On Finding finite differences *

Leonhard Euler

§44 How from the finite differences of the functions their differentials can easily be found, we explained at the beginning and even derived the principle of differentials from this source. For, if the differences, if they were assumed to be finite, vanish and go over into nothing, the differentials arise; and because in this case many and often innumerable terms, which constitute the finite difference, are neglected, the differentials can be found a lot easier and expressed both more convenient and succinct than the finite differences. And therefore there seems to be no way to ascend from differentials to finite differences. Nevertheless, by the method we will use here one will be able to define the finite differences from the differentials of all orders of any function.

§45 Let y be any function of x; because this having put $x + dx$ for x goes over into $y + dy$, if again instead of x it is put $x + dx$, the value $y + dy$ will be augmented by its differential $dy + ddy$ and it will be $= y + 2dy + ddy$ which value therefore corresponds to $x + 2dx$ put instead for x. In similar way, if we put that x is continuously augmented by its differential dx that it successively takes the values $x + dx$, $x + 2dx$, $x + 3dx$, $x + 4dx$ etc., the corresponding values of y will be those which this table indicates.

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Values of	Corresponding Values of the Function						
x	y						
$x + 1dx$	y	$+ dy$					
$x + 2dx$	y	$+ 2dy$	$+ ddy$				
$x + 3dx$	y	$+ 3dy$	$+ 3ddy$	$+ d^3y$			
$x + 4dx$	y	$+ 4dy$	$+ 6ddy$	$+ 4d^3y$	$+ d^4y$		
$x + 5dx$	y	$+ 5dy$	$+ 10ddy$	$+ 10d^3y$	$+ 5d^4y$	$+ d^5y$	
$x + 6dx$	y	$+ 6dy$	$+ 15ddy$	$+ 20d^3y$	$+ 15d^4y$	$+ 6d^5y$	$+ d^6y$
etc.				etc.			

§46 Therefore, if in general x goes over into $x + ndx$, the function y will take this form

$$y + \frac{n}{1}dy + \frac{n(n-1)}{1 \cdot 2}ddy + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}d^3y + \frac{n(n-1)(n-2)(n-3)}{1 \cdot 2 \cdot 3 \cdot 4}d^4y + \text{etc.};$$

even though in this expression any term is infinitely smaller than its preceding term, we nevertheless did not leave out any term, that this formula is rendered apt for the present task. For, we will assume an infinitely large number for n, and since we know that it can happen that the product of an infinitely large and an infinitely small quantity becomes equal to a finite quantity, the second term can certainly become homogeneous to the second or the quantity ndy can represent a finite quantity. Because of the same reason the third term $\frac{n(n-1)}{1\cdot 2}ddy$, even though ddy is infinitely smaller than ddy, nevertheless, because the one factor $\frac{n(n-1)}{1\cdot 2}$ is infinitely larger than n, can also express a finite quantity; and so having put n to be an infinite number, it is not possible to neglect any term of that expression.

§47 But having put n to be an infinite number, by whatever finite number it is either augmented or diminished, the resulting number will have the ratio of equality to n and hence one can write the number n for the single factors $n-1$, $n-2$, $n-3$, $n-4$ etc. everywhere. For, because it is $\frac{n(n-1)}{1\cdot 2}ddy = \frac{1}{2}nnddy - \frac{1}{2}nddy$, the first term $\frac{1}{2}nnddy$ will have a ratio to the second $\frac{1}{2}nddy$ of n to 1 and so it will vanish with respect to the latter; therefore, instead of $\frac{n(n-1)}{1\cdot 2}$ one will be able to write $\frac{1}{2}nn$. In similar manner, the coefficient of the fourth term $\frac{n(n-1)(n-3)}{1 \cdot 2 \cdot 3}$ can be contracted to $\frac{n^3}{6}$ and in the same manner one can neglect the numbers, by which n is diminished in the factors, in the following. But having done this, the function y, if in place of x one puts $x + ndx$ while n is an infinite number, will take the following value

$$y + \frac{ndy}{1} + \frac{nnddy}{1 \cdot 2} + \frac{n^3 d^3 y}{1 \cdot 2 \cdot 3} + \frac{n^4 d^4 y}{1 \cdot 2 \cdot 3 \cdot 4} + \frac{n^5 d^5 y}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} + \text{etc.}$$

§48 Because therefore having taken n as infinitely large number, even though dx is infinitely small, the product ndx can express a finite quantity, let us put $ndx = \omega$ that it is $n = \frac{\omega}{dx}$; n will certainly be an infinite number, because it is the quotient resulting from a division of the finite quantity ω by the infinitely small dx. But having used this value instead of n, if the variable quantity x is augmented by a certain quantity ω or if instead of x one puts $x + \omega$, then a certain function y of it will go over into this form

$$y + \frac{\omega dy}{1dx} + \frac{\omega^2 ddy}{1 \cdot 2dx^2} + \frac{\omega^3 d^3 y}{1 \cdot 2 \cdot 3dx^3} + \frac{\omega^4 d^4 y}{1 \cdot 2 \cdot 3 \cdot 4dx^4} + \text{etc.},$$

the single terms of which expression can be found by continued differentiation of y. For, because y is a function x, we showed above that these functions $\frac{dy}{dx}$, $\frac{ddy}{dx^2}$, $\frac{d^3}{dx^3}$ etc. all exhibit finite quantities.

§49 Because therefore while the variable quantity x is assumed to be augmented by the finite quantity ω, any function y of it is augmented by its first difference, which above we indicated by Δy where $\omega = \Delta x$, one will be able to find the difference of y by continued differentiation; for, it will be

$$\Delta y = \frac{\omega dy}{dx} + \frac{\omega^2 ddy}{2dx^2} + \frac{\omega^3 d^3 y}{6dx^3} + \frac{\omega^4 d^4 y}{24dx^4} + \text{etc.}$$

or

$$\Delta y = \frac{\Delta x}{1} \cdot \frac{\Delta y}{dx} + \frac{\Delta x^2}{2} \cdot \frac{ddy}{dx^2} + \frac{\Delta x^3}{6} \cdot \frac{d^3 y}{dx^3} + \frac{\Delta x^4}{24} \cdot \frac{d^4 y}{dx^4} + \text{etc.}$$

And so the finite difference Δy is expressed by a progression whose single terms proceed in powers of Δx. And hence vice versa it is clear, if the quantity x is augmented only be an infinitely small quantity, that Δx goes over into its differential dx, that all terms compared to the first vanish and that it will be $\Delta y = dy$; for, having set $\Delta x = dx$ the difference, by definition, Δy goes over into the differential dy.

§50 Because, if instead of x it is put $x+\omega$, any function y of it takes the following value

$$y+\frac{\omega dy}{dx}+\frac{\omega^2 ddy}{2dx^2}+\frac{\omega^3 d^3y}{6dx^3}+\frac{\omega^4 d^4y}{24dx^4}+\text{etc.},$$

the validity of this expression can be probed in examples of such a kind in which the higher differentials of y finally vanish; for, in these cases the number of terms of the superior expression will become finite.

Example 1

The value of the expression $xx-x$ shall be sought after, if instead of x one puts $x+1$.

Put $y=xx-x$, and because it is set that x goes over into $x+1$, it will be $\omega=1$; now, having taken the differentials, it will be

$$\frac{dy}{dx}=2x-1, \quad \frac{ddy}{dx^2}=2, \quad \frac{d^3y}{dx^3}=0 \quad \text{etc.}$$

Hence, the function $y=xx-x$ having put $x+1$ instead of x will go over into

$$xx-x+1(2x-1)+\frac{1}{2}\cdot 2=xx+x.$$

But if in $xx-x$ one actually puts $x+1$ instead of x,

$$\begin{array}{lll} xx & \text{will go over into} & xx+2x+1 \\ x & \text{will go over into} & \quad\;\; +x\;\;+1 \end{array}$$

so, in total

$$xx-x \quad \text{will go over into} \quad xx+x$$

Example 2

The value of the expression x^3+xx+x shall be sought after, if instead of x it is put $x+2$.

Put $y=x^3+xx+x$ and it will be $\omega=2$; now, because it is

$$y=x^3+xx+x,$$

it will be

$$\frac{dy}{dx} = 3xx + 2x + 1, \quad \frac{ddy}{dx^2} = 6x + 2, \quad \frac{d^3y}{dx^3} = 6, \quad \frac{d^4y}{dx^4} = 0 \quad \text{etc.}$$

From these the value of the function $y = x^3 + xx + x$, if for x one sets $x + 2$, will be the following

$$x^3 + xx + x + 2(3xx + 2x + 1) + \frac{4}{2}(6x + 2) + \frac{8}{6} \cdot 6 = x^3 + 7xx + 17x + 14,$$

which same arises, if instead of x actually $x + 2$ is substituted.

EXAMPLE 3

The value of the expression $xx + 3x + 1$ shall be sought after, if instead of x one puts $x - 3$.

Therefore, it will be $\omega = -3$, and having put

$$y = xx + 3x + 1$$

it will be

$$\frac{dy}{dx} = 2x + 3, \quad \frac{ddy}{dx^2} = 2, \quad \frac{d^3y}{dx^3} = 0 \quad \text{etc.},$$

whence having put $x - 3$ instead of x the function $x^2 + 3x + 1$ will go over into

$$x^2 + 3x + 1 - \frac{3}{1}(2x + 3) + \frac{9}{2} \cdot 2 = x^2 - 3x + 1.$$

§51 If for ω a negative number is taken, one will find the value, which any function of x takes, if the quantity x is diminished by the given quantity ω. Of course, if instead if x one puts $x - \omega$, an arbitrary function y of x will take this value

$$y - \frac{\omega dy}{dx} + \frac{\omega^2 ddy}{2dx^2} - \frac{\omega^3 d^3y}{6dx^3} + \frac{\omega^4 d^4y}{24dx^4} - \text{etc.},$$

whence all variations, the function y can undergo, while the quantity x is changed anyhow, can be found. But if y was a polynomial function of x, since one finally gets to vanishing differentials, the varied value will be expressed by means of a finite expression; but if y was not a function of this kind, the varied value will be expressed by means of an infinite series, whose sum, since, if the substitution is actually done, the varied value is easily assigned, can be exhibited by a finite expression.

§52 But as the first difference was found, so also the following differences can be exhibited by similar expressions. For, let x successively take the values $x+\omega$, $x+2\omega$, $x+3\omega$, $x+4\omega$ etc. and indicate the corresponding values of y by y^{I}, y^{II}, y^{III}, y^{IV} etc., as we put in the beginning of this book. Since therefore y^{I}, y^{II}, y^{III}, y^{IV} etc. are the values, which y obtains, if instead of y one respectively writes $x+\omega$, $x+2\omega$, $x+3\omega$, $x+4\omega$ etc., by means of the demonstrated method the values of these ys will be expressed this way:

$$\begin{aligned}
y^{\mathrm{I}} &= y + \frac{\omega dy}{dx} + \frac{\omega^2 ddy}{2dx^2} + \frac{\omega^3 d^3y}{6dx^3} + \frac{\omega^4 d^4y}{24dx^4} + \text{etc.}\\
y^{\mathrm{II}} &= y + \frac{2\omega dy}{dx} + \frac{4\omega^2 ddy}{2dx^2} + \frac{8\omega^3 d^3y}{6dx^3} + \frac{16\omega^4 d^4y}{24dx^4} + \text{etc.}\\
y^{\mathrm{III}} &= y + \frac{3\omega dy}{dx} + \frac{9\omega^2 ddy}{2dx^2} + \frac{27\omega^3 d^3y}{6dx^3} + \frac{81\omega^4 d^4y}{24dx^4} + \text{etc.}\\
y^{\mathrm{IV}} &= y + \frac{4\omega dy}{dx} + \frac{16\omega^2 ddy}{2dx^2} + \frac{64\omega^3 d^3y}{6dx^3} + \frac{256\omega^4 d^4y}{24dx^4} + \text{etc.}\\
&\text{etc.}
\end{aligned}$$

§53 Because therefore, if Δy, $\Delta^2 y$, $\Delta^3 y$, $\Delta^4 y$ etc. denote the first, second, third, fourth etc. differences, it is

$$\begin{aligned}
\Delta y &= y^{\mathrm{I}} - y\\
\Delta^2 y &= y^{\mathrm{II}} - 2y^{\mathrm{I}} + y\\
\Delta^3 y &= y^{\mathrm{III}} - 3y^{\mathrm{II}} + 3y^{\mathrm{I}} - y\\
\Delta^4 y &= y^{\mathrm{IV}} - 4y^{\mathrm{III}} + 6y^{\mathrm{II}} - 4y^{\mathrm{I}} + y\\
&\text{etc.,}
\end{aligned}$$

these differences by means of differentials will be expressed this way:

$$\begin{aligned}
\Delta y &= \frac{\omega dy}{dx} + \frac{\omega^2 ddy}{dx^2} + \frac{\omega^3 d^3y}{6dx^3} + \frac{\omega^4 d^4y}{24dx^4} + \text{etc.}\\
\Delta^2 y &= \frac{(2^2 - 2\cdot 1)\omega^2 ddy}{2dx^2} + \frac{(2^3 - 2\cdot 1)\omega^3 d^3y}{6dx^3} + \frac{(2^4 - 2\cdot 1)\omega^4 d^4y}{24dx^4} + \text{etc.}\\
\Delta^3 y &= \frac{(3^3 - 3\cdot 2^3 + 3\cdot 1)\omega^3 d^3y}{6dx^3} + \frac{(3^4 - 3\cdot 2^4 + 3\cdot 1)\omega^4 d^4y}{24dx^4} + \text{etc.}\\
\Delta^4 y &= \frac{(4^4 - 4\cdot 3^4 + 6\cdot 2^4 - 4\cdot 1)\omega^4 d^4y}{24dx^4} + \frac{(4^5 - 4\cdot 3^5 + 6\cdot 2^5 - 4\cdot 1)\omega^5 d^5y}{120dx^5} + \text{etc.}\\
&\text{etc.}
\end{aligned}$$

§54 Of how much use these expressions of differences are in the doctrine of series and progressions, is immediately clear and we will explain it in greater detail in the following. Meanwhile, in this chapter we want to consider use, which immediately follows from this for understanding of series. Although usually the indices of the terms of a certain series are assumed to constitute an arithmetic progression whose difference is the unity, that the use extends further and the application is easier, let us nevertheless set the difference $= \omega$, such that, if the general term or that corresponding to the index x, was x, the following correspond to the indices $x+\omega$, $x+2\omega$, $x+3\omega$ etc. If therefore to these indices correspond the following terms

$$\begin{array}{cccccc}
x, & x+\omega, & x+2\omega, & x+3\omega, & x+4\omega & \text{etc.}\\
y, & P, & Q, & R, & S, & \text{etc.}
\end{array}$$

the single terms will be defined from y and its differential this way:

$$\begin{aligned}
P &= y + \frac{\omega dy}{dx} + \frac{\omega^2 ddy}{2dx^2} + \frac{\omega^3 d^3y}{6dx^3} + \frac{\omega^4 d^4y}{24dx^4} + \text{etc.}\\
Q &= y + \frac{2\omega dy}{dx} + \frac{4\omega^2 ddy}{2dx^2} + \frac{9\omega^3 d^3y}{6dx^3} + \frac{16\omega^4 d^4y}{24dx^4} + \text{etc.}\\
R &= y + \frac{3\omega dy}{dx} + \frac{9\omega^2 ddy}{2dx^2} + \frac{27\omega^3 d^3y}{6dx^3} + \frac{81\omega^4 d^4y}{24dx^4} + \text{etc.}
\end{aligned}$$

$$S = y + \frac{4\omega dy}{dx} + \frac{16\omega^2 ddy}{2dx^2} + \frac{64\omega^3 d^3y}{6dx^3} + \frac{256\omega^4 d^4y}{24dx^4} + \text{etc.}$$
$$T = y + \frac{5\omega dy}{dx} + \frac{25\omega^2 ddy}{2dx^2} + \frac{125\omega^3 d^3y}{6dx^3} + \frac{625\omega^4 d^4y}{24dx^4} + \text{etc.}$$
etc.

§55 If these expressions are subtracted from each other, y will not longer go into the differences and it will be

$$P - y = \frac{\omega dy}{dx} + \frac{\omega^2 ddy}{2dx^2} + \frac{\omega^3 d^3y}{6dx^3} + \frac{\omega^4 d^4y}{24dx^4} + \text{etc.}$$
$$Q - P = \frac{\omega dy}{dx} + \frac{3\omega^2 ddy}{2dx^2} + \frac{7\omega^3 d^3y}{6dx^3} + \frac{15\omega^4 d^4y}{24dx^4} + \text{etc.}$$
$$R - Q = \frac{\omega dy}{dx} + \frac{5\omega^2 ddy}{2dx^2} + \frac{19\omega^3 d^3y}{6dx^3} + \frac{65\omega^4 d^4y}{24dx^4} + \text{etc.}$$
$$S - R = \frac{\omega dy}{dx} + \frac{7\omega^2 ddy}{2dx^2} + \frac{37\omega^3 d^3y}{6dx^3} + \frac{175\omega^4 d^4y}{24dx^4} + \text{etc.}$$
$$T - S = \frac{\omega dy}{dx} + \frac{9\omega^2 ddy}{2dx^2} + \frac{61\omega^3 d^3y}{6dx^3} + \frac{369\omega^4 d^4y}{24dx^4} + \text{etc.}$$
etc.

If these expressions are again subtracted from each other, the first differentials will also cancel each other and it will be

$$Q - 2P + y = \frac{2\omega^2 ddy}{2dx^2} + \frac{6\omega^3 d^3y}{6dx^3} + \frac{14\omega^4 d^4y}{24dx^4} + \text{etc.}$$
$$R - 2Q + P = \frac{2\omega^2 ddy}{2dx^2} + \frac{12\omega^3 d^3y}{6dx^3} + \frac{50\omega^4 d^4y}{24dx^4} + \text{etc.}$$
$$S - 2R + Q = \frac{2\omega^2 ddy}{2dx^2} + \frac{18\omega^3 d^3y}{6dx^3} + \frac{110\omega^4 d^4y}{24dx^4} + \text{etc.}$$
$$T - 2S + R = \frac{2\omega^2 ddy}{2dx^2} + \frac{24\omega^3 d^3y}{6dx^3} + \frac{194\omega^4 d^4y}{24dx^4} + \text{etc.}$$
etc.

Having subtracted them from each other again the second differentials will also go out of the computation:

$$
\begin{aligned}
R - 3Q + 3P - y &= \frac{6\omega^3 d^3 y}{6dx^3} + \frac{36\omega^4 d^4 y}{24dx^4} + \text{etc.}\\
S - 3R + 3Q - P &= \frac{6\omega^3 d^3 y}{6dx^3} + \frac{60\omega^4 d^4 y}{24dx^4} + \text{etc.}\\
T - 3S + 3R - Q &= \frac{6\omega^3 d^3 y}{6dx^3} + \frac{84\omega^4 d^4 y}{24dx^4} + \text{etc.}\\
&\text{etc.}
\end{aligned}
$$

By continuing the subtraction it will be

$$
\begin{aligned}
S - 4R + 6Q - 4P + y &= \frac{24\omega^4 d^4 y}{24dx^4} + \text{etc.}\\
T - 4S + 6R - 4Q + P &= \frac{24\omega^4 d^4 y}{24dx^4} + \text{etc.}\\
&\text{etc.}
\end{aligned}
$$

and

$$
\begin{aligned}
T - 5S + 10R - 10Q + 5P - y &= \frac{120\omega^5 d^5 y}{120dx^5} + \text{etc.}\\
&\text{etc.}
\end{aligned}
$$

§56 Therefore, if y was a polynomial function of x, since its higher differentials will finally vanish, by proceeding this way one will finally reach vanishing expressions. Because therefore these expressions are differences of y, let us consider their forms and coefficients more diligently.

$$
\begin{aligned}
y &= y\\
\Delta y &= \frac{\omega dy}{dx} + \frac{\omega^2 ddy}{2dx^2} + \frac{\omega^3 d^3 y}{6dx^3} + \frac{\omega^4 d^4 y}{24dx^4} + \frac{\omega^5 d^5 y}{120dx^5} + \text{etc.}\\
\Delta^2 y &= \frac{\omega^2 ddy}{dx^2} + \frac{3\omega^3 d^3 y}{3dx^3} + \frac{7\omega^4 d^4 y}{3\cdot 4dx^4} + \frac{15\omega^5 d^5 y}{3\cdot 4\cdot 5dx^5} + \frac{31\omega^6 d^6 y}{3\cdot 4\cdot 5\cdot 6dx^6} + \text{etc.}\\
\Delta^3 y &= \frac{\omega^3 d^3 y}{dx^3} + \frac{6\omega^4 d^4 y}{4dx^4} + \frac{25\omega^5 d^5 y}{4\cdot 5dx^5} + \frac{90\omega^6 d^6 y}{4\cdot 5\cdot 6dx^6} + \frac{301\omega^7 d^7 y}{4\cdot 5\cdot 6\cdot 7dx^7} + \text{etc.}\\
\Delta^4 y &= \frac{\omega^4 d^4 y}{dx^4} + \frac{10\omega^5 d^5 y}{5dx^5} + \frac{65\omega^6 d^6 y}{5\cdot 6dx^6} + \frac{350\omega^7 d^7 y}{5\cdot 6\cdot 7dx^7} + \frac{1701\omega^8 d^8 y}{5\cdot 6\cdot 7\cdot 8dx^8} + \text{etc.}
\end{aligned}
$$

$$\Delta^5 y = \frac{\omega^5 d^5 y}{dx^5} + \frac{15\omega^6 d^6 y}{6dx^6} + \frac{140\omega^7 d^7 y}{6\cdot 7dx^7} + \frac{1050\omega^8 d^8 y}{6\cdot 7\cdot 8dx^8} + \frac{6951\omega^9 d^9 y}{6\cdot 7\cdot 8\cdot 9dx^9} + \text{etc.}$$
$$\Delta^6 y = \frac{\omega^6 d^6 y}{dx^6} + \frac{21\omega^7 d^7 y}{7dx^7} + \frac{266\omega^8 d^8 y}{7\cdot 8dx^8} + \frac{2646\omega^9 d^9 y}{7\cdot 8\cdot 9dx^9} + \frac{22827\omega^{10} d^{10} y}{7\cdot 8\cdot 9\cdot 10dx^{10}} + \text{etc.}$$
etc.

§57 Let us also consider the same series continued backwards at the same time, which contains the terms corresponding to the indices $x-\omega$, $x-2\omega$, $x-3\omega$ etc.

$x-4\omega$	$x-3\omega,$	$x-2\omega,$	$x-\omega,$	$x,$	$x+\omega,$	$x+2\omega,$	$x+3\omega,$	$x+4\omega$	etc.
$s,$	$r,$	$q,$	$p,$	$y,$	$P,$	$Q,$	$R,$	S	etc.

Because therefore it is

$$p = y - \frac{\omega dy}{dx} + \frac{\omega^2 ddy}{2dx^2} - \frac{\omega^3 d^3 y}{6dx^3} + \frac{\omega^4 d^4 y}{24dx^4} - \text{etc.}$$
$$q = y - \frac{2\omega dy}{dx} + \frac{4\omega^2 ddy}{2dx^2} - \frac{8\omega^3 d^3 y}{6dx^3} + \frac{16\omega^4 d^4 y}{24dx^4} - \text{etc.}$$
$$r = y - \frac{3\omega dy}{dx} + \frac{9\omega^2 ddy}{2dx^2} - \frac{27\omega^3 d^3 y}{6dx^3} + \frac{81\omega^4 d^4 y}{24dx^4} - \text{etc.}$$
$$s = y - \frac{4\omega dy}{dx} + \frac{16\omega^2 ddy}{2dx^2} - \frac{64\omega^3 d^3 y}{6dx^3} + \frac{256\omega^4 d^4 y}{24dx^4} - \text{etc.}$$
etc.,

by subtracting these values from the superior ones P, Q, R, S etc. it will be

$$\frac{P-p}{2} = \frac{\omega dy}{dx} + \frac{\omega^3 d^3 y}{6dx^3} + \frac{\omega^5 d^5 y}{120dx^5} + \text{etc.}$$
$$\frac{Q-q}{2} = \frac{2\omega dy}{dx} + \frac{8\omega^3 d^3 y}{6dx^3} + \frac{32\omega^5 d^5 y}{120dx^5} + \text{etc.}$$
$$\frac{R-r}{2} = \frac{3\omega dy}{dx} + \frac{27\omega^3 d^3 y}{6dx^3} + \frac{243\omega^5 d^5 y}{120dx^5} + \text{etc.}$$
$$\frac{S-s}{2} = \frac{4\omega dy}{dx} + \frac{64\omega^3 d^3 y}{6dx^3} + \frac{1024\omega^5 d^5 y}{120dx^5} + \text{etc.}$$
etc.

But if these terms are added to the superior ones, then, as the differentials of even orders are missing here, the odd differentials will go out of the computation. For, it will be

$$\begin{aligned}
\frac{P+p}{2} &= y + \frac{\omega^2 ddy}{2dx^2} + \frac{\omega^4 d^4y}{24dx^4} + \frac{\omega^6 d^6y}{720dx^6} + \text{etc.}\\
\frac{Q+q}{2} &= y + \frac{4\omega^2 ddy}{2dx^2} + \frac{16\omega^4 d^4y}{24dx^4} + \frac{64\omega^6 d^6y}{720dx^6} + \text{etc.}\\
\frac{R+r}{2} &= y + \frac{9\omega^2 ddy}{2dx^2} + \frac{81\omega^4 d^4y}{24dx^4} + \frac{729\omega^6 d^6y}{720dx^6} + \text{etc.}\\
\frac{S+s}{2} &= y + \frac{16\omega^2 ddy}{2dx^2} + \frac{256\omega^4 d^4y}{24dx^4} + \frac{4096\omega^6 d^6y}{720dx^6} + \text{etc.}\\
&\text{etc.}
\end{aligned}$$

§58 Since the preceding terms can all be expressed, if they are collected into one sum, the summatory term of the propounded series will arise. Let the first term correspond to the index $x - n\omega$ and the first term itself will be

$$= y - \frac{n\omega dy}{dx} + \frac{n^2\omega^2}{2dx^2} - \frac{n^3\omega^3 d^3y}{6dx^3} + \frac{n^4\omega^4 d^4y}{24dx^4} + \text{etc.}$$

Since therefore the term corresponding to the index x will be $= y$ and the number of all terms is $= n+1$, the sum of all taken from the first, y included, or the summatory term will be

$$\begin{aligned}
= (n+1)y &- \frac{\omega dy}{dx}(1 + 2 + 3 + \cdots + n)\\
&+ \frac{\omega^2 ddy}{2dx^2}(1 + 2^2 + 3^2 + \cdots + n^2)\\
&- \frac{\omega^3 d^3y}{6dx^3}(1 + 2^3 + 3^3 + \cdots + n^3)\\
&+ \frac{\omega^4 d^4y}{24dx^4}(1 + 2^4 + 3^4 + \cdots + n^4)\\
&- \frac{\omega^5 d^5y}{120dx^5}(1 + 2^5 + 3^5 + \cdots + n^5)\\
&+ \text{etc.}
\end{aligned}$$

§59 Above we exhibited the sums of these single series [§ 62 of the first part]; if these are substituted here, the sum of the propounded series will be

$$\begin{aligned}
= (n+1)y - &\frac{\omega dy}{dx}\left(\frac{1}{2}nn + \frac{1}{2}n\right)\\
+ &\frac{\omega^2 ddy}{2dx^2}\left(\frac{1}{3}n^3 + \frac{1}{2}nn + \frac{1}{6}n\right)\\
- &\frac{\omega^3 d^3y}{6dx^3}\left(\frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}n^2\right)\\
+ &\frac{\omega^4 d^4y}{24dx^4}\left(\frac{1}{5}n^5 + \frac{1}{2}n^4 + \frac{1}{3}n^3 - \frac{1}{30}n\right)\\
- &\frac{\omega^5 d^5y}{120dx^5}\left(\frac{1}{6}n^6 + \frac{1}{2}n^5 + \frac{5}{12}n^4 - \frac{1}{12}n^2\right)\\
&\text{etc.},
\end{aligned}$$

where n will be given from the index of the first term from which the sum is computed. If one puts $\omega = 1$ and the index of the first term is $= 1$, of the second $= 2$ and the last $= x$ such that this series is propounded

$$\begin{array}{lllll}
1, & 2, & 3, & 4, \cdots\cdots x\\
a, & b, & c, & d, \cdots\cdots y,
\end{array}$$

the sum of this series because of $x - n = 1$ and $n = x - 1$ will be

$$\begin{aligned}
= xy - &\frac{dy}{dx}\left(\frac{1}{2}xx - \frac{1}{2}x\right)\\
+ &\frac{ddy}{2dx^2}\left(\frac{1}{3}x^3 - \frac{1}{2}xx + \frac{1}{6}x\right)\\
- &\frac{d^3y}{6dx^3}\left(\frac{1}{4}x^4 - \frac{1}{2}x^3 + \frac{1}{4}xx\right)\\
+ &\frac{d^4y}{24dx^4}\left(\frac{1}{5}x^5 - \frac{1}{2}x^4 + \frac{1}{3}xx - \frac{1}{30}x\right)
\end{aligned}$$

$$+\ \frac{d^5y}{120dx^4}\left(\frac{1}{6}x^6-\frac{1}{2}x^5+\frac{5}{12}x^4-\frac{1}{12}x^2\right)$$
$$+\ \frac{d^6y}{720dx^4}\left(\frac{1}{7}x^7-\frac{1}{2}x^6+\frac{1}{2}x^5\ -\frac{1}{6}x^3+\frac{1}{42}x\right)$$
$$\text{etc.}$$

§60 From this expression, since the coefficients will be augmented immensely, if x was a large number, hardly anything of use for the doctrine of series follows; it will nevertheless be helpful to have mentioned other properties following from there. Let the general term be x^n and the summatory term shall be indicated by $S.y$ or $S.x$. Having used this notation everywhere it will be

$$\frac{1}{2}xx-\frac{1}{2}x=S.x-x$$
$$\frac{1}{3}x^3-\frac{1}{2}x^2+\frac{1}{6}x=S.x^2-x^2$$
$$\frac{1}{4}x^4-\frac{1}{2}x^3+\frac{1}{4}xx=S.x^3-x^3$$
$$\text{etc.}$$

Therefore, it will be obtained from the superior expression

$$S.x^n=x^{n+1}-nx^{n+1}S.x+nx^n$$
$$+\frac{n(n-1)}{1\cdot 2}x^{n-2}S.x^2-\frac{n(n-1)}{1\cdot 2}x^n-\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}x^{n-3}S.x^3+\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}x^n+\text{etc.}$$

But because it is

$$(1-1)^n=0=1-n+\frac{n(n-1)}{1\cdot 2}-\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}+\text{etc.},$$

it will be

$$n-\frac{n(n-1)}{1\cdot 2}+\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}-\text{etc.}=1$$

and hence having excluded the case $n=0$ in which this expression becomes $=0$ it is

$$S.x^n = x^{n+1} + x^n - nx^{n-1}S.x + \frac{n(n-1)}{1\cdot 2}x^{n-2}S.x^2 - \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}x^{n-3}S.x^3$$
$$+\frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4}x^{n-4}S.x^4 - \text{etc.}$$

§61 To see both the validity and the power of this formula in more clarity, let us expand the single cases and at first let $n = 1$ and it will be $S.x = x^2 + x - S.x$ and hence $S.x = \frac{xx+x}{2}$, as it is sufficiently known. Therefore, let us put $n = 2$ and it will be

$$S.x^2 = x^3 + xx - 2xS.x + S.x^2,$$

and hence

$$S.x^3 = \frac{3}{2}xS.x^2 - \frac{3}{2}x^2S.x + \frac{1}{2}x^3(x+1);$$

if one puts $n = 4$, it will arise

$$S.x^4 = x^5 + x^4 - 4x^3S.x + 6x^2S.x^2 - 4xS.x^3 + S.x^4,$$

whence because of the cancelled $S.x^4$ it will be

$$S.x^3 = \frac{3}{2}xS.x^2 - x^2S.x + \frac{1}{4}x^3(x+1);$$

if from the triple of this the double of the preceding is subtracted, it will remain

$$S.x^3 = \frac{3}{2}xS.x^2 - \frac{1}{4}x^3(x+1).$$

If one puts $n = 5$, it will become

$$S.x^5 = x^5 + x^5 - 5x^4S.x + 10x^3S.x^2 - 10x^2S.x^3 + 5xS.x^4 - S.x^5$$

or

$$S.x^5 = \frac{5}{2}xS.x^4 - 5x^2S.x^3 + 5x^3S.x^2 - \frac{5}{2}x^4S.x + \frac{1}{2}x^5(x+1)$$

and from $n = 6$ it follows

$$S.x^6 = x^7 + x^6 - 6x^5S.x + 15x^4S.x^2 - 20x^3S.x^3 + 15x^2S.x^4 - 6xS.x^5 + S.x^6$$

or

$$S.x^5 = \frac{5}{2}xS.x^4 - \frac{10}{3}x^2S.x^3 + \frac{5}{2}x^3S.x^2 - x^4S.x + \frac{1}{6}x^5(x+1).$$

§62 From these we therefore conclude in general, if $n = 2m+1$, that it will be

$$S.x^{2m+1} = \frac{2m+1}{2}xS.x^{2m} - \frac{(2m+1)2m}{2\cdot 1\cdot 2}x^2S.x^{2m-1}$$
$$+\frac{(2m+1)2m(2m-1)}{2\cdot 1\cdot 2\cdot 3}x^3S.x^{2m-2} - \cdots - \frac{2m+1}{2}x^{2m}S.x + \frac{1}{2}x^{2m+1}(x+1).$$

But if it was $n = 2m+2$, since the terms $S.x^{2m+2}$ cancel each other, one will find

$$S.x^{2m+1} = \frac{2m+1}{2}xS.x^{2m} - \frac{(2m+1)2m}{2\cdot 3}x^2S.x^{2m-1}$$
$$+\frac{(2m-1)2m(2m+1)}{2\cdot 3\cdot 4}x^3S.x^{2m-2} - \cdots - x^{2m}S.x + \frac{1}{2m+2}x^{2m+1}(x+1).$$

Therefore, the sum of the odd powers can in two ways be determined from the sums of the lower powers and from the various combinations of these formulas infinitely many other can be formed.

§63 But the sum of the odd powers can be determined a lot easier from the preceding ones and for this it certainly suffices to know only the sum of the preceding even power. For, from the sums of powers exhibited above [§ 62 of the first part] it is known that the number of terms constituting the sums is only increased in the even powers, such that the sum of the odd powers consists of as many terms as the sum of the preceding even power. So, if the sum of the even power x^{2n} is

$$S.x^{2n} = \alpha x^{2n+1} + \beta x^{2n} + \gamma x^{2n-1} - \delta x^{2n-3} + \varepsilon x^{2n-5} - \text{etc.}$$

(for, we saw that after the third term each second is missing and at the same time the signs alternate), hence the sum of the following power x^{2n+1} will be found, if the single terms of it are respectively multiplied by these numbers

$$\frac{2n+1}{2n+2}x, \quad \frac{2n+1}{2n+1}x, \quad \frac{2n+1}{2n}x, \quad \frac{2n+1}{2n-1}x, \quad \frac{2n+1}{2n-2}x \quad \text{etc.}$$

not omitting the missing terms; and therefore it will be

$$S.x^{2n+1} = \frac{2n+1}{2n+2}\alpha x^{2n+2} + \frac{2n+1}{2n+1}\beta x^{2n+1} + \frac{2n+1}{2n}\gamma x^{2n} - \frac{2n+1}{2n-1}\delta x^{2n-2}$$
$$+\frac{2n+1}{2n-4}\varepsilon x^{2n-4} - \frac{2n+1}{2n-6}\zeta x^{2n-6} + \text{etc.}$$

If therefore the sum of the power x^{2n} is known, from it the sum of the following power x^{2n+1} can be formed in a convenient manner.

§64 This investigation of the following sums is also extended to the even powers; but since the sums of these receive a new term, this term is not found by means of this method, nevertheless it can always be found from the nature of the series itself, from which it is clear, if one puts $x = 1$, that the sum has also to become $= 1$. But vice versa from a the sum of certain power one will always be able to find the sum of the preceding powers. For, if it was

$$S.x^n = \alpha x^{n+1} + \because x^n + \gamma x^{n-1} - \delta x^{n-3} + \varepsilon x^{n-5} - \zeta x^{n-7} + \text{etc.},$$

it will be for the preceding power

$$S.x^{n-1} = \frac{n+1}{n}\alpha x^n + \frac{n}{n}\beta x^{n-1} + \frac{n-1}{n}\gamma x^{n-2} - \frac{n-3}{n}\delta x^{n-4} + \text{etc.}$$

and hence one can go backwards as far as one desires. But it is to be noted that it always is $\alpha = \frac{1}{2}$ and $\beta = \frac{1}{2}$ as is it already clear from the formulas given above.

§65 To anyone paying attention it will immediately become clear that the sum of x^{n-1} arises, if the sum of the powers x^n is differentiated and its differential is divided by ndx; and it will be $d.S.x^n = ndx \cdot S.x^{n-1}$, and because it is $d.x^n = nx^{n-1}dx$, it will be

$$d.S.x^n = S.nx^{n-1}dx = S.d.x^n;$$

from this it is understood that the differential of the sum becomes equal to the sum of the differentials; so in general, if the general term of a certain series was $= y$ and $S.y$ was its summatory term, it will also be $S.dy = d.S.y$, this means: the sum of all differentials becomes equal to the differential of the sum of the terms themselves. The truth of this equality is easily seen from those things we treated above on the differentiation of series. For, because it is

$$S.x^n = x^n + (x-1)^n + (x-2)^n + (x-3)^n + (x-4)^n + \text{etc.},$$

it will be

$$\frac{d.S.x^n}{ndx} = x^{n-1} + (x-1)^{n-1} + (x-2)^{n-2} + (x-3)^{n-1} + \text{etc.} = S.x^{n-1},$$

which proof extends to all other series.

§66 But let us return, from where we started, to the differences of functions, on which still several things are to be remarked. Because we saw, if y was any function of x and instead of x one everywhere puts $x \pm \omega$, that the function y will obtain the following value

$$y \pm \frac{\omega dy}{1dx} + \frac{\omega^2 ddy}{1 \cdot 2dx^2} \pm \frac{\omega^3 d^3 y}{1 \cdot 2 \cdot 3dx^3} + \frac{\omega^4 d^4 y}{1 \cdot 2 \cdot 3 \cdot 4dx^4} \pm \frac{\omega^5 d^5 y}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5dx^5} + \text{etc.},$$

this expression will be valid, whether for ω any constant quantity is taken or even a variable value depending on x. For, having found the values of $\frac{dy}{dx}$, $\frac{ddy}{dx^2}$, $\frac{d^3y}{dx^3}$ etc. in the factors ω, ω^2, ω^3 etc. by differentiation, the variability is not considered and hence it does not matter, whether ω denotes a constant quantity or a variable quantity depending on x.

§67 Therefore, let us put that it is $\omega = x$ and in the function y instead of x $x - x = 0$ is written. Therefore, if in any function of x instead of x one writes 0 everywhere, the value of the function will be this one

$$y - \frac{xdy}{1dx} + \frac{x^2 ddy}{1 \cdot 2dx^2} - \frac{x^3 d^3}{1 \cdot 2 \cdot 3dx^3} + \frac{x^4 d^4 y}{1 \cdot 2 \cdot 3 \cdot 4dx^4} - \text{etc.}$$

Therefore, this expression always indicates the value which any function y obtains, if in it one puts $x = 0$, the validity of which statement the following examples will illustrate.

Example 1

Let $y = xx + ax + ab$, whose value, if one puts $x = 0$, shall be sought after which is of course known to be $= ab$.

Because it is $y = xx + ax + ab$, it will be

$$\frac{dy}{1dx} = 2x + a, \quad \frac{ddy}{1 \cdot 2dx^2} = 1$$

and hence the value sought after arises as

$$= xx + ax + ab - x(2x + a) + xx \cdot 1 = ab.$$

Example 2

Let $y = x^3 - 2x + 3$, whose value having put $x = 0$ shall be sought after, which value is known to be $= 3$.

Because it is $y = x^3 - 2x + 3$, it will be

$$\frac{dy}{dx} = 3xx - 2, \quad \frac{ddy}{1 \cdot 2dx^2} = 3x, \quad \frac{d^3y}{1 \cdot 2 \cdot 3dx^3} = 1;$$

the value sought after will be obtained as

$$= x^3 - 2x + 3 - x(3xx - 2) + xx \cdot 3x - x^3 \cdot 1 = 3.$$

Example 3

Let $y = \frac{x}{1-x}$, whose value having put $x = 0$ shall be sought after, which is known to be $= 0$.

Because it is $y = \frac{x}{1-x}$, it will be

$$\frac{dy}{dx} = \frac{1}{(1-x)^2}, \quad \frac{ddy}{1 \cdot 2dx^2} = \frac{1}{(1-x)^3}, \quad \frac{d^3y}{1 \cdot 2 \cdot 3dx^3} = \frac{1}{(1-x)^4} \quad \text{etc.}$$

Hence, the value in question will be

$$= \frac{x}{1-x} - \frac{x}{(1-x)^2} + \frac{xx}{(1-x)^3} - \frac{x^3}{(1-x)^4} + \frac{x^4}{(1-x)^5} - \text{etc.}$$

and therefore the value of this series is $= 0$. This is also plain from the fact that this series truncated by the first term, e.g. $\frac{x}{(1-x)^2} - \frac{xx}{(1-x)^3} + \frac{x^3}{(1-x)^4} - \text{etc.}$, is a geometric series and its sum is $= \frac{x}{(1-x)^2+x(1-x)} = \frac{x}{1-x}$, whence the value found will be

$$= \frac{x}{1-x} - \frac{x}{1-x} = 0.$$

EXAMPLE 4

Let $y = e^x$ while e denotes the number whose hyperbolic logarithm is the unity and the value of y be sought after, if one puts $x = 0$, which value is known to be $= 1$.

Because it is $y = e^x$, it will be

$$\frac{dy}{dx} = e^x, \quad \frac{ddy}{dx^2} = e^x \quad \text{etc.}$$

and hence the value in question will be

$$= e^x - \frac{e^x}{1} + \frac{e^x xx}{1 \cdot 2} - \frac{e^x x^3}{1 \cdot 2 \cdot 3} + \frac{e^x x^4}{1 \cdot 2 \cdot 3 \cdot 4} - \text{etc.}$$
$$= e^x \left(1 - \frac{x}{1} + \frac{xx}{1 \cdot 2} - \frac{x^3}{1 \cdot 2 \cdot 3} + \frac{x^4}{1 \cdot 2 \cdot 3 \cdot 4} - \text{etc.}\right).$$

But above we saw that the series

$$1 - \frac{x}{1} + \frac{xx}{1 \cdot 2} - \frac{x^3}{1 \cdot 2 \cdot 3} + \text{etc.}$$

expresses the value e^{-x}; therefore, the value in question will be $e^x \cdot e^{-x} = \frac{e^x}{e^x} = 1$, of course.

EXAMPLE 5

Let $y = \sin x$ and having put $x = 0$ it is manifest that is will be $y = 0$, which also the general formula will indicate.

For, if it is $y = \sin x$, it will be

$$\frac{dy}{dx} = \cos x, \quad \frac{ddy}{dx^2} = -\sin x, \quad \frac{d^3y}{dx^3} = -\cos x, \quad \frac{d^4y}{dx^4} = \sin x \quad \text{etc.}$$

Having put $x = 0$ the value of y will be this one

$$\sin x - \frac{x}{1}\cos x - \frac{xx}{1\cdot 2}\sin x + \frac{x^3}{1\cdot 2\cdot 3}\cos x + \frac{x^4}{1\cdot 2\cdot 3\cdot 4}\sin x - \text{etc.}$$

which is

$$= \sin x\left(1 - \frac{xx}{1\cdot 2} + \frac{x^4}{1\cdot 2\cdot 3\cdot 4} - \frac{x^6}{1\cdot 2\cdot 3\cdots 6} + \text{etc.}\right)$$
$$- \cos x\left(\frac{x}{1} - \frac{x^3}{1\cdot 2\cdot 3} + \frac{x^5}{1\cdot 2\cdot 3\cdot 4\cdot 5} - \frac{x^7}{1\cdot 2\cdot 3\cdots 7} + \text{etc.}\right)$$

But the superior of these series' expresses $\cos x$, the inferior $\sin x$, whence the value in question will be

$$= \sin x\cos x - \cos x\cdot\sin x = 0.$$

§68 Hence, we therefore vice versa realize, if y was a function of x of such a kind that it vanishes having put $x = 0$, that then it will be

$$y - \frac{xdy}{1dx} + \frac{xxddy}{1\cdot 2dx^2} - \frac{x^3d^3y}{1\cdot 2\cdot 3dx^3} + \frac{x^4d^4y}{1\cdot 2\cdot 3\cdot 4dx^4} - \text{etc.} = 0.$$

Hence, this is the general equation of completely all functions of x, which, if $x = 0$, at the same time vanish themselves. And therefore this equation is of such a nature, that, no matter which function of x, as long as it vanishes as x vanishes, is substituted for y, it is always satisfied. But if therefore y was a function of such a kind of x which having put $x = 0$ shall receive a given value $= A$, then it will be

$$= y - \frac{xdy}{1dx} + \frac{x^2ddy}{1\cdot 2dx^2} - \frac{x^3d^3y}{1\cdot 2\cdot 3dx^3} + \frac{x^4d^4y}{1\cdot 2\cdot 3\cdot 4dx^4} - \text{etc.} = A,$$

in which function all functions of x are contained which having put $x = 0$ go over into A.

§69 If instead of x one writes $2x$ or $x + x$, any function of x, which shall be denoted by y, will obtain this value

$$y + \frac{xdy}{1dx} + \frac{x^2ddy}{1\cdot 2dx^2} + \frac{x^3d^3y}{1\cdot 2\cdot 3dx^3} + \frac{x^4d^4y}{1\cdot 2\cdot 3\cdot 4dx^4} + \text{etc.}$$

And if we write nx instead of x, this means $x+(n-1)x$, the function y will take the following value

$$y+\frac{(n-1)xdy}{1dx}+\frac{(n-1)^2xxddy}{1\cdot 2dx^2}+\frac{(n-1)^3x^3d^3y}{1\cdot 2\cdot 3dx^3}+\text{etc.}$$

but if we in general write t for x, any function y of x will because of $t=x+t-x$ be transformed into the following form

$$y+\frac{(t-x)dy}{1dx}+\frac{(t-x)^2ddy}{1\cdot 2dx^2}+\frac{(t-x)^3d^3y}{1\cdot 2\cdot 3dx^3}+\text{etc.}$$

If therefore v was such a function of t as y is of x, since v arises from y by putting t instead of x, it will be

$$v=y+\frac{(t-x)dy}{1dx}+\frac{(t-x)^2ddy}{1\cdot 2dx^2}+\frac{(t-x)^3d^3y}{1\cdot 2\cdot 3dx^3}+\text{etc.},$$

the validity of which formula can be probed by any arbitrary example.

Example

For, let $y=xx-x$; it is manifest that having put t instead of x that it will be $v=tt-t$, which same the found expression will also reveal.

For, because of $y=xx-x$ it will be

$$\frac{dy}{dx}=2x-1 \quad\text{and}\quad \frac{ddy}{2dx^2}=1;$$

hence, it will be

$$\begin{aligned} v&=xx-x+(t-x)(2x-1)+(t-x)^2 \\ &=xx-x+2tx-2xx-t+x+tt-2tx+xx=tt-t. \end{aligned}$$

Therefore, if y was a function of such a kind of x, which having put $x=a$ goes over into A, because of $t=a$ and $v=A$ it will be

$$A=y+\frac{(a-x)dy}{1dx}+\frac{(a-x)^2ddy}{1\cdot 2dx^2}+\frac{(a-x)^3d^3y}{1\cdot 2\cdot 3dx^3}+\text{etc.}$$

and hence all functions of x, which having put $x=a$ go over into A, satisfy this equation.