On the Conversion of Functions into Series *

Leonhard Euler

§70 In the superior chapter already partially the use was demonstrated, which the general expressions found there for the finite differences have for the investigation of series, which exhibit the value of a certain function of x. For, if y was a given function of x, its value, which it takes for $x = 0$, will be known; and if this value is put $= A$, it will be, as we found,

$$y - \frac{xdy}{dx} + \frac{x^2ddy}{1 \cdot 2dx^2} - \frac{x^3d^3y}{1 \cdot 2 \cdot 3} + \frac{x^4d^4y}{1 \cdot 2 \cdot 3 \cdot 4dx^4} - \text{etc.} = A.$$

Therefore, hence we not only have a, in most cases infinite, series, whose sum is equal to the constant quantity A, even though the variable quantity x is contained in the single terms, but we will also be able to express the function y by means of a series; for, it will be

$$y = A + \frac{xdy}{dx} - \frac{xxddy}{1 \cdot 2dx^2} + \frac{x^3d^3y}{1 \cdot 2 \cdot 3dx^3} - \frac{x^4dy^4}{1 \cdot 2 \cdot 3 \cdot 4dx^4} + \text{etc.},$$

several examples of which were already mentioned.

§71 But that this investigation extends further, let us put that the function y goes over into z, if instead of x one writes $x + \omega$ everywhere, such that z is such a function of $x + \omega$ as y is one of x, and we showed [§ 48] that it will be

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$$z = y + \frac{\omega dy}{dx} + \frac{\omega^2 ddy}{1 \cdot 2dx^2} + \frac{\omega^3 d^3y}{1 \cdot 2 \cdot 3dx^3} + \frac{\omega^4 d^4y}{1 \cdot 2 \cdot 3 \cdot 4dx^4} + \text{etc.}$$

Therefore, since the single terms of this series can be found by continued differentiation of y by putting dx constant and at the same time the value of z can actually be exhibited by means of the substitution of $x + \omega$ for x, this way one will always obtain a series equal to the value of z, which, if ω was a very small quantity, converges rapidly and by taking many terms will yield an approximately true value of z. Hence the use of this formula in the task of approximation will be very high.

§72 Therefore, that in the demonstration of the immense use of this formula we proceed in order, let us at first substitute algebraic functions of x for y. And at first let $y = x^n$ and it will be, if $x + \omega$ is put instead of x, $z = (x + \omega)^n$. Therefore, because it is

$$\frac{dy}{dx} = nx^{n-1}, \quad \frac{ddy}{dx^2} = n(n-1)x^{n-2}, \quad \frac{d^3y}{dx^3} = n(n-1)(n-2)x^{n-3},$$

$$\frac{d^4y}{dx^4} = n(n-1)(n-2)(n-3)x^{n-4} \quad \text{etc.},$$

having substituted these values it will be

$$(x+\omega)^n = x^n + \frac{n}{1}x^{n-1}\omega + \frac{n(n-1)}{1 \cdot 2}x^{n-2}\omega^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}x^{n-3}\omega^3 + \text{etc.},$$

which is the very well known Newtonian expression, by which the power of the binomial $(x + \omega)^n$ is converted into a series. And the number of terms of this series is always finite. if n was a positive integer.

§73 Hence we will also be able to find the progression, which expresses the value of the power of the binomial in such a way that it terminates, if the the exponent of the power was a negative number. For, let us set

$$\omega = \frac{-ux}{x+u};$$

it will be

$$z = (x+\omega)^n = \left(\frac{xx}{x+u}\right)^n$$

and hence one will have

$$\frac{x^{2n}}{(x+u)^n} = x^n - \frac{nx^n u}{1(x+u)} + \frac{n(n-1)x^n u^2}{1\cdot 2(x+u)^2} - \frac{n(n-1)(n-2)x^n u^3}{1\cdot 2\cdot 3(x+u)^3} + \text{etc.}$$

Divide by x^{2n} everywhere and it will be

$$(x+u)^{-n} = x^{-n} - \frac{nx^{-n}u}{1(x+u)} + \frac{n(n-1)x^{-n}u^2}{1\cdot 2(x+u)^2} - \frac{n(n-1)(n-2)x^{-n}u^3}{1\cdot 2\cdot 3(x+u)^3} + \text{etc.}$$

Now put $-n = m$ and it will arise

$$(x+u)^m = x^m + \frac{mx^m u}{1(x+u)} + \frac{m(m+1)x^m u^2}{1\cdot 2(x+u)^2} + \frac{m(m+1)(m+2)x^m u^2}{1\cdot 2\cdot 3(x+u)^3} + \text{etc.},$$

which series, if m is a negative integer, will consist of a finite number of terms. Therefore, this series is equal to the one found first, if for ω and n one writes u and m; for, hence it will be

$$(x+u)^m = x^m + \frac{mx^{m-1}u}{1} + \frac{m(m-1)x^{m-2}u^2}{1\cdot 2} + \frac{m(m-1)(m-2)x^{m-2}u^2}{1\cdot 2\cdot 3} + \text{etc.}$$

§74 This same series can also be deduced from the expression given at the beginning of § 70. For, because, if for $x=0$ y goes over into A, it is

$$y - \frac{xdy}{dx} + \frac{xxddy}{1\cdot 2dx^2} - \frac{x^3d^3y}{1\cdot 2\cdot 3dx^3} + \frac{x^4d^4y}{1\cdot 2\cdot 3\cdot 4dx^4} - \text{etc.} = A,$$

put $y = (x+a)^n$ and it will be $A = a^n$ and because of

$$\frac{dy}{dx} = n(x+a)^{n-1}, \quad \frac{ddy}{dx^2} = n(n-1)(x+a)^{n-2},$$

$$\frac{d^3y}{dx^3} = n(n-1)(n-2)(x+a)^{n-3} \quad \text{etc.}$$

it will be

$$(x+a)^n - \frac{n}{1}x(x+a)^{n-1} + \frac{n(n-1)}{1\cdot 2}x^2(x+a)^{n-2} - \text{etc.} = a^n;$$

divide by $a^n(x+a)^n$ and it will arise

$$(x+a)^{-n} = a^{-n} - \frac{na^{-n}x}{1(x+a)} + \frac{n(n-1)a^{-n}x^2}{1\cdot 2(x+a)^2} - \text{etc.},$$

which having respectively put u, x and $-m$ for x, a and n the series found before will arise.

§75 If one puts fractional numbers for m, both series will continue forever, nevertheless, if u with respect to x was a very small quantity, it will converge rapidly to the true value. Therefore, let $m = \frac{\mu}{\nu}$ and $x = a^\nu$; it will be from the series found first

$$(a^\nu + u)^{\frac{\mu}{\nu}} = a^\mu \left(1 + \frac{\mu}{\nu}\cdot\frac{u}{a^\nu} + \frac{\mu(\mu-\nu)}{\nu\cdot 2\nu}\cdot\frac{uu}{a^{2\nu}} + \frac{\mu(\mu-\nu)(\mu-2\nu)}{\nu\cdot 2\nu\cdot 3\nu}\cdot\frac{u^3}{a^{3\nu}} + \text{etc.}\right).$$

But the series found later will give

$$(a^\nu + u)^{\frac{\mu}{\nu}} = a^\mu \left(1 + \frac{\mu u}{\nu(a^\nu+u)} + \frac{\mu(\mu+\nu)u^2}{\nu\cdot 2\nu(a^\nu+u)^2} + \frac{\mu(\mu+\nu)(\mu+2\nu)u^3}{\nu\cdot 2\nu\cdot 3\nu(a^\nu+u)^3} + \text{etc.}\right).$$

But this last series converges stronger than the first, since its terms even decrease, if it was $u > a^\nu$, in which case the first series nevertheless diverges.

Therefore, let $\mu = 1$, $\nu = 2$, it will be

$$\sqrt{a^2+u} = a\left(1 + \frac{1u}{2(a^2+u)} + \frac{1\cdot 3u^2}{2\cdot 4(a^2+u)^2} + \frac{1\cdot 3\cdot 5u^3}{2\cdot 4\cdot 6(a^2+u)^3 + \text{etc.}}\right).$$

In similar manner by putting the numbers 3, 4, 5 etc. for ν, while $\mu = 1$ remains, it will be

$$\sqrt[3]{a^3+u} = a\left(1 + \frac{1u}{3(a^3+u)} + \frac{1\cdot 4u^2}{3\cdot 6(a^3+u)^2} + \frac{1\cdot 4\cdot 7u^3}{3\cdot 6\cdot 9(a^3+u)^3} + \text{etc.}\right)$$

$$\sqrt[4]{a^4+u} = a\left(1 + \frac{1u}{4(a^4+u)} + \frac{1\cdot 5u^2}{4\cdot 8(a^4+u)^2} + \frac{1\cdot 5\cdot 9u^3}{4\cdot 8\cdot 12(a^4+u)^3} + \text{etc.}\right)$$

$$\sqrt[5]{a^5+u} = a\left(1 + \frac{1u}{5(a^5+u)} + \frac{1\cdot 6u^2}{5\cdot 10(a^5+u)^2} + \frac{1\cdot 6\cdot 11u^3}{5\cdot 10\cdot 15(a^5+u)^3} + \text{etc.}\right)$$

etc.

§76 From these formulas one can therefore easily find the root of a certain power of any number. For, having propounded the number c, find the power closest to it, either larger or smaller; in the first case u will become a negative number, in the second positive. But if the resulting series does not seem to converge sufficiently fast, multiply the number c by any power, say by f^ν, if the root of the power ν has to be extracted, and find the root of the number $f^\nu c$, which divided by f will give the root in question of the number c. The greater the number f is assumed, the more the series will converge and that especially, if a similar power a^μ does not deviate much from $f^\nu c$.

Example 1

Let the square root of the number 2 *be in question.*

If without further preparation one puts $a = 1$ and $u = 1$, it will be

$$\sqrt{2} = 1 + \frac{1}{2\cdot 2} + \frac{1\cdot 3}{2\cdot 4\cdot 2^2} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 2^3} + \text{etc.};$$

even though this one already converges rapidly, it is nevertheless preferable to multiply the number 2 by a square, as 25, before, that the product 50 deviates from another square 49 as less as possible. Therefore, find the square root of 50, which divided by 5 will give $\sqrt{2}$. But it will then be $a = 7$ and $u = 1$, whence it will be

$$\sqrt{50} = 5\sqrt{2} = 7\left(1 + \frac{1}{2\cdot 50} + \frac{1\cdot 3}{2\cdot 4\cdot 50^2} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 50^3} + \text{etc.}\right)$$

or

$$\sqrt{2} = \frac{7}{5}\left(1 + \frac{1}{100} + \frac{1\cdot 3}{100\cdot 200} + \frac{1\cdot 3\cdot 5}{100\cdot 200\cdot 300} + \text{etc.}\right),$$

which is most apt for the calculation in decimal numbers. For, it will be

$$\begin{aligned}
\frac{7}{5} &= 1.4000000000000 \\
\frac{7}{5}\cdot\frac{1}{100} &= 140000000000 \\
\frac{7}{5}\cdot\frac{1}{100}\cdot\frac{3}{200} &= 2100000000 \\
\frac{7}{5}\cdot\frac{1}{100}\cdot\frac{3}{200}\cdot\frac{5}{300} &= 35000000 \\
\text{precisely by } \frac{7}{400} &= 612500 \\
\text{precisely by } \frac{9}{500} &= 11025 \\
\text{precisely by } \frac{11}{600} &= 202 \\
\text{precisely by } \frac{13}{400} &= 3 \\
\text{Therefore } \sqrt{2} &= 1.4142135623730.
\end{aligned}$$

EXAMPLE 2

Let the cube root of 3 *be in question.*

Multiply 3 by the cube 8 and find the cubic root of 24; for, it will be $\sqrt[3]{24} = 2\sqrt[3]{3}$. Therefore, put $a = 3$ and $u = -3$ and it will be

$$\sqrt[3]{24} = 3\left(1 - \frac{1\cdot 3}{3\cdot 24} + \frac{1\cdot 4\cdot 3^2}{3\cdot 6\cdot 24^2} - \frac{1\cdot 4\cdot 7\cdot 3^2}{3\cdot 6\cdot 9\cdot 24^3} + \text{etc.}\right)$$

and

$$\sqrt[3]{3} = \frac{3}{2}\left(1 - \frac{1}{3\cdot 8} + \frac{1\cdot 4}{3\cdot 6\cdot 8^2} - \frac{1\cdot 4\cdot 7}{3\cdot 6\cdot 9\cdot 8^3} + \text{etc.}\right)$$

or

$$\sqrt[3]{3} = \frac{3}{2}\left(1 - \frac{1}{24} + \frac{1}{24}\cdot\frac{4}{48} - \frac{1}{24}\cdot\frac{4}{48}\cdot\frac{7}{72} + \text{etc.}\right),$$

which series already converges rapidly, since every term is more than eighth times smaller than the preceding one. But if 3 is multiplied by the cube 729, it will be 2187 and $\sqrt[3]{2187} = \sqrt{13^3 - 10} = 9\sqrt[3]{3}$. Therefore, because of $a = 13$ and $u = -10$ it will be

$$\sqrt[3]{3} = \frac{13}{9}\left(1 - \frac{1\cdot 10}{3\cdot 2187} + \frac{1\cdot 4\cdot 10^2}{3\cdot 6\cdot 2187^2} - \frac{1\cdot 4\cdot 7\cdot 10^3}{3\cdot 6\cdot 9\cdot 2187^3} + \text{etc.}\right),$$

of which every term is more than two hundred times smaller than the preceding one.

§77 The expansion of the power of the binomial extends so far that all algebraic functions can be comprehended in it. For, if for the sake of an example the value of this function $\sqrt{a+2bx+cxx}$ expressed by means of a series is in question, this can be done by means of the preceding formulas by considering two terms as one. Further, this expansion can be done by means of the expression given first; for, if one puts $\sqrt{a+2bx+cxx} = y$, since having put $x = 0$ it is $y = a$, it will be $A = \sqrt{a}$, and since the differentials of y behave as this

$$\frac{dy}{dx} = \frac{b+cx}{\sqrt{a+2bx+cxx}}, \quad \frac{ddy}{dx^2} = \frac{ac-bb}{(a+2bx+cxx)^{\frac{3}{2}}}, \quad \frac{d^3y}{dx^3} = \frac{3(bb-ac)(b+cx)}{(a+2bx+cxx)^{\frac{5}{2}}},$$

$$\frac{d^4y}{dx^4} = \frac{3(bb-ac)(ac-5bb-8bcx-4ccxx)}{(a+2bx+cxx)^{\frac{7}{2}}} \quad \text{etc.},$$

from these one will therefore obtain

$$\sqrt{a+2bx+cxx} - \frac{(b+cx)x}{\sqrt{a+2bx+cxx}} - \frac{(bb-ac)xx}{2(a+2bx+cxx)^{\frac{3}{2}}} - \frac{(bb-ac)(b+cx)x^3}{2(a+2bx+cxx)^{\frac{5}{2}}}$$

$$-\frac{(bb-ac)(5bb-ac+8bcx+4ccxx)x^4}{8(a+2bx+cxx)^{\frac{7}{2}}} - \text{etc.} = \sqrt{a}$$

Therefore, if one multiplies by $\sqrt{a+2bx+cxx}$ everywhere, the series will be rational and it will be

$$\sqrt{a(a+2bx+cxx)} = a+2bx+cxx-(b+cx)x-\frac{(bb-ac)xx}{2(a+2bx+cxx)}$$

$$-\frac{(bb-ac)(b+cx)x^3}{2(a+2bx+cxx)^2}-\frac{(bb-ac)(5bb-ac+8bcx+4ccxx)x^4}{8(a+2bx+cxx)^3}-\text{etc.}$$

or

$$\sqrt{a+2bx+cxx} = \sqrt{a}+\frac{bx}{\sqrt{a}}-\frac{(bb-ac)xx}{2(a+2bx+cxx)\sqrt{a}}-\frac{(bb-ac)(b+cx)x^3}{2(a+2bx+cxx)^2\sqrt{a}}-\text{etc.}$$

§78 Therefore, let us go over to transcendental functions, which we want to substitute for y. Therefore, at first let $y = \log x$ and having put $x+\omega$ instead of x it will be $z = \log(x+\omega)$. But let these logarithms be such ones which to the hyperbolic ones have a ratio of $n:1$, and for the hyperbolic logarithms it will be $n=1$ and for the tabulated logarithms it will be $n = 0.4343944819032$. Hence the differentials of $y = \log x$ will be

$$\frac{dy}{dx} = \frac{n}{x}, \quad \frac{ddy}{dx^2} = -\frac{n}{x^2}, \frac{d^3y}{dx^3} = \frac{2n}{x^3} \quad \text{etc.},$$

from which one concludes

$$\log(x+\omega) = \log x + \frac{n\omega}{x} - \frac{n\omega^2}{2x^2} + \frac{n\omega^3}{3x^3} - \frac{n\omega^4}{4x^4} + \text{etc.}$$

In similar manner, if ω is set to be negative, it will be

$$\log(x-\omega) = \log x - \frac{n\omega}{x} - \frac{n\omega^2}{2x^2} - \frac{n\omega^3}{3x^3} - \frac{n\omega^4}{4x^4} - \text{etc.}$$

Therefore, if this series is subtracted from the first, it will be

$$\log\frac{x+\omega}{x-\omega} = 2n\left(\frac{\omega}{x}+\frac{\omega^3}{3x^3}+\frac{\omega^5}{5x^5}+\frac{\omega^7}{7x^7}+\text{etc.}\right).$$

§79 If in the series found first

$$\log(x+\omega) = \log x + \frac{n\omega}{x} - \frac{n\omega^2}{2x^2} + \frac{n\omega^3}{3x^3} - \frac{n\omega^4}{4x^4} + \text{etc.}$$

it is put

$$\omega = \frac{xx}{u-x},$$

it will be $x+\omega = \frac{ux}{u-x}$ and

$$\log(x+\omega) = \log u + \log x - \log(u-x) = \log x + \frac{nx}{u-x} - \frac{nxx}{2(u-x)^2} + \text{etc.}$$

and

$$\log(u-x) = \log u - \frac{nx}{u-x} + \frac{nxx}{2(u-x)^2} - \frac{nx^3}{3(u-x)^3} + \text{etc.}$$

and having taken a negative x one will have

$$\log(u+x) = \log u + \frac{nx}{u+x} + \frac{nxx}{2(u+x)^2} + \frac{nx^3}{3(u+x)^3} + \frac{nx^4}{4(u+x)^4} + \text{etc.}$$

Therefore, by means of these series the logarithms can be found in a convenient manner, if these series converge rapidly. But of this kind will be the following, which are easily deduced from the found ones,

$$\log(x+1) = \log x + n\left(\frac{1}{x} - \frac{1}{2xx} + \frac{1}{3x^3} - \frac{1}{4x^4} + \text{etc.}\right)$$
$$\log(x-1) = \log x - n\left(\frac{1}{x} + \frac{1}{2xx} + \frac{1}{3x^3} + \frac{1}{4x^4} + \text{etc.}\right);$$

because these two series differ only in the signs, if they are used for a calculation, from the known logarithm of the number x at the same time the logarithms of the two numbers $x-1$ and $x+1$ will be found. Furthermore, from the remaining series it will be

$$\log(x+1) = \log(x-1 = +2n\left(\frac{1}{x} + \frac{1}{3x^3} + \frac{1}{5x^5} + \frac{1}{7x^7} + \text{etc.}\right)$$
$$\log(x-1) = \log x - n\left(\frac{1}{x-1} - \frac{1}{2(x-1)^2} + \frac{1}{3(x-1)^3} - \frac{1}{4(x-1)^4} + \text{etc.}\right)$$
$$\log(x+1) = \log x + n\left(\frac{1}{x+1} + \frac{1}{2(x+1)^2} + \frac{1}{3(x+1)^3} + \frac{1}{4(x+1)^4} + \text{etc.}\right).$$

§80 Therefore, from a given logarithm of the number x the logarithms of the contiguous numbers $x+1$ and $x-1$ can easily be found; from the logarithm of the number $x-1$ even the logarithm of the number greater by two units and vice versa will be found. Although this was shown in much detail in the *Introductio*, we will nevertheless add certain examples here.

Example 1

From the given hyperbolic logarithm of the number 10, *which is* 2.3025850919940, *to find the hyperbolic logarithms of the numbers* 11 *and* 9.

Since this question concerns hyperbolic logarithms, it will be $n=1$ and hence one will have these series

$$\begin{aligned} \log 11 &= \log 10 + \frac{1}{10} - \frac{1}{2\cdot 10^2} + \frac{1}{3\cdot 10^3} - \frac{1}{4\cdot 10^4} + \frac{1}{5\cdot 10^5} - \text{etc.} \\ \log 9 &= \log 10 + \frac{1}{10} + \frac{1}{2\cdot 10^2} + \frac{1}{3\cdot 10^3} + \frac{1}{4\cdot 10^4} + \frac{1}{5\cdot 10^5} - \text{etc.} \end{aligned}$$

To find the sums of these series collect the even and odd terms separately and it will be

$\frac{1}{10}$	=	0.1000000000000	$\frac{1}{2\cdot 10^2}$	=	0.0050000000000
$\frac{1}{3\cdot 10^3}$	=	0.0003333333333	$\frac{1}{4\cdot 10^4}$	=	0.0000250000000
$\frac{1}{5\cdot 10^5}$	=	0.0000020000000	$\frac{1}{6\cdot 10^6}$	=	0.0000001666666
$\frac{1}{7\cdot 10^7}$	=	0.0000000142857	$\frac{1}{8\cdot 10^8}$	=	0.0000000012500
$\frac{1}{3\cdot 10^9}$	=	0.0000000001111	$\frac{1}{10\cdot 10^{10}}$	=	0.0000000000100
$\frac{1}{11\cdot 10^{11}}$	=	0.0000000000009	$\frac{1}{12\cdot 10^{12}}$	=	0.0000000000001
Sum	=	0.1003353477310	sum	=	0.05050251679267

The sum of both will be			0.1053605156577
The difference of both will be			0.0953101798043
Now it is	$\log 10$	$=$	2.3025850929940
Therefore, it will be	$\log 11$	$=$	2.397895272793
and	$\log 9$	$=$	2.1972245773363
Hence further it is	$\log 3$	$=$	1.0986122886681
and	$\log 99$	$=$	4.5951198501346

Example 2

From the hyperbolic logarithm of the number 99 *found now to find the logarithm of the number* 101.

For this apply the series found above

$$\log(x+1) = \log(x-1) + \frac{2}{x} + \frac{2}{3x^3} + \frac{2}{5x^5} + \frac{2}{7x^7} + \text{etc.},$$

in which it shall be $x = 100$, and it will be

$$\log 101 = \log 99 + \frac{2}{100} + \frac{2}{3 \cdot 100^3} + \frac{2}{5 \cdot 100^5} + \frac{2}{7 \cdot 100^7} + \text{etc.},$$

the sum of which series is calculated from this four terms to be $= 0.0200006667066$, which added to $\log 99$ will give $\log 101 = 4.6151205168412$.

Example 3

From the given tabulated logarithm of the number 10, *which is* $= 1$, *to find the logarithm of the numbers* 11 *and* 9.

Since here we look for the common tabulated logarithm, it will be

$$n = 0.434244819032;$$

therefore, having put $x = 10$ it will be

$$\log 11 = \log 10 + \frac{n}{10} - \frac{n}{2\cdot 10^2} + \frac{n}{3\cdot 10^3} - \frac{n}{4\cdot 10^4} + \text{etc.}$$

$$\log 9 \;\; = \log 10 - \frac{n}{10} - \frac{n}{2\cdot 10^2} - \frac{n}{3\cdot 10^3} - \frac{n}{4\cdot 10^4} - \text{etc.}$$

Therefore, collect the even and odd terms separately

$\frac{n}{10}$	=	0.0434294481903	$\frac{n}{2\cdot 10^2}$	=	0.0021714724095
$\frac{n}{3\cdot 10^3}$	=	0.0001447648273	$\frac{n}{4\cdot 10^4}$	=	0.0000108573620
$\frac{n}{5\cdot 10^5}$	=	0.0000008685889	$\frac{n}{6\cdot 10^6}$	=	0.0000000723824
$\frac{n}{7\cdot 10^7}$	=	0.0000000062042	$\frac{n}{8\cdot 10^8}$	=	0.0000000005428
$\frac{n}{3\cdot 10^9}$	=	0.0000000000482	$\frac{n}{10\cdot 10^{10}}$	=	0.0000000000043
$\frac{n}{11\cdot 10^{11}}$	=	0.0000000000004	$\frac{n}{12\cdot 10^{12}}$	=	0.0000000000000
Sum	=	0.0435750878593	sum	=	0.0021824027010

The aggregate of both is		=	0.0457574905603
Their difference is		=	0.0413926851583
Therefore, because it is	$\log 10$	=	1.0000000000000
it will be	$\log 11$	=	1.0413926851582
and	$\log 9$	=	0.9542425094397
hence	$\log 3$	=	0.4771212547198
and	$\log 99$	=	1.9956351945980

Example 4

From the tabulated logarithm of the number 99 *found here to find the tabulated logarithm of the number* 101.

Here, by applying the same series we used in the second example, we will have

$$\log 101 = \log 99 + 2n\left(\frac{1}{100} + \frac{1}{3\cdot 100^3} + \frac{1}{5\cdot 100^5} + \text{etc.}\right),$$

the sum of which series having put the corresponding value for n will soon be found

		$=$	0.0086861791849
having added which to	$\log 99$	$=$	1.9956351945980
it arises	$\log 101$	$=$	2.0043213737829

§81 Now, in our general expression let us to y attribute the an exponential value and let $y = a^x$; having put $x + \omega$ instead x it will be $z = a^{x+\omega}$, whose value because of the differentials

$$\frac{dy}{dx} = a^x \log a, \quad \frac{ddy}{dx^2} = a^x (\log a)^2, \quad \frac{d^3y}{dx^3} = a^x (\log a)^3 \quad \text{etc.}$$

it will be

$$a^{x+\omega} = a^x\left(1 + \frac{\omega \log a}{1} + \frac{\omega^2 (\log a)^2}{1\cdot 2} + \frac{\omega^3 (\log a)^3}{1\cdot 2\cdot 3} + \text{etc.}\right);$$

if this is divided by a^x, the series expressing the values of an exponential quantity will be arise, which we already found above in the *Introductio*, namely

$$a^\omega = 1 + \frac{\omega \log a}{1} + \frac{\omega^2 (\log a)^2}{1\cdot 2} + \frac{\omega^3 (\log a)^3}{1\cdot 2\cdot 3} + \frac{\omega^4 (\log a)^4}{1\cdot 2\cdot 3\cdot 4} + \text{etc.}$$

In similar manner for negative ω it will be

$$a^{-\omega} = 1 - \frac{\omega \log a}{1} + \frac{\omega^2 (\log a)^2}{1\cdot 2} - \frac{\omega^3 (\log a)^3}{1\cdot 2\cdot 3} + \frac{\omega^4 (\log a)^4}{1\cdot 2\cdot 3\cdot 4} - \text{etc.},$$

from whose combination it arises

$$\frac{a^\omega + a^{-\omega}}{2} = 1 + \frac{\omega^2 (\log a)^2}{1\cdot 2} + \frac{\omega^4 (\log a)^4}{1\cdot 2\cdot 3\cdot 4} + \frac{\omega^6 (\log a)^6}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6} + \text{etc.}$$

$$\frac{a^\omega - a^{-\omega}}{2} = \frac{\omega \log a}{1} + \frac{\omega^3 (\log a)^3}{1\cdot 2\cdot 3} + \frac{\omega^5 (\log a)^5}{1\cdot 2\cdot 3\cdot 4\cdot 5} + \text{etc.},$$

where it is to be noted that $\log a$ denotes the hyperbolic logarithm of the number a.

§82 By means of this formula from a given logarithm one will be able to find the number corresponding to it. For, let any canonical logarithm u be propounded, in which the logarithm of the number a is set $= 1$. In the same base find the logarithm coming closest to u and let $u = x + \omega$, but the let the number corresponding to x be $y = a^x$; the number corresponding to the logarithm $u = x + \omega$ will be $= a^{x+\omega} = z$ and it will be

$$z = y\left(1 + \frac{\omega \log a}{1} + \frac{\omega^2(\log a)^2}{1 \cdot 2} + \frac{\omega^3(\log a)^3}{1 \cdot 2 \cdot 3} + \frac{\omega^4(\log a)^4}{1 \cdot 2 \cdot 3 \cdot 4} + \text{etc.}\right),$$

which series because of the very small number ω converges rapidly, whose use we will show in the following example.

Example

Let the number equal to this power of two $2^{2^{24}}$ *be in question.*

Since it is $2^{24} = 16777216$, it will be $2^{2^{24}} = 2^{16777216}$ and by taking common logarithms the logarithm of this number will be $= 16777216 \log 2$. But because it is

$$\log 2 = 0.30102999566398119521373889,$$

the logarithm of the number in question will be

$$5050445.259733675932039063,$$

whose characteristic indicates that the number in question is expressed in 5050446 figures; since they cannot all be exhibited, it will suffice to have assigned the initial figures, which must be investigated from the mantissa

$$,259733675932029063 = u.$$

But from tables one concludes that the number, whose logarithm comes closest to this, will be $= 1.818$, which shall be put y; it logarithm is

$$
\begin{array}{lrcl}
 & x & = & 0.259593878885948644 \\
\text{whence it will be} & \omega & = & 0.000139797046090419 \\
\text{Because now it is} & a & = & 10 \\
\text{it will be} & \log a & = & 2.302585092994045684017991 4 \\
\hline
\text{and} & \omega \log a & = & 0.000321894594372400 \\
\hline
\text{Further, it will be} & y & = & 1.818000000000000000 \\
 & \frac{\omega \log a}{1} y & = & 0.000585204372569023 \quad \text{and} \\
 & \frac{\omega^2 (\log a)^2}{1 \cdot 2} y & = & 0.000000094187062066 \\
 & \frac{\omega^3 (\log a)^3}{1 \cdot 2 \cdot 3} y & = & 0.000000000010106102 \\
 & \frac{\omega^4 (\log a)^4}{1 \cdot 2 \cdot 3 \cdot 4} y & = & 0.000000000000000813 \\
\hline
 & & & 1818585298569738004
\end{array}
$$

these are the initial figures of the number in question, of which all figures except for the maybe the last are correct.

§83 Let us consider quantities depending on the circle and let, as we always put, the radius of the circle be $= 1$ and let y denote the arc of the circle, whose sine is $= x$, or let $y = \arcsin x$. Put $x + \omega$ instead of x and it will be $z = \arcsin(x + \omega)$; to express this values find the differentials of y [§ 200 of the first part]

$$
\frac{dy}{dx} = \frac{1}{\sqrt{1 - xx}}, \quad \frac{ddy}{dx^2} = \frac{x}{(1 - xx)^{\frac{3}{2}}}, \quad \frac{d^3y}{dx^3} = \frac{1 + 2xx}{(1 - xx)^{\frac{5}{2}}}, \quad \frac{d^4y}{dx^4} = \frac{9x + 6x^3}{(1 - xx)^{\frac{7}{2}}},
$$

$$
\frac{d^5y}{dx^5} = \frac{9 + 72x^2 + 24x^4}{(1 - xx)^{\frac{9}{2}}}, \quad \frac{d^6y}{dx^6} = \frac{225x + 600x^3 + 120x^5}{(1 - xx)^{\frac{11}{2}}}
$$

etc.

Therefore, one finds from these

$$\arcsin(x+\omega) = \arcsin x + \frac{\omega}{\sqrt{1-xx}} + \frac{\omega^2 x}{2(1-xx)^{\frac{3}{2}}} + \frac{\omega^3(1+2xx)}{6(1-xx)^{\frac{5}{2}}}$$

$$+\frac{\omega^4(9x+6x^3)}{24(1-xx)^{\frac{7}{2}}} + \frac{\omega^5(9+72x^2+24x^4)}{120(1-xx)^{\frac{9}{2}}} + \text{etc.}$$

§84 Therefore, if the arc was known, whose sine is $= x$, by means of this formulas one will be able to find the arc, whose sine is $x+\omega$, if ω was a very small quantity. But the series, whose sum must be added, will be expressed in parts of the radius, which will easily be reduced to an arc, as it will be seen from this example.

EXAMPLE

Let the arc of the circle be in question, whose sine is $= \frac{1}{3} = 0.3333333333$.

From tables find the arc of the sines, whose sine is approximately smaller than $\frac{1}{3}$, which will be $19^\circ 28^{\text{I}}$, whose sine is $= 0.3332584$. Therefore, put $19^\circ 28^{\text{I}} = \arcsin x = y$; it will be $x = 0.3332584$ and $\omega = 0.000749$ and from tables $\sqrt{1-xx} = \cos y = 0.9428356$. Therefore, the arc in question z, whose sine $= \frac{1}{3}$ is propounded, will be

$$= 19^\circ 28^{\text{I}} + \frac{\omega}{\cos y} + \frac{\omega\omega \sin y}{2\cos^3 y},$$

which expression already suffices; therefore, by using logarithms in the calculation it will be

$$
\begin{aligned}
\log\omega &= 5.8744818\\
\log\cos y &= 9.9744359\\
\log\frac{\omega}{\cos y} &= 5.9000459 \qquad \frac{\omega}{\cos y} = 0.0000794412\\
\log\frac{\omega^2}{\cos^2 y} &= 1.8000918\\
\log\frac{\omega^3}{\cos^3 y} &= 9.5483452\\
&\ 1.3484370\\
\log 2 &= 0.3010300\\
\log\frac{\omega^2\sin y}{2\cos^2 y} &= 1.0474070 \qquad \frac{\omega^2\sin y}{2\cos^3 y} = 0.0000000011\\
&\qquad\qquad\qquad \text{sum} = 0.000794442,
\end{aligned}
$$

which is the value of the arc to be added to $19^\circ 28^{\mathrm{I}}$, to express which in minutes and seconds let us take its logarithm, which is

	5.9000518
from which subtract	4.6855749
	1.2144769
to which logarithm corresponds the number	= 16.38615,

which is the numbers of minutes and seconds; but by expressing this fraction in thirds and quarters the arc in question will be

$$= 19^\circ 28^{\mathrm{I}} 16^{\mathrm{II}} 23^{\mathrm{III}} 10^{\mathrm{IV}} 8^{\mathrm{V}} 14^{\mathrm{VI}}.$$

§85 In similar manner the expression for the cosine will be found; for, having put $y = \arccos x$, since it is $dy = \frac{-dx}{\sqrt{1-xx}}$, the series found remains unchanged, as long as its signs are permuted. Therefore, it will be

$$
\begin{aligned}
\arccos(x+\omega) = \arccos x - \frac{\omega}{\sqrt{1-xx}} - \frac{\omega^2 x}{2(1-xx)^{\frac{3}{2}}} - \frac{\omega^3(1+2xx)}{6(1-xx)^{\frac{5}{2}}}\\
- \frac{\omega^4(9x+6x^3)}{24(1-xx)^{\frac{7}{2}}} - \frac{\omega^5(9+72x^2+24x^4)}{120(1-xx)^{\frac{9}{2}}} - \text{etc.},
\end{aligned}
$$

which series as the preceding will always converge rapidly, if from tables of sines angles close to the true one are chosen, such that in most cases the first term $\frac{\omega}{\sqrt{1-xx}}$ alone suffices. Nevertheless, if x approximately equal to 1 or to a whole sine, then because of the very small denominators the series will only slowly converge. Therefore, in these cases, in which x does not deviate much from 1. since then the differences become very small, we will more conveniently use the usual interpolation alone.

§86 Therefore, let us also for y the arc, whose tangent is given, and let $y = \arctan x$ and $z = \arctan(x+\omega)$ such that it is

$$z = y + \frac{\omega dy}{dx} + \frac{\omega^2 ddy}{2dx^2} + \frac{\omega^3 d^3y}{6dx^3} + \text{etc.}$$

To investigate the terms find the single differentials of y

$$\frac{dy}{dx} = \frac{1}{1+xx}, \quad \frac{ddy}{dx^2} = \frac{-2x}{(1+xx)^2}, \quad \frac{d^3y}{dx^3} = \frac{-2+6xx}{(1+xx)^3}, \quad \frac{d^4y}{dx^4} = \frac{24x-24x^3}{(1+xx)^4},$$

$$\frac{d^5y}{dx^5} = \frac{24.240x^2+120x^4}{(1+xx)^5}, \quad \frac{d^5y}{dx^5} = \frac{-720x+2400x^3-720x^5}{(1+xx)^5}$$

$$\text{etc.,}$$

whence one concludes that it will be

$$\arctan(x+\omega) = \arctan x$$

$$+\frac{\omega}{1+xx} - \frac{\omega^2 x}{(1+xx)^2} + \frac{\omega^3}{(1+xx)^3}\left(xx-\frac{1}{3}\right) - \frac{\omega^4}{(1+xx)^4}(x^3-x)$$

$$+\frac{\omega^5}{(1+xx)^5}\left(x^4-2x^2+\frac{1}{5}\right) - \frac{\omega^6}{(1+xx)^6}\left(x^5-\frac{10}{3}x^3+x\right) + \text{etc.}$$

§87 This series, whose law of progression is not that manifest, can be transformed into another form, whose progression is immediately clear. For this aim, put $\arctan x = 90^\circ - u$ that it is $x = \cot u = \frac{\cos u}{\sin u}$; it will be $1+xx = \frac{1}{\sin^2 u}$, whence it is $\frac{dy}{dx} = \frac{1}{1+xx} = \sin^2 u$. Further, because it is $dx = \frac{-du}{\sin^2 u}$ or $du = -dx \sin^2 u$, it will be by taking further differentials

$$\frac{ddy}{dx} = 2du \sin u \cdot \cos u = du \sin 2u = -dx \sin^2 u \cdot \sin 2u$$

and hence

$$\frac{ddy}{1dx^2} = -\sin^2 u \cdot \sin 2u,$$

$$\frac{d^3y}{2dx^2} = -du \sin u \cdot \cos u \cdot \sin 2u - du \sin^2 u \cdot \cos 2u = -du \sin u \cdot \sin 3u$$
$$= dx \sin^3 u \cdot \sin 3u$$

and hence

$$\frac{d^3y}{1 \cdot 2dx^3} = +\sin^3 u \cdot \sin 3u,$$

$$\frac{d^4y}{1 \cdot 2 \cdot 3dx^3} = du \sin^2 u(\cos u \cdot \sin 3u + \sin u \cdot \cos 3u) = du \sin^2 u \cdot \sin 4u$$
$$= -dx \sin^4 \cdot \sin 4u$$

and hence

$$\frac{d^4y}{1 \cdot 2 \cdot 3dx^4} = -\sin^4 \cdot \sin 4u,$$

$$\frac{d^5y}{1 \cdot 2 \cdot 3 \cdot 4dx^4} = -du \sin^3 u(\cos u \cdot \sin 4u + \sin u \cdot \cos 4u) = -du \cdot \sin^3 u \cdot \sin 5u$$
$$= +dx \sin^5 u \cdot 5u$$

and hence

$$\frac{d^5y}{1 \cdot 2 \cdot 3 \cdot 4dx^5} = +\sin^5 u \cdot \sin 5u$$

etc.

From these one concludes that it will be

$$\arctan(x+\omega) = \arctan x + \frac{\omega}{1}\sin u \cdot \sin u - \frac{\omega^2}{2}\sin^2 u \cdot \sin 2u + \frac{\omega^3}{3}\sin^3 u \cdot \sin 3u$$
$$-\frac{\omega^4}{4}\sin^4 u \cdot \sin 4u + \frac{\omega^5}{5}\sin^5 u \cdot \sin 5u - \frac{\omega^6}{6}\sin^6 \cdot \sin 6u + \text{etc.};$$

because here it is $\arctan x = y$ and $\arctan x = 90^\circ - u$, it will be $y = 90^\circ - u$.

§88 If one puts $\operatorname{arccot} x = y$ and $\operatorname{arccot}(x+\omega) = z$, it will be

$$z = y + \frac{\omega dy}{dx} + \frac{\omega^2 ddy}{1 \cdot 2dx^2} + \frac{\omega^3 d^3 y}{1 \cdot 2 \cdot 3dx^3} + \frac{\omega^4 d^4 y}{1 \cdot 2 \cdot 3 \cdot 4dx^4} + \text{etc.}$$

But because it is $dy = \frac{-dx}{1+xx}$, the terms of this series except for the first agree with the ones found before just with different signs. Hence, if one as before puts $\arctan x = 90^\circ - u$ or $\operatorname{arccot} x = u$, that it is, it will be

$$\operatorname{arccot}(x+\omega) = \operatorname{arccot} x - \frac{\omega}{1}\sin u \cdot \sin u + \frac{\omega^2}{2}\sin^2 u \cdot \sin 2u - \frac{\omega^3}{3}\sin^3 u \cdot \sin 3u$$
$$+ \frac{\omega^4}{4}\sin^4 u \cdot \sin 4u - \frac{\omega^5}{5}\sin^5 u \cdot \sin 5u + \text{etc.},$$

which expression follows from the preceding one immediately; for, since it is

$$\operatorname{arccot}(x+\omega) = 90^\circ - \arctan(x+\omega) \quad \text{and} \quad \operatorname{arccot} x = 90^\circ - \arctan x,$$

it will be

$$\operatorname{arccot}(x+\omega) - \operatorname{arccot} x = -\arctan(x+\omega) + \arctan x.$$

§89 From these expressions many extraordinary corollaries follow, depending on which values are substituted for the given ones x and ω. Therefore, let at first be $x = 0$, and because it is $u = 90^\circ - \arctan x$, it will be $u = 90^\circ$ and $\sin u = 1$, $\sin 2u = 0$, $\sin 3u = -1$, $\sin 4u = 0$, $\sin 5u = 1$, $\sin 6u = 0$, $\sin 7u = -1$ etc., whence it will be

$$\arctan \omega = \frac{\omega}{1} - \frac{\omega^3}{3} + \frac{\omega^5}{5} - \frac{\omega^7}{7} + \frac{\omega^9}{9} - \frac{\omega^{11}}{11} + \text{etc.},$$

which is the very well known series expressing the arc whose tangent is $= \omega$. Let $x = 1$; it will be $\arctan x = 45^\circ$ and hence $u = 45^\circ$, hence $\sin u = \frac{1}{\sqrt{2}}$, $\sin 2u = 1$, $\sin 3u = \frac{1}{\sqrt{2}}$, $\sin 4u = 0$, $\sin 5u = -\frac{1}{\sqrt{2}}$, $\sin 6u = -1$, $\sin 7u = -\frac{1}{\sqrt{2}}$, $\sin 8u = 0$, $\sin 9u = \frac{1}{\sqrt{2}}$ etc. From this it is

$$\arctan(1+\omega) = 45^\circ + \frac{\omega}{2} - \frac{\omega^2}{2 \cdot 2} + \frac{\omega^3}{3 \cdot 4} - \frac{\omega^5}{5 \cdot 8} + \frac{\omega^6}{6 \cdot 8} - \frac{\omega^7}{7 \cdot 16} + \frac{\omega^9}{9 \cdot 32} - \frac{\omega^{10}}{10 \cdot 32}$$

$$+\frac{\omega^{11}}{11\cdot 64}-\frac{\omega^{13}}{13\cdot 128}+\frac{\omega^{14}}{14\cdot 128}-\text{etc.}$$

Therefore, if it is $\omega=-1$, because of $\arctan(1+\omega)=0$ and $45^\circ=\frac{\pi}{4}$ it will be

$$\frac{\pi}{4}=\frac{1}{1\cdot 2}+\frac{1}{2\cdot 2}+\frac{1}{3\cdot 2^3}-\frac{1}{6\cdot 2^3}-\frac{1}{7\cdot 2^4}+\frac{1}{9\cdot 2^5}+\frac{1}{10\cdot 2^5}+\frac{1}{11\cdot 2^6}-\text{etc.;}$$

If this value is substituted for 45° in that expression, it will be

$$\arctan(1+\omega)=\frac{\omega+1}{1\cdot 2}-\frac{\omega^2-1}{2\cdot 2}+\frac{\omega^3+1}{3\cdot 2^2}-\frac{\omega^5+1}{5\cdot 2^3}+\frac{\omega^6-1}{6\cdot 2^3}-\frac{\omega^7+1}{7\cdot 2^4}+\text{etc.}$$

But that series is most appropriate to find the value of $\frac{\pi}{4}$ approximately.

§90 Because it is

$$\frac{\pi}{4}=\frac{1}{1\cdot 2}+\frac{1}{2\cdot 2}+\frac{1}{3\cdot 2^2}-\frac{1}{5\cdot 2^3}-\frac{1}{6\cdot 2^3}-\frac{1}{7\cdot 2^4}+\text{etc.,}$$

but the terms having 2, 6, 10 etc. in the denominators

$$\frac{1}{2\cdot 2}-\frac{1}{6\cdot 2^3}+\frac{1}{10\cdot 2^5}-\frac{1}{14\cdot 2^7}+\text{etc.}$$

express $\frac{1}{2}\arctan\frac{1}{2}$, it will be

$$\frac{\pi}{4}=\frac{1}{2}\arctan\frac{1}{2}+\frac{1}{1\cdot 2}+\frac{1}{3\cdot 2^2}-\frac{1}{5\cdot 2^3}-\frac{1}{7\cdot 2^4}+\frac{1}{9\cdot 2^5}+\frac{1}{11\cdot 2^6}-\text{etc.}$$

But because in the other formula for negative ω it is

$$\begin{aligned}\arctan(1-\omega)=&\frac{1}{1\cdot 2}+\frac{1}{2\cdot 2}+\frac{1}{3\cdot 2^2}-\frac{1}{5\cdot 2^3}-\frac{1}{6\cdot 2^3}-\frac{1}{7\cdot 2^4}+\text{etc.}\\&-\frac{\omega}{1\cdot 2}-\frac{\omega^2}{2\cdot 2}-\frac{\omega^3}{3\cdot 2^2}+\frac{\omega^5}{5\cdot 2^3}+\frac{\omega^6}{6\cdot 2^3}+\frac{\omega^7}{7\cdot 2^4}-\text{etc.,}\end{aligned}$$

if it is $\omega=\frac{1}{2}$, it will be

$$\arctan(1-\omega) = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 2} + \frac{1}{3\cdot 2^2} - \frac{1}{5\cdot 2^3} - \frac{1}{6\cdot 2^3} - \frac{1}{7\cdot 2^4} + \text{etc.}$$
$$- \frac{1}{1\cdot 2} - \frac{1}{2\cdot 2} - \frac{1}{3\cdot 2^2} + \frac{1}{5\cdot 2^3} + \frac{1}{6\cdot 2^3} + \frac{1}{7\cdot 2^4} - \text{etc.}$$

and having taken the terms divided by 2, 6, 10 etc. it will be

$$\arctan\frac{1}{2} = \frac{1}{2}\arctan\frac{1}{2} + \frac{1}{1\cdot 2} + \frac{1}{3\cdot 2^3} - \frac{1}{5\cdot 2^3} - \frac{1}{7\cdot 2^4} + \frac{1}{9\cdot 2^5} + \text{etc.}$$
$$- \frac{1}{2}\arctan\frac{1}{2} - \frac{1}{1\cdot 2^2} - \frac{1}{3\cdot 2^5} + \frac{1}{5\cdot 2^8} + \frac{1}{7\cdot 2^{11}} - \frac{1}{9\cdot 2^{14}} - \text{etc.}$$

and hence

$$\frac{1}{2}\arctan\frac{1}{2} = + \frac{1}{1\cdot 2} + \frac{1}{3\cdot 2^2} - \frac{1}{5\cdot 2^3} - \frac{1}{7\cdot 2^4} + \text{etc.}$$
$$- \frac{1}{2}\arctan\frac{1}{8} - \frac{1}{1\cdot 2^2} - \frac{1}{3\cdot 2^5} + \frac{1}{5\cdot 5^8} + \frac{1}{7\cdot 2^{11}} - \text{etc.};$$

if this value is substituted in the superior series and $\arctan\frac{1}{8}$ itself is converted into a series, one will find

$$\frac{\pi}{4} = \begin{cases} 1 + \frac{1}{3\cdot 2^1} - \frac{1}{5\cdot 2^2} - \frac{1}{7\cdot 2^3} + \frac{1}{9\cdot 2^4} + \text{etc.} \\ - \frac{1}{1\cdot 2^2} - \frac{1}{3\cdot 2^5} + \frac{1}{5\cdot 2^8} + \frac{1}{7\cdot 2^{11}} - \frac{1}{9\cdot 2^{14}} - \text{etc.} \\ - \frac{1}{1\cdot 2^4} - \frac{1}{3\cdot 2^{10}} - \frac{1}{5\cdot 2^{16}} + \frac{1}{7\cdot 2^{22}} - \frac{1}{9\cdot 2^{28}} + \text{etc.} \end{cases}$$

§ 90a These and many others follow from the position $x = 1$; but if we put $x = \sqrt{3}$ that it is $\arctan x = 60^\circ$, it will be $u = 30^\circ$ and $\sin u = \frac{1}{2}$, $\sin 2u = \frac{\sqrt{3}}{2}$, $\sin 3u = 1$, $\sin 4u = \frac{\sqrt{3}}{2}$, $\sin 5u = \frac{1}{2}$, $\sin 6u = 0$, $\sin 7u = -\frac{1}{2}$ etc., whence it will be

$$\arctan(\sqrt{3}+\omega) = 60^\circ + \frac{\omega}{1\cdot 2^2} - \frac{\omega^2\sqrt{3}}{2\cdot 2^3} + \frac{\omega^3}{3\cdot 2^3} - \frac{\omega^4\sqrt{3}}{4\cdot 2^5} + \frac{\omega^5}{5\cdot 2^6} - \frac{\omega^7}{7\cdot 2^8}$$
$$+\frac{\omega^8\sqrt{3}}{8\cdot 2^9} - \frac{\omega^9}{9\cdot 2^9} + \frac{\omega^{10}\sqrt{3}}{10\cdot 2^{11}} - \frac{\omega^{11}}{11\cdot 2^{12}} + \text{etc.}$$

But if one puts $x = \frac{1}{\sqrt{3}}$ that it is $\arctan x = 30^\circ$, it will be $u = 60^\circ$ and $\sin u = \frac{\sqrt{3}}{2}$, $\sin 2u = \frac{\sqrt{3}}{2}$, $\sin 3u = 0$, $\sin 4u = -\frac{\sqrt{3}}{2}$, $\sin 6u = 0$, $\sin 7u = \frac{\sqrt{3}}{2}$ etc., having substituted which values it will be

$$\arctan\left(\frac{1}{\sqrt{3}}+\omega\right) = 30^\circ + \frac{3\omega}{1\cdot 2^2} - \frac{3\omega^2\sqrt{3}}{2\cdot 2^3} + \frac{3^2\omega^4\sqrt{3}}{4cdot 2^5} - \frac{3^3\omega^5}{5\cdot 2^5} + \text{etc.};$$

therefore, if it is $\omega = -\frac{1}{\sqrt{3}}$, because if $30^\circ = \frac{\pi}{6}$ it will be

$$\frac{\pi}{6\sqrt{3}} = \frac{1}{1\cdot 2^2} + \frac{1}{2\cdot 2^3} - \frac{1}{4\cdot 2^5} - \frac{1}{5\cdot 2^6} + \frac{1}{7\cdot 2^8} + \frac{1}{8\cdot 2^9} - \text{etc.}$$

§91 Let us return to the general expression found

$$\arctan(x+\omega)$$
$$= \arctan x + \frac{\omega}{1}\sin u\cdot\sin u - \frac{\omega^2}{2}\sin^2 u\cdot\sin 2u + \frac{\omega^3}{3}\sin^3 u\cdot\sin 3u - \text{etc.}$$

and let us put $\omega = -x$ that it is $\arctan(x+\omega) = 0$ and it will be

$$\arctan x = \frac{x}{1}\sin u\cdot\sin u + \frac{x^2}{2}\sin^2 u\cdot\sin 2u + \frac{x^3}{3}\sin^3 u\cdot\sin 3u + \text{etc.}$$

But because it is $\arctan x = 90^\circ - u = \frac{\pi}{2} - u$, it will be $x = \cot u = \frac{\cos u}{\sin u}$. Therefore, it will be

$$\frac{\pi}{2} = u + \cos u\cdot\sin u + \frac{1}{2}\cos^2 u\cdot\sin 2u + \frac{1}{3}\cos^3 u\cdot\sin 3u + \frac{1}{4}\cos^4 u\cdot\sin 4u + \text{etc.},$$

which series is even more remarkable, since, whatever arc is taken for u, the value of the series always arises as the same $= \frac{\pi}{2}$.

But if it is $\omega = -2x$, because of $\arctan(-x) = -\arctan x$ it will be

$$2\arctan x = \frac{2x}{1}\sin u \cdot \sin u + \frac{4x^2}{2}\sin^2 u \cdot \sin 2u + \frac{8}{3}\sin^3 u \cdot \sin 3u + \text{etc.}$$

But because it is $\arctan x = \frac{\pi}{2} - u$ and $x = \frac{\cos u}{\sin u}$, it will be

$$\pi = 2u + \frac{2}{1}\cos u \cdot \sin u + \frac{2^2}{2}\cos^2 u \cdot \sin 2u + \frac{2^3}{3}\cos^3 u \cdot \sin 3u + \text{etc.}$$

Let $u = 45^\circ$; it will be $\cos u = \frac{1}{\sqrt{2}}$, $\sin u = \frac{1}{\sqrt{2}}$, $\sin 2u = 1$, $\sin 3u = \frac{1}{\sqrt{2}}$, $\sin 4u = 0$, $\sin 5u = \frac{-1}{\sqrt{2}}$, $\sin 6u = -1$, $\sin 7u = \frac{-1}{\sqrt{2}}$, $\sin 8u = 0$, $\sin 9u = \frac{1}{\sqrt{2}}$ etc. and it will be

$$\frac{\pi}{2} = \frac{1}{1} + \frac{2}{2} + \frac{2}{3} - \frac{2^5}{5} - \frac{2^3}{6} - \frac{2^3}{7} + \frac{2^4}{9} + \frac{2^5}{10} + \frac{2^5}{11} - \text{etc.},$$

which series, even though it diverges, nevertheless because of its simplicity is remarkable.

§92 In general expression found put

$$\omega = -x - \frac{1}{x} = \frac{-1}{\sin u \cdot \cos u}$$

because of $x = \frac{\cos u}{\sin u}$; it will be

$$\arctan(x+\omega) = \arctan\left(-\frac{1}{x}\right) = -\arctan\frac{1}{x} = -\frac{\pi}{2} + \arctan x.$$

Therefore, one will hence obtain the following expression

$$\frac{\pi}{2} = \frac{\sin u}{1\cos u} + \frac{\sin 2u}{2\cos^2 u} + \frac{\sin 3u}{3\cos^3 u} + \frac{\sin 4u}{4\cos^4 u} + \frac{\sin 5u}{5\cos^5 u} + \text{etc.},$$

which having put $u = 45^\circ$ gives the same series which we found last.

But if we put $\omega = -\sqrt{1+xx}$, because of $x = \frac{\cos u}{\sin u}$ it will be

$$\omega = -\frac{1}{\sin u}$$

and

$$\arctan(x-\sqrt{1+xx}) = -\arctan(\sqrt{1+xx}-x)$$
$$= -\frac{1}{2}\arctan\frac{1}{x} = -\frac{1}{2}\left(\frac{\pi}{2}-\arctan x\right) = -\frac{1}{2}u$$

and

$$\arctan x = \frac{\pi}{2} - u.$$

Therefore, it will be

$$\frac{\pi}{2} = \frac{1}{2}u + \frac{1}{1}\sin u + \frac{1}{2}\sin 2u + \frac{1}{3}\sin 3u + \frac{1}{4}\sin 4u + \text{etc.}$$

Therefore, if this equation is differentiated, it will be

$$0 = \frac{1}{2} + \cos u + \cos 2u + \cos 3u + \cos 4u + \cos 5u + \text{etc.},$$

whose correctness is understood from the nature of recurring series.

§93 If in similar manner the series found before are differentiated, new summable series will be found. At at first from the series

$$\arctan(1+\omega) = \frac{\pi}{4} + \frac{\omega}{2} - \frac{\omega^2}{2\cdot 2} + \frac{\omega^3}{3\cdot 4} - \frac{\omega^5}{5\cdot 8} + \frac{\omega^6}{6\cdot 8} - \text{etc.}$$

it follows

$$\frac{1}{2+2\omega+\omega^2} = \frac{1}{2} - \frac{\omega}{2} + \frac{\omega^2}{4} - \frac{\omega^4}{8} + \frac{\omega^5}{8} - \frac{\omega^6}{16} + \frac{\omega^8}{32} - \text{etc.},$$

which arises from the expansion of this fraction $\frac{2-2\omega+\omega^2}{4+\omega^4} = \frac{1}{2+2\omega+\omega^2}$.
Further, this series

$$\frac{\pi}{2} = u + \cos u \cdot \sin u + \frac{1}{2}\cos^2 u \cdot \sin 2u + \frac{1}{3}\cos^3 u \cdot \sin 3u + \frac{1}{4}\cos^4 \cdot \sin 4u + \text{etc.}$$

by means of differentiation will give

$$0 = 1 + \cos 2u + \cos \cdot \cos 3u + \cos^2 u \cdot \cos 4u + \cos^3 u \cdot \cos 5u + \text{etc.}$$

Finally, the series

$$\frac{\pi}{2} = \frac{\sin u}{\cos u} + \frac{\sin 2u}{2\cos^2 u} + \frac{\sin 3u}{3\cos^3 u} + \frac{\sin 4u}{4\cos^4 u} + \text{etc.}$$

gives

$$0 = \frac{1}{\cos^2 u} + \frac{\cos u}{\cos^3 u} + \frac{\cos 2u}{\cos^4 u} + \frac{\cos 3u}{\cos^5 u} + \frac{\cos 4u}{\cos^6 u} + \text{etc.}$$

or

$$0 = 1 + \frac{\cos u}{\cos u} + \frac{\cos 2u}{\cos^2 u} + \frac{\cos 3u}{\cos^3 u} + \frac{\cos 4u}{\cos^4 u} + \frac{\cos 5u}{\cos^5 u} + \text{etc.}$$

§94 But especially the expression found

$$= \arctan x + \frac{\omega}{1}\sin u \cdot \sin u - \frac{\omega^2}{2}\sin^2 u \cdot \sin 2u + \frac{\omega^3}{3}\sin^3 u \cdot \sin 3u - \text{etc.},$$

while

$$x = \cot u \quad \text{or} \quad u = \operatorname{arccot} x = 90^\circ - \arctan x,$$

will serve for finding the angle or the arc corresponding to a given certain tangent. For, let the tangent $= t$ be propounded and in tables find the tangent coming closet to this $= x$, to which the arc $= y$ shall correspond, and it will be $u = 90^\circ - y$. Then put $x + \omega = t$ or $\omega = t - x$ and the arc in question will be

$$= y + \frac{\omega}{1}\sin u \cdot \sin u - \frac{\omega^2}{2} \cdot \sin 2u + \text{etc.},$$

which rule is especially useful then, when the propounded tangent was very large and therefore the arc in question hardly deviates from 90°. For, in these cases because of the rapidly increasing tangents the usual method of interpolation leads too far away from the truth. Therefore, let this example be propounded.

EXAMPLE

Let the arc be in question, whose tangent is $= 100$ *having put the radius* $= 1$. The arc approximately equal to the one in question is $89^\circ 25^{\mathrm{I}}$, whose tangent is

$$\begin{array}{rrcl}
 & x & = & 98.217943 \\
\text{which subtract from} & t & = & 100.000000 \\
\hline
\text{it will remain} & \omega & = & 1.782057
\end{array}$$

Further, because it is $y = 89°25^{\mathrm{I}}$, it will be $u°35^{\mathrm{I}}$, $2u = 1°10^{\mathrm{I}}$, $3u = 1°45^{\mathrm{I}}$ etc. Now investigate the single terms by means of logarithms. To

$$\begin{array}{lrcl}
\text{add} & \log\omega & = & 0.2509125 \\
 & \log\sin u & = & 8.0077867 \\
 & \log\sin u & = & 8.0077867 \\
\hline
\text{subtract} & \log\omega\sin u \cdot \sin u & = & 6.2664949 \\
 & & & 4.6855749 \\
 & & = & 1.5809200 \\
\text{Therefore} & \omega\sin u \cdot \sin u & = & 38.09956 \quad \text{seconds} \\
\text{To} & \log\omega\sin^2 u & = & 6.2664949 \\
\text{add} & \log\omega & = & 0.2509215 \\
 & \log 2u & = & 8.3087941 \\
\hline
\text{subtract} & & & 4.8262105 \\
 & \log 2 & = & 0.3010300 \\
\hline
 & \log\frac{1}{2}\omega^2\sin^2 u \cdot 2u & = & 4.5251805 \\
\text{subtract} & & & 4.6855749 \\
\hline
\text{it remains} & & & 9.8396056
\end{array}$$

$$
\begin{array}{llcll}
\text{Therefore} & \frac{1}{2}\omega^2 \sin^2 u \cdot \sin 2u & = & 0.6912000 & \text{seconds} \\
\hline
\text{Further to} & \log \omega^3 & = & 0.7527645 & \\
 & \log \sin^3 u & = & 4.0233601 & \\
\text{subtract} & \log \sin 3u & = & 3.2609725 & \\
 & \log 3 & = & 0.4771213 & \\
\hline
 & & & 2.7838512 & \\
\text{subtract} & & & 4.6855749 & \\
\hline
 & & & 8.0982763 & \\
\hline
\text{Therefore} & \frac{1}{3}\omega^3 \sin^3 u \cdot \sin 3u & = & 0.0125400 & \text{seconds.}
\end{array}
$$

$$
\begin{array}{llcl}
\text{Finally, to} & \log \omega^4 & = & 1.0036860 \\
\text{add} & \log \sin^4 u & = & 2.0311468 \\
 & \log \sin 4u & = & 8.6097341 \\
\hline
 & & & 1.6445669 \\
\text{subtract} & \log 4 & = & 0.6020600 \\
\hline
 & & & 1.0425069 \\
\text{subtract} & & & 4.6855749 \\
\hline
 & & & 6.3569320
\end{array}
$$

Therefore,

$$\frac{1}{4}\omega^4 \sin^4 u = 0.00023 \quad \text{seconds}$$

Hence

	Terms to be added	Terms to be subtracted
	38.09956	0.69120
	0.01254	0.00023
subtract	0.69143	

Hence in total

$$37.4067 = 37^{\mathrm{II}}25^{\mathrm{III}}14^{\mathrm{IV}}24^{\mathrm{V}}36^{\mathrm{VI}}.$$

Therefore, the arc, whose tangent is hundred times the radius, will be

$$89^\circ 25^{\mathrm{I}}37^{\mathrm{II}}25^{\mathrm{III}}14^{\mathrm{IV}}24^{\mathrm{V}}36^{\mathrm{VI}}$$

and the error does not ascend to the fourth minute, but can only be in the fifth minute, from we will be able to say correctly that this angle is almost $= 89^\circ 25^{\mathrm{I}}37^{\mathrm{II}}25^{\mathrm{III}}14^{\mathrm{IV}}$. If an even greater tangent is propounded, even though ω might arise larger, one will nevertheless because of the still small angle u be able to define the arc in a convenient way.

§95 Since we here substituted an arc of the circle for y, let us now put reciprocal functions for y, of which kind are $\sin x$, $\cos x$, $\tan x$, $\cot x$ etc. Therefore, let $y = \sin x$ and having put $x + \omega$ instead of x it will be $z = \sin(x + \omega)$ and the equation

$$z = y + \frac{\omega dy}{dx} + \frac{\omega^2 ddy}{2dx^2} + \frac{\omega^3 d^3y}{6dx^3} + \frac{\omega^4 d^4y}{24dx^4} + \text{etc.}$$

because of

$$\frac{dy}{dx} = \cos x, \quad \frac{ddy}{dx^2} = -\sin x, \quad \frac{d^3y}{dx^3} = -\cos x, \quad \frac{d^4y}{dx^4} = \sin x \quad \text{etc.}$$

will give

$$\sin(x+\omega) = \sin x + \omega\cos x - \frac{1}{2}\omega^2 \sin x - \frac{1}{6}\omega^3 \cos x + \frac{1}{24}\omega^4 \sin x + \text{etc.}$$

and having taken a negative ω it will be

$$\sin(x-\omega) = \sin x - \omega\cos x - \frac{1}{2}\omega^2 \sin x + \frac{1}{6}\omega^3 \cos x + \frac{1}{24}\omega^4 \sin x - \text{etc.}$$

So, if one sets $y = \cos x$, because of

$$\frac{dy}{dx} = -\sin x, \quad \frac{ddy}{dx^2} = -\cos x, \quad \frac{d^3y}{dx^3} = \sin x, \quad \frac{d^4y}{dx^4} = \cos x \quad \text{etc.}$$

it will be

$$\cos(x+\omega) = \cos x - \omega \sin x - \frac{1}{2}\omega^2 \cos x + \frac{1}{6}\omega^3 \sin x + \frac{1}{24}\omega^4 \cos x - \text{etc.}$$

and for a negative ω it will be

$$\cos(x-\omega) = \cos x + \omega \sin x - \frac{1}{2}\omega^2 \cos x - \frac{1}{6}\omega^3 \sin x + \frac{1}{24}\omega^4 \cos x + \text{etc.}$$

§96 The use of these formulas is immense both for constructing and interpolation of tables of sines and cosines. For, if the sines and cosines of a certain arc x were known, from them in an easy task the sines and cosines of the angles $x+\omega$ and $x-\omega$ can be found, if the difference ω was sufficiently small; for, in this case the found series converge rapidly. For this it is necessary that the arc ω is expressed in parts of the radius; this, because the arc of 180° is

$$3.14159265358979323846,$$

will be done easy; for, having done a division by 180 it will be

$$\begin{aligned} \text{arc of } 1^\circ &= 0.01745329251994329576 9\\ \text{arc of } 1^{\mathrm{I}} &= 0.00029088820866572159 6\\ \text{arc of } 1^{\mathrm{II}} &= 0.00000484813681109535 99. \end{aligned}$$

Example 1

To find the sine and the cosine of the angles $45^\circ 1^I$ and $44^\circ 59^I$ from the given sine and cosine of the angle 45°, both of which are $= \frac{1}{\sqrt{2}} = 0.707167811865$.

Therefore, since it is

$$\sin x = \cos x = 0.7071067811865$$

and

$$\omega \qquad = 0.0002908882086$$

it will be to perform the multiplications in an easier manner

$$\begin{aligned}
2\omega &= 0.0005817764173\\
3\omega &= 0.0008726646259\\
4\omega &= 0.0011635528346\\
5\omega &= 0.0014544410433\\
6\omega &= 0.0017453292519\\
7\omega &= 0.0020362174606\\
8\omega &= 0.0023271056693\\
9\omega &= 0.0026179938779
\end{aligned}$$

Therefore, $\omega \sin x$ and $\omega \cos x$ will be found this way:

7	·	0.00020362174606
0	·	
7	·	0.00000203621746
1	·	2908882
0	·	
6	·	174532
7	·	20362
8	·	2327
1	·	29
1	·	2
8	·	2
6	·	0

In total

$$\omega \sin x = \omega \cos x = 0.00020568902488$$

Therefore,

$$
\begin{array}{llcr}
 & \frac{1}{2}\omega\cos x & = & 0.00010284451244 \\
\hline
\text{by } \omega & 1 & \cdot & 0.00000002908882 \\
 & 0 & \cdot & \\
 & 2 & \cdot & 58177 \\
 & 8 & \cdot & 23271 \\
 & 4 & \cdot & 1163 \\
 & 4 & \cdot & 116 \\
 & 5 & \cdot & 14 \\
\hline
 & \frac{1}{2}\omega^2\cos x & = & 0.00000002991623 \\
 & \frac{1}{6}\omega^3\cos x & = & 0.00000000997208 \\
\hline
\text{by } \omega & 9 & \cdot & 0.0000000000261 \\
 & 9 & \cdot & 26 \\
 & 7 & \cdot & 2 \\
\hline
 & \frac{1}{6}\omega^3\cos x & = & 0.0000000000289
\end{array}
$$

Therefore, to find $\sin 45^\circ 1^{\mathrm{I}}$ to

$$
\begin{array}{llcrcl}
 & \sin x & = & 0.7071067811865 & & \\
\text{add} & \omega\cos x & = & 2056890249 & & \\
\hline
 & & & 0.7073124702114 & & \\
\text{subtract} & \frac{1}{2}\omega^2\sin x & = & 299162 & & \\
\hline
 & & & 0.7073124402952 & & \\
\text{subtract} & \frac{1}{6}\omega^3\cos x & = & 29 & & \\
\hline
 & \sin 45^\circ 1^{\mathrm{I}} & = & 0.7073124402923 & = & \cos 44^\circ 59^{\mathrm{I}}
\end{array}
$$

But to find $\cos 45^\circ 1^{\mathrm{I}}$ from

$$\begin{array}{lrcr}
 & \cos x & = & 0.7071067811865 \\
\text{subtract} & \omega \sin x & = & 2056890249 \\ \hline
 & & & 0.70769010921616 \\
\text{subtract} & \frac{1}{2}\omega^2 \cos x & = & 299162 \\ \hline
 & & & 0.7069010622454 \\
\text{add} & \frac{1}{6}\omega^3 \cos x & = & 29 \\ \hline
 & \cos 45^\circ 1^{\text{I}} & = & 0.7069010622483 = \sin 44^\circ 59^{\text{I}}
\end{array}$$

EXAMPLE 2

From the given sine and cosine of the arc $67^\circ 30^I$ to find the sine and the cosine of the arcs $67^\circ 31^I$ and $67^\circ 29^I$.

Let us perform this calculation in decimal fractions up to 7 digits, as the common tables are usually constructed, and so the task will easily be done using logarithms. Because it is $x = 67^\circ 30^{\text{I}}$ and $\omega = 0.000290888$, it will be

$$\log \omega = 6.4637259$$

$$\begin{array}{lrcrrcr}
\text{and} & \log \sin x & = & 9.9656153 & \log \cos x & = & 9.5828397 \\
 & \log \omega & = & 6.6437259 & \log \omega & = & 6.6437259 \\ \hline
 & \log \omega \sin x & = & 6.4293412 & \log \omega \cos x & = & 6.0465656 \\
 & \log \frac{1}{2}\omega & = & 6.1626959 & \log \frac{1}{2}\omega^2 & = & 6.1626959 \\
 & \log \frac{1}{2}\omega^2 \sin x & = & 2.59200371 & \log \frac{1}{2}\omega^2 \cos x & = & 2.2092615 \\
\text{Therefore} & \omega \sin x & = & 0.00026874 & \omega \cos x & = & 0.00011132 \\
 & \frac{1}{2}\omega^2 \sin x & = & 0.00000004 & \frac{1}{2}\omega^2 \cos x & = & 0.00000001 \\
\text{whence} & \sin 67^\circ 31^{\text{I}} & = & 0.9239908 & \cos 67^\circ 31^{\text{I}} & = & 0.3824147 \\
 & \sin 67^\circ 29^{\text{I}} & = & 0.9237681 & \cos 67^\circ 29^{\text{I}} & = & 0.3829522
\end{array}$$

where not even the terms $\frac{1}{2}\omega^2 \sin x$ and $\frac{1}{2}\omega^2 \cos x$ were necessary.

§97 From the series, which we found above,

$$\begin{aligned}
\sin(x+\omega) &= \sin x + \omega\cos x - \frac{1}{2}\omega^2 \sin x - \frac{1}{6}\omega^3 \cos x + \frac{1}{24}\omega^4 \sin x + \text{etc.}\\
\cos(x+\omega) &= \cos x - \omega\sin x - \frac{1}{2}\omega^2 \cos x + \frac{1}{6}\omega^3 \sin x + \frac{1}{24}\omega^4 \cos x - \text{etc.}\\
\sin(x-\omega) &= \sin x - \omega\cos x - \frac{1}{2}\omega^2 \sin x + \frac{1}{6}\omega^3 \cos x + \frac{1}{24}\omega^4 \sin x - \text{etc.}\\
\cos(x-\omega) &= \cos x + \omega\sin x - \frac{1}{2}\omega^2 \cos x - \frac{1}{6}\omega^3 \sin x + \frac{1}{24}\omega^4 \cos x + \text{etc.}
\end{aligned}$$

it follows by combination that it will be

$$\frac{\sin(x+\omega)+\sin(x-\omega)}{2}$$
$$= \sin x - \frac{1}{2}\omega^2 \sin x + \frac{1}{24}\sin x - \frac{1}{720}\omega^6 \sin x + \text{etc.} = \sin x \cdot \cos\omega$$

and

$$\frac{\sin(x+\omega)-\sin(x-\omega)}{2}$$
$$= \omega\cos x - \frac{1}{6}\omega^3 \cos x + \frac{1}{120}\omega^5 \cos x - \frac{1}{5040}\omega^7 \cos x + \text{etc.} = \cos x \cdot \sin\omega,$$

whence the series found above for the sines and cosines arise

$$\begin{aligned}
\cos\omega &= 1 - \frac{1}{2}\omega^2 + \frac{1}{24}\omega^4 - \frac{1}{720}\omega^6 + \text{etc.}\\
\sin\omega &= \omega - \frac{1}{6}\omega^3 + \frac{1}{120}\omega^5 - \frac{1}{5040}\omega^7 + \text{etc.},
\end{aligned}$$

which same series follow from the first by putting $x = 0$; for, because it is $\cos x = 1$ and $\sin x = 0$, the first series will exhibit $\sin\omega$, the second on the other hand $\cos\omega$.

§98 Now let us also put $y = \tan x$ that it is $z = \tan(x+\omega)$; because of $y = \frac{\sin x}{\cos x}$ [§ 206 of the first part]

$$\frac{dy}{dx} = \frac{1}{\cos^2 x}, \quad \frac{ddy}{2dx^2} = \frac{\sin x}{\cos^3 x}, \quad \frac{d^3y}{2dx^3} = \frac{1}{\cos^2 x} + \frac{3\sin^2 x}{\cos^4 x} = \frac{3}{\cos^4 x} - \frac{2}{\cos^2 x},$$
$$\frac{d^4y}{2\cdot 4dx^4} = \frac{3\sin x}{\cos^5 x} - \frac{\sin x}{\cos^3 x}, \quad \frac{d^5y}{2\cdot 4} = \frac{15}{\cos^6 x} - \frac{15}{\cos^4 x} + \frac{2}{\cos^2 x},$$

whence it follows that it will be

$$\tan(x+\omega) = \tan x + \begin{cases} \dfrac{\omega}{\cos^2 x} + \dfrac{\omega^2 \sin x}{\cos^3 x} + \dfrac{\omega^3}{\cos^4 x} + \dfrac{\omega^4 \sin x}{\cos^5 x} + \text{etc.} \\ \qquad\qquad\qquad\qquad - \dfrac{2\omega^3}{3\cos^2 x} - \dfrac{\omega^4 \sin x}{3\cos^3 x} - \text{etc.} \end{cases}$$

by means of which formula from a given tangent of any angle one can find the tangents of very closes angles. Since the superior series is a geometric one, collected into one sum it will be

$$\tan(x+\omega) = \tan x + \frac{\omega + \omega^2 \tan x}{\cos^2 x - \omega^2} - \frac{2\omega^3}{3\cos^2 x} - \frac{\omega^4 \sin x}{3\cos^3 x} - \text{etc.}$$

or

$$\tan(x+\omega) = \frac{\sin x \cdot \cos x + \omega}{\cos^2 x - \omega^2} - \frac{2\omega^3}{3\cos^2 x} - \frac{\omega^4 \sin x}{3\cos^3 x} - \text{etc.},$$

which formula is used more conveniently for this aim.

§99 Similar expressions can also be found for the logarithms of sines, cosines and tangents. For, let $y =$ a logarithm of the sine of the angle x, which we want to express this way $y = \log \sin x$, and $z = \log \sin(x+\omega)$; because of $\frac{dy}{dx} = \frac{n\cos x}{\sin x}$ it will be $\frac{ddy}{dx^2} = \frac{-n}{\sin^2 x}$, $\frac{d^3y}{dx^3} = \frac{+2n\cos x}{\sin^3 x}$ etc., whence it will be

$$z = \log \sin(x+\omega) = \log \sin x + \frac{n\omega \cos x}{\sin x} - \frac{n\omega^2}{2\sin^2 x} + \frac{n\omega^3 \cos x}{3\sin^3 x} - \text{etc.},$$

where n denotes the number, by which the hyperbolic must be multiplied that the propounded logarithms arise. But if it is $y = \tan x$ and $z = \log \tan(x+\omega)$, it will be $\frac{dy}{dx} = \frac{n}{\sin x \cdot \cos x} = \frac{2n}{\sin 2x}$, $\frac{ddy}{2dx^2} = \frac{-2n\cos 2x}{(\sin 2x)^2}$ etc. and hence

$$\log\tan(x+\omega) = \log\tan x + \frac{2n\omega}{\sin 2x} - \frac{2n\omega^2\cos 2x}{(\sin 2x)^2} + \text{etc.},$$

by means of which formulas the logarithms of sines and tangents can be interpolated.

§100 Let us put that y denotes the arc, whose logarithm of the sine is $= x$, or that it is $y = \mathrm{A}\,.\log x$, and that z is the arc, whose logarithm of the sine shall be $= x+\omega$ or $z = \mathrm{A}\,.\log\sin(x+\omega)$; it will be $x = \log\sin y$ and

$$\frac{dx}{dy} = \frac{n\cos y}{\sin y}, \quad \text{whence} \quad \frac{dy}{dx} = \frac{\sin y}{n\cos y};$$

it will be

$$\frac{ddy}{dx} = \frac{dy}{n\cos^2 y} = \frac{dx\sin y}{n^2\cos^3 y}, \quad \text{therefore} \quad \frac{ddy}{dx^2} = \frac{\sin y}{n^2\cos^3 y}.$$

As a logical consequence

$$z = y + \frac{\omega\sin y}{n\cos y} + \frac{\omega^2\sin y}{2n^2\cos^3 y} + \text{etc.}$$

In similar manner, if the logarithm of a cosine is given, the expression will be found.

But if it is $y = \mathrm{A}\,.\log\tan x$ and $z = \mathrm{A}\,.\log(x+\omega)$, because it is $x = \log\tan y$, it will

$$\frac{dx}{dy} = \frac{n}{\sin y\cdot\cos y} \quad \text{and} \quad \frac{dy}{dx} = \frac{\sin y\cdot\cos y}{n} = \frac{\sin 2y}{2n},$$

whence

$$\frac{ddy}{dx} = \frac{2dy\cos 2y}{2n} = \frac{dx\sin 2y\cdot\cos 2y}{2nn}$$

and

$$\frac{ddy}{dx^2} = \frac{\sin 2y\cdot\cos 2y}{2nn} = \frac{\sin 4y}{4nn}, \quad \frac{d^3y}{dx^3} = \frac{\sin 2y\cdot 4y}{2n^3} \quad \text{etc.};$$

hence

$$z = y + \frac{\omega\sin 2y}{2n} + \frac{\omega^2\sin 2y\cdot\cos 2y}{4nn} + \frac{\omega^3\sin 2y\cdot\cos 4y}{12n^3} + \text{etc.}$$

§101 Since the use of these expressions for the construction of tables of logarithms of sines and cosines can easily be seen from the preceding paragraphs, we will not treat this here any longer. Therefore, finally let us consider the value $y = e^x \sin nx$ and let $z = e^{x+\omega} \sin n(x+\omega)$; since it is

$$\begin{aligned}
\frac{dy}{dx} &= e^x(\sin nx + n\cos nx)\\
\frac{ddy}{dx^2} &= e^x((1-nn)\sin nx + 2n\cos nx)\\
\frac{d^3y}{dx^3} &= e^x((1-3nn)\sin nx + n(3-nn)\cos nx)\\
\frac{d^4y}{dx^4} &= e^x((1-6nn+n^4)\sin nx + n(4-4nn)\cos nx)\\
\frac{d^5y}{dx^5} &= e^x((1-10nn+5n^4)\sin nx + n(5-10nn+n^4)\cos nx),
\end{aligned}$$

etc.,

having substituted these values and having divided by e^x it will be

$$e^\omega \sin n(x+\omega) = \sin nx$$

$$+\omega \sin nx + \frac{1-nn}{2}\omega^2 \sin nx + \frac{1-3nn}{6}\omega^3 \sin nx + \frac{1-6nn+n^4}{24}\omega^4 \sin nx + \text{etc.}$$

$$+n\omega \cos nx + \frac{2n}{2}\omega^2 \cos nx + \frac{n(3-nn)}{6}\omega^3 \cos nx + \frac{n(4-4nn)}{24}\omega^4 \cos nx + \text{etc.}$$

§102 Hence many extraordinary corollaries can be deduced; but it shall suffice for us to have mentioned these things here. If it was $x = 0$, it will be

$$e^\omega \sin n\omega = n\omega + \frac{2n}{2}\omega^2 + \frac{n(3-nn)}{6}\omega^3 + \frac{n(4-nn)}{24}\omega^4 + \frac{n(5-10n^2+n^4)}{120}\omega^5 + \text{etc.}$$

If it is $\omega = -x$, because of $\sin n(x+\omega) = 0$ it will be

$$\tan nx = \frac{nx - \frac{2n}{2}x^2 + \frac{n(3-nn)}{6}x^3 - \frac{n(4-4nn)}{24}x^4 + \frac{n(5-10n^2+n^4)}{120}x^5 - \text{etc.}}{1 - x + \frac{1-nn}{2}x^2 - \frac{1-3nn}{6}x^3 + \frac{1-6nn+n^4}{24}x^4 - \text{etc.}}$$

But in general, if it is $n = 1$, one will have

$$e^{\omega}\sin(x+\omega) = \sin x\left(1+\omega-\frac{1}{3}\omega^3-\frac{1}{6}\omega^4-\frac{1}{30}\omega^5+\frac{1}{630}\omega^7+\text{etc.}\right)$$

$$+\omega\cos x\left(1+\omega+\frac{1}{3}\omega^2-\frac{1}{30}\omega^4-\frac{1}{90}\omega^5-\frac{1}{630}\omega^6+\text{etc.}\right)$$

But if it is $n = 0$, because $\sin n(x+\omega) = n(x+\omega)$ and $\sin nx = nx$ and $\cos nx = 1$, if in divides by n everywhere, it will arise

$$e^{\omega}(x+\omega) = x+\omega x+\frac{1}{2}\omega^2 x+\frac{1}{6}\omega^3 x+\frac{1}{24}\omega^4 x+\text{etc.}$$

$$+\omega+\omega^2+\frac{1}{2}\omega^3+\frac{1}{6}\omega^4+\frac{1}{24}\omega^5+\text{etc.},$$

the correctness of which is manifest.