\alpha\cos a\sin x\cos y + \cos\alpha\sin x\sin y = \sin\alpha\sin a\cos x\ ,$$

or

$$\operatorname{tang}\alpha\cos a\operatorname{tang}x\cos y + \operatorname{tang}x\sin y = \operatorname{tang}\alpha\sin a\ ,$$

just as in paragraph 19.

Corollary 8

25. This combination

$$\text{(II)}\cdot\sin a\cos\alpha - \text{(III)}\cdot\cos a\sin\alpha\sin\phi$$

gives

$$(1-CC)\,(\cos a\sin\alpha\sin s\sin\phi - \sin a\cos\alpha\cos y) = \cos\phi\left(\sin a\cos^2\alpha + \cos^2 a\sin\alpha\sin x\sin\phi\right)\ .$$

whose value, since $\sin x\sin\phi = \sin a\sin\alpha$, is

$$\cos a\cos\phi\left(\cos^2\alpha + \cos^2 a\sin^2\alpha\right) = (1-CC)\sin a\cos\phi\ .$$

Thus, dividing by $1-CC$, one will have

$$\cos a\sin\alpha\sin s\sin\phi - \sin a\cos\alpha\cos y = \sin a\cos\phi.$$

Now, having $\sin s = \dfrac{\sin a\sin y}{\sin\phi}$, one will obtain

$$\cos a\sin\alpha\sin y - \cos\alpha\cos y = \cos\phi,$$

just as in paragraph 20.

Corollary 9

26. Now this combination

$$\text{(II)}\cdot\sin a\sin x - \text{(IV)}\cdot 1$$

gives

$$(1-CC)(\sin a\sin x\cos y - \cos s) = \cos a\cos x(\sin a\sin\alpha\sin x\sin\phi - 1)\ ,$$

which since $\sin x \sin\phi = \sin a \sin\alpha$, has the value

$$\cos a \cos x \left(\sin^2 a \sin^2 \alpha - 1\right) = -(1 - CC) \cos a \cos x\ .$$

Thus, dividing by $-(1 - CC)$, one will have

$$\cos s - \sin a \sin x \cos y = \cos a \cos x\ .$$

Corollary 10

27. This combination:

$$(\mathrm{II}) \cdot 1 - (\mathrm{IV}) \cdot \sin\alpha \sin\phi$$

gives

$$(1 - CC) (\cos y - \sin\alpha \sin\phi \cos s) = \cos\alpha \cos\phi (\sin a \sin\alpha \sin x \sin\phi - 1)\ ,$$

so that

$$\sin\alpha \sin\phi \cos s - \cos y = \cos\alpha \cos\phi\ .$$

Corollary 11

28. The combination:

$$(\mathrm{III}) \cdot \sin a \cos\alpha + (\mathrm{IV}) \cdot \cos a$$

gives

$$(1 - CC) (\sin a \cos\alpha \sin s + \cos a \cos s) = \cos x \left(\sin^2 a \cos^2 \alpha + \cos^2 a\right);$$

thus

$$\sin a \cos\alpha \sin s + \cos a \cos s = \cos x,$$

just as in paragraph 22.

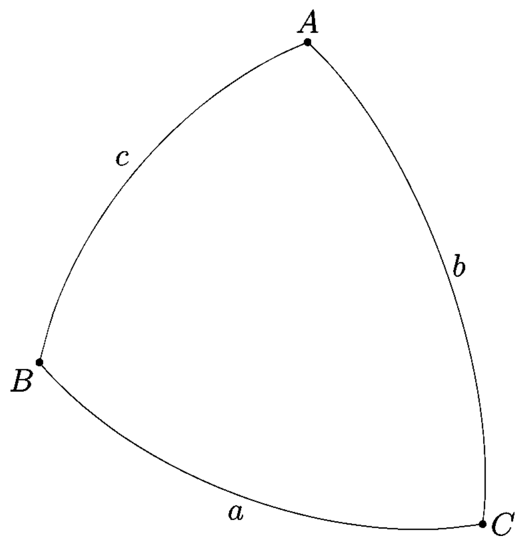

Figure 4

Corollary 12

29. Finally, the combination

$$(\mathrm{III}) \cdot \cos a - (\mathrm{IV}) \cdot \sin a \cos \alpha$$

gives

$$(1 - CC)\,(\cos a \sin s - \sin a \cos \alpha \cos s) = \sin x \cos \phi \left(\cos^2 a + \sin^2 a \cos^2 \alpha\right) \ ,$$

thus

$$\cos a \sin s - \sin a \cos \alpha \cos s = \sin x \cos \phi = \frac{\sin a \sin \alpha \cos \phi}{\sin \phi},$$

or

$$\operatorname{tang} \phi \sin s - \operatorname{tang} a \operatorname{tang} \phi \cos \alpha \cos s = \operatorname{tang} a \sin \alpha \ ,$$

just as in paragraph 23.

Problem V

30. *Find all the properties* (Fig. 4) *between the sides and the angles of an arbitrary spherical triangle.*

Solution

Whatever be the proposed spherical triangle ABC, one of the angles A can be regarded as the pole of the sphere; then the sides AB and AC will each be meridians, and the third side BC the shortest line that can be drawn on the surface of the sphere from point B to point C, so that this triangle ABC can be compared with the figure ECM, which we have just considered in the previous problem. Thus, if we use the letters A, B, C to denote the angles of the same name, and if we set the sides $AB = c$, $AC = b$, and $BC = a$, the previous designations are reduced to the present ones in this fashion:

Former designations.............	a,	x,	s,	y,	α,	ϕ
New designations...............	c,	b,	a,	A,	B,	C

The formulae found in the corollaries of the preceding problem will now furnish us the following properties for the spherical triangle ABC:

I.	$\sin a : \sin A = \sin b : \sin B = \sin c : \sin C$	by paragraph 21
	$\cos C = \cos c \sin A \sin B - \cos A \cos B$	by paragraph 20
II.	$\cos B = \cos b \sin A \sin C - \cos A \cos C$	by analogy
	$\cos A = \cos a \sin B \sin C - \cos B \cos C$	by paragraph 27
	$\cos c = \cos C \sin a \sin b + \cos a \cos b$	by analogy
III.	$\cos b = \cos B \sin a \sin c + \cos a \cos c$	by paragraph 22
	$\cos a = \cos A \sin b \sin c + \cos b \cos c$	by paragraph 26
	$\sin a \operatorname{tang} C - \sin B \operatorname{tang} c = \cos a \cos B \operatorname{tang} C \operatorname{tang} c$	by paragraph 23
IV.	$\sin b \operatorname{tang} A - \sin C \operatorname{tang} a = \cos b \cos C \operatorname{tang} A \operatorname{tang} a$	by analogy
	$\sin c \operatorname{tang} B - \sin A \operatorname{tang} b = \cos c \cos A \operatorname{tang} B \operatorname{tang} b$	by paragraph 19

And it is to these four properties, which all the formulae that we found in the previous problem are reduced.

Corollary 1

31. The first property contains a well-known quality of all spherical triangles, by which we know that the sines of the sides have the same ratio as the sines of their opposite angles.

Corollary 2

32. Thus, in a spherical triangle, if we know one side with its opposite angle, and besides this another angle, or side, we shall immediately find the side, or the angle, opposite.

Corollary 3

33. Each of the formulae which we have just found contains only four quantities belonging to the triangle. Therefore, if three of these are known, the fourth may be determined.

Corollary 4

34. Therefore, from these [formulae] one can derive the rules for the resolution of all spherical triangles. Now as there are six objects in each triangle, namely, the three sides and the three angles, if three of these are known, the other three can be found, as we shall see in the following problems.

Problem 6

35. *Given the three sides of a spherical triangle* (Fig. 4), *find the angles.*

Solution

In the spherical triangle ABC let the three sides $AB = c$, $AC = b$, and $BC = a$ be given. It is required to find the three angles A, B, and C; this is done by means of the third property, which furnishes us:

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c},$$
$$\cos B = \frac{\cos b - \cos a \cos c}{\sin a \sin c},$$
$$\cos C = \frac{\cos c - \cos a \cos b}{\sin a \sin b}.$$

Corollary 1

36. Thus we have

$$1 - \cos A = \frac{\sin b \sin c + \cos b \cos c - \cos a}{\sin b \sin c},$$

or

$$1 - \cos A = \frac{\cos(b-c) - \cos a}{\sin b \sin c},$$

because

$$\cos(b-c) = \cos b \cos c + \sin b \sin c.$$

Corollary 2

37. Now, since in general

$$\cos p - \cos q = 2 \sin \frac{1}{2}(q-p) \sin \frac{1}{2}(p+q)\ ,$$

our formula will change into

$$1 - \cos A = \frac{2 \sin \frac{1}{2}(a-b+c) \sin \frac{1}{2}(a+b-c)}{\sin b \sin c}.$$

Thus, since

$$1 - \cos A = 2(\sin \frac{1}{2} A)^2,$$

we shall have

$$\sin \frac{1}{2} A = \sqrt{\frac{\sin \frac{1}{2}(a-b+c) \sin \frac{1}{2}(a+b-c)}{\sin b \sin c}},$$

and in the same way

$$\sin\frac{1}{2}B = \sqrt{\frac{\sin\frac{1}{2}(b-a+c)\sin\frac{1}{2}(b+a-c)}{\sin a\sin c}},$$
$$\sin\frac{1}{2}C = \sqrt{\frac{\sin\frac{1}{2}(c-a+b)\sin\frac{1}{2}(c+a-b)}{\sin a\sin b}}.$$

Corollary 3

38. In adding unity to the cosines found, one has

$$1+\cos A = \frac{\cos a - \cos b\cos c + \sin b\sin c}{\sin b\sin c} = \frac{\cos a - \cos(b+c)}{\sin b\sin c}.$$

Thus, since $1+\cos A = 2(\cos\frac{1}{2}A)^2$, the same conversion will give

$$\cos\frac{1}{2}A = \sqrt{\frac{\sin\frac{1}{2}(b+c-a)\sin\frac{1}{2}(b+c+a)}{\sin b\sin c}}$$
$$\cos\frac{1}{2}B = \sqrt{\frac{\sin\frac{1}{2}(a+c-b)\sin\frac{1}{2}(a+c+b)}{\sin a\sin c}}$$
$$\cos\frac{1}{2}C = \sqrt{\frac{\sin\frac{1}{2}(a+b-c)\sin\frac{1}{2}(a+b+c)}{\sin a\sin b}}$$

Corollary 4

39. From these are derived the tangents of the half-angles A, B, C:

$$\operatorname{tang}\frac{1}{2}A = \sqrt{\frac{\sin\frac{1}{2}(a-b+c)\sin\frac{1}{2}(a+b-c)}{\sin\frac{1}{2}(b+c-a)\sin\frac{1}{2}(b+c+a)}}$$

$$\operatorname{tang}\frac{1}{2}B = \sqrt{\frac{\sin\frac{1}{2}(b-a+c)\sin\frac{1}{2}(b+a-c)}{\sin\frac{1}{2}(a+c-b)\sin\frac{1}{2}(a+c+b)}}$$

$$\operatorname{tang}\frac{1}{2}C = \sqrt{\frac{\sin\frac{1}{2}(c-a+b)\sin\frac{1}{2}(c+a-b)}{\sin\frac{1}{2}(a+b-c)\sin\frac{1}{2}(a+b+c)}}$$

Corollary 5

40. These formulae are very useful for performing the computation by means of logarithms. Now, having found one of the angles, say A, the other two are found very easily; by the first property one has

$$\sin B = \frac{\sin b \sin A}{\sin a} \qquad \text{and} \qquad \sin C = \frac{\sin c \sin A}{\sin a},$$

provided that it is known whether the angles are larger or smaller than a right angle. But, in using the formulae found, this ambiguity disappears, since one finds the half-angles, which are always smaller than a right angle.

Corollary 6

41. The tangents of the half-angles provide other significant formulae, because, multiplying two together, one will have:

$$\operatorname{tang}\frac{1}{2}A\ \operatorname{tang}\frac{1}{2}B = \frac{\sin\frac{1}{2}(a+b-c)}{\sin\frac{1}{2}(a+b+c)},$$

and since

$$\sin p + \sin q = 2\sin\frac{1}{2}(p+q)\cos\frac{1}{2}(p-q)$$

and

$$\sin p - \sin q = 2 \sin \frac{1}{2}(p-q) \cos \frac{1}{2}(p+q)$$

one will obtain

$$1 + \operatorname{tang} \frac{1}{2}A \ \operatorname{tang} \frac{1}{2}B = \frac{2 \sin \frac{1}{2}(a+b) \cos \frac{1}{2}c}{\sin \frac{1}{2}(a+b+c)}$$

and

$$1 - \operatorname{tang} \frac{1}{2}A \ \operatorname{tang} \frac{1}{2}B = \frac{2 \sin \frac{1}{2}c \ \cos \frac{1}{2}(a+b)}{\sin \frac{1}{2}(a+b+c)}.$$

Corollary 7

42. Likewise, in adding or subtracting two of these tangents, one obtains

$$\operatorname{tang} \frac{1}{2}A \pm \operatorname{tang} \frac{1}{2}B = \frac{\left(\sin \frac{1}{2}(a+c-b) \pm \sin \frac{1}{2}(b+c-a)\right) \sqrt{\sin \frac{1}{2}(a+b-c)}}{\sqrt{\sin \frac{1}{2}(b+c-a)}\sqrt{\sin \frac{1}{2}(a+c-b)}\sqrt{\sin \frac{1}{2}(a+b+c)}}$$

or

$$\operatorname{tang} \frac{1}{2}A \pm \operatorname{tang} \frac{1}{2}B = \frac{\sin \frac{1}{2}(a+c-b) \pm \sin \frac{1}{2}(b+c-a)}{\operatorname{tang} \frac{1}{2}C \sin \frac{1}{2}(a+b+c)},$$

if one introduces the value of the tangent of C. Thus, using the reduction shown above [paragraph 41], we shall have these two equations:

$$\begin{aligned} \operatorname{tang} \frac{1}{2}A + \operatorname{tang} \frac{1}{2}B &= \frac{2 \sin \frac{1}{2}c \cos \frac{1}{2}(a-b)}{\operatorname{tang} \frac{1}{2}C \sin \frac{1}{2}(a+b+c)} \\ \operatorname{tang} \frac{1}{2}A - \operatorname{tang} \frac{1}{2}B &= \frac{2 \sin \frac{1}{2}(a-b) \cos \frac{1}{2}c}{\operatorname{tang} \frac{1}{2}C \sin \frac{1}{2}(a+b+c)}. \end{aligned}$$

Corollary 8

43. Now, since

$$\operatorname{tang}\frac{1}{2}(A+B)=\frac{\operatorname{tang}\frac{1}{2}A+\operatorname{tang}\frac{1}{2}B}{1-\operatorname{tang}\frac{1}{2}A\operatorname{tang}\frac{1}{2}B},$$

we shall find by the formulae of the two previous corollaries:

$$\operatorname{tang}\frac{1}{2}(A+B)=\frac{\cos\frac{1}{2}(a-b)}{\operatorname{tang}\frac{1}{2}C\cos\frac{1}{2}(a+b)}$$

and similarly

$$\operatorname{tang}\frac{1}{2}(A+C)=\frac{\cos\frac{1}{2}(a-c)}{\operatorname{tang}\frac{1}{2}B\cos\frac{1}{2}(a+c)}$$
$$\operatorname{tang}\frac{1}{2}(B+C)=\frac{\cos\frac{1}{2}(b-c)}{\operatorname{tang}\frac{1}{2}A\cos\frac{1}{2}(b+c)}.$$

Corollary 9

44. In the same way, since

$$\operatorname{tang}\frac{1}{2}(A-B)=\frac{\operatorname{tang}\frac{1}{2}A-\operatorname{tang}\frac{1}{2}B}{1+\operatorname{tang}\frac{1}{2}A\operatorname{tang}\frac{1}{2}B},$$

we shall have:

$$\operatorname{tang}\frac{1}{2}(A-B)=\frac{\sin\frac{1}{2}(a-b)}{\operatorname{tang}\frac{1}{2}C\sin\frac{1}{2}(a+b)}$$

and similarly

$$\operatorname{tang}\frac{1}{2}(A-C)=\frac{\sin\frac{1}{2}(a-c)}{\operatorname{tang}\frac{1}{2}B\sin\frac{1}{2}(a+c)}$$
$$\operatorname{tang}\frac{1}{2}(B-C)=\frac{\sin\frac{1}{2}(b-c)}{\operatorname{tang}\frac{1}{2}A\sin\frac{1}{2}(b+c)}.$$

Problem 7

45. *In a spherical triangle, being given the three angles, find the three sides.*

Solution

Let ABC be a spherical triangle, whose angles A, B, and C are given. It is required to find the three sides

$$AB = c, \qquad AC = b, \qquad \text{and} \qquad BC = a.$$

Now property (II) of paragraph 30 gives us the cosines of these sides expressed as follows:

$$\cos a = \frac{\cos A + \cos B \cos C}{\sin B \sin C}$$
$$\cos b = \frac{\cos B + \cos A \cos C}{\sin A \sin C}$$
$$\cos c = \frac{\cos C + \cos A \cos B}{\sin A \sin B}.$$

Corollary 1

46. From this we derive from this the following formulae:

$$1 - \cos a = -\frac{\cos A + \cos(B + C)}{\sin B \sin C},$$
$$1 + \cos a = \frac{\cos A + \cos(B - C)}{\sin B \sin C}.$$

Now, since in general $\cos p + \cos q = 2\cos\frac{1}{2}(p+q)\cos\frac{1}{2}(p-q)$, we shall have

$$1 - \cos a = -\frac{2\cos\frac{1}{2}(A + B + C)\cos\frac{1}{2}(B + C - A)}{\sin B \sin C}$$
$$1 + \cos a = \frac{2\cos\frac{1}{2}(A + B - C)\cos\frac{1}{2}(A - B + C)}{\sin B \sin C}\ .$$

Corollary 2

47. Thus, since $1-\cos a = 2(\sin\frac{1}{2}a)^2$ and $1+\cos a = 2(\cos\frac{1}{2}a)^2$, we obtain the following formulae:

$$\sin\frac{1}{2}a = \sqrt{-\frac{\cos\frac{1}{2}(A+B+C)\cos\frac{1}{2}(B+C-A)}{\sin B\sin C}}$$

$$\sin\frac{1}{2}b = \sqrt{-\frac{\cos\frac{1}{2}(A+B+C)\cos\frac{1}{2}(A+C-B)}{\sin A\sin C}}$$

$$\sin\frac{1}{2}c = \sqrt{-\frac{\cos\frac{1}{2}(A+B+C)\cos\frac{1}{2}(A+B-C)}{\sin A\sin B}},$$

where it must be noted, that since the sum of the angles $A+B+C$ is always greater that two right angles, the half-sum is greater than a right angle, and therefore its cosine is negative.

Corollary 3

48. For the cosines of the half-sides one will have:

$$\cos\frac{1}{2}a = \sqrt{\frac{\cos\frac{1}{2}(A+B-C)\cos\frac{1}{2}(A-B+C)}{\sin B\sin C}}$$

$$\cos\frac{1}{2}b = \sqrt{\frac{\cos\frac{1}{2}(B+A-C)\cos\frac{1}{2}(B-A+C)}{\sin A\sin C}}$$

$$\cos\frac{1}{2}c = \sqrt{\frac{\cos\frac{1}{2}(C+A-B)\cos\frac{1}{2}(C-A+B)}{\sin A\sin B}},$$

and these formulae facilitate the use of logarithms.

Corollary 4

49. From the sine and cosine of the half-sides their tangents are easily de-

rived; these will be

$$\operatorname{tang}\frac{1}{2}a = \sqrt{-\frac{\cos\frac{1}{2}(A+B+C)\cos\frac{1}{2}(B+C-A)}{\cos\frac{1}{2}(A+B-C)\cos\frac{1}{2}(A-B+C)}},$$

$$\operatorname{tang}\frac{1}{2}b = \sqrt{-\frac{\cos\frac{1}{2}(A+B+C)\cos\frac{1}{2}(A+C-B)}{\cos\frac{1}{2}(B+A-C)\cos\frac{1}{2}(B-A+C)}},$$

$$\operatorname{tang}\frac{1}{2}c = \sqrt{-\frac{\cos\frac{1}{2}(A+B+C)\cos\frac{1}{2}(A+B-C)}{\cos\frac{1}{2}(C+A-B)\cos\frac{1}{2}(C-A+B)}},$$

where again, one can easily use the calculus of logarithms.

Corollary 5

50. Multiplying two of these tangents together, one will obtain

$$\operatorname{tang}\frac{1}{2}a\operatorname{tang}\frac{1}{2}b = -\frac{\cos\frac{1}{2}(A+B+C)}{\cos\frac{1}{2}(A+B-C)}.$$

Now from this the two following formulae are derived:

$$1-\operatorname{tang}\frac{1}{2}a\operatorname{tang}\frac{1}{2}b = \frac{2\cos\frac{1}{2}(A+B)\cos\frac{1}{2}C}{\cos\frac{1}{2}(A+B-C)},$$

$$1+\operatorname{tang}\frac{1}{2}a\operatorname{tang}\frac{1}{2}b = \frac{2\sin\frac{1}{2}C\sin\frac{1}{2}(A+B)}{\cos\frac{1}{2}(A+B-C)}.$$

Corollary 6

51. And if we add or subtract together two of these formulae, we shall obtain:

$$\operatorname{tang}\frac{1}{2}a \pm \operatorname{tang}\frac{1}{2}b = \frac{\left(\cos\frac{1}{2}(B+C-A)\pm\cos\frac{1}{2}(A+C-B)\right)\sqrt{-\cos\frac{1}{2}(A+B+C)}}{\sqrt{\cos\frac{1}{2}(A+B-C)}\sqrt{\cos\frac{1}{2}(A+C-B)}\sqrt{\cos\frac{1}{2}(B+C-A)}},$$

Now having

$$\operatorname{tang}\frac{1}{2}c = \sqrt{-\frac{\cos\frac{1}{2}(A+B+C)\cos\frac{1}{2}(A+B-C)}{\cos\frac{1}{2}(C+A-B)\cos\frac{1}{2}(C-A+B)}}\,,$$

one will have

$$\operatorname{tang}\frac{1}{2}a \pm \operatorname{tang}\frac{1}{2}b = \frac{\left(\cos\frac{1}{2}(B+C-A) \pm \cos\frac{1}{2}(A+C-B)\right)\operatorname{tang}\frac{1}{2}c}{\cos\frac{1}{2}(A+B-C)}\,.$$

Corollary 7

52. From this is obtained, by the simplifications shown:

$$\operatorname{tang}\frac{1}{2}a + \operatorname{tang}\frac{1}{2}b = \frac{2\cos\frac{1}{2}C\cos\frac{1}{2}(A-B)\operatorname{tang}\frac{1}{2}c}{\cos\frac{1}{2}(A+B-C)}$$

and

$$\operatorname{tang}\frac{1}{2}a - \operatorname{tang}\frac{1}{2}b = \frac{2\sin\frac{1}{2}(A-B)\sin\frac{1}{2}C\operatorname{tang}\frac{1}{2}c}{\cos\frac{1}{2}(A+B-C)}\,.$$

Corollary 8

53. Thus we find as above:

$$\begin{aligned}
\operatorname{tang}\frac{1}{2}(a+b) &= \frac{\cos\frac{1}{2}(A-B)}{\cos\frac{1}{2}(A+B)}\operatorname{tang}\frac{1}{2}c\\
\operatorname{tang}\frac{1}{2}(a+c) &= \frac{\cos\frac{1}{2}(A-C)}{\cos\frac{1}{2}(A+C)}\operatorname{tang}\frac{1}{2}b\\
\operatorname{tang}\frac{1}{2}(b+c) &= \frac{\cos\frac{1}{2}(B-C)}{\cos\frac{1}{2}(B+C)}\operatorname{tang}\frac{1}{2}a\,.
\end{aligned}$$

Corollary 9

54. In the same way, the tangents of the half-difference of the sides will be:

$$\operatorname{tang}\frac{1}{2}(a-b) = \frac{\sin\frac{1}{2}(A-B)}{\sin\frac{1}{2}(A+B)}\operatorname{tang}\frac{1}{2}c$$
$$\operatorname{tang}\frac{1}{2}(a-c) = \frac{\sin\frac{1}{2}(A-C)}{\sin\frac{1}{2}(A+C)}\operatorname{tang}\frac{1}{2}b$$
$$\operatorname{tang}\frac{1}{2}(b-c) = \frac{\sin\frac{1}{2}(B-C)}{\cos\frac{1}{2}(B+C)}\operatorname{tang}\frac{1}{2}a\ .$$

The use of these formulas will be of great importance in the problems which follow.

Problem 8

55. *In a spherical triangle* (Fig. 4)*, given two sides with the angle between them, find the third side and the other two angles*

Solution

Let ABC be a triangle, for which are given the two sides $AB = c$ and $AC = b$, together with the angle A between them, and it is required to find the side $BC = a$ and the angles B and C.

To begin, the third formula of the third property [paragraph 30] yields

$$\cos a = \cos A \sin b \sin c + \cos b \cos c,$$

and the third formula of the fourth property gives the angle B,

$$\operatorname{tang} B = \frac{\sin A \operatorname{tang} b}{\sin c - \operatorname{tang} b \cos c \cos A}$$

from which one takes by analogy:

$$\operatorname{tang} C = \frac{\sin A \operatorname{tang} c}{\sin b - \operatorname{tang} c \cos b \cos A}\ .$$

Now the expressions for the cotangents will be more convenient, so that one has the following formulae for the solution:

$$\begin{aligned}
\cos a &= \cos A \sin b \sin c + \cos b \cos c \\
\cot B &= \frac{\sin c \cot b - \cos c \cos A}{\sin A} \\
\cot C &= \frac{\sin b \cot c - \cos b \cos A}{\sin A} \ .
\end{aligned}$$

Corollary 1

56. Since

$$\cos b \cos c = \frac{1}{2}\cos(b-c) + \frac{1}{2}\cos(b+c)$$

and

$$\sin b \sin c = \frac{1}{2}\cos(b-c) - \frac{1}{2}\cos(b+c),$$

the cosine of side a can be expressed by the addition and subtraction of simple cosines in this manner:

$$\begin{aligned}
\cos a = \frac{1}{4}\cos(A-b+c) + \frac{1}{4}\cos(A+b-c) - \frac{1}{4}\cos(A-b-c) \\
- \frac{1}{4}\cos(A+b+c) + \frac{1}{2}\cos(b-c) + \frac{1}{2}\cos(b+c) \ .
\end{aligned}$$

Corollary 2

57. But if one wishes to use logarithms, this formula is less convenient. However, logarithms could be applied here by introducing a new angle u, setting

$$\operatorname{tang} u = \frac{\cos A \sin b}{\cos b} \ ,$$

or $\operatorname{tang} u = \cos A \operatorname{tang} b$, and having found this angle u, one will have:

$$\cos a = \operatorname{tang} u \cos b \sin c + \cos b \cos c = \frac{\cos b \cos(c-u)}{\cos u} \ ,$$

from which the side a will easily be found by means of logarithms.

Corollary 3

58. The same introduction of the angle u, so that

$$\operatorname{tang} u = \cos A \operatorname{tang} b,$$

also makes the other formulae appropriate for the application of logarithms, for one will have:

$$\operatorname{tang} B = \frac{\sin A \operatorname{tang} b}{\sin c - \operatorname{tang} u \cos c} = \frac{\sin A \operatorname{tang} b \cos u}{\sin(c-u)} = \frac{\operatorname{tang} A \sin u}{\sin(c-u)} \ .$$

As for the other angle C, it will be found by the rule

$$\sin C = \frac{\cos A \sin c}{\sin a} \ .$$

Corollary 4

59. But the most convenient search for the angles B and C will be drawn from the formulae given in paragraphs 43 and 44, whence one will have

$$\operatorname{tang} \frac{1}{2}(B+C) = \frac{\cos \frac{1}{2}(b-c)}{\cos \frac{1}{2}(b+c)} \cot \frac{1}{2} A \ ,$$
$$\operatorname{tang} \frac{1}{2}(B-C) = \frac{\sin \frac{1}{2}(b-c)}{\sin \frac{1}{2}(b+c)} \cot \frac{1}{2} A \ .$$

For having half of the sum along with half of the difference, one will have each separately, and from this the side a can then be obtained by the rule

$$\sin a = \frac{\sin b}{\sin B} \sin A = \frac{\sin c}{\sin C} \sin A \ .$$

Problem 9

60. *In a spherical triangle* (Fig. 4), *being given two angles with the side between them, find the third angle with the two sides.*

Solution

Let ABC be a triangle, in which are given the two angles A and B, with the side between them $AB = c$; it is required to find the third angle C together with the two other sides $AC = b$ and $BC = a$.

To begin, the first formula of the second property(paragraph 30) gives:

$$\cos C = \cos c \sin A \sin B - \cos A \cos B$$

and the third formula of the fourth property gives

$$\operatorname{tang} b = \frac{\sin c \operatorname{tang} B}{\sin A + \cos c \cos A \operatorname{tang} B},$$

and by analogy

$$\operatorname{tang} a = \frac{\sin c \operatorname{tang} A}{\sin B + \cos c \cos B \operatorname{tang} A} .$$

From which, taking the cotangents, one will have the following solution:

$$\begin{aligned} \cos C &= \cos c \sin A \sin B - \cos A \cos B \\ \cot a &= \frac{\cot A \sin B + \cos c \cos B}{\sin c} \\ \cot b &= \frac{\cot B \sin A + \cos c \cos A}{\sin c} . \end{aligned}$$

Corollary 1

61. The two sides will be found more easily from the formulae of paragraphs

52 and 53, from which one takes:

$$\begin{aligned}
\operatorname{tang}\frac{1}{2}(a+b) &= \frac{\cos\frac{1}{2}(A-B)}{\cos\frac{1}{2}(A+B)}\operatorname{tang}\frac{1}{2}c \\
\operatorname{tang}\frac{1}{2}(a-b) &= \frac{\sin\frac{1}{2}(A-B)}{\sin\frac{1}{2}(A+B)}\operatorname{tang}\frac{1}{2}c\,,
\end{aligned}$$

where it is easy to use logarithms.

Corollary 2

62. After having found the sides a and b, one will easily find the angle C, since it is

$$\sin C = \frac{\sin A}{\sin a}\sin c = \frac{\sin B}{\sin b}\sin c\,,$$

or one could also, if desired, express $\cos C$ by simple cosines in this fashion:

$$\begin{aligned}
\cos C = \frac{1}{4}\cos(c+A-B) + \frac{1}{4}\cos(c-A+B) - \frac{1}{4}\cos(c-A-B) \\
- \frac{1}{4}\cos(c+A+B) - \frac{1}{2}\cos(A-B) + \frac{1}{2}\cos(A+B)\,.
\end{aligned}$$

Problem X

63. *In a spherical triangle, given two sides with an angle not between them* (Fig. 4) *or equivalently, given two angles with a side not between them, find the other quantities belonging to the triangle.*

Solution

Let ABC be the triangle in which is given, for the first case, the two sides $BC = a$ and $AC = b$ together with the angle A; one immediately knows the angle B because

$$\sin B = \frac{\sin A}{\sin a}\sin b\,.$$

For the other case, let A and B be the given angles, together with the side $BC = a$, one will immediately have the side b because

$$\sin b = \frac{\sin a}{\sin A} \sin B \ .$$

Consequently, in the one and the other case, one will be able to regard as given the two sides $BC = a$ and $AC = b$, as well as the two opposite angles A and B. Thus, it is a matter of finding the side $AB = c$ and the angle C. Now the first formula of the fourth property gives:

$$\sin a \operatorname{tang} C - \sin B \operatorname{tang} c = \cos a \cos B \operatorname{tang} C \operatorname{tang} c \ ,$$

from which, transposing the sides a and b with the angles A and B, we shall have

$$\sin b \operatorname{tang} C - \sin A \operatorname{tang} c = \cos b \cos A \operatorname{tang} C \operatorname{tang} c \ .$$

From these two equations, eliminating either $\operatorname{tang} C$ or $\operatorname{tang} c$, we have either

$$\operatorname{tang} c = \frac{\sin A \sin a - \sin B \sin b}{\sin A \cos B \cos a - \cos A \sin B \cos b}$$

or

$$\operatorname{tang} C = \frac{\sin A \sin a - \sin B \sin b}{\cos B \cos a \sin b - \cos A \sin a \cos b} \ ,$$

to which must be added this equation

$$\sin A \sin b = \sin B \sin a \ .$$

Corollary 1

64. Since $\sin A \colon \sin B = \sin a \colon \sin b$, we shall have also

$$\operatorname{tang} c = \frac{\sin^2 a - \sin^2 b}{\cos B \sin a \cos a - \cos A \sin b \cos b}$$

and

$$\text{tang}\, C = \frac{\sin^2 A - \sin^2 B}{\sin B \cos B \sin a - \sin A \cos A \cos b}\,.$$

Corollary 2

65. But paragraphs 43, 44, 53, and 54 again furnish us more convenient solutions, which are these:

$$\text{tang}\,\frac{1}{2}c = \frac{\cos\frac{1}{2}(A+B)}{\cos\frac{1}{2}(A-B)}\,\text{tang}\,\frac{1}{2}(a+b) = \frac{\sin\frac{1}{2}(A+B)}{\sin\frac{1}{2}(A-B)}\,\text{tang}\,\frac{1}{2}(a-b)$$

and

$$\text{tang}\,\frac{1}{2}C = \frac{\cos\frac{1}{2}(a-b)}{\cos\frac{1}{2}(a+b)}\cot\frac{1}{2}(A+B) = \frac{\sin\frac{1}{2}(a-b)}{\sin\frac{1}{2}(a+b)}\cot\frac{1}{2}(A-B)\,,$$

to which the use of logarithms can easily be applied.

Problem 9[4]

66. *Find* (Fig. 3) *the area of an arbitrary spherical triangle.*

Solution

Let EOM be the spherical triangle under consideration; let us name as above (paragraph 17) the side $OE = a$, the angle $OEM = \alpha$, the angle $EOM = y$, side $OM = x$ and the angle $OME = \phi$. This being set, the trilinear figure MOm will represent the differential of the area which we seek, and since $mn = dx$ and $Mn = dy \sin x$, the product $dy\, dx \sin x$ expresses the differential of MOm, whence

$$MOm = dy \int dx \sin x = dy(1 - \cos x)$$

[4]In both the original publication and the *Opera Omnia*, a second "Problem 9" occurs at this point in the text[*tr*]

and hence the area

$$EOM = y - \int dy \cos x\ .$$

Now we have found

$$dy = \frac{C\, dx}{\sin x \sqrt{x^2 - CC}}\ ,$$

so that

$$\text{the area } EOM = y - \int \frac{C\, dx \cos x}{\sin x \sqrt{\sin^2 x - CC}}\ .$$

Then having found $\sin\phi = \dfrac{C}{\sin x}$, because $C = \sin a\ \sin\alpha$ and since

$$\cos\phi = \frac{\sqrt{\sin^2 x - CC}}{\sin x}\ ,$$

we shall have

$$d\phi \cos\phi = -C\, dx \frac{\cos x}{\sin^2 x}\ ,$$

thus

$$d\phi = -\frac{C\, dx \cos x}{\sin x \sqrt{\sin^2 x - CC}} \quad \text{and} \quad -\int \frac{C\, dx \cos x}{\sin x \sqrt{\sin^2 x - CC}} = \phi + \text{Const.}$$

Consequently, the area of the triangle EOM will be

$$= y + \phi + \text{Const.} = \alpha + y + \phi - \text{Const.}$$

To know this constant, suppose $y = 0$ and since ϕ then becomes $= 180° - \alpha$, the area of this vanishing triangle will be $= 180° - \text{Const.}$ and hence $\text{Const.} = 180°$. So we have

$$\text{the area of the triangle } EOM = \alpha + y + \phi - 180°\ .$$

Corollary 1

67. Therefore, to find the area of any spherical triangle, one only needs to take the excess of the sum of its three angles over two right angles, when the radius of the sphere is expressed by 1. Now in an arbitrary sphere one will take a great circle arc which is the measure of the said excess, and the product of this arc by the radius of the sphere will give the desired area of the spherical triangle.

Corollary 2

68. Thus, the larger a spherical triangle, the more will the sum of its angles exceed two right angles, and when the area of the triangle occupies one-eighth of the surface of the sphere, this excess will equal one right angle. For an arc of a great circle of 90° multiplied by the radius gives half the area of the great circle, hence the eighth part of the surface of the sphere. From this one will obtain this rule to find the area of the entire spherical triangle. One will say, as 8 right angles, or 720°, is to the excess of the sum of the three angles over two right angles, so is the whole surface of the sphere to the area of the triangle under consideration.