On the Summation of Progressions by means of infinite Series *

Leonhard Euler

§140 The general expression, we found in the preceding chapter for the summatory term of a certain series, whose general term or the term corresponding to the index x is $= z$,

$$Sz = \int zdx + \frac{1}{2}z + \frac{\mathfrak{A}dz}{1 \cdot 2dx} - \frac{\mathfrak{B}d^3z}{1 \cdot 2 \cdot 3 \cdot 4dx^3} + \frac{\mathfrak{C}d^5z}{1 \cdot 2 \cdots 6dx^5} - \text{etc.}$$

especially serves for the summation of series whose general terms are any polynomial functions of the index x, since in these cases one finally gets to vanishing differentials. But if z was not a function of x of such a kind, then its differentials proceed to infinity and so an infinite series expressing the sum of the propounded series up to the given term, whose index is $= x$, results. Therefore, the sum of the propounded series continued to infinity will arise, if is put $x = \infty$; and this way another infinite series equal to the first arises.

§141 But if one puts $x = 0$, then the expression exhibiting the sum has to vanish, as we already mentioned; if this does not happen, a constant quantity has either to be added or subtracted that this condition is satisfied. If, having done this, one puts $x = 1$, the found sum will yield the first term of the series; but if $x = 2$, the aggregate of the first and the second, if $x = 3$, the aggregate

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§142 Since, if a constant of such a kind was added to the sum that it vanishes for $x = 0$, then the sum satisfies all remaining cases, whatever numbers are substituted for x, it is manifest, as long as to the found sums a constant quantity of such a kind is added that in one special case the true sum is indicated, that in all remaining cases the true sum has to arise. Hence, if by putting $x = 0$ it is not clear, a value of which kind the expression of the sum receives, and hence to constant to be added cannot be found, then any other number can be set for x and by adding a constant it can be caused that the correct sum is indicated; how this has to happen, will become more perspicuous from the following.

§142[a] At first, let us consider this harmonic progression

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdot + \frac{1}{x} = s;$$

because the general term of it is $= \frac{1}{x}$, it will be $z = \frac{1}{x}$ and the summatory term s will be found this way. First, it will be $\int z dx = \int \frac{dx}{x} = \ln x$; further, the differentials will behave as this

$$\frac{dz}{dx} = -\frac{1}{x^2}, \quad \frac{ddz}{2dx^2} = \frac{1}{x^3}, \quad \frac{d^3z}{6dx^3} = -\frac{1}{x^4}, \quad \frac{d^4z}{24dx^4} = \frac{1}{x^5}, \quad \frac{d^5z}{120dx^5} = -\frac{1}{x^6} \quad \text{etc.}$$

Hence, it will therefore be

$$s = \ln x + \frac{1}{2x} - \frac{\mathfrak{A}}{2x^2} + \frac{\mathfrak{B}}{4x^4} - \frac{\mathfrak{C}}{6x^6} + \frac{\mathfrak{D}}{8x^8} - \text{etc.} + \text{Constant.}$$

Therefore, the constant to be added here cannot be defined from the case $x = 0$. Therefore, put $x = 1$, since then $s = 1$; it will be

$$1 = \frac{1}{2} - \frac{\mathfrak{A}}{2} + \frac{\mathfrak{B}}{4} - \frac{\mathfrak{C}}{6} + \frac{\mathfrak{D}}{8} - \text{etc.} + \text{Const.},$$

whence this constant becomes

$$= \frac{1}{2} + \frac{\mathfrak{A}}{2} - \frac{\mathfrak{B}}{4} + \frac{\mathfrak{C}}{6} - \frac{\mathfrak{D}}{8} + \text{etc.},$$

and hence the summatory term sought after

$$\begin{aligned} s = \ln x &+ \frac{1}{2x} - \frac{\mathfrak{A}}{2x^2} + \frac{\mathfrak{B}}{4x^4} - \frac{\mathfrak{C}}{6x^6} + \frac{\mathfrak{D}}{8x^8} - \text{etc.} \\ &+ \frac{1}{2} + \frac{\mathfrak{A}}{2} - \frac{\mathfrak{B}}{4} + \frac{\mathfrak{C}}{6} - \frac{\mathfrak{C}}{8} + \text{etc.} \end{aligned}$$

§143 Since the Bernoulli numbers $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ etc. constitute a divergent series, this value of the constant cannot be found. But if instead of x a greater number is substituted and the sum of any many terms is actually sought after, the value of the constant will be conveniently investigated. For this purpose, set $x = 10$ and by collecting the first ten terms, one will find their sum

$$= 2.928968253968253968,$$

to which the expression of the sum has to be equal, if in it one puts $x = 10$, which is

$$\ln 10 + \frac{1}{20} - \frac{\mathfrak{A}}{200} + \frac{\mathfrak{B}}{40000} - \frac{\mathfrak{C}}{6000000} + \frac{\mathfrak{D}}{800000000} - \text{etc.} + C.$$

Therefore, having taken for $\ln 10$ the hyperbolic logarithm of ten and having substituted the values found above [§ 122] instead of $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ etc. one will find that constant

$$C = 0.5772156649015325,$$

which sum therefore expresses the sum of the series

$$\frac{1}{2} + \frac{\mathfrak{A}}{2} - \frac{\mathfrak{B}}{4} + \frac{\mathfrak{C}}{6} - \frac{\mathfrak{D}}{8} + \frac{\mathfrak{E}}{10} - \text{etc.}$$

§144 If for x sufficiently small numbers are substituted, since the sum of the series is actually easily found, one will obtain the sum of this series

$$\frac{1}{2x} - \frac{\mathfrak{A}}{2x^2} + \frac{\mathfrak{B}}{4x^4} - \frac{\mathfrak{C}}{6x^6} + \frac{\mathfrak{D}}{8x^8} - \text{etc.} = s - \ln x - C.$$

But if x denotes a very large number, since then the value of this expression running to infinity is easily assigned in decimal numbers, vice versa the sum of the series will be defined. And first it is certainly clear, if the series is continued to infinity, that its sum will be infinitely large; for, having put $x = \infty$ also $\ln x$ becomes infinite, even though $\ln x$ has an infinitely small ratio to x. But that the sum of an arbitrary number of terms of the series is conveniently assigned, let us express the values of the letters $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ etc. in decimal fractions.

$$\begin{aligned}
\mathfrak{A} &= 0.1666666666666 \\
\mathfrak{B} &= 0.0333333333333 \\
\mathfrak{C} &= 0.0238095238095 \\
\mathfrak{D} &= 0.0333333333333 \\
\mathfrak{E} &= 0.0757575757575 \\
\mathfrak{F} &= 0.2531135531135 \\
\mathfrak{G} &= 1.1666666666666 \\
\mathfrak{H} &= 7.0921568627451 \quad \text{etc.}
\end{aligned}$$

whence it will therefore be

$$\begin{aligned}
\frac{\mathfrak{A}}{2} &= 0.0833333333333 \\
\frac{\mathfrak{B}}{4} &= 0.0083333333333 \\
\frac{\mathfrak{C}}{6} &= 0.0039682539682 \\
\frac{\mathfrak{D}}{8} &= 0.0041666666666 \\
\frac{\mathfrak{E}}{10} &= 0.0075757575757 \\
\frac{\mathfrak{F}}{12} &= 0.0210927960928 \\
\frac{\mathfrak{G}}{14} &= 0.0833333333333 \\
\frac{\mathfrak{H}}{16} &= 0.4432598039316
\end{aligned}$$

EXAMPLE 1

To find the sum of thousand terms of the series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} +$ *etc.*

Therefore, put $x = 1000$, and because it is

$$\ln 10 \qquad = 2.302585092994045684 0,$$

it will be

$$\begin{aligned} \ln x &= 6.9077553789821 \\ \text{Const.} &= 0.5772156649015 \\ \frac{1}{2x} &= 0.0005000000000 \end{aligned}$$

and in total

$$\begin{aligned} &= 7.4854709438836 \\ \text{subtr. } \frac{\mathfrak{A}}{2xx} &= 0.0000000833333 \end{aligned}$$

which yields

$$\begin{aligned} &= 7.4854708605503 \\ \text{add } \frac{\mathfrak{B}}{4x^4} &= 0.0000000000000 \end{aligned}$$

Therefore

$$= 7.4854708605503$$

is the sought after sum of thousand terms, which is still smaller than seven and a half units.

EXAMPLE 2

To find the sum of a million terms of the series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} +$ *etc.*

Since it is $x = 10000$, it will be $\ln x = 6 \cdot \ln 10$, therefore

$$\begin{aligned}\ln x &= 12.8155105579642\\ \text{Const.} &= 0.05772156649015\\ \frac{1}{2x} &= 0.0000005000000\end{aligned}$$

in total

$$= 14.3927267228657 = \text{Sum sought after}$$

§145 If therefore for x one sets a very large number, the sum is found sufficiently exact from the first term $\ln x$ alone augmented by the constant C; hence, extraordinary corollaries can be deduced. So, if x was a very large number and one puts

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots + \frac{1}{x} = s$$

and

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{x} + \cdots + \frac{1}{x+y} = t,$$

since it is approximately $s = \ln x + C$ and $t = \ln(x+y) + C$, it will be

$$t - s = \ln(x+y) - \ln x = \frac{x+y}{x}$$

and hence this logarithm is approximately expressed by means a harmonic series consisting of a finite numbers of terms in this way

$$\ln\frac{x+y}{y} = \frac{1}{x+1} + \frac{1}{x+2} + \frac{1}{x+3} + \cdots + \frac{1}{x+y}.$$

But this logarithm is exhibited more accurately, if the superior sums s and t are taken more exact. So, because it is

$$s = \ln x + C + \frac{1}{2x} - \frac{1}{12xx} \quad \text{and} \quad t = \ln(x+y) + C + \frac{1}{2(x+y)} - \frac{1}{12(x+y)^2}$$

it will be

$$t - s = \ln\frac{x+y}{x} - \frac{1}{2x} + \frac{1}{2(x+y)} + \frac{1}{12xx} - \frac{1}{12(x+y)^2},$$

and hence

$$\ln\frac{x+y}{x} = \frac{1}{x+1} + \frac{1}{x+2} + \frac{1}{x+3} + \cdots + \frac{1}{x+y} + \frac{1}{2x} - \frac{1}{2(x+y)} - \frac{1}{12xx} + \frac{1}{12(x+y)^2}.$$

But if x is such a large number that the two last terms can be rejected, it will approximately be

$$\ln\frac{x+y}{x} = \frac{1}{x+1} + \frac{1}{x+2} + \frac{1}{x+3} + \cdots + \frac{1}{x+y} + \frac{1}{2}\left(\frac{1}{x} - \frac{1}{x+y}\right).$$

§ 145a From this harmonic series we will also be able to define the sum of this series, in which only the odd numbers occur,

$$\frac{1}{1} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \cdots + \frac{1}{2x+1}.$$

For, because taking all terms it is

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{2x} + \frac{1}{2x+1}$$
$$= \ln(2x+1) + C + \frac{1}{2(2x+1)} - \frac{\mathfrak{A}}{2(2x+1)^2} + \frac{\mathfrak{B}}{4(2x+1)^4} - \frac{\mathfrak{C}}{6(2x+1)^6} + \text{etc.},$$

the sum of the even terms

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \cdots + \frac{1}{2x}$$

is the half of the superior one, namely

$$\frac{1}{2}C + \frac{1}{2}\ln x + \frac{1}{4x} - \frac{\mathfrak{A}}{4x^2} + \frac{\mathfrak{B}}{8x^4} - \frac{\mathfrak{C}}{12x^6} + \frac{\mathfrak{D}}{16x^8} - \text{etc.},$$

having subtracted this series from the latter

$$1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots + \frac{1}{2x+1}$$
$$= \frac{1}{2}C + \ln\frac{2x+1}{\sqrt{x}} + \frac{1}{2(2x+1)} - \frac{\mathfrak{A}}{2(2x+1)^2} + \frac{\mathfrak{B}}{4(2x+1)^4} - \text{etc.}$$
$$- \frac{1}{4x} + \frac{\mathfrak{A}}{4x^2} - \frac{\mathfrak{B}}{8x^4} + \text{etc.}$$

§146 One can indeed even find the sum of any harmonic series by means of the same general expression; for, let be

$$\frac{1}{m+n}+\frac{1}{2m+n}+\frac{1}{3m+n}+\frac{1}{4m+n}+\cdots+\frac{1}{mx+n}=s;$$

since the general term is $z=\frac{1}{mx+n}$, it will be

$$\int zdx=\frac{1}{m}\ln(mx+n),\quad \frac{dz}{dx}=-\frac{m}{(mx+n)^2},\quad \frac{ddz}{2dx^2}=\frac{mm}{(mx+n)^3},$$

$$\frac{d^3z}{6dx^3}=-\frac{m^3}{(mx+n)^4},\quad \frac{d^4z}{24dx^4}=\frac{m^4}{(mx+n)^5},\quad \frac{d^5z}{120dx^5}=-\frac{m^5}{(mx+n)^6}\quad \text{etc.}$$

From these it is therefore found

$$s=D+\frac{1}{m}\ln(mx+n)+\frac{1}{2(mx+n)}-\frac{\mathfrak{A}m}{2(mx+n)^2}+\frac{\mathfrak{B}m^3}{4(mx+n)^4}$$

$$-\frac{\mathfrak{C}m^5}{6(mx+n)^6}+\frac{\mathfrak{D}m^7}{8(mx+n)^8}-\text{etc.}$$

Therefore, having put $x=0$ the constant to be added will be

$$D=-\frac{1}{m}\ln n-\frac{1}{2n}+\frac{\mathfrak{A}m}{2n^2}-\frac{\mathfrak{B}m^3}{4n^4}+\frac{\mathfrak{C}m^5}{6n^6}-\text{etc.}$$

§147 But if $n=0$, since the sum of the series

$$\frac{1}{m}+\frac{1}{2m}+\frac{1}{3m}+\frac{1}{4m}+\cdots+\frac{1}{mx}$$

is

$$=\frac{1}{m}C+\frac{1}{m}\ln x+\frac{1}{2mx}-\frac{\mathfrak{A}}{2mx^2}+\frac{\mathfrak{B}}{4mx^4}-\text{etc.},$$

but the sum of this series

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\cdots+\frac{1}{mx}$$

is

$$= C + \ln mx + \frac{1}{2mx} - \frac{\mathfrak{A}}{2m^2x^2} + \frac{\mathfrak{B}}{4m^4x^4} - \text{etc.},$$

if from this series the latter taken m times is subtracted that this series arises

$$\begin{aligned} 1 + \frac{1}{2} + \cdots + \frac{1}{m} + \cdots + + \frac{1}{2m} + \cdots + \frac{1}{3m} + \cdots + \frac{1}{mx} \\ - \frac{m}{m} \qquad\qquad - \frac{m}{2m} \qquad\qquad - \frac{m}{3m} \qquad\qquad - \frac{m}{mx}, \end{aligned}$$

its sum will be

$$\begin{aligned} = \ln m + \frac{1}{2mx} - \frac{\mathfrak{A}}{2m^2x^2} + \frac{\mathfrak{B}}{4m^4x^4} - \text{etc.} \\ - \frac{1}{2x} + \frac{\mathfrak{A}}{2xx} - \frac{\mathfrak{B}}{4x^4} + \text{etc.}, \end{aligned}$$

and if one sets $x = \infty$, the sum will be $= \ln m$. Hence, by taking the numbers 2, 3, 4 etc. for m it will be

$$\begin{aligned} \ln 2 &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \text{etc.} \\ \ln 3 &= 1 + \frac{1}{2} - \frac{2}{3} + \frac{1}{4} + \frac{1}{5} - \frac{2}{6} + \frac{1}{7} + \frac{1}{8} - \frac{2}{9} + \text{etc.} \\ \ln 4 &= 1 + \frac{1}{2} + \frac{1}{3} - \frac{3}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} - \frac{3}{8} + \text{etc.} \\ \ln 5 &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} - \frac{4}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} - \frac{4}{10} + \text{etc.} \\ &\text{etc.} \end{aligned}$$

§148 But having left the harmonic series let us proceed to the reciprocal series of the squares and let

$$s = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots + \frac{1}{xx};$$

since in it the general term is $z = \frac{1}{xx}$, it will be $\int z dx = -\frac{1}{x}$ and the differential of z will behave this way

$$\frac{dz}{2dx} = -\frac{1}{x^3}, \quad \frac{ddz}{2 \cdot 3dx^2} = \frac{1}{x^4}, \quad \frac{d^3z}{2 \cdot 3 \cdot 4dx^3} = -\frac{1}{x^5} \quad \text{etc.},$$

whence the sum will be

$$s = C - \frac{1}{x} + \frac{1}{2xx} - \frac{\mathfrak{A}}{x^3} + \frac{\mathfrak{B}}{x^5} - \frac{\mathfrak{C}}{x^7} + \frac{\mathfrak{D}}{x^9} - \frac{\mathfrak{E}}{x^{11}} + \text{etc.},$$

in which the constant C to be added is to be defined from a single case, in which the sum is known. Therefore, let us put $x = 1$; since $x = 1$, it has to be

$$C = 1 + 1 - \frac{1}{2} + \mathfrak{A} - \mathfrak{B} + \mathfrak{C} - \mathfrak{D} + \mathfrak{E} - \text{etc.},$$

which series, since it is most divergent, does not show the value of the constant C. But since we demonstrated above [§ 125] that the sum of this series continued to infinity is $= \frac{\pi\pi}{6}$, having put $x = \infty$, if one puts $s = \frac{\pi\pi}{6}$, it will be $C = \frac{\pi\pi}{6}$ because all remaining terms vanish. Therefore, it will be

$$1 + 1 - \frac{1}{2} + \mathfrak{A} - \mathfrak{B} + \mathfrak{C} - \mathfrak{D} + \mathfrak{E} - \text{etc.} = \frac{\pi\pi}{6}.$$

§149

	s	=1.549767731166540690
then it is		
add	$\frac{1}{x}$	=0.1
subtr.	$\frac{1}{2xx}$	=0.0005
		1.644767731166540690
add	$\frac{\mathfrak{A}}{x^3}$	=0.000166666666666666
		1.644934397833207356
subtr.	$\frac{\mathfrak{B}}{x^5}$	=0.000000333333333333
		1.644934064499874023
add	$\frac{\mathfrak{C}}{x^7}$	=0.000000002380952381
		1.644934066880826404
subtr.	$\frac{\mathfrak{D}}{x^9}$	=0.000000000033333333
		1.6444066847493071

$$
\begin{array}{lr}
\text{add} & \frac{\mathfrak{E}}{x^{11}}=0.000000000000757575 \\
 & 1.644934066848250646 \\
\text{subtr.} & \frac{\mathfrak{F}}{x^{13}}=0.000000000000025311 \\
 & 1.644934066848225335 \\
\text{add} & \frac{\mathfrak{G}}{x^{15}}=0.000000000000001166 \\
\text{subtr.} & \frac{\mathfrak{H}}{x^{17}}=0.000000000000000071 \\
 & 1.644934066848226430 = C.
\end{array}
$$

And this value at the same time is the value of the expression $\frac{\pi\pi}{6}$, as is will become plain to anyone carrying out the calculation from the known value of π. Hence, it is understood at the same time, even though the series $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ etc., that nevertheless the true sum arises this way.

§150 Now let $z = \frac{1}{x^3}$ and

$$s = 1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \cdots + \frac{1}{x^3};$$

since it is

$$\int z dx = -\frac{1}{2xx}, \quad \frac{dz}{1 \cdot 2 \cdot 3dx} = -\frac{1}{2x^4}, \quad \frac{ddz}{1 \cdot 2 \cdot 3 \cdot 4dx^2} = \frac{1}{2x^5},$$
$$\frac{d^3z}{1 \cdot 2 \cdots 5dx^3} = -\frac{1}{2x^6}, \quad \frac{d^4z}{1 \cdot 2 \cdots 6dx^4} = \frac{1}{2x^7}, \quad \frac{d^5z}{1 \cdot 2 \cdots 7dx^5} = -\frac{1}{2x^6} \quad \text{etc.},$$

it will be

$$s = C - \frac{1}{2xx} + \frac{1}{2x^3} - \frac{3\mathfrak{A}}{2x^4} + \frac{5\mathfrak{B}}{2x^6} - \frac{7\mathfrak{C}}{2x^6} + \text{etc.}$$

and hence having put $x = 1$ because of $s = 1$ it will be

$$C = 1 + \frac{1}{2} - \frac{1}{2} + \frac{3}{2}\mathfrak{A} - \frac{5}{2}\mathfrak{B} + \frac{7}{2}\mathfrak{C} - \frac{9}{2}\mathfrak{D} + \text{etc.}.$$

and this value of C at the same time will show the sum of the propounded series continued to infinity. Since the sums of the odd powers are not known as the sum of the even ones, this value of C has to be defined from the known sum of some terms. Therefore, let $x = 10$; it will be

$$C = s + \frac{1}{2xx} - \frac{1}{2x^3} + \frac{2\mathfrak{A}}{2x^4} - \frac{5\mathfrak{B}}{2x^6} + \frac{7\mathfrak{C}}{2x^8} - \text{etc.}$$

But, in order to perform the calculation more easily, it is

$$\begin{aligned}
\frac{3\mathfrak{A}}{2} &= 0.250000000000 \\
\frac{5\mathfrak{B}}{2} &= 0.833333333333 \\
\frac{7\mathfrak{C}}{2} &= 0.833333333333 \\
\frac{9\mathfrak{D}}{2} &= 0.150000000000 \\
\frac{11\mathfrak{E}}{2} &= 0.416666666666 \\
\frac{13\mathfrak{F}}{2} &= 1.645280952380 \\
\frac{15\mathfrak{G}}{2} &= 8.750000000000 \\
\frac{17\mathfrak{H}}{2} &= 60.283333333333 \\
&\text{etc.}
\end{aligned}$$

Hence, the terms to be added to s will become

$$\begin{aligned}
\tfrac{1}{2xx} &= 0.005000000000000000 \\
\tfrac{3\mathfrak{A}}{2x^4} &= 0.000025000000000000 \\
\tfrac{7\mathfrak{C}}{2x^8} &= 0.000000000833333333 \\
\tfrac{11\mathfrak{E}}{2x^{12}} &= 0.000000000000416666 \\
\tfrac{13\mathfrak{F}}{2x^{16}} &= 0.000000000000000875 \\
\hline
& \, 0.005025000833750875
\end{aligned}$$

but the terms to be subtracted are

$$
\begin{aligned}
\tfrac{1}{2x^3} &= 0.005000000000000000\\
\tfrac{5\mathfrak{B}}{2x^6} &= 0.000000000833333333\\
\tfrac{9\mathfrak{D}}{2x^{10}} &= 0.000000000001500000\\
\tfrac{13\mathfrak{F}}{2x^{14}} &= 0.000000000000016452\\
\tfrac{17\mathfrak{F}}{2x^{18}} &= 0.000000000000000060\\
\hline
&\,0.000500083348349845\\
\text{from} &\,0.005025000833750875\\
\hline
&\,0.004524917485401030\\
s &= 1.197531985674193251\\
\hline
C &= 1.202056903159594281
\end{aligned}
$$

§151 If we continued this way, we will find the sum of all series of reciprocal powers expressed in decimal fractions.

$$
\begin{aligned}
1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\text{etc.} &= 1.6449340668482264 = \frac{2\mathfrak{A}}{1\cdot 2}\pi^2\\
1+\frac{1}{2^3}+\frac{1}{3^3}+\frac{1}{4^3}+\text{etc.} &= 1.2020569031595942\\
1+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\text{etc.} &= 1.0823232337111381 = \frac{2^3\mathfrak{B}}{1\cdot 2\cdot 3\cdot 4}\pi^4\\
1+\frac{1}{2^5}+\frac{1}{3^5}+\frac{1}{4^5}+\text{etc.} &= 1.0369277551433699\\
1+\frac{1}{2^6}+\frac{1}{3^6}+\frac{1}{4^6}+\text{etc.} &= 1.0173430619844491 = \frac{2^5\mathfrak{C}}{1\cdot 2\cdots 6}\pi^6\\
1+\frac{1}{2^7}+\frac{1}{3^7}+\frac{1}{4^7}+\text{etc.} &= 1.0083492773819288\\
1+\frac{1}{2^8}+\frac{1}{3^8}+\frac{1}{4^8}+\text{etc.} &= 1.0040773561979443 = \frac{2^7\mathfrak{D}}{1\cdot 2\cdots 8}\pi^8\\
1+\frac{1}{2^9}+\frac{1}{3^9}+\frac{1}{4^9}+\text{etc.} &= 1.0020083928260822
\end{aligned}
$$

$$1+\frac{1}{2^{10}}+\frac{1}{3^{10}}+\frac{1}{4^{10}}+\text{etc.}=1.0009945751278180=\frac{2^9\mathfrak{E}}{1\cdot 2\cdots 10}\pi^{10}$$
$$1+\frac{1}{2^{11}}+\frac{1}{3^{11}}+\frac{1}{4^{11}}+\text{etc.}=1.0004941886041194$$
$$1+\frac{1}{2^{12}}+\frac{1}{3^{12}}+\frac{1}{4^{12}}+\text{etc.}=1.0002460865533080=\frac{2^{11}\mathfrak{F}}{1\cdot 2\cdots 12}\pi^{12}$$
$$1+\frac{1}{2^{13}}+\frac{1}{3^{13}}+\frac{1}{4^{13}}+\text{etc.}=1.0001227133475784$$
$$1+\frac{1}{2^{14}}+\frac{1}{3^{14}}+\frac{1}{4^{14}}+\text{etc.}=1.0000612481350587=\frac{2^{13}\mathfrak{G}}{1\cdot 2\cdots 14}\pi^{14}$$
$$1+\frac{1}{2^{15}}+\frac{1}{3^{15}}+\frac{1}{4^{15}}+\text{etc.}=1.0000305882363070$$
$$1+\frac{1}{2^{16}}+\frac{1}{3^{16}}+\frac{1}{4^{16}}+\text{etc.}=1.0000152822594086=\frac{2^{15}\mathfrak{H}}{1\cdot 2\cdots 16}\pi^{16}$$

etc.

§152 From these vice versa the sums of those series consisting of the Bernoulli numbers can be exhibited. For, it will be

$$1+-\frac{1}{2}+\frac{\mathfrak{A}}{2}-\frac{\mathfrak{B}}{4}+\frac{\mathfrak{C}}{6}-\frac{\mathfrak{D}}{8}+\text{etc.}=0.57721\text{ etc.}$$
$$1+1-\frac{1}{2}+\mathfrak{A}-\mathfrak{B}+\mathfrak{C}-\mathfrak{D}+\text{etc.}=\frac{2\mathfrak{A}}{1\cdot 2}\pi^2$$
$$1+\frac{1}{2}-\frac{1}{2}+\frac{2\mathfrak{A}}{2}-\frac{5\mathfrak{B}}{2}+\frac{7\mathfrak{C}}{2}-\frac{9\mathfrak{D}}{2}+\text{etc.}=1.2020\text{ etc.}$$
$$1+\frac{1}{3}-\frac{1}{2}+\frac{3\cdot 4\mathfrak{A}}{2\cdot 3}-\frac{5\cdot 5\mathfrak{B}}{2\cdot 3}+\frac{7\cdot 8\mathfrak{C}}{2\cdot 3}-\frac{9\cdot 10\mathfrak{D}}{2\cdot 3}+\text{etc.}=\frac{2^3\mathfrak{B}}{1\cdot 2\cdot 3\cdot 4}\pi^4$$
$$1+\frac{1}{4}-\frac{1}{2}+\frac{3\cdot 4\cdot 5\mathfrak{A}}{2\cdot 3\cdot 4}-\frac{5\cdot 6\cdot 7\mathfrak{B}}{2\cdot 3\cdot 4}+\frac{7\cdot 8\cdot 9\mathfrak{C}}{2\cdot 3\cdot 4}-\frac{9\cdot 10\cdot 11\mathfrak{D}}{2\cdot 3\cdot 4}+\text{etc.}=1.0369\text{ etc.}$$
$$1+\frac{1}{5}-\frac{1}{2}+\frac{3\cdot 4\cdot 5\cdot 6\mathfrak{A}}{2\cdot 3\cdot 4\cdot 5}-\frac{4\cdot 5\cdot 6\cdot 7\mathfrak{B}}{2\cdot 3\cdot 4\cdot 5}+\frac{5\cdot 6\cdot 7\cdot 8\mathfrak{C}}{2\cdot 3\cdot 4\cdot 5}-\text{etc.}=\frac{2^5\mathfrak{C}}{1\cdot 2\cdots 6}\pi^6$$

etc.

Therefore, each second of these series can be summed by means of the quadrature of the circle; on which transcendental quantity the remaining depend, is not known to this day; for, they cannot be reduced to powers of π with odd exponent, such that the coefficients would be rational numbers. But that this at least becomes approximately clear, how the coefficients of powers of π will behave for odd exponents, we added the following table.

$$
\begin{array}{lll}
1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\text{etc. to infinity} = \frac{\pi}{0.0000} & = \infty \\
1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\text{etc. to infinity} = \frac{\pi^2}{6.0000} & \text{exactly} \\
1+\frac{1}{2^3}+\frac{1}{3^3}+\frac{1}{4^3}+\text{etc. to infinity} = \frac{\pi^3}{25.79436} & \text{approximately} \\
1+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\text{etc. to infinity} = \frac{\pi^2}{90.00000} & \text{exactly} \\
1+\frac{1}{2^5}+\frac{1}{3^5}+\frac{1}{4^5}+\text{etc. to infinity} = \frac{\pi^5}{295.1215} & \text{approximately} \\
1+\frac{1}{2^6}+\frac{1}{3^6}+\frac{1}{4^6}+\text{etc. to infinity} = \frac{\pi^6}{945.000} & \text{exactly} \\
1+\frac{1}{2^7}+\frac{1}{3^7}+\frac{1}{4^7}+\text{etc. to infinity} = \frac{\pi^2}{2995.284} & \text{approximately} \\
1+\frac{1}{2^8}+\frac{1}{3^8}+\frac{1}{4^8}+\text{etc. to infinity} = \frac{\pi^2}{9450.0000} & \text{exactly} \\
1+\frac{1}{2^9}+\frac{1}{3^9}+\frac{1}{4^9}+\text{etc. to infinity} = \frac{\pi^9}{29749.35} & \text{approximately} \\
\text{etc.}
\end{array}
$$

§153 From these source the series of Bernoulli numbers

$$
\begin{array}{cccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & \\
\mathfrak{A}, & \mathfrak{B}, & \mathfrak{C}, & \mathfrak{D}, & \mathfrak{E}, & \mathfrak{F}, & \mathfrak{G}, & \mathfrak{H}, & \mathfrak{I} & \text{etc.,}
\end{array}
$$

how irregular it might seem, can be interpolated or the terms constituted in the middle of any two can be assigned; for, if the term falling in the middle between the first $\mathfrak{A}$ and the second $\mathfrak{B}$ or the one corresponding to the index $1\frac{1}{2}$ was $= p$, it will be

$$1+\frac{1}{2^3}+\frac{1}{3^3}+\text{etc.} = \frac{2^2 p}{1\cdot 2\cdot 3}\pi^3$$

and hence

$$p = \frac{3}{2\pi^3}\left(1+\frac{1}{2^3}+\frac{1}{3^3}+\text{etc.}\right) = 0.05815227.$$

If in similar manner the term falling in the middle between $\mathfrak{B}$ and $\mathfrak{C}$ or having the index $2\frac{1}{2}$ is put $= q$, since it will be

$$1 + \frac{1}{2^5} + \frac{1}{3^5} + \text{etc.} = \frac{2^4 q}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}\pi^5,$$

it will be

$$q = \frac{15}{2\pi^5}\left(1 + \frac{1}{2^5} + \frac{1}{3^5} + \text{etc.}\right) = 0.02541327.$$

Therefore, if the sums of these series, in which the exponents of the powers are odd numbers could be exhibited, then also the series of the Bernoulli numbers could be interpolated.

§154 Now, let us put $z = \frac{1}{nn+xx}$ and search after the sum of this series

$$s = \frac{1}{nn+1} + \frac{1}{nn+4} + \frac{1}{nn+9} + \cdots + \frac{1}{nn+xx}.$$

Since it is $\int zdx = \int \frac{dx}{nn+xx}$, it will be

$$\int zdx = \frac{1}{n}\arctan\frac{x}{n}.$$

Put $\operatorname{arccot}\frac{x}{n} = u$; it will be

$$\int zdx = \frac{1}{n}\left(\frac{\pi}{2} - u\right)$$

and

$$\frac{x}{n} = \cot u = \frac{\cos u}{\sin u} \quad \text{and} \quad \frac{nn+xx}{nn} = \frac{1}{\sin^2 u} \quad \text{and} \quad z = \frac{\sin^2 u}{nn} \quad \text{and} \quad \frac{dx}{n} = -\frac{du}{\sin^2 u},$$

whence it is

$$du = -\frac{dx\sin^2 u}{n}.$$

Hence, the differentials of z will be found this way

$$dz = \frac{2du\sin u \cdot \cos u}{nn} = -\frac{dx\sin^2 u\sin 2u}{n^3} \quad \text{and} \quad \frac{dz}{dx} = -\frac{\sin^2 u \cdot \sin 2u}{n^3},$$

$$\frac{ddz}{dx^2} = -\frac{du(\sin u \cdot \cos u \cdot \sin 2u + \sin^2 u \cdot \cos 2u)}{n^3} = \frac{dx \sin^3 \cdot 3u}{n^4}$$

and

$$\frac{ddz}{2dx^2} = \frac{\sin^3 u \cdot \sin 3u}{n^4}.$$

In similar way, as we already found above [§ 87] for the same case, it will be

$$\frac{d^3z}{2 \cdot 3dx^3} = -\frac{\sin^4 \cdot \sin 4u}{n^5}, \quad \frac{d^4z}{2 \cdot 3 \cdot 4dx^4} = \frac{\sin^5 \cdot \sin 5u}{n^6} \quad \text{etc.},$$

from which the sum in question will be formed

$$s = \frac{\pi}{2n} - \frac{u}{n} + \frac{\sin u \cdot \sin u}{2nn} - \frac{\mathfrak{A}}{2} \cdot \frac{\sin^2 u \cdot \sin 2u}{n^3} + \frac{\mathfrak{B}}{4} \cdot \frac{\sin^4 u \cdot \sin 4u}{n^5}$$
$$-\frac{\mathfrak{C}}{6} \cdot \frac{\sin^6 u \cdot \sin 6u}{n^7} + \frac{\mathfrak{D}}{8} \cdot \frac{\sin^8 u \cdot \sin 8u}{n^9} - \text{etc.} + \text{Const.}$$

If here to determine the constant one sets $x = 0$, in which case $s = 0$, it will be $\cot u = 0$ and hence u the angle of 90° and therefore $\sin u = 1$, $\sin 2u = 0$, $\sin 4u = 0$, $\sin 6u = 0$ etc.; therefore, it seems that it will be

$$0 = \frac{\pi}{2n} - \frac{\pi}{2n} + \frac{1}{2nn} + C \quad \text{and hence} \quad C = -\frac{1}{2nn};$$

but on the other hand it is to be noted, even though the remaining terms vanish, that nevertheless, since the coefficients $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ etc. eventually grow to infinity, that their sum can be finite.

§155 To determine this constant in the right manner let us put that $x = \infty$; for, we defined the sum of this series running to infinity already in the *Introdctio* and showed that it is

$$= -\frac{1}{2nn} + \frac{\pi}{2n} + \frac{\pi}{n(e^{2\pi b} - 1)}.$$

But having put $x = \infty$ it will be $u = 0$ and hence $\sin u = 0$ and at the same time the sines of all multiple arcs will vanish. But since in this series the powers of $\sin u$ grow, the divergence of the series cannot impede that the value of the series vanishes in this case. Therefore, it will become $s = \frac{\pi}{2n} + C$; hence, it will be

$$\frac{\pi}{2n}+C=-\frac{1}{2nn}+\frac{\pi}{2n}+\frac{\pi}{n(e^{2n\pi})-1}\quad\text{and}\quad C=-\frac{1}{2nn}+\frac{\pi}{n(e^{2n\pi}-1)}.$$

Therefore, the sum of the series in question will be

$$s=\frac{\pi}{2n}-\frac{u}{n}-\frac{1}{2nn}+\frac{\sin^2 u}{2nn}-\frac{\mathfrak{A}}{2}\cdot\frac{\sin^2 u\cdot\sin 2u}{n^3}$$
$$+\frac{\mathfrak{B}}{4}\cdot\frac{\sin^4 u\cdot\sin 4u}{n^5}-\frac{\mathfrak{C}}{6}\cdot\frac{\sin^6 u\cdot\sin 6u}{n^7}+\text{etc.}+\frac{\pi}{n(e^{2n\pi}-1)}.$$

Where it is to be noted, if n was a mediocre large number, that the last term $\frac{\pi}{n(e^{2\pi n}-1)}$ will become so small that it can be neglected.

§156 Let us put that it is $x=n$ such that it denotes

$$s=\frac{1}{nn+1}+\frac{1}{nn+4}+\frac{1}{nn+9}+\cdots+\frac{1}{nn+nn}.$$

Then, it will be $\cot u=1$ and $u=45^\circ=\frac{\pi}{4}$. Therefore, one will have $\sin u=\frac{1}{\sqrt{2}}$, $\sin 2u=1$, $\sin 4u=0$, $\sin 6u=-1$, $\sin 8u=0$, $\sin 10u=1$ etc. Therefore, it will be

$$s=\frac{\pi}{4n}-\frac{1}{2nn}+\frac{1}{4nn}-\frac{\mathfrak{A}}{2\cdot 2n^3}+\frac{\mathfrak{C}}{6\cdot 8n^7}-\frac{\mathfrak{E}}{10\cdot 2^5n^{11}}+\frac{\mathfrak{G}}{14\cdot 2^7n^{15}}-\text{etc.}+\frac{\pi}{n(e^{2n\pi}-1)},$$

in which expression only each second Bernoulli appears. Therefore, if the value of s was already found by actually performed calculation, hence the quantity π can be defined; for, it will be

$$\pi=4ns+\frac{1}{n}+\frac{\mathfrak{A}}{1\cdot n^2}-\frac{\mathfrak{C}}{3\cdot 2^2n^6}+\frac{\mathfrak{E}}{5\cdot 2^4n^{10}}-\frac{\mathfrak{G}}{7\cdot 2^6n^{14}}+\text{etc.}-\frac{4\pi}{e^{2n\pi}-1}.$$

For, even though in the last term π is contained, nevertheless, since it is so small, it suffices to determine the value of π approximately.

Example

Let $n = 5$; it will be

$$s = \frac{1}{26} + \frac{1}{29} + \frac{1}{34} + \frac{1}{41} + \frac{1}{50};$$

these terms actually added will give

$$s = 0.146746306590549494;$$

Hence the terms will be

$$\begin{array}{rl}
4ns = & 2.93492611381098988 \\
\frac{1}{n} = & 0.20000000000000000 \\
\frac{\mathfrak{A}}{nn} = & 0.00666666666666666 \\
\hline
 & 3.14159278047765654 \\
\frac{\mathfrak{C}}{3\cdot 2^2\cdot n^6} = & 0.00000012698412698 \\
\hline
 & 3.14159265349352956 \\
\frac{\mathfrak{D}}{5\cdot 2^4\cdot n^{10}} = & 0.00000000009696969 \\
\hline
 & 3.14159265359049925 \\
\frac{\mathfrak{E}}{7\cdot 2^6\cdot n^{14}} = & 0.00000000000042666 \\
\hline
 & 3.14159265359007259 \\
\frac{\mathfrak{F}}{9\cdot 2^8\cdot n^{18}} = & 0.00000000000000625 \\
\hline
 & 3.14159265359007884.
\end{array}$$

This value already comes so close to the truth that one has to wonder why by means of such a simple calculation one can get this far. This expression is indeed a little bit larger than the correct value; for, one has still to subtract $\frac{4\pi}{e^{2n\pi}-1}$, whose value, as long as π is sufficiently accurately known, can be exhibited; this will be achieved by means of logarithms.

Since it is $\pi \log e = 1.3643763538$, it will be

$$\log e^{2n\pi} = 10\pi \log e = 13.6437635.$$

Because it is

$$\frac{4\pi}{e^{2n\pi}-1} = \frac{4\pi}{e^{2n\pi}} + \frac{4\pi}{e^{4n\pi}} + \text{etc.},$$

it is easily understood that for our calculation it suffices to have taken the first of the terms. Therefore, let us augment the characteristic by the number 17, since we have the same number of decimal places; it will be

$$\begin{array}{llll}
 & \log \pi & = & 17.4971498 \\
 & \log 4 & = & 0.6020600 \\
\hline
 & & & 18.0992098 \\
 & \text{subtr. } \log e^{2n\pi} & = & 13.6437635 \\
\text{Therefore} & & & 4.4554463 \\
\hline
 & \frac{4\pi}{e^{2n\pi}} & = & 28539 \\
\text{subtract from} & & & \\
 & & & 3.14159265369007884 \\
\hline
\text{it will be} & \pi & = & 3.14159265358979345
\end{array}$$

which expression just in the penultimate figure recedes from the truth; this is not to be wondered about, since we would have to subtract the term $\frac{\mathfrak{L}}{11\cdot 2^{10}\cdot n^{22}}$, which gives 22, and so not even the last figure would have been wrong. Moreover, it is understood, if for n we would have taken a greater number as 10, then in an easy task the periphery π could have been found up to 25 and more figures.

§157 Now let us also put transcendental functions of x for z and let be $z = \ln x$ by taking hyperbolic logarithms, since ordinary ones are easily reduced to them, and let

$$s = \ln 1 + \ln 2 + \ln 3 + \ln 4 + \cdots + \ln x.$$

Since it therefore is $z = \ln x$, it will be

$$\int z dx = x \ln x - x;$$

for, its differential gives $dx \ln x$. Furthermore, it is

$$\frac{dz}{dx} = \frac{1}{x}, \quad \frac{ddz}{dx^2} = -\frac{1}{x^2}, \quad \frac{d^3z}{1\cdot 2dx^3} = \frac{1}{x^3}, \quad \frac{d^4z}{1\cdot 2\cdot 3dx^4} = -\frac{1}{x^4}, \quad \frac{d^5z}{1\cdot 2\cdot 3\cdot 4dx^5} = \frac{1}{x^5} \quad \text{etc.}$$

Therefore, one will conclude that it will be

$$s = x\ln x - x + \frac{1}{2}\ln x + \frac{\mathfrak{A}}{1\cdot 2x} - \frac{\mathfrak{B}}{3\cdot 4x^3} + \frac{\mathfrak{C}}{5\cdot 6x^5} - \frac{\mathfrak{D}}{7\cdot 8x^7} + \text{etc.} + \text{Const.}$$

But this constant by putting $x = 1$, since $s = \ln 1 = 0$, will be defined in such a way that it is

$$C = 1 - \frac{\mathfrak{A}}{1\cdot 2} + \frac{\mathfrak{B}}{3\cdot 4} - \frac{\mathfrak{C}}{5\cdot 6} + \frac{\mathfrak{D}}{7\cdot 8} - \text{etc.},$$

which series because of the too strong divergence is inept to find the value of C at least approximately.

§158 But we will not only find an approximate value, but even even the true value of C, if we consider Wallis's expression found for the value of π and demonstrated in the *Introductio*, which was

$$\frac{\pi}{2} = \frac{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6\cdot 8\cdot 8\cdot 10\cdot 10\cdot 12\cdot \text{etc.}}{1\cdot 3\cdot 3\cdot 5\cdot 5\cdot 7\cdot 7\cdot 9\cdot\ 9\ \cdot 11\cdot 11\cdot \text{etc.}}$$

For, by taking logarithms, it will therefore be

$$\ln\pi - \ln 2 = 2\ln 2 + 2\ln 4 + 2\ln 6 + 2\ln 8 + 2\ln 10 + \ln 12 + \text{etc.}$$

$$- \ln 1 - 2\ln 3 - 2\ln 5 - 2\ln 7 - 2\ln 9 - 2\ln 11 - \text{etc.}$$

Therefore, in the assumed series let us put $x = \infty$, and because it is

$$\ln 1 + \ln 2 + \ln 3 + \ln 4 + \cdots + \ln x = C + \left(x + \frac{1}{2}\right)\ln x - x,$$

it will be

$$\ln 1 + \ln 2 + \ln 3 + \ln 4 + \cdots + \ln 2x = C + \left(2x + \frac{1}{2}\right)\ln 2x - 2x$$

and

$$\ln 2+\ln 4+\ln 6+\ln 8+\cdots+\ln 2x = C+\left(x+\frac{1}{2}\right)\ln x - x\ln 2 - x,$$

hence

$$\ln 1+\ln 3+\ln 5+\ln 7+\cdots+\ln(2x-1) = C+\left(x+\frac{1}{2}\right)\ln 2 - x.$$

Since it therefore is

$$\begin{aligned}\ln\frac{\pi}{2} &= 2\ln 2+2\ln 4+2\ln 6+\cdots+2\ln 2x-\ln 2x\\ &\quad -2\ln 1\ -2\ln 3-2\ln 5-\cdots-2\ln(2x-1),\end{aligned}$$

having put $x=\infty$ it will be

$$\ln\frac{\pi}{2} = 2C+(2x+1)\ln x+2x\ln 2-2x-\ln 2-\ln x-2x\ln x-(2x+1)\ln 2+2x$$

and hence

$$\ln\frac{\pi}{2} = 2C-2\ln 2,\quad \text{therefore}\quad 2C=\ln 2\pi \quad\text{and}\quad C=\frac{1}{2}\ln 2\pi,$$

whence in decimal fractions it is found

$$C = 0.9189385332046727417803297,$$

and at the same time the following series is summed

$$1-\frac{\mathfrak{A}}{1\cdot 2}+\frac{\mathfrak{B}}{3\cdot 4}-\frac{\mathfrak{C}}{5\cdot 6}+\frac{\mathfrak{D}}{7\cdot 8}-\frac{\mathfrak{E}}{9\cdot 10}+\text{etc.}=\frac{1}{2}\ln 2\pi.$$

§159 Now, having known this constant $C=\frac{1}{2}\ln 2\pi$ the sum of any number of logarithms from this series $\ln 1+\ln 2+\ln 3+\text{etc.}$ can be exhibited. For, if one puts

$$s=\ln 1+\ln 2+\ln 3+\ln 4+\cdots+\ln x,$$

it will be

$$s = \frac{1}{2}\ln 2\pi + \left(x + \frac{1}{2}\right)\ln x - x + \frac{\mathfrak{A}}{1\cdot 2x} - \frac{\mathfrak{B}}{3\cdot 4x^3} + \frac{\mathfrak{C}}{5\cdot 6x^5} - \frac{\mathfrak{D}}{7\cdot 8x^7} + \text{etc.},$$

if the propounded logarithms were hyperbolic; but if ordinary logarithms are propounded, then in the terms $\frac{1}{2}\ln 2\pi + \left(x+\frac{1}{2}\right)\ln x$ for $\ln 2\pi$ and $\ln x$ one has to take ordinary logarithms, but the remaining terms of the series

$$-x + \frac{\mathfrak{A}}{1\cdot 2x} - \frac{\mathfrak{B}}{3\cdot 4x^3} + \text{etc.}$$

have to be multiplied by $0.434294481903251827 = n$. Therefore, in this case for ordinary values it will be

$$\begin{aligned} \log\pi &= 0.497149872694133854351268 \\ \log 2 &= 0.301029995663981195213738 \\ \hline \log 2\pi &= 0.798179868358115049565006 \\ \tfrac{1}{2}\log 2\pi &= 0.399089934179057524782503. \end{aligned}$$

Example

Let the aggregate of thousand tabled logarithms be sought after

$$s = \log 1 + \log 2 + \log 3 + \cdots + \log 1000.$$

Therefore, it will be $x = 1000$ and

$$\begin{aligned} \log x &= 3.000000000000 \\ \text{whence it is}\quad x\log x &= 3000.000000000000 \\ \tfrac{1}{2}\log x &= 1.500000000000 \\ \tfrac{1}{2}\log 2\pi &= 0.399089341790 \\ \hline &\ 3001.8990899341790 \\ \text{subtr.}\quad nx &= 2567.6046080309272 \end{aligned}$$

Furthermore, it is

	$\frac{n\mathfrak{A}}{1\cdot 2x} =$	0.0000361912068
subtr.	$\frac{n\mathfrak{B}}{3\cdot 4x^3} =$	0.0000000000012
	add. $=$	2567.6046080309272
sum sought after	$s =$	2567.6046442221328.

Since therefore s is the logarithm of the product of the numbers

$$1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdots 1000,$$

it is clear that this product, if actually multiplied, consists of 2568 figures and the first numbers will be 4023872, which will be followed by 2561 other numbers.

§160 Therefore, by means of this summation of logarithms the products of arbitrary many factors, which proceed in the natural numbers, can be assigned approximately. To this one can mainly refer the problem, in which the middle term or the largest term in any power of the binomial $(a+b)^m$ is sought after, where it is certainly to be noted, if m is an odd number that two equal middle terms are given which taken together yield the middle term in the following even power. Because hence the largest coefficient in any even power is twice as large as the middle coefficient in the preceding odd power, it will be sufficient to have determined the largest middle term for the even powers. Therefore, let $m = 2n$ and the middle coefficient will be expressed in such a way that it is

$$\frac{2n(2n-1)(2n-2)(2n-3)\cdots(n+1)}{1\cdot 2\cdot 3\cdot 4\cdots n}.$$

Let us call this middle coefficient which is in question $= u$ and one will be able to represent it in this way that it is

$$u = \frac{1\cdot 2\cdot 3\cdot 4\cdot 5\cdots 2n}{(1\cdot 2\cdot 3\cdot 4\cdots 2n)^2},$$

and having taken logarithms it will be

$$\ln u = \ln 1 + \ln 2 + \ln 3 + \ln 4 + \ln 5 + \cdots + \ln 2n$$
$$-2\ln 1 - 2\ln 2 - 2\ln 3 - 2\ln 4 - 2\ln 5 - \cdots - 2\ln 2n.$$

§161 Now, by taking these logarithms as hyperbolic logarithms it will be

$$\ln 1+\ln 2+\ln 3+\ln 4+\cdots+\ln 2n = \frac{1}{2}\ln 2\pi + \left(2n+\frac{1}{2}\right)\ln n + \left(2n+\frac{1}{2}\right)\ln 2 - 2n$$
$$+\frac{\mathfrak{A}}{1\cdot 2\cdot 2n} - \frac{\mathfrak{B}}{3\cdot 4\cdot 2^3 n^3} + \frac{\mathfrak{C}}{5\cdot 6\cdot 2^5 n^5} - \text{etc.}$$

and

$$2\ln 1+2\ln 2+2\ln 3+2\ln 4+\cdots+2\ln n$$
$$= \ln 2\pi + (2n+1)\ln n - 2n + \frac{2\mathfrak{A}}{1\cdot 2n} - \frac{2\mathfrak{B}}{3\cdot 4n^3} + \frac{2\mathfrak{C}}{5\cdot 6n^5} - \text{etc.},$$

having subtracted which expression from the latter it will remain

$$\ln u = -\frac{1}{2}\ln\pi - \frac{1}{2}\ln n + 2n\ln 2 + \frac{\mathfrak{A}}{1\cdot 2\cdot 2n} - \frac{\mathfrak{B}}{3\cdot 4\cdot 2^3 n^3} + \frac{\mathfrak{C}}{5\cdot 6\cdot 2^5 n^5} - \text{etc.}$$
$$-\frac{2\mathfrak{A}}{1\cdot 2n} + \frac{2\mathfrak{B}}{3\cdot 4n^3} - \frac{2\mathfrak{C}}{5\cdot 6n^5} + \text{etc.};$$

by collecting each two terms it will be

$$\ln u = \ln\frac{2^{2n}}{\sqrt{n\pi}} - \frac{3\mathfrak{A}}{1\cdot 2\cdot 2n} + \frac{15\mathfrak{B}}{3\cdot 4\cdot 2^3 n^3} - \frac{63\mathfrak{C}}{5\cdot 6\cdot 2^5 n^5} + \frac{255\mathfrak{D}}{7\cdot 8\cdot 2^7 n^7} - \text{etc.}$$

Let

$$\frac{3\mathfrak{A}}{1\cdot 2\cdot 2^2 n^2} - \frac{15\mathfrak{B}}{3\cdot 4\cdot 2^4 n^4} + \frac{63\mathfrak{C}}{5\cdot 6\cdot 2^6 n^6} - \frac{255\mathfrak{D}}{7\cdot 8\cdot 2^8 n^8} + \text{etc.}$$
$$= \ln\left(1 + \frac{A}{2^2 n^2} + \frac{B}{2^4 n^4} + \frac{C}{2^6 n^6} + \frac{D}{2^8 n^8} + \text{etc.}\right);$$

it will be

$$\ln u = \ln\frac{2^{2n}}{\sqrt{n\pi}} - 2n\ln\left(1 + \frac{A}{2^2 n^2} + \frac{B}{2^4 n^4} + \frac{C}{2^6 n^6} + \text{etc.}\right)$$

and hence

$$\frac{u^{2n}}{\left(1 + \frac{A}{2^2 n^2} + \frac{B}{2^4 n^4} + \frac{C}{2^6 n^6} + \text{etc.}\right)^{2n}\sqrt{n\pi}}.$$

Having put $2n = m$ it will be

$$\begin{aligned}
&\ln\left(1 + \frac{A}{2^2n^2} + \frac{B}{2^4n^4} + \frac{C}{2^6n^6} + \frac{D}{2^8n^8} + \text{etc.}\right)\\
&= \frac{A}{m^2} + \frac{B}{m^4} + \frac{C}{m^6} + \frac{D}{m^8} + \frac{E}{m^{10}} + \text{etc.}\\
&\qquad - \frac{A^2}{2m^4} - \frac{AB}{m^6} - \frac{AC}{m^8} - \frac{AD}{m^{10}} - \text{etc.}\\
&\qquad\qquad - \frac{BB}{2m^8} - \frac{BC}{m^{10}} - \text{etc.}\\
&\qquad + \frac{A^3}{3m^6} + \frac{A^2B}{m^8} + \frac{A^2C}{m^{10}} + \text{etc.}\\
&\qquad\qquad + \frac{AB^2}{m^{10}} + \text{etc.}\\
&\qquad\qquad - \frac{A^4}{4m^8} - \frac{A^3B}{m^{10}} - \text{etc.}\\
&\qquad\qquad + \frac{A^5}{5m^{10}} + \text{etc.};
\end{aligned}$$

since this expression has to be equal to this one

$$\frac{2\mathfrak{A}}{1\cdot 2m^2} - \frac{15\mathfrak{B}}{3\cdot 4} + \frac{63\mathfrak{C}}{5\cdot 6m^6} - \frac{255\mathfrak{D}}{7\cdot 8m^8} + \text{etc.},$$

it will be

$$\begin{aligned}
A &= \frac{3\mathfrak{A}}{1\cdot 2}\\
B &= \frac{A^2}{2} - \frac{25\mathfrak{B}}{3\cdot 4}\\
C &= AB - \frac{1}{3}A^3 + \frac{63\mathfrak{C}}{5\cdot 6}\\
D &= AC + \frac{1}{2}B^2 - A^2B + \frac{1}{4}A^4 - \frac{255\mathfrak{D}}{7\cdot 8}\\
E &= AD + BC - A^2C - AB^2 + A^3B - \frac{1}{5}A^5 + \frac{1023\mathfrak{E}}{9\cdot 10}\\
&\qquad \text{etc.}
\end{aligned}$$

§162 Because it is $\mathfrak{A} = \frac{1}{6}$, $\mathfrak{B} = \frac{1}{30}$, $\mathfrak{C} = \frac{1}{42}$, $\mathfrak{D} = \frac{1}{30}$, $\mathfrak{E} = \frac{5}{66}$, it will be

$$A = \frac{1}{4}, \quad B = -\frac{1}{96}, \quad C = \frac{27}{640}, \quad D = -\frac{90031}{2^{11} \cdot 3^2 \cdot 5 \cdot 7} \quad \text{etc.}$$

Hence, one causes

$$u = \frac{2^{2n}}{\left(1 + \frac{1}{2^4 n^2} - \frac{1}{2^9 \cdot 3 n^4} + \frac{27}{2^{13} \cdot 5 n^6} - \frac{90031}{2^{19} \cdot 3^2 \cdot 5 \cdot 7 n^8} + \text{etc.}\right)^{2n} \sqrt{n\pi}}$$

or

$$u = \frac{2^{2n}\left(1 - \frac{1}{2^4 n^2} + \frac{7}{2^9 \cdot 3 n^4} - \frac{121}{2^{13} \cdot 3 \cdot 5 n^6} + \frac{107489}{2^{19} \cdot 3^2 \cdot 5 \cdot 7 n^8} - \text{etc.}\right)^{2n}}{\sqrt{n\pi}},$$

or if this expansion of the series is actually done, it will approximately be

$$u = \frac{2^{2n}}{\sqrt{n\pi}\left(1 + \frac{1}{4n} + \frac{1}{32n^2} - \frac{1}{128n^5} - \frac{5}{16 \cdot 128 n^4} + \text{etc.}\right)};$$

hence, the middle term in $(1+1)^{2n}$ will behave to the sum of all terms 2^{2n}

$$\text{as} \quad 1 \quad \text{to} \quad \sqrt{n\pi\left(1 + \frac{1}{4n} + \frac{1}{32n^2} - \frac{1}{128n^3} - \frac{5}{16 \cdot n^4} + \text{etc.}\right)};$$

or having put $4n = \nu$ for the sake of brevity this ratio will be

$$\text{as} \quad \text{to} \quad \sqrt{n\pi\left(1 + \frac{1}{\nu} + \frac{1}{2\nu^2} - \frac{1}{2\nu^3} - \frac{5}{8\nu^4} + \frac{23}{8\nu^5} + \frac{53}{16\nu^6} - \text{etc.}\right)}.$$

Example 1

Let the middle term in the expanded binomial $(a+b)^{10}$ be sought after, which is known to be

$$= \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} = 252.$$

Applying the last formula found for u it will be $n = 5$ and hence

	$\frac{1}{4n} = 0.0500000$
	$\frac{1}{32n^2} = 0.0012500$
	0.0512500
subtract	$\frac{1}{128n^3} = 0.0000625$
	0.0511875
Therefore	$1 + \frac{1}{4n} + \text{etc.} = 1.0511836$
the log. of this	$= 0.0216784$
	$\ln n = 0.6989700$
	$\ln \pi = 0.4971498$
	1.2177982
	$\ln \sqrt{n\pi(1+\text{etc.})} = 0.6088991$
from	$\ln 2^{2n} = 3.0102999$
	$\ln u = 2.4014008$
whence it is	$u = 252.$

EXAMPLE 2

Investigate the ratio which in the hundredth power of the binomial $1+1$ *the middle term has to sum of all* 2^{100}.

For this let us use the formula found first

$$\ln u = \ln \frac{2^{2n}}{\sqrt{n\pi}} - \frac{3\mathfrak{A}}{1\cdot 2\cdot 2n} + \frac{15\mathfrak{B}}{3\cdot 4\cdot 2^3 n^3} - \frac{63\mathfrak{C}}{5\cdot 6\cdot 2^5 n^5} + \text{etc.},$$

in which having put $2n = m$ that one has this power $(1+1)^m$ and having substituted the values for $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ etc. it will be

$$\ln u = \ln \frac{2^m}{\sqrt{\frac{1}{2}m\pi}} - \frac{1}{4m} + \frac{1}{24m^3} - \frac{1}{20m^5} + \frac{17}{112m^7} - \frac{31}{36m^9} + \frac{691}{88m^{11}} - \text{etc.};$$

since these logarithms are hyperbolic, multiply them by

$$k = 0.434294481903251,$$

that they are transformed into tabled ones, and it will be

$$\log u = \log \frac{2^m}{\sqrt{\frac{1}{2}m\pi}} - \frac{k}{4m} + \frac{k}{24m^3} - \frac{k}{20m^5} + \frac{17}{112m^7} - \frac{31}{36m^9} + \text{etc.},$$

whence, because the middle term is u, the ratio in question will be $2^m : u$ and hence

$$\log \frac{2^m}{u} = \log \frac{1}{2}m\pi + \frac{k}{4m} - \frac{k}{24m^3} + \frac{k}{20m^5} - \frac{17k}{122m^7} + \frac{31k}{36m^9} - \frac{691k}{88m^{11}} + \text{etc.}$$

Hence, because it is because of the exponent $m = 100$

$$\frac{k}{m} = 0.0043429448, \quad \frac{k}{m^3} = 0.0000004343, \quad \frac{k}{m^5} = 0.0000000000,$$

it will be

$$\begin{aligned} \tfrac{k}{4m} &= 0.0010857362 \\ \tfrac{k}{24m^3} &= 0.0000000181 \\ \hline &\ 0.0010857181 \end{aligned}$$

Then it is

$$\begin{aligned} \log \pi &= 0.4971498726 \\ \log \tfrac{1}{2}m &= 1.6989700043 \\ \hline \log \tfrac{1}{2}m\pi &= 2.1961198769 \\ \hline \log \sqrt{\tfrac{1}{2}m\pi} &= 1.0980599384 \\ \tfrac{k}{4m} - \tfrac{k}{24m^3} + \text{etc.} &= 0.0010857181 \\ \hline &\ 1.0991456565 = \ln \tfrac{2^{100}}{u}. \end{aligned}$$

Therefore, it will be $\frac{2^{100}}{u} = 12.56451$ and hence in the expanded power $(1 + 1)^{100}$ the middle term will behave to the sum of all 2^{100} as 1 to 12.56451.

§163 Now, let the general term z denote the exponential function a^x such that this geometric series has to be summed

$$s = a + a^2 + a^3 + a^4 + \cdots + a^x;$$

since it is a geometric series, its sum is already known; for, it will be $s = \frac{(a^x-1)a}{a-1}$. But let us investigate this sum in the way explained here. Since it is $z = a^x$, it will be $\int zdx = \frac{a^x}{\ln a}$; for, the differential of this is $a^x dx$; but then it will be

$$\frac{dz}{dx} = a^x \ln a, \quad \frac{ddz}{dx^2} = a^x (\ln a)^2, \quad \frac{d^3z}{dx^3} = a^x (\ln a)^3 \quad \text{etc.},$$

whence it follows that it will be

$$s = a^x \left(\frac{1}{\ln a} + \frac{1}{2} + \frac{\mathfrak{A}}{1\cdot 2} \ln a - \frac{\mathfrak{B}}{1\cdot 2\cdot 3\cdot 4} (\ln a)^3 + \frac{\mathfrak{C}}{1\cdot 2\cdot 3\cdots 6} - \text{etc.} \right) + C.$$

To define the constant C put $x = 0$ and because of $s = 0$ it will be

$$C = -\frac{1}{\ln a} - \frac{1}{2} - \frac{\mathfrak{A}}{1\cdot 2} \ln a + \frac{\mathfrak{B}}{1\cdot 2\cdot 3\cdot 4} (\ln a)^3 - \text{etc.}$$

and hence it will be

$$s = (a^x - 1) \left(\frac{1}{\ln a} + \frac{1}{2} + \frac{\mathfrak{A}}{1\cdot 2} \ln a - \frac{\mathfrak{B}}{1\cdot 2\cdot 3\cdot 4} (\ln a)^3 + \frac{\mathfrak{C}}{1\cdot 2\cdot 3\cdots 6} (\ln a)^5 - \text{etc.} \right)$$

Because therefore the sum is $\frac{(a^x-1)a}{a-1}$ it will be

$$\frac{a}{a-1} = \frac{1}{\ln a} + \frac{1}{2} + \frac{\mathfrak{A}}{1\cdot 2} \ln a - \frac{\mathfrak{B}}{1\cdot 2\cdot 3\cdot 4} (\ln a)^3 + \frac{\mathfrak{C}}{1\cdot 2\cdot 3\cdots 6} (\ln a)^5 - \text{etc.},$$

where $\ln a$ denotes the hyperbolic logarithm of a; hence, it becomes

$$\frac{(a+1)\ln a}{2(a-1)} = 1 + \frac{\mathfrak{A}(\ln a)^2}{1\cdot 2} - \frac{\mathfrak{B}(\ln a)^4}{1\cdot 2\cdot 3\cdot 4} + \frac{\mathfrak{C}(\ln a)^5}{1\cdot 2\cdot 3\cdots 6} - \text{etc.}$$

and so one will be able to exhibit the sum of this series.

§164 Let the general term be $z = \sin ax$ and

$$s = \sin a + \sin 2a + \sin 3a + \cdots + \sin ax;$$

this series, since it is recurring, can also be summed; for, it will be

$$s = \frac{\sin a + \sin ax - \sin(ax + a)}{1 - 2\cos a + 1} = \frac{\sin a + (1 - \cos a)\sin ax - \sin a \cdot \cos ax}{2(1 - \cos a)}.$$

It will be

$$\int z dx = \int dx \sin ax = -\frac{1}{a}\cos ax$$

and

$$\frac{dz}{dx} = a\cos ax, \quad \frac{d^3z}{dx^3} = -a^3\cos ax, \quad \frac{d^5z}{dx^5} = a^5\cos ax \quad \text{etc.}$$

Therefore

$$s = C - \frac{1}{a}\cos ax + \frac{1}{2}\sin ax + \frac{\mathfrak{A}a\cos ax}{1\cdot 2} + \frac{\mathfrak{B}a^3\cos ax}{1\cdot 2\cdot 3\cdot 4} + \frac{\mathfrak{C}a^5\cos ax}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6} + \frac{\mathfrak{D}a^7\cos ax}{1\cdot 2\cdots 8} + \text{etc.}$$

Put $x = 0$ that it is $s = 0$ and it will be

$$C = \frac{1}{a} - \frac{\mathfrak{A}a}{1\cdot 2} - \frac{\mathfrak{B}a^3}{1\cdot 2\cdot 3\cdot 4} - \frac{\mathfrak{C}a^5}{1\cdot 2\cdots 6} - \text{etc.},$$

therefore

$$s = \frac{1}{2}\sin ax + (1 - \cos sx)\left(\frac{1}{a} - \frac{\mathfrak{A}a}{1\cdot 2} - \frac{\mathfrak{B}a^3}{1\cdot 2\cdot 3\cdot 4} - \frac{\mathfrak{C}a^5}{1\cdot 2\cdots 6} - \text{etc.}\right).$$

But because it is

$$s = \frac{1}{2}\sin ax + \frac{(1 - \cos ax)\sin a}{2(1 - \cos a)},$$

it will become

$$\frac{\sin a}{2(1 - \cos a)} = \frac{1}{2}\cot\frac{1}{2}a = \frac{1}{a} - \frac{\mathfrak{A}a}{1\cdot 2} - \frac{\mathfrak{B}a^3}{1\cdot 2\cdot 3\cdot 4} - \frac{\mathfrak{C}a^5}{1\cdot 2\cdots 6} - \text{etc.},$$

which same series we already had above (§ 127).

§165 Now let $z = \cos ax$ and the series to be summed

$$s = \cos a + \cos 2a + \cos 3a + \cdots + \cos ax;$$

the sum of this series, since it is recurring, will be

$$s = \frac{\cos a - 1 + \cos ax - \cos(ax + a)}{1 - 2\cos a + 1} = -\frac{1}{2} + \frac{1}{2}\cos ax + \frac{1}{2}\cot\frac{1}{2}a \cdot \sin ax.$$

But on the other hand to express the sum by means of our method it will be

$$\int z dx = \int dx \cos ax = \frac{1}{a}\sin ax$$

and

$$\frac{dz}{dx} = -a\sin ax, \quad \frac{d^3z}{dx^3} = a^3 \sin ax, \quad \frac{d^5z}{dx^5} = -a^5 \sin ax \quad \text{etc.}$$

Therefore,

$$s = C + \frac{1}{a}\sin ax + \frac{1}{2}\cos ax - \frac{\mathfrak{A}a\sin ax}{1\cdot 2} - \frac{\mathfrak{B}a^3\sin ax}{1\cdot 2\cdot 3\cdot 4} - \text{etc.}$$

Let $x = 0$, it will be $s = 0$ and $C = -\frac{1}{2}$ and hence it will be

$$s = -\frac{1}{2} + \frac{1}{2}\cos ax + \frac{1}{a}\sin ax - \frac{\mathfrak{A}a\sin ax}{1\cdot 2} - \frac{\mathfrak{B}a^3\sin ax}{1\cdot 2\cdot 3\cdot 4} - \text{etc.}$$

Hence, because it is

$$s = -\frac{1}{2} + \frac{1}{2}\cos ax + \frac{1}{2}\cot\frac{1}{2}a \cdot \sin ax,$$

it will be as we already just found [§ 164]

$$\frac{1}{2}\cot\frac{1}{2}a = \frac{1}{a} - \frac{\mathfrak{A}a}{1\cdot 2} - \frac{\mathfrak{B}a^3}{1\cdot 2\cdot 3\cdot 4} - \frac{\mathfrak{C}a^5}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6} - \text{etc.}$$

§166 Since we found above [§ 166], if a denotes any arc, that it is

$$\frac{\pi}{2} = \frac{a}{2} + \sin a + \frac{1}{2}\sin 2a + \frac{1}{3}\sin 3a + \frac{1}{4}\sin 4a + \text{etc.},$$

let us consider this series and let $z = \frac{1}{x}\sin ax$ that it is

$$s = \sin a + \frac{1}{2}\sin 2a + \frac{1}{3}\sin 3a + \cdots + \frac{1}{x}\sin ax.$$

But in this case it is $\int zdx = \int \frac{dx}{x}\sin ax$, which integral cannot be exhibited. Therefore, it will be

$$\frac{dz}{dx} = \frac{a}{x}\cos ax - \frac{1}{xx}\sin ax, \quad \frac{ddz}{dx^2} = -\frac{a^2}{x}\sin ax - \frac{2a}{xx}\cos ax + \frac{2}{x^3}\sin ax,$$

$$\frac{d^3z}{dx^3} = -\frac{a^3}{x}\cos ax + \frac{3a^2}{x^2}\sin ax + \frac{6a}{x^3}\cos ax - \frac{6}{x^4}\sin ax,$$

$$\frac{d^4z}{dx^4} = \frac{a^4}{x}\sin ax + \frac{4a^3}{xx}\cos ax - \frac{12a^2}{x^3}\sin ax - \frac{24a}{x^4}\cos ax + \frac{24}{x^5}\sin ax.$$

Since therefore neither the integral formula $\int zdx$ can be exhibited nor is it possible to express this differential sufficiently convenient, we are not able to define the sum of this series by means of this series, such that anything could be concluded from there. The same inconvenience occurs in many other series, if the general term is not sufficiently simple that it differentials can be expressed in general. Therefore, in the following chapter we will find other general expressions for the sums of the series whose general terms are either to composite or cannot be given at all; these can be used with happy success. But the insufficiency of the method treated here is especially revealed, if the signs of the terms of the propounded series alternate; for, then, even though the general terms are simple, the summatory terms can nevertheless not be expressed in a convenient way by means of this method.