Investigation of the sum of series from the general Term *

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§103 Let the general term corresponding to the index x of a certain series be $= y$, such that y is an arbitrary function of x. Further, let Sy be the sum or the summatory term of the series expressing the aggregate of all terms from the first or another fixed term up to y, inclusively. But we will compute the sum of the series from the first term, whence, if it is $x = 1$, y will give the first term and Sy will exhibited this first term y; but if one puts $x = 0$, the summatory term Sy has to go over into nothing, because there are no terms to be summed. Therefore, the summatory term Sy will be a function of x of such a kind which vanishes for $x = 0$.

§104 If the general term y consists of several parts, that it is $y = p + q + r +$ etc., then one can consider the series itself as conflated of several other series, whose general terms are p, q, r etc. Hence, if the sums of these series are known, one will also be able to assign the sum of the propounded series; for, it will be the aggregate of all single series. Therefore, if $y = p + q + r +$ etc., it will be $Sy = Sp + Sq + Sq +$ etc. Therefore, because above we exhibited the sums of series, whose general terms are any arbitrary powers of x having positive integer coefficients, hence one will be able to find the summatory term of any series, whose general term is $ax^\alpha + bx^\beta + cx^\gamma +$ etc. while α, β, γ etc. are positive integer numbers or whose general term is a polynomial function of x.

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§105 In this series whose general term or the term corresponding to the exponent x is $= y$ let the term preceding this one or the term corresponding to the index $x-1$ be $= v$; since v arises from y, if instead of x one writes $x-1$, it will be

$$v = y - \frac{dy}{dx} + \frac{ddy}{2dx^2} - \frac{d^3y}{6dx^3} + \frac{d^4y}{24dx^4} - \frac{d^5y}{120dx^5} + \text{etc.}$$

Therefore, if y was the general term of this series

$$\begin{array}{cccccc} 1 & 2 & 3 & 4\cdots & \cdots x-1 & x \\ a+ & b+ & c+ & d+ & \cdots + v+ & y \end{array}$$

and the term corresponding to the index 0 of this series was $= A$, v, as it is a function of x, will be the general term of this series

$$\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5\cdots\cdots\cdots x \\ A+ & a+ & b+ & c+ & d+\cdots+v, \end{array}$$

whence, if Sv denotes the sum of this series, it will be $Sv = Sy - y + A$. And so having put $x = 0$, since it is $Sy = 0$ and $y = A$, also Sv will vanish.

§106 Therefore, since it is

$$v = y - \frac{dy}{dx} + \frac{ddy}{2dx^2} - \frac{d^3y}{6dx^3} + \text{etc.},$$

it will be by means of the things shown before

$$Sv = Sy - S\frac{dy}{dx} + S\frac{ddy}{2dx^2} + S\frac{d^4y}{24dx^4} - \text{etc.}$$

and because of $Sv = Sy - y + A$ it will be

$$y - A = S\frac{dy}{dx} - S\frac{ddy}{2dx^2} + S\frac{d^3y}{6dx^3} - S\frac{d^4y}{24dx^4} + \text{etc.}$$

and hence one will have

$$S\frac{dy}{dx} = y - A + S\frac{ddy}{2dx^2} - S\frac{d^3y}{6dx^3} + S\frac{d^4y}{24dx^4} - \text{etc.}$$

Therefore, if one has the summatory terms of the series, whose general terms are $\frac{ddy}{dx^2}$, $\frac{d^3y}{dx^3}$, $\frac{d^4y}{dx^4}$ etc., from them one will obtain the summatory term of the series, whose general term is $\frac{dy}{dx}$. The quantity A has to be of such a nature that having put $x = 0$ the summatory $S\frac{dy}{dx}$ term vanishes, and by this condition it is easier determined than if we would say that it is the term corresponding to the index 0 in the series whose general term is $= y$.

§107 From this source usually the sums of the powers of natural numbers are investigated. For, let $y = x^{n+1}$; since it is

$$\frac{dy}{dx} = (n+1)x^n, \quad \frac{ddy}{2dx^2} = \frac{(n+1)n}{1 \cdot 2}x^{n-1}, \quad \frac{d^3y}{6dx^3} = \frac{(n+1)n(n-1)}{1 \cdot 2 \cdot 3}x^{n-2},$$
$$\frac{d^4y}{24dx^4} = \frac{(n+1)n(n-1)(n-2)}{1 \cdot 2 \cdot 3 \cdot 4}x^{n-3} \quad \text{etc.,}$$

it will be having substituted these values

$$(n+1)Sx^n = x^{n+1} - A + \frac{(n+1)n}{1 \cdot 2}Sx^{n-1} - \frac{(n+1)n(n-1)}{1 \cdot 2 \cdot 3}Sx^{n-2} + \text{etc.};$$

and if one divides by $n+1$ on both sides, it will be

$$Sx^n = \frac{1}{n+1}x^{n+1} + \frac{n}{2}Sx^{n-1} - \frac{n(n-1)}{2 \cdot 3}Sx^{n-2} + \frac{n(n-1)(n-2)}{2 \cdot 3 \cdot 4}Sx^{n-3} - \text{etc.} - \text{Const.},$$

which constant has to be taken in such a way that having put $x = 0$ the total summatory term vanishes. Therefore, by means of this formula from the already known sums of inferior powers, whose general terms are x^{n-1}, x^{n-2} etc. one will be able to find the sum of the superior powers expressed by the general term x^n.

§108 If in this expression n denotes a positive integer, the number of terms will be finite. And hence the sum of infinitely many powers will be absolutely found; for, if $n = 0$, it will be

$$Sx^0 = x.$$

And having known this one it will be possible to proceed to the superior one; for, having put $n = 1$, it will become

$$Sx^1 = \frac{1}{2}x^2 + \frac{1}{2}Sx^0 = \frac{1}{2}x^2 + \frac{1}{2}x;$$

if one further sets $n = 2$, it will arise

$$Sx^2 = \frac{1}{3}x^3 + Sx - \frac{1}{3}Sx^0 = \frac{1}{3}x^3 + \frac{1}{2}x^2 + \frac{1}{6}x,$$

moreover

$$Sx^3 = \frac{1}{4}x^4 + \frac{3}{2}Sx^2 - Sx + \frac{1}{4}Sx^0 = \frac{1}{4}x^4 + \frac{1}{2}x^3 + \frac{1}{4}x^2,$$

$$Sx^4 = \frac{1}{5}x^5 + \frac{4}{2}Sx^3 - \frac{4}{2}Sx^2 + Sx - \frac{1}{5}Sx^0$$

or

$$Sx^4 = \frac{1}{5}x^5 + \frac{1}{2}x^4 + \frac{1}{3}x^3 - \frac{1}{30}x.$$

And so forth the successive sums of any higher powers are derived from the inferior ones; but this achieved a lot easier by the following means.

§109 Since we found above that it is

$$S\frac{dy}{dx} = y + \frac{1}{2}S\frac{ddy}{dx^2} - \frac{1}{6}S\frac{d^3y}{dx^3} + \frac{1}{24}S\frac{d^4y}{dx^4} - \frac{1}{120}S\frac{d^5y}{dx^5} + \text{etc.}$$

if we put $\frac{dy}{dx} = z$, it will be $\frac{ddy}{dx^2} = \frac{dz}{dx}$, $\frac{d^3y}{dx^3} = \frac{ddz}{dx^2}$ etc. But then because of $dy = zdx$ y will be the quantity whose differential is $= zdx$ which we indicate this way that it is $y = \int zdx$. Although this invention of y from the given z depends on integral calculus, we will nevertheless be able to use this formula $\int zdx$ here, if for z we substitute only functions of x of such a kind that this function whose differential is $= zdx$ can be exhibited from the preceding ones. Therefore, having substituted these values, it will be

$$Sz = \int zdx + \frac{1}{2}S\frac{dz}{dx} - \frac{1}{6}S\frac{ddz}{dx^2} + \frac{1}{24}S\frac{d^3z}{dx^3} - \text{etc.}$$

by adding a constant of such a kind that having put $x = 0$ the sum Sz itself vanishes.

§110 But by substituting the letter z instead of y in the superior expression or, which is the same, by differentiating this equation it will be

$$S\frac{dz}{dx} = z + \frac{1}{2}S\frac{ddz}{dx^2} - \frac{1}{6}S\frac{d^3z}{dx^3} + \frac{1}{24}S\frac{d^4z}{dx^4} - \text{etc.};$$

but if one puts $\frac{dz}{dx}$ instead of y, it will be

$$S\frac{ddz}{dx^2} = \frac{dz}{dx} + \frac{1}{2}S\frac{d^3z}{dx^3} - \frac{1}{6}S\frac{d^4z}{dx^4} + \frac{1}{24}S\frac{d^5z}{dx^5} - \text{etc.}$$

But if in similar manner for y one successively puts the values $\frac{ddz}{dx^2}$, $\frac{d^3z}{dx^3}$ etc., one will find

$$S\frac{d^3z}{dx^3} = \frac{ddz}{dx^2} + \frac{1}{2}S\frac{d^4z}{dx^4} - \frac{1}{6}S\frac{d^5z}{dx^5} + \frac{1}{24}S\frac{d^6z}{dx^6} - \text{etc.}$$
$$S\frac{d^4z}{dx^4} = \frac{d^3}{dx^3} + \frac{1}{2}S\frac{d^5z}{dx^5} - \frac{1}{6}S\frac{d^6z}{dx^6} + \frac{1}{24}S\frac{d^7z}{dx^7} - \text{etc.}$$

and so forth to infinity.

§111 If now these values are successively substituted for $S\frac{dz}{dx}$, $S\frac{ddz}{dx^2}$, $S\frac{d^3z}{dx^3}$ etc. in the expression

$$Sz = \int zdx + \frac{1}{2}S\frac{dz}{dx} - \frac{1}{6}S\frac{ddz}{dx^2} + \frac{1}{24}S\frac{d^3z}{dx^3} - \text{etc.},$$

one will find an expression for Sz which will consist of these terms $\int zdx$, z, $\frac{dz}{dx}$, $\frac{ddz}{dx^2}$, $\frac{d^3z}{dx^3}$ etc., whose coefficients are easier investigated the following way. Put

$$Sz = \int zdx + \alpha z + \frac{\beta dz}{dx} + \frac{\gamma ddz}{dx^2} + \frac{\delta d^3z}{dx^3} + \frac{\varepsilon d^4z}{dx^4} + \text{etc.}$$

and substitute its values for these terms which they obtain from the preceding series from which it is

$$\begin{aligned}
\int z dx &= Sz - \frac{1}{2}S\frac{dz}{dx} + \frac{1}{6}S\frac{ddz}{dx^2} - \frac{1}{24}S\frac{d^3z}{dx^3} + \frac{1}{120}S\frac{d^4z}{dx^4} - \text{etc.}\\
\alpha z &= \qquad + \alpha\, S\frac{dz}{dx} - \frac{\alpha}{2}S\frac{ddz}{dx^2} + \frac{\alpha}{6}\; S\frac{d^3z}{dx^3} - \frac{\alpha}{24}\; S\frac{d^4z}{dx^4} + \text{etc.}\\
\frac{\beta dz}{dx} &= \qquad\qquad + \beta\, S\frac{ddz}{dx^2} - \frac{\beta}{2}\; S\frac{d^3z}{dx^3} + \frac{\beta}{6}\; S\frac{d^4z}{dx^4} + \text{etc.}\\
\frac{\gamma ddz}{dx^2} &= \qquad\qquad\qquad + \gamma\; S\frac{d^3z}{dx^3} - \frac{\gamma}{2}\; S\frac{d^4z}{dx^4} + \text{etc.}\\
\frac{\delta d^3z}{dx^3} &= \qquad\qquad\qquad\qquad + \frac{\delta}{2}\; S\frac{d^4z}{dx^4} + \text{etc.}\\
&\qquad \text{etc.}
\end{aligned}$$

Since added together these values have to produce Sz, the coefficients α, β, γ, δ etc. will be defined from the following equations

$$\alpha - \frac{1}{2} = 0, \quad \beta - \frac{\alpha}{2} + \frac{1}{6} = 0, \quad \gamma - \frac{\beta}{2} + \frac{\alpha}{6} - \frac{1}{24} = 0,$$
$$\delta - \frac{\gamma}{2} + \frac{\beta}{6} - \frac{\alpha}{24} + \frac{1}{120} = 0, \quad \varepsilon - \frac{\delta}{2} + \frac{\beta}{24} + \frac{\alpha}{120} - \frac{1}{720} = 0,$$
$$\zeta - \frac{\varepsilon}{2} + \frac{\delta}{6} - \frac{\gamma}{24} + \frac{\beta}{120} - \frac{\alpha}{720} + \frac{1}{5040} = 0 \quad \text{etc.}$$

and continuing this way one will continuously find terms of which each second vanishes. Therefore, the third, fifth, seventh letter and in general all odd ones will be $= 0$ except the first, because of which the law of continuity seems to be violated. Therefore, it is even more necessary that it is rigorously proved that all odd terms except the first necessarily vanish.

§113 Since the single letters are determined according to a constant law from the preceding ones, they will constitute a recurring series. To make this explicit assume this series

$$1 + \alpha u + \beta u^2 + \gamma u^3 + \delta u^4 + \varepsilon u^5 + \zeta u^6 + \text{etc.},$$

whose value shall be $= V$, and it is manifest that this recurring series arises from the expansion of this fraction

$$V = \frac{1}{1 - \frac{1}{2}u + \frac{1}{6}u^2 - \frac{1}{24}u^3 + \frac{1}{240}u^4 - \text{etc.}}$$

And if this fraction in another way can be resolved into a power series in u, it is necessary that always the same series

$$V = 1 + \alpha u + \beta u^2 + \gamma u^3 + \delta u^4 + \varepsilon u^5 + \text{etc.}$$

results; and this way another law, by which the same values α, β, γ, δ etc. are determined, will be found.

§114 Since, if e denotes the number, whose hyperbolic logarithm is equal to the unity, it will be

$$e^{-u} = 1 - u + \frac{1}{2}u^2 - \frac{1}{6}u^3 + \frac{1}{24}u^4 - \frac{1}{120}u^5 + \text{etc.}$$

it will be

$$\frac{1-e^{-u}}{u} = 1 - \frac{1}{2}u + \frac{1}{6}u^2 - \frac{1}{24}u^3 + \frac{1}{120}u^4 - \text{etc.}$$

and hence

$$V = \frac{u}{1-e^{-u}}.$$

Now cancel the second term $\alpha u = \frac{1}{2}u$ from the series that it is

$$V - \frac{1}{2}u = 1 + \beta u^2 + \gamma u^3 + \delta u^4 + \varepsilon u^5 + \zeta u^6 + \text{etc.};$$

it will be

$$V - \frac{1}{2}u = \frac{\frac{1}{2}u(1+e^{-u})}{1-e^{-u}}.$$

Multiply the numerator and denominator by $e^{\frac{1}{2}u}$ and it will

$$V - \frac{1}{2}u = \frac{u\left(e^{\frac{1}{2}u} + e^{-\frac{1}{2}u}\right)}{\left(e^{\frac{1}{2}u} - e^{-\frac{1}{2}u}\right)}$$

and having converted the quantities $e^{\frac{1}{2}u}$ and $e^{-\frac{1}{2}u}$ into series it will be

$$V - \frac{1}{2}u = \frac{1 + \frac{u^2}{2\cdot4} + \frac{u^4}{2\cdot4\cdot6\cdot8} + \frac{u^6}{2\cdot4\cdot6\cdot8\cdot10\cdot12} + \text{etc.}}{2\left(\frac{1}{2} + \frac{u^2}{2\cdot4\cdot6} + \frac{u^4}{2\cdot4\cdot6\cdot8\cdot10} + \text{etc.}\right)}$$

or

$$V-\frac{1}{2}u=\frac{1+\frac{u^2}{2\cdot 4}+\frac{u^4}{2\cdot 4\cdot 6\cdot 8}+\frac{u^8}{2\cdot 4\cdots 12}+\frac{u^8}{2\cdot 4\cdots 16}+\text{etc.}}{1+\frac{u^2}{4\cdot 6}+\frac{u^4}{4\cdot 6\cdot 8\cdot 10}+\frac{u^8}{4\cdot 6\cdots 14}+\frac{u^8}{4\cdot 6\cdots 18}+\text{etc.}}$$

§115 Because therefore in this fraction the odd powers are completely missing, in its expansion into a power series no odd powers will go into in it; hence, because $V-\frac{1}{2}u$ becomes equal to this series

$$1+\beta u^2+\gamma u^3+\delta u^4+\varepsilon u^5+\zeta u^6+\text{etc.},$$

the coefficients of the odd powers γ, ε, η, ι etc. will all vanish. And so the reason is manifest why in the series $1+\alpha u+\beta u^2+\gamma u^3+\delta u^4+\text{etc.}$ all even terms except the first are $=0$ and the law of continuity is nevertheless not violated. Therefore, it will be

$$V=1+\frac{1}{2}u+\beta u^2+\delta u^4+\zeta u^6+\theta u^8+\varkappa x^{10}+\text{etc.}$$

and having determined the letters β, δ, ζ, θ, $\varkappa$ etc. by expansion of the fraction above we will obtain the summatory term Sz of the series whose general term corresponding to the index x is $=z$ expressed this way

$$Sz=\int zdx+\frac{1}{2}z+\frac{\beta dz}{dx}+\frac{\delta d^3z}{dx^3}+\frac{\zeta d^5z}{dx^5}+\frac{\theta d^7z}{dx^7}+\text{etc.}$$

§116 Since the series $1+\beta u^2+\delta u^4+\zeta u^6+\theta u^8+\text{etc.}$ arises from the expansion of this fraction

$$\frac{1+\frac{u^2}{2\cdot 4}+\frac{u^4}{2\cdot 4\cdot 6\cdot 8}+\frac{u^6}{2\cdot 4\cdot 6\cdot 8\cdot 10\cdot 12}+\text{etc.}}{1+\frac{u^2}{4\cdot 6}+\frac{u^4}{4\cdot 6\cdot 8\cdot 10}+\frac{u^8}{4\cdot 6\cdot 8\cdot 10\cdot 12\cdot 14}+\text{etc.}},$$

the letters β, δ, ζ, θ etc. will follow this law that it is

$$\begin{aligned}\beta&=\frac{1}{2\cdot 4}-\frac{1}{4\cdot 6}\\ \gamma&=\frac{1}{2\cdot 4\cdot 6\cdot 8}-\frac{\beta}{4\cdot 6}-\frac{1}{4\cdot 6\cdot 8\cdot 10}\end{aligned}$$

$$\delta = \frac{1}{2\cdot 4\cdot 6\cdots 12} - \frac{\delta}{4\cdot 6} - \frac{\beta}{4\cdot 6\cdot 8\cdot 10} - \frac{1}{4\cdot 6\cdots 11}$$
$$\theta = \frac{1}{2\cdot 4\cdot 6\cdots 16} - \frac{\zeta}{4\cdot 6} - \frac{\delta}{4\cdot 6\cdot 8\cdot 10} - \frac{\beta}{4\cdot 6\cdots 14} - \frac{1}{4\cdot 6\cdots 18}$$
$$\text{etc.}$$

But this values are alternately positive and negative.

§117 If therefore each second of these letters is taken negative such that it is

$$Sz = \int zdx + \frac{1}{2}z - \frac{\beta dz}{dx} + \frac{\delta d^3z}{dx^3} - \frac{\zeta d^5z}{dx^5} + \frac{\theta d^7z}{dx^7} - \text{etc.},$$

the letters β, δ, ζ, θ will be defined from this fraction

$$\frac{1 - \frac{u^2}{2\cdot 4} + \frac{u^4}{2\cdot 4\cdot 6\cdot 8} - \frac{u^6}{2\cdot 4\cdot 6\cdot 8\cdot 10\cdot 12} - \text{etc.}}{1 - \frac{u^2}{4\cdot 6} + \frac{u^4}{4\cdot 6\cdot 8\cdot 10} - \frac{u^8}{4\cdot 6\cdot 8\cdot 10\cdot 12\cdot 14} - \text{etc.}},$$

by expanding it into the series

$$1 + \beta u^2 + \delta u^4 + \zeta u^6 + \theta u^8 + \text{etc.};$$

therefore, it will be

$$\beta = \frac{1}{4\cdot 6} - \frac{1}{2\cdot 4}$$
$$\delta = \frac{\beta}{4\cdot 6} - \frac{1}{4\cdot 6\cdot 8\cdot 10} + \frac{1}{2\cdot 4\cdot 6\cdot 8}$$
$$\zeta = \frac{\delta}{4\cdot 6} - \frac{\beta}{4\cdot 6\cdot 8\cdot 10} + \frac{1}{4\cdot 6\cdots 14} - \frac{1}{2\cdot 4\cdots 12}$$
$$+ \text{etc.};$$

but now all terms will become negative.

§118 Therefore, let us put $\beta = -A$, $\delta = -B$, $\zeta = -C$ etc. such that it is

$$Sz = \int zdx + \frac{1}{2}z + \frac{Adz}{dx} - \frac{Bd^3z}{dx^3} + \frac{Cd^5z}{dx^5} - \frac{Dd^7z}{dx^7} + \text{etc.},$$

and to define the letters A, B, C, D etc. consider this series

$$1 - Au^2 - Bu^4 - Cu^6 - Du^8 - Eu^{10} - \text{etc.},$$

which arise from the expansion of this fraction

$$\frac{1-\frac{u^2}{2\cdot 4}+\frac{u^4}{2\cdot 4\cdot 6\cdot 8}-\frac{u^6}{2\cdot 4\cdots 12}+\frac{u^8}{2\cdot 4\cdots 16}-\text{etc.}}{1-\frac{u^2}{4\cdot 6}+\frac{u^4}{4\cdot 6\cdot 8\cdot 10}-\frac{u^6}{4\cdot 6\cdots 10}+\frac{u^8}{4\cdot 6\cdots 18}-\text{etc.}},$$

or consider this series

$$\frac{1}{u}-Au-Bu^3-Cu^5-Du^7-Eu^9-\text{etc.}=s,$$

which arise from the expansion of this fraction

$$s=\frac{1-\frac{u^2}{2\cdot 4}+\frac{u^4}{2\cdot 4\cdot 6\cdot 8}-\frac{u^6}{2\cdot 4\cdots 12}+\text{etc.}}{u-\frac{u^3}{4\cdot 6}+\frac{u^5}{4\cdot 6\cdot 8\cdot 10}-\frac{u^7}{4\cdot 6\cdots 14}+\text{etc.}}.$$

But because it is

$$\cos\frac{1}{2}u=1-\frac{u^2}{2\cdot 4}+\frac{u^4}{2\cdot 4\cdot 6\cdot 8}-\frac{u^6}{2\cdot 4\cdots 12}+\text{etc.},$$
$$\sin\frac{1}{2}u=\frac{u}{2}-\frac{u^3}{2\cdot 4\cdot 6}+\frac{u^5}{2\cdot 4\cdot 6\cdot 8\cdot 10}-\frac{u^7}{2\cdot 4\cdots 14}+\text{etc.},$$

it follows that it will be

$$s=\frac{\cos\frac{1}{2}u}{2\sin\frac{1}{2}u}=\frac{1}{2}\cot\frac{1}{2}u.$$

Therefore, if the cotangent of the arc $\frac{1}{2}u$ is converted into a series whose single terms proceed in the powers of u, from it one will find the values of the letters A, B, C, D, E etc.

§119 Therefore, because it is $s=\frac{1}{2}\cot\frac{1}{2}u$, it will be $\frac{1}{2}u=\operatorname{arccot}2s$ and by differentiating it will be $\frac{1}{2}du=\frac{-2ds}{1+4ss}$ or $4ds+du+4ssdu=0$ or

$$\frac{4ds}{du}+1+4ss=0.$$

But because it is

$$s=\frac{1}{u}-Au-Bu^3-Cu^5-\text{etc.},$$

it will be

$$\begin{aligned}
\frac{4ds}{du} &= -\frac{4}{uu} - 4A - 3\cdot 4Bu^2 - 5\cdot 4Cu^4 - 7\cdot 4Du^6 - \text{etc.}\\
1 &= \qquad +1\\
4ss &= +\frac{4}{uu} - 8A - 8Bu^2 \quad - 8Cu^4 \quad - 8Du^6 \quad - \text{etc.}\\
&\qquad + 4A^2u^2 \quad + 8ABu^4 \quad + 8ACu^6 \quad + \text{etc.}\\
&\qquad\qquad + 4BBu^6 \quad + \text{etc.}
\end{aligned}$$

Having equated these homogeneous terms to zero it will be

$$A = \frac{1}{12}, \quad B = \frac{A^2}{5}, \quad C = \frac{2AB}{7}, \quad D = \frac{2AC + BB}{9}, \quad E = \frac{2AD + 2BD}{11},$$
$$F = \frac{2AE + 2BD + CC}{13}, \quad G = \frac{2AF + 2BE + 2CD}{15}, \quad H = \frac{2AG + 2BF + 2CE + DD}{17},$$
$$\text{etc.}$$

From these formulas it now manifestly follows that these single values are positive.

§120 But since the denominators of these values become immensely large and impede the calculation quite a lot, instead of the letters A, B, C, D etc. let us introduce these new letters

$$A = \frac{\alpha}{1\cdot 2\cdot 3}, \quad B = \frac{\beta}{1\cdot 2\cdot 3\cdot 4\cdot 5}, \quad C = \frac{\gamma}{1\cdot 2\cdot 3\cdots 7},$$
$$D = \frac{\delta}{1\cdot 2\cdot 3\cdots 9}, \quad E = \frac{\varepsilon}{1\cdot 2\cdot 3\cdots 11} \quad \text{etc.}$$

And one will find that it will be

$$\alpha = \frac{1}{2}, \quad \beta = \frac{2}{3}\alpha^2, \quad \gamma = 2\cdot\frac{2}{3}\alpha\beta, \quad \delta = 2\cdot\frac{4}{3}\alpha\gamma + \frac{8\cdot 7}{4\cdot 5}\beta^2,$$
$$\varepsilon = 2\cdot\frac{5}{3}\alpha\delta + 2\cdot\frac{10\cdot 9\cdot 8}{1\cdot 2\cdots 5}\beta\gamma, \quad \zeta = 2\cdot\frac{12}{1\cdot 2\cdot 3}\alpha\varepsilon + 2\frac{12\cdot 11\cdot 10}{1\cdot 2\cdots 5}\beta\delta + \frac{12\cdot 11\cdot 10\cdot 9\cdot 8}{1\cdot 2\cdots 7}\gamma\gamma,$$
$$\eta = 2\cdot\frac{14}{1\cdot 2\cdot 3}\alpha\zeta + 2\cdot\frac{14\cdot 13\cdot 12}{1\cdot 2\cdots 5}\beta\varepsilon + 2\cdot\frac{14\cdot 13\cdot 12\cdot 11\cdot 10}{1\cdot 2\cdots 7}\gamma\delta$$
$$\text{etc.}$$

§121 But we will more conveniently use these formulas

$$\alpha = \frac{1}{2}, \quad \beta = \frac{4}{3}\cdot\frac{\alpha\alpha}{2}, \quad \gamma = \frac{6}{3}\cdot\alpha\beta, \quad \delta = \frac{8}{3}\cdot\alpha\gamma + \frac{8\cdot 7\cdot 6}{3\cdot 4\cdot 5}\cdot\frac{\beta\beta}{2},$$

$$\varepsilon = \frac{10}{3}\cdot\alpha\delta + \frac{10\cdot 9\cdot 8}{3\cdot 4\cdot 5}\cdot\beta\gamma, \quad \zeta = \frac{12}{3}\cdot\alpha\varepsilon + \frac{12\cdot 11\cdot 10}{3\cdot 4\cdot 5}\cdot\beta\delta + \frac{12\cdot 11\cdot 10\cdot 9\cdot 8}{3\cdot 4\cdot 5\cdot 6\cdot 7}\cdot\frac{\gamma\gamma}{2},$$

$$\eta = \frac{14}{3}\cdot\alpha\zeta + \frac{14\cdot 13\cdot 12}{3\cdot 4\cdot 5}\cdot\beta\varepsilon + \frac{14\cdot 13\cdot 12\cdot 11\cdot 10}{3\cdot 4\cdot 5\cdot 6\cdot 7}\cdot\gamma\delta,$$

$$\theta = \frac{16}{3}\cdot\alpha\eta + \frac{16\cdot 15\cdot 14}{3\cdot 4\cdot 5}\cdot\beta\zeta + \frac{16\cdot 15\cdots 12}{3\cdot 4\cdots 7}\gamma\varepsilon + \frac{16\cdot 15\cdots 10}{3\cdot 4\cdots 9}\cdot\frac{\delta\delta}{2}$$

etc.

From this law, according to which the calculus is easily done, if the values of the letters α, β, γ, δ etc. were found, then the summatory term of any arbitrary series whose general term or the term corresponding to the index x was $= z$, will be expressed in such a way that it is

$$Sz = \int zdx + \frac{1}{2}z + \frac{\alpha dz}{1\cdot 2\cdot 3\cdot dx} - \frac{\beta d^3z}{1\cdot 2\cdot 3\cdot 4\cdot 5dx^5} + \frac{\gamma d^5z}{1\cdot 2\cdot s7dx^5}$$

$$-\frac{\delta d^7z}{1\cdot 2\cdots 9dx^7} + \frac{\varepsilon d^9z}{1\cdot 2\cdots 11dx^9} - \frac{\zeta d^{11}z}{1\cdot 2\cdots 13dx^{11}} + \text{etc.}$$

But these letters α, β, γ, δ etc. were found to have the following values:

$\alpha = \frac{1}{2}$	or	$1\cdot 2\alpha = 1$
$\beta = \frac{1}{6}$		$1\cdot 2\cdot 3\beta = 1$
$\gamma = \frac{1}{6}$		$1\cdot 2\cdot 3\cdot 4\gamma = 4$
$\delta = \frac{3}{10}$		$1\cdot 2\cdot 3\cdots 5\delta = 36$
$\varepsilon = \frac{5}{6}$		$1\cdot 2\cdot 3\cdots 6\varepsilon = 600$
$\zeta = \frac{691}{210}$		$1\cdot 2\cdot 3\cdots 7\zeta = 24\cdot 691$
$\eta = \frac{35}{2}$		$1\cdot 2\cdot 3\cdots 8\eta = 20160\cdot 35$

$$\theta = \frac{3617}{30} \qquad 1 \cdot 2 \cdot 3 \cdots 9\theta = 12096 \cdot 3617$$
$$\iota = \frac{43867}{42} \qquad 1 \cdot 2 \cdot 3 \cdots 10\iota = 86400 \cdot 43867$$
$$\varkappa = \frac{1222277}{110} \qquad 1 \cdot 2 \cdot 3 \cdots 11\varkappa = 362880 \cdot 1222277$$
$$\lambda = \frac{854513}{6} \qquad 1 \cdot 2 \cdot 3 \cdots 12\lambda = 79833600 \cdot 854513$$
$$\mu = \frac{1181820455}{546} \qquad 1 \cdot 2 \cdot 3 \cdots 13\mu = 11404800 \cdot 1181820455$$
$$\nu = \frac{76977927}{2} \qquad 1 \cdot 2 \cdot 3 \cdots 14\nu = 43589145600 \cdot 76977927$$
$$\xi = \frac{23749461029}{30} \qquad 1 \cdot 2 \cdot 3 \cdots 15\xi = 43589145600 \cdot 23749461029$$
$$\pi = \frac{8615841276005}{462} \qquad 1 \cdot 2 \cdot 3 \cdots 16\pi = 45287424000 \cdot 8615841276005$$

§122 These numbers have the greatest use throughout the whole doctrine of series. For, first one can from these numbers form the last terms in the sums of the even powers, which we remarked above [§ 63 of the first part] that they cannot be found in the same way as the remaining terms from the sums of the preceding. For, in the even powers the last terms x of the sums are multiplied by certain numbers which numbers for the powers II, IV, VI, VII etc. are $\frac{1}{6}$, $\frac{1}{30}$, $\frac{1}{42}$, $\frac{1}{30}$ etc. affected with alternating signs. But these numbers arise, if the values of the letters α, β, γ, δ etc. found above are respectively divided by the odd numbers 3, 5, 7, 9 etc. whence these numbers which after its discoverer Jacob Bernoulli are usually called Bernoulli numbers, will be

$$\frac{\alpha}{3} = \frac{1}{6} = \mathfrak{A}, \qquad \frac{\iota}{19} = \frac{43867}{798} = \mathfrak{I}$$
$$\frac{\beta}{5} = \frac{1}{30} = \mathfrak{B}, \qquad \frac{\varkappa}{21} = \frac{174611}{330} = \mathfrak{K} = \frac{283 \cdot 617}{330}$$
$$\frac{\gamma}{7} = \frac{1}{42} = \mathfrak{C}, \qquad \frac{\lambda}{21} = \frac{854513}{138} = \mathfrak{L} = \frac{11 \cdot 131 \cdot 593}{2 \cdot 3 \cdot 23}$$
$$\frac{\alpha}{9} = \frac{1}{30} = \mathfrak{D}, \qquad \frac{\mu}{25} = \frac{236364091}{2730} = \mathfrak{M}$$

$$\frac{\varepsilon}{11} = \frac{5}{66} = \mathfrak{E}, \qquad \frac{\nu}{27} = \frac{8553103}{6} = \mathfrak{N} = \frac{13 \cdot 657931}{6}$$
$$\frac{\zeta}{13} = \frac{691}{2730} = \mathfrak{F}, \qquad \frac{\xi}{27} = \frac{23749461029}{870} = \mathfrak{O}$$
$$\frac{\eta}{15} = \frac{7}{6} = \mathfrak{G}, \qquad \frac{\pi}{31} = \frac{8615841276005}{14322} = \mathfrak{P}$$
$$\frac{\theta}{17} = \frac{3617}{510} = \mathfrak{H}, \qquad \text{etc.}$$

§123 Therefore, one will be able to find these Bernoulli numbers $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ etc. immediately from the following equations

$$\begin{aligned}
\mathfrak{A} &= \frac{1}{6} \\
\mathfrak{B} &= \frac{4 \cdot 3}{1 \cdot 2} \cdot \frac{1}{5}\mathfrak{A}^2 \\
\mathfrak{C} &= \frac{6 \cdot 5}{1 \cdot 2} \cdot \frac{2}{7}\mathfrak{A}\mathfrak{B} \\
\mathfrak{D} &= \frac{8 \cdot 7}{1 \cdot 2} \cdot \frac{2}{9}\mathfrak{A}\mathfrak{C} + \frac{8 \cdot 7 \cdot 6 \cdot 5}{1 \cdot 2 \cdot 3 \cdot 4} \cdot \frac{1}{9}\mathfrak{B}^2 \\
\mathfrak{E} &= \frac{10 \cdot 9}{1 \cdot 2} \cdot \frac{2}{11}\mathfrak{A}\mathfrak{D} + \frac{12 \cdot 11 \cdot 10 \cdot 9}{1 \cdot 2 \cdot 3 \cdot 4} \cdot \frac{2}{13}\mathfrak{B}\mathfrak{D} + \frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} \cdot \frac{1}{13}\mathfrak{C}^2 \\
\mathfrak{G} &= \frac{14 \cdot 13}{1 \cdot 2} \cdot \frac{2}{15}\mathfrak{A}\mathfrak{F} + \frac{14 \cdot 13 \cdot 12 \cdot 11}{1 \cdot 2 \cdot 3 \cdot 4} \cdot \frac{2}{15}\mathfrak{B}\mathfrak{E} + \frac{14 \cdot 13 \cdot 12 \cdot 12 \cdot 11 \cdot 9}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} \cdot \frac{2}{15}\mathfrak{C}\mathfrak{D} \\
&\qquad \text{etc.,}
\end{aligned}$$

the law of which equation is clear per se, if one only notes, where the square of a certain letter occurs, that its coefficients are half as small as it seems to have to be according to the rule. But, the terms, which contain the products of different letters, are to be considered to occur twice; for, for the sake of an example it will be

$$\begin{aligned}
13\mathfrak{F} = \frac{12 \cdot 11}{1 \cdot 2}\mathfrak{A}\mathfrak{E} + \frac{12 \cdot 11 \cdot 10 \cdot 9}{1 \cdot 2 \cdot 3 \cdot 4}\mathfrak{B}\mathfrak{D} + \frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}\mathfrak{C}\mathfrak{C} \\
+ \frac{12 \cdot 11 \cdot 10 \cdots 5}{1 \cdot 2 \cdot 3 \cdots 8}\mathfrak{D}\mathfrak{B} + \frac{12 \cdot 11 \cdot 10 \cdots 3}{1 \cdot 2 \cdot 3 \cdots 10}\mathfrak{E}\mathfrak{A}.
\end{aligned}$$

§124 Further, the same numbers α, β, γ, δ etc. also go into the expressions of the sums of the series of fractions contained in this general form

$$1+\frac{1}{2^n}+\frac{1}{3^n}+\frac{1}{4^n}+\frac{1}{5^n}+\frac{1}{6^n}+\text{etc.},$$

if n is a positive even number. For, we gave these sums in the *Introductio* expressed by means of the half of the circumference of the circle π while the radius is $=1$ and in the coefficients of these powers these numbers α, β, γ, δ etc. are detected to go in. But to understand that this circumstance does not happen accidentally but has to happen let us investigate the same sums in a singular way by means of which the law of those sums will easier become clear. Since we found above (§ 43) that it is

$$\frac{\pi}{n}\cot\frac{m}{n}\pi=\frac{1}{m}-\frac{1}{n-m}+\frac{1}{n+m}-\frac{1}{2n-m}+\frac{1}{2n+m}-\frac{1}{3n-m}+\text{etc.},$$

by connecting each two terms we will have

$$\frac{\pi}{n}\cot\frac{m}{n}\pi=\frac{1}{m}-\frac{2m}{nn-m^2}-\frac{2m}{4n^2-m^2}-\frac{2m}{9n^2-m^2}-\frac{2m}{16n^2-m^2}-\text{etc.},$$

whence we conclude that it will be

$$\frac{1}{n^2-m^2}+\frac{1}{4n^2-m^2}+\frac{1}{9n^2-m^2}+\frac{1}{16n^2-m^2}+\text{etc.}=\frac{1}{2mm}-\frac{\pi}{2mn}\cot\frac{m}{n}\pi.$$

Now, let us set $n=1$ and for m let us put u that it is

$$\frac{1}{1-u^2}+\frac{1}{4-u^2}+\frac{1}{9-u^2}+\frac{1}{16-u^2}+\text{etc.}=\frac{1}{2uu}-\frac{\pi}{2u}\cot\pi u.$$

Resolve these single fractions into series:

$$\begin{aligned}
\frac{1}{1-u^2}&=1+u^2+u^4+u^6+u^8+\text{etc.}\\
\frac{1}{4-u^2}&=\frac{1}{2^2}+\frac{u^2}{2^4}+\frac{u^4}{2^6}+\frac{u^6}{2^8}+\frac{u^8}{2^8}+\text{etc.}\\
\frac{1}{9-u^2}&=\frac{1}{3^2}+\frac{u^2}{3^4}+\frac{u^4}{3^6}+\frac{u^6}{3^8}+\frac{u^8}{3^{10}}+\text{etc.}\\
\frac{1}{16-u^2}&=\frac{1}{4^2}+\frac{u^2}{4^4}+\frac{u^6}{4^8}+\frac{u^8}{4^{10}}+\text{etc.}\\
&\text{etc.}
\end{aligned}$$

§125 If one therefore puts

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\text{etc.}=\mathfrak{a} \qquad 1+\frac{1}{2^8}+\frac{1}{3^8}+\frac{1}{4^8}+\text{etc.}=\mathfrak{b}$$

$$1+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\text{etc.}=\mathfrak{c} \qquad 1+\frac{1}{2^{10}}+\frac{1}{3^{10}}+\frac{1}{4^{10}}+\text{etc.}=\mathfrak{d}$$

$$1+\frac{1}{2^6}+\frac{1}{3^6}+\frac{1}{4^6}+\text{etc.}=\mathfrak{c} \qquad 1+\frac{1}{2^{12}}+\frac{1}{3^{12}}+\frac{1}{4^{12}}+\text{etc.}=\mathfrak{f}$$

etc.

the superior series will be transformed into this one

$$\mathfrak{a}+\mathfrak{b}u^2+\mathfrak{c}u^4+\mathfrak{d}u^6+\mathfrak{e}u^8+\mathfrak{f}u^{10}+\text{etc.}=\frac{1}{2uu}-\frac{\pi}{2u}\cot \pi u.$$

Therefore, because in § 118 the letters A, B, C, D etc. were found to be of such a nature that having put

$$s=\frac{1}{u}-Au-Bu^3-Cu^5-Du^7-Eu^9-\text{etc.}$$

it is $s=\frac{1}{2}\cot\frac{1}{2}u$, having put πu instead of $\frac{1}{2}u$ or $2\pi u$ instead of u it will be

$$\frac{1}{2}\cot \pi u=\frac{1}{2\pi u}-2A\pi u-2^3B\pi^3u^3-2^5C\pi^5u^5-2^7D\pi^7u^7-\text{etc.},$$

whence by multiplying by $\frac{\pi}{u}$ it will be

$$\frac{\pi}{2u}\cot \pi u=\frac{1}{2uu}-2A\pi^2-2^3B\pi^4u^2-2^5C\pi^6u^4-2^7D\pi^8u^6-\text{etc.},$$

and hence it follows that it will be

$$\frac{1}{2uu}-\frac{\pi}{2u}\cot \pi u=2A\pi^2+2^3B\pi^4u^2+2^5C\pi^6u^4+2^7D\pi^8u^6+\text{etc.}$$

Since we just found that it is

$$\frac{1}{2uu}-\frac{\pi}{2u}\cot \pi u=\mathfrak{a}+\mathfrak{b}u^2+\mathfrak{c}u^4+\mathfrak{d}u^6+\text{etc.},$$

it is necessary that it is

$$
\begin{aligned}
\mathfrak{a} &= 2\ A\pi^2 = \frac{2\alpha}{1\cdot 2\cdot 3}\pi^2 = \frac{2\mathfrak{A}}{1\cdot 2}\pi^2\\
\mathfrak{b} &= 2^3\ A\pi^4 = \frac{2^3\beta}{1\cdot 2\cdot 3\cdot 4\cdot 5}\pi^4 = \frac{2^3\mathfrak{B}}{1\cdot 2\cdot 3\cdot 4}\pi^4\\
\mathfrak{c} &= 2^5\ A\pi^6 = \frac{2^5\gamma}{1\cdot 2\cdot 3\cdots 7}\pi^6 = \frac{2^3\mathfrak{C}}{1\cdot 2\cdot 3\cdots 6}\pi^6\\
\mathfrak{d} &= 2^7\ A\pi^8 = \frac{2^7\delta}{1\cdot 2\cdot 3\cdots 9}\pi^8 = \frac{2^5\mathfrak{D}}{1\cdot 2\cdot 3\cdots 8}\pi^8\\
\mathfrak{e} &= 2^9\ E\pi^{10} = \frac{2^9\varepsilon}{1\cdot 2\cdot 3\cdots 11}\pi^{10} = \frac{2^9\mathfrak{E}}{1\cdot 2\cdot 3\cdots 10}\pi^{10}\\
\mathfrak{f} &= 2^{11}F\pi^{12} = \frac{2^{11}\zeta}{1\cdot 2\cdot 3\cdots 13}\pi^{12} = \frac{2^{11}\mathfrak{F}}{1\cdot 2\cdot 3\cdots 12}\pi^{12}\\
&\text{etc.}
\end{aligned}
$$

§126 Therefore, from this so easy reasoning not only all series of reciprocal powers which we exhibited in the preceding paragraph are conveniently summed but at the same time it is also understood how these sums are formed from the known values of the letters α, β, γ, δ, ε etc. or even from the Bernoulli numbers $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ etc. Therefore, since we defined fifteen of these numbers in § 122, from these one will be able to assign the sums of all even [reciprocal] powers up to the sum of this series inclusively:

$$1+\frac{1}{2^{30}}+\frac{1}{3^{30}}+\frac{1}{4^{30}}+\frac{1}{5^{30}}+\text{etc.};$$

for, the sum of this series will be

$$=\frac{2^{29}\pi}{1\cdot 2\cdot 3\cdots 31}\pi^{31}=\frac{2^{29}\mathfrak{P}}{1\cdot 2\cdots 30}\pi^{30}.$$

And if somebody wants to determine these letters further, this is very easily done by continuing these numbers α, β, γ etc. or these $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ etc.

§127 Therefore, the origin of these numbers α, β, γ, δ etc. or the one formed from them $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ etc. are mainly due to the expansion of the cotangent of a certain angle into an infinite series. For, if it is

$$\frac{1}{2}\cot\frac{1}{2}u=\frac{1}{u}-Au-Bu^3-Cu^5-Du^7-Eu^9-\text{etc.},$$

it will be

$$Au^2 + Bu^4 + Cu^6 + Du^8 + \text{etc.} = 1 - \frac{u}{2}\cot\frac{1}{2}u;$$

therefore, if instead of the coefficients A, B, C, D etc. the values of the letters are substituted, it will be found

$$\frac{\alpha u^2}{1\cdot 2\cdot 3} + \frac{\beta u^4}{1\cdot 2\cdots 5} + \frac{\gamma u^7}{1\cdot 2\cdots 7} + \frac{\delta u^8}{1\cdot 2\cdots 9} + \text{etc.} = 1 - \frac{u}{2}\cot\frac{1}{2}u$$

and by using the Bernoulli number it will be

$$\frac{\mathfrak{A}u^2}{1\cdot 2} + \frac{\mathfrak{B}u^4}{1\cdot 2\cdot 3\cdot 4} + \frac{\mathfrak{C}u^6}{1\cdot 2\cdots 6} + \frac{\mathfrak{D}u^8}{1\cdot 2\cdots 8} + \text{etc.} = 1 - \frac{u}{2}\cot\frac{1}{2}u,$$

from which series by differentiation innumerable others can be deduced and so infinite series can be summed into which these most remarkable number go in.

§128 Let us take the first equation which we want to multiply by u that it is

$$\frac{\alpha u^3}{1\cdot 2\cdot 3} + \frac{\beta u^5}{1\cdot 2\cdots 5} + \frac{\gamma u^7}{1\cdot 2\cdots 7} + \frac{\delta u^9}{1\cdot 2\cdots 9} + \text{etc.} = u - \frac{uu}{2}\cot\frac{1}{2}u,$$

which differentiated and divided by du gives

$$\frac{\alpha u^2}{1\cdot 2} + \frac{\beta u^4}{1\cdot 2\cdot 3\cdot 4} + \frac{\gamma u^6}{1\cdot 2\cdots 6} + \frac{\delta u^8}{1\cdot 2\cdots 8} + \text{etc.} = 1 - u\cot\frac{1}{2}u + \frac{uu}{4(\sin\frac{1}{2}u)^2};$$

and if it is differentiated again, it will be

$$\frac{\alpha u}{1} + \frac{\beta u^3}{1\cdot 2\cdot 3} + \frac{\gamma u^5}{1\cdot 2\cdot 3\cdot 4\cdot 5} + \text{etc.} = -\cot\frac{1}{2}u + \frac{u}{(\sin\frac{1}{2}u)^2} - \frac{uu\cos\frac{1}{2}u}{4(\sin\frac{1}{2}u)^2}.$$

But if the other equation is differentiated, it will be

$$\frac{\mathfrak{A}u}{1} + \frac{\mathfrak{B}u^3}{1\cdot 2\cdot 3} + \frac{\mathfrak{C}u^5}{1\cdot 2\cdots 5} + \frac{\mathfrak{D}u^7}{1\cdot 2\cdots 7} + \text{etc.} = -\frac{1}{2}\cot\frac{1}{2}u + \frac{u}{4(\sin\frac{1}{2}u)^2}.$$

From these, if one puts $u = \pi$, because of $\cot \frac{1}{2}\pi = 0$ and $\sin \frac{1}{2}\pi = 1$ these summations follows

$$\begin{aligned} 1 &= \frac{\alpha\pi^2}{1\cdot 2\cdot 3} + \frac{\beta\pi^4}{1\cdot 2\cdot 3\cdot 4\cdot 5} + \frac{\gamma\pi^6}{1\cdot 2\cdot 3\cdots 7} + \frac{\delta\pi^8}{1\cdot 2\cdot 3\cdots 9} + \text{etc.} \\ 1 + \frac{\pi^2}{4} &= \frac{\alpha\pi^2}{1\cdot 2} + \frac{\beta\pi^4}{1\cdot 2\cdot 3\cdot 4} + \frac{\gamma\pi^6}{1\cdot 2\cdot 3\cdots 6} + \frac{\delta\pi^8}{1\cdot 2\cdot 3\cdots 8} + \text{etc.} \\ \pi &= \frac{\alpha\pi}{1} + \frac{\beta\pi^3}{1\cdot 2\cdot 3} + \frac{\gamma\pi^5}{1\cdot 2\cdot 3\cdot 4\cdot 5} + \frac{\delta\pi^7}{1\cdot 2\cdot 3\cdots 7} + \text{etc.} \end{aligned}$$

or

$$1 = \alpha + \frac{\beta\pi^2}{1\cdot 2\cdot 3} + \frac{\gamma\pi^4}{1\cdot 2\cdot 3\cdot 4\cdot 5} + \frac{\delta\pi^6}{1\cdot 2\cdot 3\cdots 7} + \text{etc.};$$

if from this one the first is subtracted, it will remain

$$\alpha = \frac{(\alpha-\beta)\pi^2}{1\cdot 2\cdot 3} + \frac{(\beta-\gamma)\pi^4}{1\cdot 2\cdot 3\cdot 4\cdot 5} + \frac{(\alpha-\gamma)\pi^6}{1\cdot 2\cdot 3\cdots 7} + \text{etc.}$$

But then it will be

$$\begin{aligned} 1 &= \frac{\mathfrak{A}\pi^2}{1\cdot 2} + \frac{\mathfrak{B}\pi^4}{1\cdot 2\cdot 3\cdot 4} + \frac{\mathfrak{C}\pi^6}{1\cdot 2\cdot 3\cdots 6} + \frac{\mathfrak{D}\pi^8}{1\cdot 2\cdot 3\cdots 8} \\ \frac{\pi}{4} &= \frac{\mathfrak{A}\pi}{1} + \frac{\mathfrak{B}\pi^3}{1\cdot 2\cdot 3} + \frac{\mathfrak{C}\pi^5}{1\cdot 2\cdot 3\cdot 4\cdot 5} + \frac{\mathfrak{D}\pi^7}{1\cdot 2\cdot 3\cdots 7} \end{aligned}$$

or

$$\frac{1}{4} = \frac{\mathfrak{A}}{1} + \frac{\mathfrak{B}\pi^2}{1\cdot 2\cdot 3} + \frac{\mathfrak{C}\pi^4}{1\cdot 2\cdot 3\cdot 4\cdot 5} + \frac{\mathfrak{D}\pi^6}{1\cdot 2\cdot 3\cdots 7} + \text{etc.}$$

§129 From the table of values of the numbers α, β, γ, δ etc. which we exhibited above (§ 121) it is plain that they decrease at first, but then increase and do so to infinity. Therefore, it will be worth the effort to investigate, how these numbers, after they were already continued a long time, behave. Therefore, let φ be any number of this series of the numbers α, β, γ, δ etc. removed very far from the beginning and let ψ be the following of those numbers. Since by means of these numbers the sum of the reciprocal powers are defined, let $2n$ be the exponent of the power, into whose sum the number φ goes into; $2n+2$ will be the exponent corresponding to number ψ and the number n will already by immensely huge. Hence, from § 125 one will have

$$1+\frac{1}{2^{2n}}+\frac{1}{3^{2n}}+\frac{1}{4^{2n}}+\text{etc.}=\frac{2^{2n-1}\varphi}{1\cdot 2\cdot 3\cdots(2n+1)}\pi^{2n},$$
$$1+\frac{1}{2^{2n+2}}+\frac{1}{3^{2n+2}}+\frac{1}{4^{2n+2}}+\text{etc.}=\frac{2^{2n+1}\varphi}{1\cdot 2\cdot 3\cdots(2n+3)}\pi^{2n+2}.$$

Therefore, if this one is divided by that one, it will be

$$\frac{1+\frac{1}{2^{2n+2}}+\frac{1}{3^{2n+2}}+\text{etc.}}{1+\frac{1}{2^{2n}}+\frac{1}{3^{2n}}+\text{etc.}}=\frac{4\psi\pi^2}{(2n+2)(2n+3)\varphi}.$$

but since n is an immensely huge number and since both series are very close to 1, it will be

$$\frac{\psi}{\varphi}=\frac{(2n+2)(2n+3)}{4\pi^2}=\frac{nn}{\pi\pi}.$$

Therefore, because n denotes, how far away was from the first number α the number φ was computed, this number φ will behave to the following ψ as π^2 to n^2 which ratio, if n was an infinite number, will be completely in agreement with the truth. Since it is almost $\pi\pi=10$, if one puts $n=100$, the hundredths term will be thousand times smaller than its following term. Therefore, the numbers α, β, γ, δ etc. in the same way as the Bernoulli numbers $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ etc. constitute a highly divergent series which increases even more than a geometric series proceeding in increasing terms.

§130 Therefore, having found the values of the numbers α, β, γ, δ etc. or $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ etc., if a series is propounded whose general term z was any function of the index x, the summatory term Sz of this series will be expressed the following way that it is

$$Sz=\int zdx+\frac{1}{2}z+\frac{1}{6}\cdot\frac{dz}{1\cdot 2}-\frac{1}{30}\cdot\frac{d^3z}{1\cdot 2\cdot 3\cdot 4dx^3}$$
$$+\frac{1}{42}\cdot\frac{d^5}{1\cdot 2\cdot 3\cdots 6dx^5}-\frac{1}{30}\cdot\frac{d^7z}{1\cdot 2\cdot 3\cdots 8dx^7}$$

$$+\frac{5}{66}\cdot\frac{d^9z}{1\cdot 2\cdot 3\cdot 10dx^9}-\frac{691}{2730}\cdot\frac{d^{11}z}{1\cdot 2\cdot 3\cdots 12dx^{11}}$$
$$+\frac{7}{6}\cdot\frac{d^{13}z}{1\cdot 2\cdot 3\cdots 14dx^{13}}-\frac{3617}{510}\cdot\frac{d^{15}z}{1\cdot 2\cdot 3\cdots 16dx^{16}}$$
$$\frac{854513}{138}\cdot\frac{d^{21}z}{1\cdot 2\cdot 3\cdots 22dx^{21}}-\frac{236364091}{2730}\cdot\frac{d^{23}z}{1\cdot 2\cdot 3\cdots 24dx^{23}}$$
$$+\frac{8553103}{6}\cdot\frac{d^{25}z}{1\cdot 2\cdot 3\cdots 26dx^{25}}-\frac{23749461029}{870}\cdot\frac{d^{27}z}{1\cdot 2\cdot 3\cdots 28dx^{27}}$$
$$+\frac{8615841276005}{14322}\cdot\frac{d^{29}z}{1\cdot 2\cdot 3\cdots 30dx^{29}}-\text{etc.}$$

Therefore, if the integral $\int zdx$ or the quantity whose differential is $= zdx$ is known, the summatory term will be found by means of continued differentiation. But it is always to be noted that to this expression always a constant of such a kind is to be added that the sum becomes $= 0$, if the index x is put to go over into nothing.

§128 Therefore, if z was a polynomial function of x, since its differentials finally vanish, the summatory terms will be expressed by a finite expression; we will illustrate this in the following examples.

EXAMPLE 1

The summatory term of this series shall be sought after

$$\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & \qquad x \\ 1+ & 9+ & 25+ & 49+ & 81+\cdots+ & (2x-1)^2. \end{array}$$

Since here it is $z = (2x-1)^2 = 4xx - 4x + 1$, it will be

$$\int zdx = \frac{4}{3}x^3 - 2x^2 + x;$$

for, from the differentiation of this series $4xxdx - 4xdx + dx = zdx$ arises. Furthermore, by differentiation it will be

$$\frac{dz}{dx} = 8x-4, \quad \frac{ddz}{dx^2} = 8, \quad \frac{d^3z}{dx^3} = 0 \quad \text{etc.}$$

Hence, the summatory term sought after will be

$$\frac{4}{3}x^3 - 2x^2 + x + 2xx - 2x + \frac{1}{2} + \frac{2}{3}x \pm \text{Const.},$$

by which constant the terms $\frac{1}{2} - \frac{1}{3}$ have to be cancelled; hence, it will be

$$S(2x-1)^2 = \frac{4}{3}x^3 - \frac{1}{3}x = \frac{x}{3}(2x-1)(2x+1).$$

So, having put $x = 4$ the sum of the first four terms will be

$$1 + 9 + 25 + 49 = \frac{4}{3} \cdot 7 \cdot 9 = 84.$$

EXAMPLE 2

The summatory term of this series shall be sought after

$$\begin{array}{l} 1 \quad 2 \quad\; 3 \quad\;\; 4 \qquad\qquad\qquad\quad x \\ 1 + 27 + 125 + 343 + \cdots + (2x-1)^3. \end{array}$$

Since it is $z = (2x-1)^3 = 8x^3 - 12x^2 + 6x - 1$, it will be

$$\frac{dz}{dx} = 24x^2 - 24x + 6, \quad \frac{ddz}{dx^2} = 48x - 24, \quad \frac{d^3z}{dx^3} = 48;$$

the following vanish. Hence, it will be

$$\begin{aligned} S(2x-1)^3 = 2x^4 - 4x^3 + 3x^3 - 1x& \\ + 4x^3 - 6x^2 + 3x - \frac{1}{2}& \\ + 2x^2 - 2x + \frac{1}{2}& \\ - \frac{1}{15}& \pm \text{Const.}, \end{aligned}$$

this means

$$S(2x-1)^3 = 2x^4 - x^2 = x^2(2xx-1).$$

So, having put $x = 4$ it will be

$$1 + 27 + 125 + 343 = 16 \cdot 31 = 496.$$

§132 From this general expression found for the summatory term immediately that summatory term follows, which we gave in the superior part [§ 29 and 61] for the powers of the natural numbers and whose demonstration could not be given at that point. For, if we put $z = x^n$, it will certainly be $\int zdx = \frac{1}{n+1}x^{n+1}$; the differentials on the other hand will behave this way

$$\frac{dz}{dx} = nx^{n-1}, \quad \frac{ddz}{dx^2} = n(n-1)x^{n-2}, \quad \frac{d^3z}{dx^3} = n(n-1)(n-2)x^{n-3},$$

$$\frac{d^5z}{dx^5} = n(n-1)(n-2)(n-3)(n-4)x^{n-5}, \quad \frac{d^7z}{dx^7} = n(n-1)\cdots(n-6)x^{n-7} \quad \text{etc.}$$

From these therefore the following summatory term corresponding to the general term x^n will be deduced, of course

$$Sx^n = \frac{1}{n+1}x^{n+1} + \frac{1}{2}x^n + \frac{1}{6}\cdot\frac{n}{2}x^{n-1} - \frac{1}{30}\cdot\frac{n(n-1)(n-2)}{2\cdot3\cdot4}x^{n-3}$$

$$\begin{aligned}
&+\frac{1}{42}\cdot\frac{n(n-1)(n-2)(n-3)(n-4)}{2\cdot3\cdot4\cdot5\cdot6}x^{n-5}\\
&-\frac{1}{30}\cdot\frac{n(n-1)\cdots\cdots\cdots(n-6)}{2\cdot3\cdots8}x^{n-7}\\
&+\frac{5}{66}\cdot\frac{n(n-1)\cdots\cdots\cdots(n-8)}{2\cdot3\cdots10}x^{n-9}\\
&-\frac{691}{2730}\cdot\frac{n(n-1)\cdots\cdots\cdots(n-10)}{2\cdot3\cdots12}x^{n-11}\\
&+\frac{7}{6}\cdot\frac{n(n-1)\cdots\cdots\cdots(n-12)}{2\cdot3\cdots14}x^{n-13}\\
&-\frac{3617}{510}\cdot\frac{n(n-1)\cdots\cdots\cdots(n-14)}{2\cdot3\cdots16}x^{n-13}\\
&+\frac{43867}{798}\cdot\frac{n(n-1)\cdots\cdots\cdots(n-16)}{2\cdot3\cdots18}x^{n-17}\\
&-\frac{174611}{330}\cdot\frac{n(n-1)\cdots\cdots\cdots(n-18)}{2\cdot3\cdots20}x^{n-19}\\
&+\frac{854513}{138}\cdot\frac{n(n-1)\cdots\cdots\cdots(n-20)}{2\cdot3\cdots22}x^{n-21}
\end{aligned}$$

$$-\frac{236364091}{2730}\cdot\frac{n(n-1)\cdots\cdots(n-22)}{2\cdot3\cdots24}x^{n-23}$$
$$+\frac{8553103}{6}\cdot\frac{n(n-1)\cdots\cdots(n-24)}{2\cdot3\cdots26}x^{n-25}$$
$$-\frac{23749461029}{870}\cdot\frac{n(n-1)\cdots\cdots(n-26)}{2\cdot3\cdots28}x^{n-27}$$
$$+\frac{8615841276005}{14322}\cdot\frac{n(n-1)\cdots\cdots(n-28)}{2\cdot3\cdots30}x^{n-29}$$
$$\text{etc.};$$

this expression does not differ from that one we gave above except that we here introduced the Bernoulli numbers $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ etc., whereas above we used the numbers α, β, γ, δ etc.; nevertheless, the agreement is immediately clear. Hence, it is possible to exhibit the summatory term of all terms up to the thirtieth powers inclusively; this investigation, if it was undertaken another way, could have only have done with very long and most tedious calculations.

§133 Above (§ 59) we already gave an almost equal expression for defining the summatory term from the general term. For, it proceeded according to the differences of the general term; from that one it was mainly different in that aspect that the latter did not require the integral $\int zdx$, but the single differences of the general term were multiplied by certain functions of x. Therefore, let us find the same expression again in the following way more accommodated to the nature of series, from which at the same time the law will become more clear, according to which the coefficients of the differentials proceed. Therefore, let the general term of the series be z, a function of the index x; the summatory term sought after on the other hand shall be s; since this term, as we saw, will be a function of x of such a kind that it vanishes having put $x=0$, it will be by means of the things we demonstrated above [§ 68] about the nature of functions of this kind

$$s-\frac{xds}{1dx}+\frac{x^2dds}{1\cdot2dx^2}-\frac{x^3d^3s}{1\cdot2\cdot3dx^3}+\frac{x^4d^4s}{1\cdot2\cdot3\cdot4dx^4}-\text{etc.}=0.$$

§134 Since s denotes the sum of all terms of the series from the first to the last z, it is perspicuous, if in s instead of x one puts $x-1$, that then the first sum does not have the last term z; it will be

$$s - z = s - \frac{ds}{dx} + \frac{dds}{2dx^2} - \frac{d^3s}{6dx^3} + \frac{d^4s}{24dx^4} - \text{etc.}$$

and hence

$$z = \frac{ds}{dx} - \frac{dds}{2dx^2} + \frac{d^3s}{6dx^3} - \frac{d^4s}{24dx^4} + \text{etc.},$$

which equation provides a way to define the general term from the given summatory term which is per se very easy. But from an appropriate combination of this equation with that we found in the preceding paragraph one will be able to define the value of s by means of x and z. For this aim, let us put that it is

$$s - Az + \frac{Bdz}{dx} - \frac{Cddz}{dx^2} + \frac{Ddz^3}{dx^3} - \frac{Ed^4z}{dx^4} + \text{etc.} = 0,$$

where A, B, C, D etc. denote the necessary coefficients, either constant or variable; for, because it is

$$z = \frac{ds}{dx} - \frac{dds}{2dx^2} + \frac{d^3s}{6dx^3} - \frac{d^4s}{24dx^4} + \frac{d^5s}{120dx^5} - \text{etc.},$$

if hence the values for z, $\frac{dz}{dx}$, $\frac{ddz}{dx^2}$, $\frac{d^3z}{dx^3}$ etc. are substituted in the superior equation, it will arise

$$\begin{array}{rl}
+ \; s & = +s \\
- \; Az & = -\frac{Ads}{dx} + \frac{Adds}{2dx^2} - \frac{Ad^3s}{6dx^3} + \frac{Ad^4s}{24dx^4} - \frac{Ad^5s}{120dx^5} + \text{etc.} \\
+ \frac{Bdz}{dx} & = \qquad\quad + \frac{Bdds}{dx^2} - \frac{Bd^3s}{2dx^3} + \frac{Bd^4s}{6dx^4} - \frac{Bd^5s}{24dx^5} + \text{etc.} \\
- \frac{Cddz}{dx^2} & = \qquad\qquad\qquad - \frac{Cd^3s}{dx^3} + \frac{Cd^4s}{2dx^4} - \frac{Cd^5s}{6dx^5} + \text{etc.} \\
+ \frac{Dd^3z}{dx^3} & = \qquad\qquad\qquad\qquad\quad + \frac{Dd^4s}{dx^4} - \frac{Dd^5s}{2dx^5} + \text{etc.} \\
- \frac{Ed^4z}{dx^4} & = \qquad\qquad\qquad\qquad\qquad\qquad - \frac{Ed^5s}{dx^5} + \text{etc.} \\
& \text{etc.,}
\end{array}$$

which series collected together therefore will be equal to nothing.

§135 Therefore, since we found before that it is

$$0 = s - \frac{xds}{dx} + \frac{x^2dds}{2dx^2} - \frac{x^3d^3s}{6dx^3} + \frac{x^4d^4s}{24dx^4} - \frac{x^5d^5s}{120dx^5} + \text{etc.},$$

if the superior equation is put equal to this one, the following determinations of the letters A, B, C, D etc. will arise

$$A = x, \quad B = \frac{x^2}{2} - \frac{A}{2}, \quad C = \frac{x^3}{6} - \frac{B}{2} - \frac{A}{6},$$
$$D = \frac{x^4}{24} - \frac{C}{2} - \frac{B}{6} - \frac{A}{24}, \quad E = \frac{x}{120} - \frac{D}{2} - \frac{C}{6} - \frac{B}{24} - \frac{A}{120} \quad \text{etc..}$$

Therefore, having found the values of the letters A, B, C, D etc. from the general term z the summatory term $s = Sz$ will be determined in such a way that it is

$$Sz = Az - \frac{Bdz}{dx} + \frac{Cddz}{dx^2} - \frac{Dd^3z}{dx^3} + \frac{Ed^4z}{dx^4} - \frac{Fd^5z}{dx^5} + \text{etc.}$$

§136 But because it is

$$A = x, \quad B = \frac{1}{2}x^2 - \frac{1}{2}x, \quad C = \frac{1}{6}x^3 - \frac{1}{4}x^2 + \frac{1}{12}x,$$
$$D = \frac{1}{24}x^4 - \frac{1}{12}x^3 + \frac{1}{24}xx \quad \text{etc.},$$

it is clear that these coefficients are the same as those we had above (§ 59); hence that expression of the summatory term is the same as the one we found there and therefore it will be

$$A = Sx^0 = S1, \quad B = \frac{1}{1}Sx^1 - \frac{1}{1}x, \quad C = \frac{1}{2}Sx^2 - \frac{1}{2}x^2,$$
$$D = \frac{1}{6}Sx^3 - \frac{1}{6}x^3, \quad E = \frac{1}{24}Sx^4 - \frac{1}{24}x^4 \quad \text{etc.}$$

Hence, it will be

$$Sz = xz - \frac{dz}{dx}Sx + \frac{ddz}{2dx^2}Sx^2 - \frac{d^3z}{6dx^3}Sx^3 + \frac{d^4z}{24dx^4}Sx^4 - \text{etc.}$$
$$+ \frac{xdz}{dx} - \frac{x^2ddz}{2dx^2} + \frac{x^3d^3z}{6dx^3} - \frac{x^4d^4z}{24dx^4} + \text{etc.}$$

But if in the general term z is put $= 0$, the term corresponding to the index $= 0$ will arise; if it is put $= a$, it will be

$$a = z - \frac{xdz}{dx} + \frac{x^2ddz}{2dx^2} - \frac{x^3d^3z}{6dx^3} + \text{etc.}$$

and hence

$$\frac{xdz}{dx} - \frac{x^2ddz}{2dx^2} + \frac{x^3d^3z}{6dx^3} - \frac{x^4d^4z}{24dx^4} + \text{etc.} = z - a,$$

having substituted which value one will have

$$Sz = (x+1)z - a - \frac{dz}{dx}Sx + \frac{ddz}{2dx^2}Sx^2 - \frac{d^3z}{6dx^3} + \frac{d^4z}{24dx^4}Sx^4 - \text{etc.}$$

Therefore, having found the sums of the powers from a certain given general term one can then exhibit the summatory term corresponding to it.

§137 Therefore, since we found two expressions of the summatory term Sz for the general term z and one of the formulas contains the integral $\int zdx$, if these two expressions are put equal to each other, one will obtain the value of $\int zdx$ expressed by means of a series. For, because it is

$$\begin{aligned}\int zdx = &+\frac{1}{2}z + \frac{\mathfrak{A}dz}{1\cdot 2dx} - \frac{\mathfrak{B}d^3z}{1\cdot 2\cdot 3\cdot 4dx^3} + \frac{\mathfrak{C}d^5z}{1\cdot 2\cdots 6dx^5} - \text{etc.}\\ &= (x+1)z - a - \frac{dz}{1dx}Sx + \frac{ddz}{1\cdot 2dx^2}Sx^2 - \frac{d^3z}{1\cdot 2\cdot 3}Sx^3 + \text{etc.},\end{aligned}$$

it will be

$$\int zdx = \left(x + \frac{1}{2}\right)z - a - \frac{dz}{dx}\left(Sx + \frac{1}{2}\mathfrak{A}\right) + \frac{ddz}{2dx^2}Sx^2 - \frac{d^3z}{6dx^3}\left(Sx^3 - \frac{1}{4}\mathfrak{B}\right)$$

$$+\frac{d^4z}{24dx^4}Sx^4 - \frac{d^5z}{120dx^5}\left(Sx^5 + \frac{1}{6}\mathfrak{C}\right) + \frac{d^6z}{720dx^6}Sx^6 - \frac{d^7z}{5040dx^7}\left(Sx^7 - \frac{1}{8}\mathfrak{D}\right) + \text{etc.},$$

where $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ etc. denote the Bernoulli numbers exhibited above (§ 122).

For the sake of an example, let $z = xx$; it will be $a = 0$, $\frac{dz}{dx} = 2x$ and $\frac{ddz}{2dx^2} = 1$; it will hence be

$$\int xxdx = \left(x+\frac{1}{2}\right)xx - 2x\left(\frac{1}{2}xx+\frac{1}{2}x+\frac{1}{12}\right)+1\left(\frac{1}{3}x^3+\frac{1}{2}x^2+\frac{1}{6}x\right)$$

or $\int xxdx = \frac{1}{3}x^3$; but $\frac{1}{3}x^3$ differentiated gives $xxdx$, of course.

§138 Hence, there is a new way to find the summatory terms of the series of powers; for, since from the coefficients A, B, C, D etc. assumed before these summatory terms are most easily formed, but any of these coefficients is conflated of the preceding ones, if in the formulas given in § 135 instead of these letters the values given in § 136 are substituted, it will be

$$\begin{aligned}
Sx^1 - x &= \frac{1}{2}xx - \frac{1}{2}x \\
Sx^2 - x^2 &= \frac{1}{3}x^3 - \frac{1}{3} \quad - \frac{2}{2}(Sx-x) \\
Sx^3 - x^3 &= \frac{1}{4}x^4 - \frac{1}{4}x - \frac{3}{2}(Sx^2-x^2) - \frac{3\cdot 2}{2\cdot 3}(Sx-x) \\
Sx^4 - x^4 &= \frac{1}{5}x^5 - \frac{1}{5}x - \frac{4}{2}(Sx^3-x^3) - \frac{4\cdot 3}{2\cdot 3}(Sx^2-x^2) - \frac{4\cdot 3\cdot 2}{2\cdot 3\cdot 4}(Sx-x) \\
&\text{etc.}
\end{aligned}$$

Hence, one will be able to form the sums of the superior powers from the sums of the inferior ones.

§139 But if we consider the law, which the coefficients A, B, C, D etc. above (§ 135) were found to follow, with more attention, we will detect that they constitute a recurring series. For, if we expand this fraction

$$y = \frac{x+\frac{1}{2}xxu+\frac{1}{6}x^3u^2+\frac{1}{24}x^4u^3+\frac{1}{120}x^5u^4+\text{etc.}}{1+\frac{1}{2}u \quad +\frac{1}{6}u^2 \quad +\frac{1}{24}u^3 \quad +\frac{1}{120}u^4 \quad +\text{etc.}}$$

into a power series in u and assume this series to result

$$A + Bu + Cu^2 + Du^3 + Eu^4 + \text{etc.},$$

it will be, as we found before,

$$A = x, \quad B = \frac{1}{2}xx - \frac{1}{2}A \quad \text{etc.}$$

and so having found this series one will obtain the summatory terms of the series of powers. But that fraction from whose expansion this series arises will go over into this form $\frac{e^{xu}-1}{e^{u}-1}$ which, if x was a positive integer, goes over into

$$1+e^{u}+e^{2u}+e^{3u}+\cdots+e^{(x-1)u};$$

because it therefore is

$$\begin{aligned}
1 &= 1 \\
e^{u} &= 1+\frac{u}{1}+\frac{u^2}{1\cdot 2}+\frac{u^3}{1\cdot 2\cdot 3}+\frac{u^4}{1\cdot 2\cdot 3\cdot 4}+\text{etc.} \\
e^{2u} &= 1+\frac{2u}{1}+\frac{4u^2}{1\cdot 2}+\frac{8u^3}{1\cdot 2\cdot 3}+\frac{16u^4}{1\cdot 2\cdot 3\cdot 4}+\text{etc.} \\
e^{3u} &= 1+\frac{3u}{1}+\frac{9u^2}{1\cdot 2}+\frac{27u^3}{1\cdot 2\cdot 3}+\frac{81u^4}{1\cdot 2\cdot 3\cdot 4}+\text{etc.} \\
e^{(x-1)u} &= 1+\frac{(x-1)u}{1}+\frac{(x-1)^2u^2}{1\cdot 2}+\frac{(x-1)^3u^3}{1\cdot 2\cdot 3}+\frac{(x-1)^4u^4}{1\cdot 2\cdot 3\cdot 4}+\text{etc.}
\end{aligned}$$

and hence it will be

$$\begin{aligned}
A &= x \\
B &= S(x-1) = Sx - x \\
C &= \frac{1}{2}S(x-1)^2 = \frac{1}{2}Sx^2 - \frac{1}{2}x^2 \\
D &= \frac{1}{6}S(x-1)^3 = \frac{1}{6}Sx^3 - \frac{1}{6}x^3 \\
& \text{etc.}
\end{aligned}$$

Hence, the connection already mentioned before of these coefficients to the sums of the powers is completely confirmed and demonstrated.