On Maxima and Minima *

Leonhard Euler

§250 If a function of x was of such a nature, that while the values of x increase the function itself continuously increases or decreases, then this function will have no maximum or minimum. For, whatever value of this function is considered, the following will be larger, the preceding on the other hand will be smaller. A function of this kind is $x^3 + x$, whose value for increasing x increases continuously, but for decreasing x decreases continuously; therefore, this function cannot obtain another maximal value, if not the maximal value, this means an infinite, is attributed to x; and in similar manner, it will obtain the minimal value, if one puts $x = -\infty$. But if the function was not of such a nature, that with growing x it either increases or decreases continuously, it will have a maximum or minimum somewhere else, this means a value of such a kind, which is either greater or smaller than the preceding and following. So, this function $xx - 2x + 3$ obtains a minimal value, if one puts $x = 1$; for, whatever other value is attributed to x, the function will always have a greater value.

§251 But that the nature of maxima and minima is better understood, let us put that y is a function of x of such a kind, which shall obtain a maximal value, if one puts $x = f$, and it is seen, if x is put either greater or smaller than f, that then the value of y to arise from there will be smaller than the latter, which it obtains, if one puts $x = f$. in similar manner, if for $x = f$ the function y shall obtain a minimal value, it is necessary, that, no matter whether x is

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§252 But before we show the way to find these maxima and minima, it is convenient to note that this investigation is only for those functions of x, which we called *uniform* above and which are of such a nature, that for the single values of x they similarly receive single values. But we called functions *biform* and *multiform*, which for single values of x induce two or several values, functions of which kind roots of quadratic equations and higher dimensional equations are. Therefore, if y was a biform or multiform function of such a kind of x, then it cannot be said to receive a maximal or minimal value for $x = f$; for, since for $x = f$ it obtains either two or more values at the same time and let the preceding and the following lead to the same number, then one cannot decide for a maximum or minimum so easily, if not by coincidence all values of the function y, which correspond to the single values of x, are imaginary except one; in this case functions of this kind are counted to the class of uniform functions. Therefore, we will at first consider the class of uniform functions; but then we will show, how the consider maxima and minima of multiform functions.

§253 Therefore, let y be a uniform function, which hence, no matter which value is substituted for x, always receives one single real value, and let x denote the value, which induces the maximal or minimal value to the function y. Therefore, in the first case, no matter whether one substitutes $x + \alpha$ or $x - \alpha$ for x, the value of y will be smaller than for $\alpha = 0$, in the second case on the other hand larger. Therefore, because for having put $x + \alpha$ for x the function y goes over into

$$y + \frac{\alpha dy}{dx} + \frac{\alpha^2 ddy}{2dx^2} + \frac{\alpha^3 d^3 y}{6dx^3} + \text{etc.},$$

but having put $x-\alpha$ for x into

$$x-\frac{\alpha dy}{dx}+\frac{\alpha^2 ddy}{2dx^2}-\frac{\alpha^3 d^3y}{6dx^3}+\text{etc.},$$

it is necessary, that it is in the case of a maximum

$$y>y+\frac{\alpha dy}{dx}+\frac{\alpha^2 ddy}{2dx^2}+\frac{\alpha^3 d^3y}{6dx^3}+\text{etc.}$$

and

$$y>y-\frac{\alpha dy}{dx}+\frac{\alpha^2 ddy}{2dx^2}-\frac{\alpha^3 d^3y}{6dx^3}+\text{etc.}$$

But in the case, in which the value of y becomes minimal, it will be

$$y<y+\frac{\alpha dy}{dx}+\frac{\alpha^2 ddy}{2dx^2}+\frac{\alpha^3 d^3y}{6dx^3}+\text{etc.}$$
$$y<y-\frac{\alpha dy}{dx}+\frac{\alpha^2 ddy}{2dx^2}-\frac{\alpha^3 d^3y}{6dx^3}+\text{etc.}$$

§254 Since these things have to happen, if α denotes a very small quantity, let us set α so small, that its higher powers can be rejected, and so for the case of the maximum as the minimum it has to be $\frac{\alpha dy}{dx}=0$. For, if $\frac{\alpha dy}{dx}$ was not $=0$, the value of y could be neither maximal nor minimal. Hence, for the investigation so for maxima as for minima one has the general rule, the the differential of propounded y is put equal to zero, and that value of x, which renders the function either maximal or minimal, will be a root of the last equation. But whether the value of y found this way is a maximum or a minimum, is left uncertain; it can even happen, that y is neither a maximum nor a minimum; for, we only found that in both cases it will be $\frac{dy}{dx}=0$ and we did not vice versa affirmed, if $\frac{dy}{dx}=0$, that also a maximal or minimal value for y arises.

§255 Nevertheless, to investigate cases, in which the value of y becomes either maximal or minimal, this first operation is to be done, that the differential of the propounded function becomes equal to nothing and from the equation $\frac{dy}{dx} = 0$ all values of x are found. Having found these, afterwards it will to be checked, whether for those values the function y obtains a maximal or minimal value or is none of both. For, we will show that it can happen that there is neither a maximum or minimum, even though it is $\frac{dy}{dx} = 0$.
Let f be one of the values of x, which it the function obtains from the equation

$$\frac{dy}{dx} = 0,$$

and substitute this value in the expressions $\frac{ddy}{dx^2}$, $\frac{d^3y}{dx^3}$ etc. and by this substitution let

$$\frac{ddy}{dx^2} = p, \quad \frac{d^3y}{dx^3} = q, \quad \frac{d^4y}{dx^4} = r \quad \text{etc.}$$

But let the function y having put f instead of x go over into F, and if one puts $f + \alpha$ instead of x, this function will go over into

$$F + \frac{1}{2}\alpha^2 p + \frac{1}{6}\alpha^3 q + \frac{1}{24}\alpha^4 r + \text{etc.};$$

but if one puts $f - \alpha$ instead of x, it will arise

$$F + \frac{1}{2}\alpha^2 p - \frac{1}{6}\alpha^3 q + \frac{1}{24}\alpha^4 r - \text{etc.};$$

hence it is plain, if p was a positive quantity, that both values will be smaller than F, at least if α denotes a very small quantity, and therefore the value F, which the function y takes for $x = f$, will be a minimum. But if p is a negative quantity, then the value $x = f$ will induce a maximal value to the function y.

§256 But if it was $p = 0$, then the value if q is to be considered; if it was not $= 0$, the value of y will be neither maximal nor minimal; for, having put $x = f + \alpha$ it will be $F + \frac{1}{6}\alpha^3 q > F$ and having put $x = f - \alpha$ it will be $F - \frac{1}{6}\alpha^3 q < F$. But if it also was $q = 0$, the quantity r is to be considered; if it had a positive value, the value of the function F, which is receives for $x = f$, will be a minimum; but if r has a negative value, F will be a maximum. But if also r vanishes, the judgement is to be rendered from the value of the following letter s, which will be similar to the one, which was rendered from the letter q. if s was not $= 0$, then the function F will be neither a maximum nor a minimum; but if also $s = 0$, then the following letter t, if it has a positive value, will indicate a minimum; but if it has a negative value, it will indicate a maximum. But if also this letter t vanishes, then in the judgement one has to proceed further in completely the same way, as we did in the preceding cases. And so for any root of the equation $\frac{dy}{dx} = 0$ it is decided, whether the function y obtains a maximal or minimal value or none of both; and this way all maxima and minima, which the function y can receive, will be found.

§257 Therefore, if the equation $\frac{dy}{dx} = 0$ has two equal roots, such that it has the quadratic factor $(x - f)^2$, then for $x = f$ at the same time $\frac{ddy}{dx^2}$ will vanish and it will be $p = 0$, but not $q = 0$. In this case the function y will obtain neither a maximal nor a minimal value. But if the equation $\frac{dy}{dx} = 0$ has three equal roots or $\frac{dy}{dx}$ has the cubic factor $(x - f)^3$, then having put $x = f$ it will be $\frac{ddy}{dx^2} = 0$ and $\frac{d^3y}{dx^3} = 0$, but not $\frac{d^4y}{dx^4}$. Therefore, if the value of this term was positive, it will indicate a minimum, if negative, a maximum. Therefore, the way of decision explained before reduces to this, that, if the fraction $\frac{dy}{dx}$ had a factor $(x - f)^n$, while n is an odd number, the function y, if in it one puts $x = f$, will obtain an either maximal or minimal value, but if the exponent n was an even number, then the substitution $x = f$ will produce neither a maximal nor a minimal value.

§258 Further, the invention of a maximum or minimum often is tremendously aided by the following considerations. In cases, in which the function y becomes maximal or minimal, each multiple of it ay, if a was a positive quantity, will also become maximal or minimal, and in the same way y^3, y^5, y^7 etc. and in general y^n, if n was a positive odd number, since formulas of this kind are of such a nature, that for increasing y they also increase and for decreasing y they decrease. But in these cases, in which y becomes a maximum or minimum, $-y$, $-ay$, $b - ay$ and in general $b - ay^n$, while n is an odd positive integer, in reversed order will become either minimal or maximal. In similar way in the cases, in which y becomes maximal or minimal, these formulas $\frac{a}{y}$, $\frac{a}{y^3}$, $\frac{a}{y^5}$ etc. and in general $\frac{a}{y^n} \pm b$, while a denotes a positive quantity and n a positive odd number, will in reverse order become either a minimum or a maximum; but if a was a negative quantity, then these formulas will obtain a maximal value, if y was a maximum, and a minimal value, if y is a minimum.

§259 But these rules cannot be transferred to even powers the same way; for, since, if y receives a negative value, its even powers y^2, y^4 etc. induce positive values, it can happen, that, if y receives a minimal value, a negative one of course, that its even powers become maximal. Therefore, having taken this into account, we will be able to affirm, if y was a maximum or minimum, while its value is positive, that then its even powers y^2, y^4 etc. will also be maximal or minimal, but if a negative value of y was a maximum, that then its square qq will obtain a positive value, and otherwise, if a negative value of y was a minimum, that then y^2, y^4 etc. will be a maximum. But if the even exponents of y were negative, then the opposite will happen. Furthermore, what we mentioned here about the even and odd exponents, will not only hold for integer numbers, but also for fractional ones, whose denominators are odd numbers; for, in this task the fractions $\frac{1}{3}$, $\frac{5}{3}$, $\frac{7}{3}$, $\frac{1}{5}$, $\frac{3}{5}$ etc. are equivalent to odd numbers, but $\frac{2}{3}$, $\frac{4}{3}$, $\frac{2}{5}$, $\frac{4}{5}$, $\frac{6}{7}$ etc. are equivalent to even numbers.

§260 But if the denominators were even numbers, then, because, if y has a negative value, its powers $y^{\frac{1}{2}}$, $y^{\frac{3}{4}}$ etc. become imaginary, here one can say only the following about them: If a positive value of y was a maximum or a minimum, then also $y^{\frac{1}{2}}$, $y^{\frac{3}{2}}$, $y^{\frac{1}{4}}$ etc. will also be either maximal or minimal, but on the other hand $y^{-\frac{1}{2}}$, $y^{-\frac{3}{2}}$, $y^{-\frac{1}{4}}$ etc. become minimal or maximal. But if these irrationalities at the same have double values, one positive, the other negative, about the negative ones the contrary of that, what we said about the positive ones here, is to be said. But if a negative value of y becomes a maximum or minimum, since all powers if this kind become imaginary, one will not be able to count them to maxima or minima. Therefore, by means of these auxiliary remarks, the investigation of maxima and minima is often rendered very easy, which otherwise would be extremely difficult.

§261 Because these things extend only to rational functions, which are the only uniform functions, at first let us expand polynomial functions and find the maxima and minima, which occur in them. Therefore, because functions of this kind are reduced to this form

$$x^n + Ax^{n-1} + Bx^{n-2} + Cx^{n-3} + \text{etc.},$$

at first it is plain that their values cannot be greater than if one sets $x = \infty$; then on the other hand, if $x = -\infty$, the value of these formulas arises as $= \infty^n$, if n is an even number, but $-\infty^n$, if n is an odd number, which value therefore will be the smallest of all. But furthermore often other maxima and minima are given in the sense, in which we understand those terms, which we will illustrate in the following examples.

Example 1

To find the values of x, for which the function $(x-a)^n$ becomes a maximum or a minimum.

Having put $(x-a)^n = y$ it will be

$$\frac{dy}{dx} = n(x-a)^{n-1};$$

having put it $= 0$ it will be $x = a$. Therefore, because $\frac{dy}{dx}$ has the factor $(x-a)^{n-1}$, from § 257 it is understood that y cannot be a maximum or a minimum, if $n-1$ is not an odd number or n is not even. But since then it is

$$\frac{d^y}{dx^n} = n(n-1)(n-2)\cdots 1,$$

this means a positive number, it follows that the value of y for $x = a$ will arise as minimum. This is easily clear, of course; for, having put $x = a$ it is $y = 0$, and if x is put either greater or smaller than a, because of the even number n the function y will be positive, this means greater than nothing; but if n was an odd number, then the function $y = (x-a)^n$ admits neither a maximum nor a minimum. But is is perspicuous, that the same holds, if n was a fractional number, odd or even. $(x-a)^{\frac{\mu}{\nu}}$ having put $x = a$ will become a minimum, if μ was an even number and ν was an odd number; but if both were odd, neither a maximum nor a minimum will be given.

Example 2

To find the cases, in which the value of this formula $xx + 3x + 2$ becomes a maximum or a minimum.

Put $xx + 3x + 2 = y$; it will be

$$\frac{dy}{dx} = 2x + 3, \quad \frac{ddy}{2dx^2} = 1.$$

Therefore, set $2x + 3 = 0$; it will be $x = -\frac{3}{2}$. Whether this case produces a maximum or minimum, will become known from the value $\frac{ddy}{2dx^2} = 1$; since it is affirmative, whatever x is, it indicates a minimum. But having put $x = -\frac{3}{2}$, it is $y = -\frac{1}{4}$, and if any other values are attributed to x, the value of y to arise from there will always be larger than $-\frac{1}{4}$. From the nature of the formula $xx + 3x + 2$ it is also seen, that it has to have a minimal value; for, because it grows to infinity, if one puts $x = +\infty$ or $x = -\infty$, it is necessary, that a certain value of x causes the a smallest quantity of y.

EXAMPLE 3

To find the cases, in which this expression $x^3 - axx + bx - c$ obtains the maximal or minimal value.

Having put $y = x^3 - axx + bx - c$ it will be

$$\frac{dy}{dx} = 3xx - 2ax + b \quad \text{and} \quad \frac{ddy}{2dx^2} = 3x - a, \quad \frac{d^3y}{6dx^3} = 1.$$

Therefore, set $\frac{dy}{dx} = 3xx - 2ax + b = 0$; it will be

$$x = \frac{a \pm \sqrt{aa - 3b}}{3},$$

from which it it clear, if it is not $aa > 3b$, that the propounded formula will neither have a maximum nor a minimum. Hence, it arises

$$\frac{ddy}{2dx^2} = \pm\sqrt{aa - 3bb},$$

whence it is understood, if it not $aa = 3b$, that the value $x = \frac{a+\sqrt{aa-3b}}{3}$ renders the formula $y = x^3 - axx + bx - c$ minimal, the other value $x = \frac{a-\sqrt{aa-3b}}{3}$ on the other hand maximal. But how large will these value of y be? Because it is $3xx - 2ax + b = 0$ or $x^3 - \frac{2}{3}axx + \frac{1}{3}bx = 0$, it will be

$$y = -\frac{1}{3}axx + \frac{2}{3}bx - c$$

and because of $\frac{1}{3}axx - \frac{2aa}{9}x + \frac{ab}{9} = 0$ it is

$$y = \frac{2}{9}(3b - aa)x + \frac{ab}{9} - c = -\frac{2a(aa - 3b)}{27} - \mp\frac{2(aa - 3b)\sqrt{aa - 3b}}{27} + \frac{ab}{9} - c$$

or

$$y = -\frac{2a^3}{27} + \frac{ab}{3} - c \mp \frac{2}{27}(aa - 3b)^{\frac{3}{2}},$$

where the superior sign holds for the minimum, but the inferior for the maximum.

Therefore, the case remains, in which it is $aa = 3b$; because in it it is $\frac{ddy}{dx^2} = 0$, but the following term $\frac{d^3y}{6dx^3} = 1$ is not $= 0$, it follows that in this case the propounded formula receives neither a maximum nor a minimum.

EXAMPLE 4

To find the cases, in which this function of x, $x^4 - 8x^3 + 22x^2 - 24x + 12$ *becomes maximal or minimal.*

Having put $y = x^4 - 8x^3 + 22x^2 - 24x + 12$ it will be

$$\frac{dy}{dx} = 4x^3 - 24x^2 + 44x - 24, \quad \frac{ddy}{2dx^2} = 6x^2 - 24x + 22.$$

Now set

$$\frac{dy}{dx} = 4x^3 - 24x^2 + 44x - 24 = 0 \quad \text{or} \quad x^3 - 6x^2 + 11x - 6 = 0;$$

three real values for x will arise

$$\text{I. } x = 1, \quad \text{II. } x = 2, \quad \text{III.,} \quad x = 3.$$

From the first value it is $\frac{ddy}{2dx^2} = 4$ and hence having put $x = 1$ the propounded function becomes a minimum. From the second value $x = 2$ it is $\frac{ddy}{2dx^2} = -2$ and hence the propounded function becomes a maximum. From the third value $x = 2$ it is $\frac{ddy}{2dx^2} = +4$ and hence the propounded function is again a minimum.

EXAMPLE 5

Let this function be propounded $y = x^5 - 5x^4 + 5x^3 + 1$; *in which cases it becomes a maximum or a minimum, is in question.*

Because it is

$$\frac{dy}{dx} = 5x^4 - 20x^3 + 15xx,$$

form the equation $x^4 - 4x^3 + 3xx = 0$, whose roots are

$$\text{I. and II. } x = 0, \quad \text{III. } x = 1, \quad \text{IV. } x = 3.$$

Since the first and second root are equal, from them neither a maximum nor a minimum follows; for, it is $\frac{ddy}{dx^2} = 0$, but $\frac{d^3y}{dx^3}$ does not vanish. But the third root $x = 1$ because of $\frac{ddy}{2dx^2} = 10x^3 - 30x^2 + 15x$ yields $\frac{ddy}{2dx^2} = -5$ and in this case the function becomes a maximum. From the fourth root $x = 3$ it is $\frac{ddy}{2dx^2} = 45$ and hence the propounded function a minimum.

EXAMPLE 6

To find the cases, in which this formula $y = 10x^6 - 12x^5 + 15x^4 - 20x^3 + 20$ *becomes a maximum or a minimum.*

Therefore, it will be

$$\frac{dy}{dx} = 60x^5 - 60x^4 + 60x^3 - 60x^2 \quad \text{and} \quad \frac{ddy}{60dx^4} = 5x^4 - 4x^3 + +3x^2 - 2x.$$

Form the equation $x^5 - x^4 + x^3 - xx = 0$; since it resolved into factors is $x^2(x-1)(xx+1) = 0$, it has two equal roots $x = 0$ and furthermore the root $x = 1$ and additionally two imaginary ones from $xx + 1 = 0$. Therefore, since the two equal roots $x = 0$ exhibit neither a maximum nor a minimum, only the root $x = 1$ is to be considered, from which it is $\frac{ddy}{60dx^2} = 2$, whose positive value indicates a minimum.

§262 Therefore, the determination of maxima and minima depends on the root of the differential equation $\frac{dy}{dx} = 0$; because its highest power is one degree lower than in the propounded equation itself, it is manifest, if in general this function is propounded

$$x^n + Ax^{n-1} + Bx^{n-2} + Cx^{n-3} + Dx^{n-4} + \text{etc.} = y,$$

that its maxima and minima are determined by means of the roots of this equation

$$nx^{n-1} + (n-1)Ax^{n-2} + (n-2)Bx^{n-3} + (n-3)Cx^{n-4} + \text{etc.} = 0.$$

Let us put that the real roots of this equation ordered according to their magnitude are α, β, γ, δ etc. such that α is the largest, $\beta < \alpha$, $\gamma < \beta$ etc. And first, if these roots are all different, every single one will lead to a maximum or minimum value of the propounded formula y and hence the function y will have as many maxima or minima as the equation $\frac{dy}{dx} = 0$ had real different roots. But if two or more roots were equal to each other, it will behave as follows: two equal roots will exhibit neither a maximum nor a minimum, but three on the other hand will be equivalent to a single one; and in general, if the number of equal roots was an even number, hence no maximum nor minimum results; but if the number is odd, one maximum or minimum arises from this.

§263 But which roots produce maxima and which produce minima, can be defined without using the rule given before this way. Since the function y having put $x = \infty$ equally becomes infinite and value of x between the limits ∞ and α produce neither a maximum nor a minimum, it is perspicuous that the values of the function y, if for x successively values from ∞ up to α are substituted, that the have to decrease continuously; and hence the value $x = \alpha + \omega$ will lead to a larger value of the function y than the value $x = \alpha$; hence, because $x = \alpha$ produces a maximum or minimum, it is necessary that in this cases the function y becomes a minimum. Therefore, diminishing x or putting $x = \alpha - \omega$ the value of y will increase again, until it finally is $x = \beta$, which is the second root of the equation $\frac{dy}{dx} = 0$ producing a maximum or minimum; hence, this second root $x = \beta$ will yield a maximum and the value $x = \beta - \omega$ will cause the function y to be smaller than for $x = \beta$, until one gets to $x = \gamma$, which as a logical consequence will generate a minimum again. From this reasoning it is understood, that the first, third, fifth etc. root of the equation $\frac{dy}{dx} = 0$ will exhibit minima but the first, second, fourth, sixth etc. exhibit maxima. But at the same time it is hence understood that in the case of two equal roots maximum and minimum coalesce and so none of both actually occurs.

§264 Therefore, if in the propounded function

$$y = x^n + Ax^{n-1} + Bx^{n-2} + Cx^{n-3} + \text{etc.}$$

the greatest exponent n was an even number, the equation

$$\frac{dy}{dx} = x^{n-1} + (n-1)Ax^{n-2} + \text{etc.} = 0$$

will be of odd degree and will hence have one or three or five or any odd number of real roots. If just one root was real, it will give a minimum; if three were real, the largest will yield a minimum, the middle one a maximum and the smallest a minimum again; and if five roots were real, the function y will have three minima and two maxima; and so forth.

But if the exponent n was an odd number, the equation $\frac{dy}{dx} = 0$ will extend to even grade and will have either no or two or four or six etc. real roots. In the first case the function y will have neither a maximum nor a minimum; in the other case, in which two roots are given, the greater one will indicate a minimum, the smaller a maximum; but the first of four roots (which is the largest) and the third will produce a minimum, the second and the fourth on the other hand a maximum. But no matter how many real roots were there, they will in an alternating manner give maxima and minima.

§265 Let us proceed to rational function, by which the other kind of uniform functions is constituted. Therefore, let

$$y = \frac{P}{Q}$$

where P and Q are any polynomial functions of x; and at first it is certainly clear, if to x a value of such a kind is attributed, that it is $Q = 0$, if not at the same time P vanishes, that the function y becomes infinite, was appears to be a maximum. Nevertheless, this case cannot be treated as a maximum; for, because the inverse fraction $\frac{P}{Q}$ in the same cases becomes a minimum, in which the propounded $\frac{P}{Q}$ becomes a maximum, the fraction $\frac{Q}{P}$ would have to become a minimum, if Q vanishes; but this not always happens, since even smaller values, negative ones of course, could occur. Therefore, having removed the doubt about this, at the same time the rule given before is confirmed, that maxima and minima must be found from the equation $\frac{dy}{dx} = 0$. Therefore, in the propounded cases it will be

$$\frac{dy}{dx} = \frac{QdP - PdQ}{QQdx}$$

and hence the roots of this equation

$$QdP - PdP = 0$$

will cause either a maximum or a minimum of the function y. And if there is any doubt, whether a maximum or a minimum takes place, one has to check the value $\frac{ddy}{dx^2}$; if it was positive, it will indicate a minimum, but if it was negative a minimum. If also this value $\frac{ddy}{dx^2}$ vanishes, which happens, if the equation $\frac{dy}{dx} = 0$ has two or more equal roots, it is always to be kept that an equal number of equal roots produce neither a maximum nor a minimum.

EXAMPLE 1

To find the cases, in which the function $\frac{x}{1+xx}$ becomes a maximum or a minimum.

At first it is certainly clear that this function goes over into nothing in the three cases $x = \infty$, $x = 0$ and $x = -\infty$, whence it will receive at least either two maxima or two minima. To find them put $y = \frac{x}{1+xx}$ and it will be

$$\frac{dy}{dx} = \frac{1 - xx}{(1 + xx)^2} \quad \text{and} \quad \frac{ddy}{dx^2} = \frac{-6x + 2x^3}{(1 + xx)^3}.$$

Now set $\frac{dy}{dx} = 0$; it will be $1 - xx = 0$ and either $x = +1$ or $x = -1$. In the first case $x = +1$ it is $\frac{ddy}{dx^2} = -\frac{4}{2^3}$ and hence y a maximum $= \frac{1}{2}$; in the second case $x = -1$ it is $\frac{ddy}{dx^2} = +\frac{4}{2^3}$ and hence y a minimum $= -\frac{1}{2}$.

These are also found in an easier manner, if the propounded fraction $\frac{x}{1+xx}$ is inverted by putting $y = \frac{1+xx}{x} = x + \frac{1}{x}$, if we recall that then all values, which were found to be maxima, are are to be turned into minima and vice versa. But it will be

$$\frac{dy}{dx} = 1 - \frac{1}{xx} \quad \text{and} \quad \frac{ddy}{dx^2} = \frac{2}{x^3}.$$

Therefore, having set $\frac{dy}{dx} = 0$ it is $xx - 1 = 0$ and hence either $x = +1$ or $x = -1$ as before. And in the case $x = +1$ it is $\frac{ddy}{dx^2} = 2$ and hence y a minimum and the propounded formula $\frac{1}{y}$ a maximum. But in the case $x = -1$ it is $\frac{ddy}{dx^2} = -2$, whence y is a maximum and $\frac{1}{y}$ a minimum.

EXAMPLE 2

To find the cases, in which the formula $\frac{2-3x+xx}{2+3x+xx}$ *becomes a maximum or a minimum.*

Having put $y = \frac{xx-3x+2}{xx+3x+2}$ it will be

$$\frac{dy}{dx} = \frac{6x^2 - 12}{(xx+3x+2)^2} \quad \text{and} \quad \frac{ddy}{dx^2} = \frac{-12x^3 + 72x + 72}{(xx+3x+2)^3}.$$

Set $\frac{dy}{dx} = 0$; it will be either $x = +\sqrt{2}$ or $x = -\sqrt{2}$. In the first case it will be

$$\frac{ddy}{dx^2} = \frac{48\sqrt{2} + 72}{(4+3\sqrt{2})^3}$$

and hence affirmative because of the affirmative numerator and denominator; hence, y will be a minimum

$$= \frac{4 - \sqrt{3}2}{4 + \sqrt{3}2} = 12\sqrt{2} - 17 = -0.02943725.$$

In the second case $x = -\sqrt{2}$ it is

$$\frac{ddy}{dx^2} = \frac{-48\sqrt{2} + 72}{(4-3\sqrt{2})^3} = \frac{24(3-2\sqrt{2})}{(4-3\sqrt{2})^3},$$

whose value because of the affirmative numerator and negative denominator will be negative and hence y will become a maximum

$$= \frac{4+3\sqrt{2}}{4-3\sqrt{2}} = -12\sqrt{2} - 17 = -33.97056274.$$

This value, even though it is smaller than the first minimal one, is nevertheless a maximum, since it is larger the its closest neighbors, which arise, if for x a little bit greater or smaller values then $-\sqrt{2}$ are substituted. Therefore, because $\sqrt{2}$ is contained within the limits $\frac{4}{3}$ and $\frac{3}{2}$, the crosscheck will easily be done this way:

$$\begin{array}{llllll} \text{if} & x = & \frac{4}{3}, & \text{it is} & y = & -\frac{2}{70} = -0.0285 \\ \text{if} & x = & \sqrt{2}, & \text{it is} & y = & +12\sqrt{2} - 17 = 0.0294 \qquad \text{minimum} \\ \text{if} & x = & \frac{3}{2}, & \text{it is} & y = & -\frac{1}{35} = -0.0285 \end{array}$$

$$\begin{array}{llllll}
\text{if} & x = & -\frac{4}{3}, & \text{it is} & y = & -35 \\
\text{if} & x = & -\sqrt{2}, & \text{it is} & y = & -33.970 \qquad \text{maximum} \\
\text{if} & x = & -\frac{3}{2}, & \text{it is} & y = & -35.
\end{array}$$

EXAMPLE 3

To find the cases, in which the formula $\frac{xx-x+1}{xx+x-1}$ becomes a maximum or a minimum.

Put $y = \frac{xx-x+1}{xx+x-1}$ and it will be

$$\frac{dy}{dx} = \frac{2xx-4x}{(xx+x-1)^2} \quad \text{and} \quad \frac{ddy}{dx^2} = \frac{-4x^3+12xx+4}{(xx+x-1)^3}.$$

Set $\frac{dy}{dx} = 0$; it will be either $x = 0$ or $x = 1$; in the first case it is $\frac{ddy}{dx^2} = \frac{4}{-1}$ and hence y will be a maximum $= -1$. In the second case it is $\frac{ddy}{dx^2} = \frac{20}{5^2}$ and hence y a minimum $= \frac{3}{5}$, even though that maximum is smaller than this minimum. Crosschecking will be plain from these positions:

$$\begin{array}{llllll}
\text{if} & x = & -\frac{1}{3}, & \text{it will be} & y = & -\frac{13}{11} \\
\text{if} & x = & 0, & \text{it will be} & y = & -1 \qquad \text{maximum} \\
\text{if} & x = & +\frac{1}{3}, & \text{it will be} & y = & -\frac{7}{5}
\end{array}$$

$$\begin{array}{llllll}
\text{if} & x = & 2-\frac{1}{3}, & \text{it will be} & y = & -\frac{19}{13} \\
\text{if} & x = & 2, & \text{it will be} & y = & \frac{3}{5} \qquad \text{minimum} \\
\text{if} & x = & 2+\frac{1}{3}, & \text{it will be} & y = & -\frac{37}{61}.
\end{array}$$

But that, if one puts $x = 1$, it is $y = 1$ and hence > -1, is the reason why between the values 0 and 1 of x one is contained, for which it is $y = \infty$.

EXAMPLE 4

To find the the cases, in which this fraction $\frac{x^3+x}{x^4-xx+1}$ becomes a maximum or a minimum.

Having put $y = \frac{x^3+x}{x^4-xx+1}$ it will be

$$\frac{dy}{dx} = \frac{-x^6 - 4x^4 + 4xx + 1}{(x^4 - xx + 1)^2} \quad \text{and} \quad \frac{ddy}{dx^2} = \frac{2x^9 + 18x^7 - 30x^5 - 16x^3 + 12x}{(x^4 - xx + 1)^3}.$$

Therefore, we will have this equation

$$x^6 + 4x^4 - 4xx - 1 = 0,$$

which is resolved into these two

$$xx - 1 = 0 \quad \text{and} \quad x^4 + 5x^2 + 1 = 0;$$

the roots of the first of this equations are $x = +1$ and $x = -1$, the other resolved gives $x = -\frac{5\pm\sqrt{21}}{2}$, from which no real root emerges. Therefore, the first of the two roots found $x = +1$ gives $\frac{ddy}{dx^2} = -14$ and therefore y is a minimum $= 2$; the other root $x = -1$ gives $\frac{ddy}{dx^2} = +14$ and therefore y is a minimum $= -2$.

EXAMPLE 5

To find the cases, in which this fraction $\frac{x^3-x}{x^4-xx+1}$ becomes a maximum or a minimum.

Having put $y = \frac{x^3-x}{x^4-xx+1}$ it will be

$$\frac{dy}{dx} = \frac{-x^6 + 2x^4 + 2x^2 - 1}{(x^4 - x^2 + 1)^2} \quad \text{and} \quad \frac{ddy}{dx^2} = \frac{2x^9 - 6x^7 - 18x^5 + 10x^3}{(x^4 - x^2 + 1)^3}.$$

But having put $\frac{dy}{dx} = 0$ it will be

$$x^6 - 2x^4 - 2x^2 + 1 = 0,$$

which divided by $xx + 1$ gives

$$x^4 - 3x^2 + 1 = 0,$$

and this is further resolved into

$$xx - x - 1 = 0 \quad \text{and} \quad xx + x - 1 = 0,$$

whence the following four real roots arise

$$\begin{aligned} &\text{I.} \quad x = \frac{1+\sqrt{5}}{2} \quad &&\text{II.} \quad x = \frac{1-\sqrt{5}}{2}, \\ &\text{III.} \quad x = -\frac{1+\sqrt{5}}{2} \quad &&\text{II.} \quad x = -\frac{1-\sqrt{5}}{2}. \end{aligned}$$

Since all are contained in the equation $x^4 - 3xx + 1 = 0$, having put $x^4 = 3xx - 1$ it will be for all

$$\frac{ddy}{dx^2} = \frac{2x(10-20xx)}{8x^6} = \frac{5(1-2xx)}{2x^5} = \frac{5(1-2xx)}{2x(3xx-1)} \quad \text{and} \quad y = \frac{x^3 - x}{2xx} = \frac{xx-1}{2x}.$$

But for the two first arsing from the equation $xx = x + 1$ it will be

$$\frac{ddy}{dx^2} = -\frac{5(2x+1)}{2x(3x+2)} = -\frac{5(2x+1)}{2(5x+3)} \quad \text{and} \quad y = \frac{1}{2}.$$

Therefore, the first root $x = \frac{1+\sqrt{5}}{2}$ gives

$$\frac{ddy}{dx^2} = -\frac{5(2+\sqrt{5})}{11+5\sqrt{5}}$$

and hence y is a maximum. The second root $x = \frac{1-\sqrt{5}}{2}$ gives

$$\frac{ddy}{dx^2} = -\frac{5(2-\sqrt{5})}{11-5\sqrt{5}} = -\frac{5(\sqrt{5}-2)}{5\sqrt{5}-11}$$

and hence $y = \frac{1}{2}$ will also be a maximum. The two remaining roots give $y = -\frac{1}{2}$, a minimum.

§262 Therefore, in these examples the exploration, whether a certain found value produces a maximum or a minimum, can be done easier; for, because it is $\frac{dy}{dx} = 0$, the value of the term $\frac{ddy}{dx^2}$ having taken into account its equation can be expressed in an easier way. For, let the fraction $y = \frac{P}{Q}$ be propounded; because it is

$$dy = \frac{QdP - PdQ}{QQ} \quad \text{and} \quad QdP - PdQ = 0,$$

it will be

$$ddy = \frac{d(QdP - PdQ)}{Q^2} - \frac{2dQ(QdP - PdQ)}{Q^3}.$$

But because of $QdP - PdQ = 0$ this last term vanishes and it will be

$$ddy = \frac{d(QdP - PdQ)}{QQ} = \frac{QddP - PddQ}{Q^2}.$$

But because the decision is made from the either affirmative or negative value of this term, and the denominator Q^2 is always affirmative, the task can be done from the numerator alone in such a way, that, if $QddP - PddQ$ or $\frac{d(QdP-PdQ)}{dx^2}$ was positive, a minimum will be indicated, if it is negative, a maximum. Or after $\frac{dy}{dx}$ was found, whose form will be of this kind $\frac{R}{QQ}$, only find $\frac{dR}{dx}$, and which roots leads to an affirmative value of this expression, from it a minimum will arise and otherwise a maximum.

§267 If the denominator of the propounded fraction was a square or any higher power, such that it is $y = \frac{P}{Q^n}$, it will be

$$dy = \frac{QdP - nPdQ}{Q^{n+1}}$$

and having put $\frac{QdP-nPdQ}{dx} = R$ it will be

$$\frac{dy}{dx} = \frac{R}{Q^{n+1}}$$

and the maxima and minima will be determined from the roots of the equation $R = 0$. Further, because it is

$$\frac{ddy}{dx} = \frac{QdR - (n+1)RdQ}{Q^{n+2}},$$

because of $R = 0$ it will be

$$\frac{ddy}{dx} = \frac{dR}{Q^{n+1}};$$

its positive value will indicate a minimum, a negative a maximum. But it is perspicuous, if n was an odd number, that because of the always positive Q^{n+1} the decision can be made alone from $\frac{dR}{dx}$; but if n is an even number, use the formula $\frac{QdR}{dx}$.

But let us further put that a fraction of this kind is propounded $\frac{P^m}{Q^n} = y$; it will be

$$dy = \frac{(mQdP - nPdQ)P^{m-1}}{Q^{n+1}};$$

therefore, if one puts $\frac{mQdP - nPdQ}{dx} = R$, the roots of the equation $R = 0$ will indicate the cases, in which the function y becomes a maximum or a minimum. Therefore, because it is

$$\frac{dy}{dx} = \frac{P^{m-1}R}{Q^{n+1}},$$

it will be

$$\frac{ddy}{dx} = \frac{P^{m-2}R((m-1)QdP - (n+1)PdQ)}{Q^{n+2}} + \frac{P^{m-1}dR}{Q^{n+1}},$$

and because of $R = 0$ it will be

$$\frac{ddy}{dx^2} = \frac{P^{m-1}dR}{Q^{n+1}dx};$$

this can additionally be divided by any square $\frac{P^{2\mu}}{Q^{2\nu}}$ to make the decision. Furthermore, also the equation $P = 0$ will give a maximum or a minimum, if m was an even number; and in similar manner by considering the inverse formula $\frac{Q^n}{P^m}$ a maximum or minimum will arise by putting $Q = 0$, if n was an even number, as we showed above (§ 257); but here we do not consider the maxima or minima to arise from there, but only, to explain the use of the method, find those, which arise from the equation $R = 0$.

Example 1

Let the fraction $\frac{(\alpha+\beta x)^m}{(\gamma+\delta x)^n}$; in which case it becomes a minimum or a maximum, is in question.

Having put $y = \frac{(\alpha+\beta x)^m}{(\gamma+\delta x)^n}$ at first it is certainly clear that it will be $y = 0$, if $x = -\frac{\alpha}{\beta}$, and $y = \infty$, if $x = -\frac{\gamma}{\delta}$; the latter of these cases will give a minimum, the first a maximum, if m and n were even numbers. Furthermore, it will be

$$\frac{dy}{dx} = \frac{(\alpha+\beta x)^{m-1}}{(\gamma+\delta x)^{n+1}}((m-n)\beta\delta x + m\beta\gamma - n\alpha\delta)$$

and hence

$$R = (m-n)\beta\delta x + m\beta\gamma - n\alpha\delta.$$

Hence, having put $R = 0$ it will be

$$x = \frac{n\alpha\delta - m\beta\gamma}{(m-n)\beta\delta}.$$

Further, because of $\frac{dR}{dx} = (m-n)\beta\delta$ it is perspicuous, whether

$$\frac{P^{m-1}dR}{Q^{n+1}dx} = \frac{m^{m-1}\beta^{n+1}}{n^{n+1}\delta^{m-1}}\left(\frac{\alpha\delta-\beta\gamma}{m-n}\right)^{m-n-2}\frac{dR}{dx}$$

is a positive or negative quantity. In the first case, the propounded formula will be a minimum, in the second a maximum.

Do, if it was $y = \frac{(x+3)^3}{(x+2)^2}$, it will be $\frac{P^{m+1}dR}{Q^{n+1}dx} = \frac{9}{8}$ and hence the formula $\frac{(x+3)^3}{(x+2)^2}$ will become a minimum, if one puts $x = 0$.

But if it is $y = \frac{(x-1)^m}{(x+1)^m}$, it will be

$$\frac{P^{m-1}dR}{Q^{n+1}dx} = \frac{m^{m-1}}{n^{n+1}}\left(\frac{n-m}{2}\right)^{n-m+2}(m-n)$$

and $x = \frac{n+m}{n-m}$. But because m and n are put to be positive numbers, the decision will be to made from the formula $(n-m)^{n-m+2}(m-n)$ or $(n-m)^{n-m}(m-n)$. Therefore, if it was $n > m$, the found value $x = \frac{n+m}{n-m}$ will always give a maximum; but if $n < m$, the number $m-n$ will give a minimum, but an odd a maximum; so $\frac{(x-1)^3}{(x+1)^2}$ will be a maximum having put $x = -5$; for, it is $y = -\frac{6^3}{4^2} = -\frac{27}{2}$.

EXAMPLE 2

Let the formula $y = \frac{(1+x)^3}{(1+xx)^2}$ *be propounded.*

It will be

$$\frac{dy}{dx}=\frac{(1+x)^2}{(1+xx)^3}(3-4x-xx) \quad \text{and} \quad \frac{P^{m-1}}{Q^{n+1}}\cdots\frac{dR}{dx}=-\frac{(1+x)^2}{(1+xx)^3}(2x+4);$$

because here $(1+x)^2$ and $(1+xx)^3$ always have a positive value, the decision is to be made from the formula $-x-2$; it, if it was positive, indicates a minimum, if negative, a maximum. But from the equation $3-4x-xx=0$ it follows either

$$x=-2+\sqrt{7} \quad \text{and} \quad x=-2-\sqrt{7}.$$

In the first case it is $-x-2=-\sqrt{7}$ and hence the propounded fraction will be a maximum, in the other case a minimum because of $-x-2=+\sqrt{7}$. But having put $x=-2+\sqrt{7}$ it will be $1+x=-1+\sqrt{7}$ and $1+xx=12-4\sqrt{7}$, whence

$$y=\left(\frac{-1+\sqrt{7}}{12-4\sqrt{7}}\right)^2(\sqrt{7}-1)=\frac{(2+\sqrt{7})^2(\sqrt{7}-1)}{16}=\frac{17+7\sqrt{7}}{16}=2.220-$$

But having put $x=-2-\sqrt{7}$ it will be

$$y=\frac{17-7\sqrt{7}}{16}=-0.0950.$$

§268 Also irrational and transcendental functions are given, which have the property of uniform functions, and therefore maxima and minima can be found the same way. For, the cube and all odd roots are indeed uniform, since they exhibit only one single real value; but square roots and those of all even powers even though they actually, if they are real, indicate a double value, the one positive, the other negative, every single one can nevertheless be considered separately and in this sense one can even investigate the maxima and minima. So, if y was any function of x, even though $\sqrt{y}$ has a double value, one can nevertheless can treat each single one separately. $+\sqrt{y}$ will have a maximal or minimal value, if y had such a one, if it was affirmative, since otherwise $\sqrt{y}$ would become imaginary. But vice versa $-\sqrt{y}$ will become a maximum or minimum in the same cases, in which $+\sqrt{y}$ becomes a maximum or minimum. But nay power $y^{\frac{m}{n}}$ becomes a maximum or minimum in the same cases, if n was an odd number; but if n was an even number, only those cases hold, in which y obtains a positive value, and in these cases because of the double value a double maximum or minimum will arise.

§269 Since the differential equation, which arises from the power of the function y^m, is $\frac{y^{m-1}dy}{dx} = 0$, whose roots at the same time indicate the cases, in which a surdic power $y^{\frac{m}{n}}$ becomes a maximum or minimum, to investigate this one has two equations, the one $y^{m-1} = 0$, the other $\frac{dy}{dx} = 0$, the latter of which goes over into $y = 0$ and then only exhibits maxima and minima, if $m-1$ was an odd number or if m was an even number, because of the reasons mentioned in § 257. Hence, because n is an odd number, if m was an even number, if we indicate the even numbers by 2μ and the odd numbers by $2\nu - 1$, the function $y^{2\mu:(2\nu-1)}$ will become maximal or minimal by attributing values to x, which it obtains so from this equation $y = 0$ as from this $\frac{dy}{dx} = 0$. But if m is an odd number, the function $y^{(2\mu-1):2\nu}$ or $y^{(2\mu-1):(2\nu)}$ then it only becomes maximal or minimal, if for x a value from this equation $\frac{dy}{dx} = 0$ is substituted. And in the second case $y^{(2\mu-1):2\nu}$ maxima and minima only arise, if y from the values found from the equation $\frac{dy}{dx} = 0$ receives affirmative values.

§270 So, this formula $x^{\frac{2}{3}}$ becomes a minimum by putting $x = 0$, because in this case x^2 becomes a minimum. But if we do not reduce the formula $x^{\frac{2}{3}}$ to the form x^2, the method given before would not indicate this at all, since in the case $x = 0$ the terms of the series

$$y + \frac{\omega dy}{dx} + \frac{\omega^2 ddy}{2dx^2} + \frac{\omega^3 d^3y}{6dx^3} + \text{etc.},$$

whence the decision is to be made, except for the first all become infinite. For, having put $y = x^{\frac{2}{3}}$ it will be

$$\frac{dy}{dx} = \frac{2}{3x^{\frac{1}{3}}}, \quad \frac{ddy}{dx^2} = \frac{-2}{9x^{\frac{4}{3}}}, \quad \frac{d^3y}{dx^3} = \frac{2\cdot 4}{27x^{\frac{7}{3}}} \quad \text{etc.}$$

And hence neither does the equation $\frac{dy}{dx} = \frac{2}{3x^{\frac{1}{3}}} = 0$ show the value $x = 0$ nor do the following terms indicate whether it is a maximum or a minimum. Therefore, since we assumed that the series

$$y + \frac{\omega dy}{dx} + \frac{\omega^2 ddy}{2dx^2} + \frac{\omega^3 d^3y}{6dx^3} + \text{etc.}$$

becomes convergent, if ω is set to be a very small quantity, to those cases, in which this series becomes divergent, the general method is not applicable, which happens in the example $y = x^{\frac{2}{3}}$ mentioned here, if one puts $x = 0$. Therefore, in these cases the same reduction, we used before, will be necessary to reduce the propounded expression to another form, which is not subjected to this inconvenience. But this only happens in very few cases, which are contained in the formula $y^{\frac{2\mu}{2\nu-1}}$ or are easily reduced to it. So, if the maxima and minima of the formula $y^{\frac{2\mu}{2\nu-1}}z$ are in question, where z is any function of x, investigate this form $y^{2\mu}z^{2\nu-1}$, which becomes maximal or minimal in the same cases as the propounded one itself.

§271 Having excluded this case, which is now easily handled, functions, which contain irrational quantities, can be treated the same way as rational functions and their maxima and minima can be determined, what we will illustrate in the following examples.

EXAMPLE 1

Let the formula $\sqrt{aa+xx}-x$ be propounded; in which cases is becomes maximal or minimal is in question.

Having put $y = \sqrt{aa+xx}-x$ it will be

$$\frac{dy}{dx} = \frac{x}{\sqrt{aa+xx}} - 1 \quad \text{and} \quad \frac{ddy}{dx^2} = \frac{aa}{(aa+xx)^{3:2}}.$$

Having put $\frac{dy}{dx} = 0$ it will be $x = \sqrt{aa+xx}$ and hence $x = \infty$ and it is $\frac{ddy}{dx^2} = 0$. In a similar way the following terms $\frac{d^3y}{dx^3}$, $\frac{d^4y}{dx^4}$ etc. all become $= 0$; from this a decision cannot be made, whether it is a maximum or a minimum. The reason is, that it actually is so $x = -\infty$ as $x = +\infty$. By putting $x = \infty$ because of

$$\sqrt{aa+xx} = x + \frac{aa}{2x}$$

it is $y = 0$, which value is the smallest of all.

EXAMPLE 2

Let the cases be in question, in which this form $\sqrt{aa+2bx+mxx}-nx$ becomes a maximum or a minimum.

Having put $y = \sqrt{aa + 2bx + mxx} - nx$ it will be

$$\frac{dy}{dx} = \frac{b + mx}{\sqrt{aa + 2bx + mxx}} - n;$$

having put $= 0$ it will be

$$bb + 2mbx + mmxx = nnaa + 2nnbx + mnnxx$$

or

$$xx = \frac{2bx(nn - m) + nnaa - bb}{mm - mnn}$$

and hence

$$x = \frac{(nn - m)b \pm \sqrt{mnn(m - nn)aa - nn(m - nn)bb}}{m(m - nn)}$$

or

$$x = -\frac{b}{m} \pm \frac{n}{m}\sqrt{\frac{maa - bb}{m - nn}};$$

hence, it is

$$\sqrt{aa + 2bx + mxx} = \frac{b + mx}{n} = \pm\sqrt{\frac{maa - bb}{m - nn}}.$$

Therefore, because it is

$$\frac{ddy}{dx^2} = \frac{maa - bb}{(aa + 2bx + mxx)^{\frac{3}{2}}},$$

it will be

$$\frac{ddy}{dx^2} = \frac{maa - bb}{\pm\left(\frac{maa-bb}{m-nn}\right)^{\frac{3}{2}}} = \frac{\pm(m - nn)\sqrt{m - nn}}{\sqrt{maa - bb}}.$$

Therefore, only if $\frac{m-nn}{maa-bb}$ was a positive quantity, a maximum or minimum is given. But if it is a positive quantity, the superior sign will give a minimum, if $m > nn$, a maximum on the other hand, if $m < nn$; the contrary happens, if the inferior sign holds. Therefore, if it is $m = 2$, $n = 1$ and $b = 0$, the formula $\sqrt{aa + 2xx} - x$ becomes a minimum by putting $x = +\frac{1}{2}\sqrt{2aa} = \frac{a}{\sqrt{2}}$, but a maximum by putting $x = -\frac{a}{\sqrt{2}}$. Therefore, the minimum will be $= a\sqrt{2} - \frac{a}{\sqrt{2}} = \frac{a}{\sqrt{2}}$ and the maximum $= a\sqrt{2} + \frac{a}{\sqrt{2}} = \frac{3a}{\sqrt{2}}$.

EXAMPLE 3

To find the cases, in which this expression $\sqrt[4]{1+mx^4} + \sqrt[4]{1-nx^4}$ *becomes a maximum or a minimum.*

Because it is $\frac{dy}{dx} = \frac{mx^3}{(1+mx^4)^{\frac{3}{4}}} - \frac{nx^3}{(1-nx^4)^{\frac{3}{4}}}$, it will be

$$mx^3(1-nx^4)^{\frac{3}{4}} = nx^3(1+mx^4)^{\frac{3}{4}} \quad \text{and hence} \quad m^4(1-nx^4)^3 = n^4(1+mx^4)^3$$

or

$$n^4 - m^4 + 3mn(n^3+m^3)x^4 + 3m^2n^2(n^2-m^2)x^8 + m^3n^3(n+m)x^{12} = 0.$$

Therefore, only if this equation has a positive root for x^4, a maximum or minimum is given. For, this equation in general cannot be solved in a convenient manner, since it will be

$$x^4 = \frac{m^{\frac{4}{3}} - n^{\frac{4}{3}}}{mn(\sqrt[3]{m}+\sqrt[3]{n})} \quad \text{or} \quad x^4 = \frac{m - \sqrt[3]{m^2n} + \sqrt[3]{mn^2} - n}{mn}$$

let us for a special case put $m = 8n$ and it will be

$$-4095 + 24 \cdot 513x^4 - 3 \cdot 63 \cdot 64n^2x^8 + 9 \cdot 512x^{12} = 0$$

or

$$512n^3x^{12} - 1344n^2x^8 + 1368nx^4 - 455 = 0;$$

put $8nx^4 = z$; it will be

$$z^3 - 21z^2 + 171z - 455 = 0,$$

which has the divisor $z - 5$, and the other factor will be $zz - 16zz + 91 = 0$ containing imaginary roots. Therefore, it will only be $z = 8nx^4 = 5$ and hence $x = \sqrt[4]{\frac{5}{8n}}$, which value will render the expression $\sqrt[4]{1+8nx^4} + \sqrt[4]{1-nx^4}$ a maximum or a minimum. To find out which of both happens, consider

$$\frac{ddy}{dx^2} = \frac{3mxx}{(1+mx^4)^{\frac{7}{4}}} - \frac{3nxx}{(1-nx^4)^{\frac{7}{4}}}.$$

But because of $m = 8n$ having put $x^4 = \frac{5}{8n}$ it will be

$$\frac{ddy}{dx^2} = \left(\frac{24n}{6^{\frac{7}{4}}} - \frac{3n}{(3:8)^{\frac{7}{4}}}\right) xx = -\frac{360nxx}{6^{\frac{7}{4}}}$$

and hence negative; therefore, $\sqrt[4]{1+8nx^4} + \sqrt[4]{1-nx^4}$ will become a maximum having put $x = \sqrt[4]{\frac{5}{8n}}$. This maximum will be $= \sqrt[4]{6} + \sqrt[4]{\frac{3}{8}} = \frac{3\sqrt[4]{6}}{2}$. If instead of nx^4 we put u, it is plain that this expression $\sqrt[4]{1+8u} + \sqrt[4]{1-u}$ becomes a maximum having put $u = \frac{5}{8}$ and that this maximal value will be $= \frac{3\sqrt[4]{6}}{2} =$ 2.347627. Therefore, whatever value except $\frac{5}{8}$ is written fro u, the expression will receive a smaller value.

§272 In similar manner maxima and minima will be determined, if also transcendental quantities are contained in the propounded expression. For, in the propounded function was not multiform and one has not to consider several meanings of it at the same time, the roots of the differential equation will show maxima or minima, if they were not equal roots, whose number is even. Therefore, we will demonstrate this investigation in several examples.

EXAMPLE 1

To find the number, which has the smallest ratio to its logarithm.

That a smallest ratio $\frac{x}{\log x}$ is given is plain, because this ratio having put so $x = 1$ as $x = \infty$ becomes infinite. Therefore, vice versa the fraction $\frac{\ln x}{x}$ will somewhere has a maximal value, of course in the same case, in which $\frac{x}{\ln x}$ becomes a minimum. To find this cases put $y = \frac{\log x}{x}$ and it will be

$$\frac{dy}{dx} = \frac{1}{xx} - \frac{\log x}{xx}.$$

Having put this equal to zero it will be $\log x = 1$, and since we assume the hyperbolic logarithm here, if e is put for the number, whose hyperbolic logarithm is $= 1$, it will be $x = e$. Therefore, because all hyperbolic logarithms are in given ratio to the hyperbolic ones, $\frac{e}{\log e}$ will also be a minimum for the usual logarithm or $\frac{\log e}{e}$ will be a maximum. Since in tabled logarithms it is $\log e = 0.4342944819$, the fraction $\frac{\log x}{x}$ will always be smaller than $\frac{4342944819}{27182818284}$ or approximately than $\frac{47}{305}$ and no other number is given, which to its logarithm has a smaller ratio than 305 to 47. That in this case $\frac{\log x}{x}$ is a maximum is plain, because because of $\frac{dy}{dx} = \frac{1-\log x}{xx}$ it is

$$\frac{ddy}{dx^2} = -\frac{1}{x^3} - \frac{2(1-\log x)}{x^3} = -\frac{1}{x^3}$$

because of $1 - \log x = 0$ and hence negative.

Example 2

To find the number x, that this power $x^{1:x}$ becomes a maximum.

That a maximal value of this formula is given is plain, because by substituting numbers for x it is

$$\begin{aligned} 1^{1:1} &= 1.000000 \\ 2^{1:2} &= 1.414213 \\ 3^{1:3} &= 1.442250 \\ 1^{1:4} &= 1.414213. \end{aligned}$$

Therefore, put $x^{1:x} = y$ and it will be

$$\frac{dy}{dx} = x^{1:x}\left(\frac{1}{xx} - \frac{\log x}{xx}\right).$$

Having put this value equal to zero it will be $\log x = 1$ and $x = e$ where $e = 2.718281828$. And because it is $\frac{dy}{dx} = (1-\log x)\frac{x^{1:x}}{xx}$, it will be

$$\frac{ddy}{dx^2} = -\frac{x^{1:x}}{x^3} + (1-\log x)\frac{d}{dx}\cdot\frac{x^{1:x}}{xx} = -\frac{x^{1:x}}{x^3}$$

because $1 - \log x = 0$- Hence, because $\frac{ddy}{dx^2}$ is a negative quantity, $x^{1:x}$ becomes a maximum in the case $x = e$. But because it is $e = 2.718281828$, one finds that it will $e^{\frac{1}{e}} = 1.444667861009764$, which value is easily obtained from the series

$$e^{\frac{1}{e}} = 1 + \frac{1}{e} + \frac{1}{2e^2} + \frac{1}{6e^3} + \frac{1}{24e^4} + \text{etc.}$$

This example is also resolved from the preceding; for, if $x^{1:x}$ is a maximum, also its logarithm, which is $\frac{\log x}{x}$, will have to be a maximum; that this happens, as has to be $x = e$, as we found.

EXAMPLE 3

To find the arc x that its sine is maximal or minimal.

Having put $\sin x = y$ it will be $\frac{dy}{dx} = \cos x$ and hence $\cos x = 0$, whence the following values for x arise: $\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}$ etc. But it is $\frac{ddy}{dx^2} = -\sin x$. because therefore these values substituted for x give either $+1$ or -1 for $\sin x$, the latter will be maxima, the first minima, as it is known.

EXAMPLE 4

To find the arc x, that the rectangle $x \sin x$ becomes a maximum.

That a maximum is given is plain, because having put either $x = 0°$ or $180°$ the propounded rectangle vanishes in both cases. Therefore, let $y = x \sin x$; it will be

$$\frac{dy}{dx} = \sin x + x\cos x$$

and hence

$$\tan x = -x.$$

Let $x = 90° + u$; it will be $\tan x = -\cot u$, therefore $\cot u = 90° + u$. To resolve the equation in the way explained above [§ 234] put $z = 90° + u - \cot u$ and let f be the value in question of the arc u. Because it is $dz = du + \frac{du}{\sin^2 u}$, it will be

$$p = \frac{du}{dz} = \frac{\sin^2 u}{1+\sin^2 u} \quad dp = \frac{2du \sin u \cos u}{(1+\sin u)^2}$$

and hence

$$\frac{dp}{dz} = q = \frac{2\sin^3 u \cos u}{(1+\sin^2 u)^3}, \quad dq = \frac{6du\sin^2 u\cos^2 u - 2du\sin^4 u}{(1+\sin^2 u)^3} - \frac{12du\sin^4 u\cos^2 u}{(1+\sin^2 u)^4}.$$

Therefore,

$$\frac{dq}{dz} = r = \frac{6\sin^4 u\cos^2 u - 2\sin^6 u}{(1+\sin^2 u)^4} - \frac{12\sin^2 u^6\cos^2 u}{(1+\sin^2 u)^5} = \frac{6\sin^4 - 14\sin u^6 + 4\sin^8 u}{(1+\sin^2 u)^5}.$$

From these it will be

$$f = u - pz + \frac{1}{2}qzz - \frac{1}{6}rz^3 + \text{etc.}$$

Put, after by trying several values an approximate value of f was detected, $u = 26°15'$; it will be $90° + u = 116°15'$ and the arc equal to the cotangent u is defined this way. From

	$\log\cot u$	$=$	10.3070250
subtract			4.6855749
			5.6214501

Therefore	$\cot u$	$=$	$418263.7''$
or	$\cot u$	$=$	$116°11'3\frac{7}{10}''$
hence	z	$=$	$3'56\frac{3}{10}'' = 256.3''$.

Now to find the value of the term pz perform this calculation:

	$\log\sin u$	$=$	9.6457058
	$\log\sin^2 u$	$=$	9.2914116
	$1+\sin^2 u$	$=$	1.19561
	$\log(1+\sin^2 u)$	$=$	0.0775895
	$\log p$	$=$	9.2138221
	$\log z$	$=$	2.2724637
	$\log pz$	$=$	1.5872858
Therefore	pz	$=$	38.6621 seconds
or	pz	$=$	$38''39'''43''''$
from	u	$=$	$26°15'$
it will be	f	$=$	$26°14'21''20'''17''''$
and the arc in question	x	$=$	$116°14'21''20'''17''''$

But the third term $\frac{1}{2}qzz = \frac{\sin^3 u \cos u}{(1+\sin^2 u)^3}zz$ has to added additionally. To find its value, one z must be expressed in parts of the radius this way:

	$\log \sin z$	$=$	2.3734637
add			4.6855749
			7.0590386
add	$\log \frac{\sin^2 u}{1+\sin^2 u} z$	$=$	1.5872858
			8.6463244
add	$\log \sin u$	$=$	9.6457058
	$\log \cos u$	$=$	9.9527308
			8.2447600
subtract	$\log(1+\sin^2 u)^2$	$=$	0.1551790
	$\log \frac{1}{2}qzz$	$=$	8.0895810

Therefore

$$\frac{1}{2}qzz = 0.012291$$

or

$$\frac{1}{2}qzz = 44''''15'''''.$$

Hence, having also this term the arc in question will be

$$x = 116^\circ 14' 21'' 21''' 0'''';$$

but having used larger logarithms one finds

$$x = 116^\circ 14' 21'' 20''' 35'''' 47'''''.$$