On the transformation of series *

Leonhard Euler

§1 Since it is propounded to us to show the use of differential calculus both in whole analysis and in the doctrine of series, several auxiliary tools from common algebra which are usually are not treated will have to be repeated here. Although we covered a very large part of it already in the *Introductio*, some things were nevertheless left aside there, either on purpose because it is convenient to explain them just then when the necessity demands it, or because all the things which will be necessary could not have been foreseen. This concerns the transformation of series, to which we devote this chapter and by means of which a certain series is transformed into innumerable others such that, if the sum of the propounded series is known, the resulting ones can all be summed at the same time. But having covered this chapter in advance we will be able to amplify the doctrine of series by means of differential and integral calculus even further.

§2 But, we will mainly consider series of such a kind whose single terms are multiplied by successive powers of a certain undetermined quantity, since these extend further and are of greater utility.

Therefore, let the following general series be propounded, whose sum, either known or not, we want to put $= S$, and let

$$S = ax + bx^2 + cx^3 + dx^4 + ex^5 + \text{etc.}$$

*Original title: " De Transformatione serierum", first published as part of the book „*Institutiones calculi differentialis cum eius usu in analysi finitorum ac doctrina serierum*, 1755", reprinted in in „*Opera Omnia*: Series 1, Volume 10, pp. 217 - 234 ", Eneström-Number E212, translated by: Alexander Aycock for „Euler-Kreis Mainz"

Now, put $x = \frac{y}{1+y}$ and because it is by means of infinite series

$$\begin{aligned}
x &= y - y^2 + y^3 - y^4 + y^5 - y^6 + \text{etc.}\\
x^2 &= y^2 - 2y^3 + 3y^4 - 4y^5 + 5y^6 - 6y^7 + \text{etc.}\\
x^3 &= y^3 - 3y^4 + 6y^5 - 10y^6 + 15y^7 - 21y^8 + \text{etc.}\\
x^4 &= y^4 - 4y^4 + 10y^6 - 20y^7 + 35y^8 - 35y^8 + \text{etc.}\\
&\text{etc.,}
\end{aligned}$$

these values substituted, and having ordered the series according to powers of y, will give

$$\begin{array}{lllll}
S = ay & - ay^2 & + ay^3 & - ay^4 & + ay^5 \quad \text{etc.}\\
& + b & - 2b & + 3b & - 4b\\
& & + c & - 3c & + 6c\\
& & & + d & - 3d\\
& & & & + e
\end{array}$$

§3 Since we put $x = \frac{y}{1+y}$, it will be $y = \frac{x}{1-x}$; having substituted this value for y the propounded series

$$S = ax + bx^2 + cx^3 + dx^4 + ex^5 + \text{etc.}$$

will be transformed into this one:

$$S = a\frac{x}{1-x} + (b-a)\frac{x^2}{(1-x)^2} + (c-2b+a)\frac{x^3}{(1-x)^3} + \text{etc.},$$

in which the coefficient of the second term $b-a$ is the first difference of a from the series a, b, c, d, e etc., which we denoted by Δa above; the coefficient of the third term $c-2b+a$ is the second difference $\Delta^2 a$; the coefficient of the fourth is the third difference of $\Delta^3 a$ etc. Hence, using the continued differences of a which are formed from the series a, b, c, d, e etc. the transformed series will be transformed into this one

$$S = \frac{x}{1-x}a + \frac{x^2}{(1-x)^2}\Delta a + \frac{x^3}{(1-x)^3}\Delta^2 a + \frac{x^4}{(1-x)^4}\Delta^3 a + \text{etc.},$$

the sum of which series one will therefore have, if the sum of the propounded was known.

§4 Therefore, if the series a, b, c, d etc. was of such a nature that it finally has constant differences which happens if its general term was a polynomial function then the latter series $\frac{x}{1-x}a + \frac{x^2}{(1-x)^2}\Delta a +$etc. finally will have vanishing terms and so its sum can be exhibited by means of a finite expression. So, if the first differences of the series a, b, c, d etc. already were constant, then the sum of this series $ax + bx^2 + cx^3 + dx^4 +$ etc. will be

$$= \frac{x}{1-x}a + \frac{x^2}{(1-x)^2}\Delta a.$$

But if the second differences of the coefficients of that series become constant, then the sum of the propounded series itself will be

$$= \frac{x}{1-x}a + \frac{x^2}{(1-x)^2}\Delta a + \frac{x^3}{(1-x)^3}\Delta\Delta a.$$

Hence, the sums of series of this kind are easily found from the differences of the coefficients.

I. *Let the sum of this series be sought after*

$$1x + 3x^2 + 5x + 7x^4 + 9x^5 + \text{etc.},$$

$$\text{Diff. I} \quad 2, \quad 2, \quad 2, \quad 2 \quad \text{etc.}$$

Since therefore the first differences are constant, because of $a = 1$ and $\Delta a = 2$ the sum of the propounded series will be

$$= \frac{x}{1-x} + \frac{2xx}{(1-x)^2} = \frac{x+xx}{(1-x)^2}.$$

II. *Let the sum of this series be sought after*

$$1x + 4xx + 9x^3 + 16x^4 + 25x^5 + \text{etc.}$$

$$\begin{array}{llllll} \text{Diff. I} & 3, & 5, & 7, & 9, & \text{etc.} \\ \text{Diff. II} & 2 & 2 & 2 & \text{etc.} & \end{array}$$

Since therefore it is $a = 1$, $\Delta a = 3$, $\Delta^2 a = 2$, the sum of the propounded series will be

$$= \frac{x}{1-x} + \frac{3xx}{(1-x)^2} + \frac{2x^3}{(1-x)^3} = \frac{x+xx}{(1-x)^3}.$$

III. *Let the sum of this series be sought after*

$$S = 4x + 15x^2 + 40x^3 + 85x^4 + 156x^5 + 259x^6 + \text{etc.}$$

Diff. I	11,		25,		45,		71,		103		etc.
Diff. II		14,		20,		26,		32,		etc.	
Diff. III			6,		6,		6,		etc.		

Because it is $a = 4$, $\Delta a = 11$, $\Delta^2 a = 14$, $\Delta^3 a = 6$, the sum will be

$$S = \frac{4x}{1-x} + \frac{11xx}{(1-x)^2} + \frac{14x^3}{(1-x)^3} + \frac{6x^4}{(1-x)^4}$$

or

$$S = \frac{4x - xx + 4x^3 - x^4}{(1-x)^4} = \frac{x(1+xx)(4-x)}{(1-x)^4}.$$

§5 Although this way the sums of these series continuing to infinity are found, nevertheless from the same principles these series can also be summed up to a given term. For, let this series be propounded

$$S = ax + bx^2 + cx^3 + dx^4 + \cdots\cdots + ox^n,$$

and let its sum be sought after, if its proceeds to infinity, which will be

$$= \frac{x}{1-x}a + \frac{x^2}{(1-x)^2}\Delta a + \frac{x^3}{(1-x)^3}\Delta^2 a + \text{etc.}$$

Now, consider the terms of the same series following after the last ox^n which shall be

$$px^{n+1} + qx^{n+2} + rx^{n+3} + sx^{n+4} + \text{etc.};$$

the sum of this series, if it is divided by x^n, can be found as before; this, multiplied by x^n again, will be

$$\frac{x^{n+1}}{1-x}p + \frac{x^{n+2}}{(1-x)^2}\Delta p + \frac{x^{n+3}}{(1-x)^3}\Delta^2 p + \text{etc.};$$

if the sum of this series is subtracted from the sum of the series continued to infinity, the sum of the propounded portion sought after will remain

$$S = \frac{x}{1-x}(a - x^p) + \frac{x^2}{(1-x)^2}(\Delta a - x^n \Delta p) + \frac{x^3}{(1-x)^3}(\Delta^2 a - x^n \Delta^2 p) + \text{etc.}$$

I. Let the sum of this finite series be sought after

$$S = 1x + 2x^2 + 3x^3 + 4x^4 + \cdots\cdots nx^n.$$

Seek for the differences so of these coefficients as of the ones following the last term

$$\begin{array}{ccccc|cccc} 1, & 2, & 3, & 4, & \text{etc.} & n+1, & n+2, & n+3, & \text{etc.} \\ & 1, & 1, & 1, & \text{etc.} & & 1, & 1, & \text{etc.} \end{array}$$

and it will be $a = 1$, $\Delta a = 1$, $p = n+1$, $\Delta p = 1$, whence the sum sought after is

$$s = \frac{x}{1-x}(1 - (n+1)x^n) + \frac{x^2}{(1-x)^2}(1 - x^n)$$

or

$$S = \frac{x - (n+1)x^{n+1} + nx^{n+2}}{(1-x)^2}.$$

I. Let the sum of this finite series be sought after

$$S = 1 + x + 4x + 9x^3 + 16x^4 + \cdots\cdots + n^2x^n.$$

At first, investigate the differences this way

$$\begin{array}{ccccc|cccc} 1, & 4, & 9, & 16, & \text{etc.} & (n+1)^2, & (n+2)^2, & (n+3)^2, & \text{etc.} \\ & 3, & 5, & 7, & \text{etc.} & & 2n+3, & 2n+5, & \text{etc.} \\ & & 2, & 2, & \text{etc.} & & & 2, & \text{etc.} \end{array}$$

having found which the sum sought after will be

$$S = \frac{x}{1-x}(1-(n+1)^2x^n) + \frac{x^2}{(1-x)^2}(3-(2n+3)x^n) + \frac{x^3}{(1-x)^3}(2-2x^n)$$

or

$$S = \frac{x + xx - (n+1)^2x^{n+1} + (2nn+2n-1)x^{n+2} - nnx^{n+3}}{(1-x)^3}.$$

§6 But if the propounded series does not have coefficients of such a kind, which are finally led to constant differences, then the transformation exhibited here is not of any use to determine its sum. Nevertheless the sum can be approximated in a more convenient way by means of it than it is possible by addition of the propounded series itself. For, if in the series $ax + bx^2 + cx^3 + dx^4 + \text{etc.}$ it was $x < 1$, in which case alone the summation in the sense explained above can hold, it will be $\frac{x}{1-x} > x$ and hence the new series converges less the propounded one. But if in the propounded series it was $x = 1$, then all the terms of the new series become infinite, in which case this transformation will therefore will be completely useless.

§7 Let us consider the series in which the signs $+$ and $-$ alternate which will be deduced from the preceding by putting x negative. If it therefore was

$$S = ax - bx^2 + cx^3 - dx^4 + ex^5 - \text{etc.},$$

the negative of which series arises, if in the preceding x is put negative; therefore, as before let us take the differences Δa, $\Delta^2 a$, $\Delta^3 a$ etc. from the series of coefficients a, b, c, d, e etc., having related the signs solely to the powers of x, and the propounded series will be transformed into this one

$$S = \frac{x}{1+x}a - \frac{x^2}{(1+x)^2}\Delta a + \frac{x^3}{(1+x)^3}\Delta^2 a - \frac{x^4}{(1+x)^4}\Delta^3 a + \text{etc.},$$

whence it is seen that the propounded equation can be summed in the same cases as the preceding, of course, if the series a, b, c, d etc. is finally led to constant differences.

§8 But in this case, this transformation yields a convenient approximation to the value of the propounded series $ax - bx^2 + cx^3 - dx^4 + ex^5 - fx^6 +$ etc.; for, no matter how large the number x is, the fraction $\frac{x}{1+x}$, in powers of which the other series is expanded, becomes smaller than unity; and if $x = 1$, it will be $\frac{x}{1+x} = \frac{1}{2}$. But if it is $x < 1$, say $x = \frac{1}{n}$, it will be $\frac{x}{1+x} = \frac{1}{n+1}$ and hence the series found by means of the transformation will always converge more than the propounded one. Let us especially consider the case, in which $x = 1$, which offers a huge amount of help for the summation of series, and let be

$$S = a - b + c - d + e - f + \text{etc.},$$

and denote the first, second and following differences of a, which the progression a, b, c, d, e etc. yields, by Δa, $\Delta^2 a$, $\Delta^3 a$ etc.; having found these, it will be

$$S = \frac{1}{2}a - \frac{1}{4}\Delta a + \frac{1}{8}\Delta^2 a - \frac{1}{16}\Delta^3 a + \text{etc.},$$

which, if it is not actually terminated, exhibits the approximate sum sufficiently convenient.

§9 Therefore, let us show the use of this last transformation, in which we took $x = 1$, in some examples and at first certainly in examples of such a kind, in which the true sum can be expressed finitely. Such series are divergent series, in which the numbers a, b, c, d etc. finally lead to constant differences; since the sums of these in the usual reception of the word sum can not be exhibited, we understand the word sum here in this sense, which attributed to it above [§ 111 of the first part], such that it denotes the value of the finite expression, from the expansion of which the propounded series arises.

I. *Therefore, let this series due to Leibniz be propounded*

$$S = 1 - 1 + 1 - 1 + 1 - 1 + \text{etc.};$$

because in this series all terms are equal, all differences will become $= 0$ and hence because of $a = 1$ it will be $S = \frac{1}{2}$.

II. *Let this series be propounded*

$$S = 1 - 2 + 3 - 4 + 5 - 6 + \text{etc.}$$

$$\text{Diff. I} \quad 1, \quad 1, \quad 1, \quad 1, \quad 1 \quad \text{etc.}$$

Because it therefore is $a = 1$, $\Delta a = 1$, it will be $S = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$.

III. *Let this series be propounded*

$$S = 1 - 3 + 5 - 7 + 9 - \text{etc.}$$

$$\text{Diff. II} \quad 2, \quad 2, \quad 2, \quad 2, \quad \text{etc.}$$

Because of $a = 1$ and $\Delta a = 2$ it is $S = \frac{1}{2} - \frac{2}{4} = 0$.

IV. *Let this series of the triangular numbers be propounded*

$$1 - 3 + 6 - 10 + 15 - 21 + \text{etc.}$$

$$\begin{array}{lcccccc} \text{Diff. I} & 2, & 3, & 4, & 5, & 6, & \text{etc.} \\ \text{Diff. II} & & 1 & 1, & 1, & 1 & \text{etc.} \end{array}$$

Here, because of $a = 1$, $\Delta a = 2$ and $\Delta\Delta a = 1$ it will therefore be $S = \frac{1}{2} - \frac{2}{4} + \frac{1}{8} = \frac{1}{8}$.

V. *Let the series of the squares be propounded*

$$S = 1 - 4 + 9 - 16 + 25 - 36 + \text{etc.}$$

$$\begin{array}{lcccccc} \text{Diff. I} & 3, & 5, & 7, & 9, & 11, & \text{etc.} \\ \text{Diff. II} & & 2 & 2, & 2, & 2 & \text{etc.} \end{array}$$

Because of $a = 1$, $\Delta a = 3$, $\Delta\Delta a = 2$ it will be $S = \frac{1}{2} - \frac{3}{4} + \frac{2}{8} = 0$.

VI. *Let this series of the bisquares be propounded*

$$S = 1 - 16 + 81 - 256 + 625 - 1296 + \text{etc.}$$

Diff. I	15,		65,		175,		175,		369,		671,	etc.
Diff. II		50,		110,		194,		302,		etc.		
Diff. III			60,		84,		108,		etc.			
Diff. IV				24,		24,		etc.				

Therefore, it will be $S = \frac{1}{2} - \frac{15}{4} + \frac{50}{8} - \frac{60}{16} + \frac{24}{32} = 0$.

§10 If the series diverges more, as the geometric series and other similar ones, this way these are immediately transformed into a more convergent series which, if they did not already converge sufficiently enough, in the same manner will be converted into another more convergent ones.

I. *Let this geometric series be propounded*

$$s = 1 - 2 + 4 - 8 + 16 - 32 + \text{etc.}$$

Diff. I	1,		2,		4,		8,		16,		etc.
Diff. II		1,		2,		4,		8,		etc.	
Diff. III			1,		2,		4,		etc.		

Because therefore in all these differences the first term is $=1$, the sum of the series will be expressed this way

$$S = \frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \frac{1}{32} - \frac{1}{64} + \text{etc.},$$

the sum of which is $= \frac{1}{3}$; for, it arises from the expansion of the fraction $\frac{1}{2+1}$, whereas the propounded arises from $\frac{1}{1+2}$.

II. *Let this recurring series be propounded*

$$S = 1 - 2 + 5 - 12 + 29 - 70 + 169 - \text{etc.}$$

Diff. I	1,	3,	7,	17,	41,	99,	etc.	
Diff. II	2,	4,	10,	24,	58,	etc.		
Diff. III	2,	6,	14,	34,	etc.			
Diff. IV	4,	8,	20,	etc.				
Diff. V	4,	12,	etc.					
Diff. VI	8,	etc.						
	etc.							

Therefore, the first terms of the continued differences constitute this doubled geometric series 1, 1, 2, 2, 4, 4, 8, 8, 16, 16 etc., whence it will be

$$S = \frac{1}{2} - \frac{1}{4} + \frac{2}{8} - \frac{2}{16} + \frac{4}{32} - \frac{4}{64} + \frac{8}{128} - \text{etc.};$$

because therefore except from the first each two remaining cancel each other, it will be $S = \frac{1}{2}$. But the propounded series arises from the expansion of the fraction $\frac{1}{1+2-1} = \frac{1}{2}$, as we showed in the expression of the nature of recurring series.

III. *Let the hypergeometric series be propounded*

$$S = 1 - 2 + 6 - 24 + 120 - 720 + 5040 - \text{etc.},$$

whose continued differences we will investigate more convenient this way:

	Diff. I	Diff. II	Diff. III	
1	1	3	11	
2	4	14	64	
6	18	78	426	
24	96	504	3216	
120	600	3720	27240	etc.
720	4320	30960	256320	
5040	35280	287280	2656080	
40320	322560	2943360		
362880	3265920			
3628800				

Having further continued these differences it will be

$$S = \frac{1}{2} - \frac{1}{4} + \frac{3}{8} - \frac{11}{16} + \frac{53}{32} - \frac{309}{64} + \frac{2119}{128} - \frac{16687}{256} + \frac{148329}{512} - \frac{1468457}{1024} + \frac{16019531}{2048} - \frac{190899411}{4096} + \text{etc.}$$

Collect the two initial terms and it will be $S = \frac{1}{4} + A$ where

$$A = \frac{3}{8} - \frac{11}{16} + \frac{53}{32} - \frac{309}{64} + \frac{2119}{128} - \text{etc.}$$

If now in the same way the differences are taken, it will be

$$A = \frac{3}{2^4} - \frac{5}{2^6} + \frac{21}{2^8} - \frac{99}{2^{10}} + \frac{615}{2^{12}} - \frac{4401}{2^{14}} + \frac{36585}{2^{16}} - \frac{342207}{2^{18}} + \frac{3565323}{2^{20}} - \frac{40866525}{2^{22}} + \text{etc.}$$

Collect the two initial terms, because they converge, and it will be $A = \frac{7}{2^6} + B$ while $B = \frac{21}{2^8} - \frac{99}{2^{10}} + \text{etc.}$; by again taking the differences of this series it will be

$$B = \frac{21}{2^9} - \frac{15}{2^{12}} + \frac{159}{2^{15}} - \frac{429}{2^{18}} + \frac{5241}{2^{21}} - \frac{26283}{2^{24}} + \frac{338835}{2^{27}} - \frac{2771097}{2^{30}} + \text{etc.}$$

Collect the four initial terms together to one and put $B = \frac{153}{2^{12}} + \frac{843}{2^{20}} + C$ while

$$C = \frac{5241}{2^{21}} - \frac{26283}{2^{24}} + \text{etc.}$$

and by actually summing up some terms it will approximately be $C = \frac{15645}{2^{24}} - \frac{60417}{2^{30}}$. From these therefore finally the sum of the series will be concluded to be $S = 0.40082055$, which can nevertheless only be deemed accurate hardly further than three or four figures because of the the divergence of the series; it is nevertheless certainly smaller than the correct value. For, I elsewhere found this sum to be $= 0.4036524077$, where not even the last digit deviates from the true value.

§11 But this transformation is especially highly useful to transform already but slowly converging series into others which converge a lot more rapidly. Since indeed the following terms are smaller than the preceding ones, the first differences become negative; hence, in the following the nature of the sign is carefully to be taken into account.

I. *Let this series be propounded*

$$S = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \text{etc.}$$

$$\begin{array}{lllllll} \text{Diff. I} & -\frac{1}{2}', & -\frac{1}{2\cdot 3}', & -\frac{1}{3\cdot 4}', & -\frac{1}{4\cdot 5}', & \frac{1}{5\cdot 6} & \text{etc.} \\ \text{Diff. II} & +\frac{1}{3}', & \frac{2}{2\cdot 3\cdot 4}', & \frac{2}{3\cdot 4\cdot 5}', & \frac{2}{4\cdot 5\cdot 6}\text{etc.} & & \\ \text{Diff. III} & -\frac{1}{4}', & -\frac{2\cdot 3}{2\cdot 3\cdot 4\cdot 5}', & -\frac{2\cdot 3}{3\cdot 4\cdot 5\cdot 6} & & & \\ \text{Diff. II} & +\frac{1}{5} \quad \text{etc.} & & & & & \\ & \text{etc.} & & & & & \end{array}$$

Hence, it will therefore be

$$S = \frac{1}{2} + \frac{1}{2\cdot 4} + \frac{1}{3\cdot 8} + \frac{1}{4\cdot 16} + \frac{1}{5\cdot 32} + \text{etc.};$$

but that both series exhibit the hyperbolic logarithm of two we already showed in the *Introductio*.

II. *Let this series for the circle be propounded*

$$\begin{array}{llllll} \text{Diff. I} & -\frac{2}{1\cdot 3}', & -\frac{2}{3\cdot 5}', & -\frac{2}{5\cdot 7}', & -\frac{2}{7\cdot 9}', & -\frac{2}{9\cdot 11} \quad \text{etc.} \\ \text{Diff. II} & +\frac{2\cdot 4}{1\cdot 3\cdot 5}', & \frac{2\cdot 4}{3\cdot 5\cdot 7}', & \frac{2\cdot 4}{5\cdot 7\cdot 9}', & \frac{2\cdot 4}{7\cdot 9\cdot 11} \quad \text{etc.} & \\ \text{Diff. III} & -\frac{2\cdot 4\cdot 6}{1\cdot 3\cdot 5\cdot 7}', & \frac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7\cdot 9} \quad \text{etc.} & & & \end{array}$$

Hence, it is therefore concluded that also be the sum of the series will also be

$$S = \frac{1}{2} + \frac{1}{3 \cdot 2} + \frac{1 \cdot 2}{3 \cdot 5 \cdot 2} + \frac{1 \cdot 2 \cdot 3}{3 \cdot 5 \cdot 7 \cdot 2} + \text{etc.}$$

or

$$2S = 1 + \frac{1}{3} + \frac{1 \cdot 2}{3 \cdot 5} + \frac{1 \cdot 2 \cdot 3}{3 \cdot 5 \cdot 7} + \frac{1 \cdot 2 \cdot 3 \cdot 4}{3 \cdot 5 \cdot 7 \cdot 9} + \text{etc.}$$

III. *Let the value of this infinite series be sought after*

$$S = \ln 2 - \ln 3 + \ln 4 - \ln 5 + \ln 6 - \ln 7 + \ln 8 - \ln 9 + \text{etc.}$$

Since the differences at the beginning become too unequal, actually collect the terms up to $\ln 10$ from tables, whose value will be found to be $= -0.3911005$, and it will be

$$S = -0.3911005 + \ln 10 - \ln 11 + \ln 12 - \ln 13 + \ln 14 - \ln 15 + \text{etc.}$$

up to infinity. Take those logarithms from tables and look for their differences this way:

	Diff. I	Diff. II	Diff. III	Diff. IV	Diff. V
$\ln 10 = 1.0000000$	$+$	$-$	$+$	$-$	$+$
$\ln 11 = 1.0413927$	413927				
		36042			
$\ln 12 = 1.0791812$	377885		5779		
		30263		1292	
$\ln 13 = 1.1139434$	347622		4487		368
		25776		924	
$\ln 14 = 1.1461280$	321846		3563		
		22213			
$\ln 15 = 1.1760913$	299633				

From these it is found

$$\ln 10 - \ln 11 + \ln 12 - \ln 13 + \text{etc.}$$
$$= \frac{1.0000000}{2} - \frac{413927}{4} - \frac{36042}{8} - \frac{5779}{16} - \frac{1292}{32} - \frac{368}{64} = 0.4891606.$$

Hence, the value of the propounded series will be

$$S = \ln 2 - \ln 3 + \ln 4 - \ln 5 + \text{etc.} = 0.0980601,$$

to which logarithm corresponds the number 1.253315.

§12 As we obtained these transformations by putting the fraction $\frac{y}{1 \pm y}$ instead of x in the series, so innumerable other transformations will arise, if for x other functions of y are substituted. Let again this series be propounded

$$S = ax + bx^2 + cx^3 + dx^4 + ex^5 + fx^6 + \text{etc.}$$

and put $x = y(1-y)$ having done which the following series will arise

$$\begin{aligned} S = ay - ayy & \\ + byy - 2by^3 + by^4 & \\ + cy^3 \quad - 3cy^4 + 3cy^5 - cy^6 & \\ + dy^4 \quad - 4dy^5 + 6y^6 & \\ + ey^5 \quad - 5y^6 & \\ + fy^6 \quad \text{etc.} & \end{aligned}$$

Therefore, if the one of these series was summable, at the same time the sum of the other will be known. So, if one puts

$$S = x + x^2 + x^3 + x^4 + x^5 + \text{etc.} = \frac{x}{1-x},$$

it will be

$$S = y - y^3 - y^4 + y^6 + y^7 - y^9 - y^{10} + \text{etc.}$$

The sum of this series will therefore be $= \frac{y - yy}{1 - y + yy}$.

§13 When the one series is truncated anywhere, then the sum of the first can be exhibited explicitly. Let us put that it is $a = 1$ and in the found series all terms after the first vanish that it is $S = y$, and hence because of $x = y - yy$ the sum of the first will be $= \frac{1}{2} - \sqrt{\frac{1}{4} - x}$. But because of $a = 1$ it will be as follows

$$\begin{aligned}
b &= 1 && = \frac{1}{4} \cdot 2^2 \\
c &= 2 && = \frac{1 \cdot 3}{4 \cdot 6} \cdot 2^4 \\
d &= 5 && = \frac{1 \cdot 3 \cdot 5}{4 \cdot 6 \cdot 8} \cdot 2^6 \\
e &= 14 && = \frac{1 \cdot 3 \cdot 5 \cdot 7}{4 \cdot 6 \cdot 8 \cdot 10} \cdot 2^8 \\
f &= 42 && = \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdot 9}{4 \cdot 6 \cdot 8 \cdot 10 \cdot 12} \cdot 2^{10} \\
g &= 132 && = \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdot 9 \cdot 11}{4 \cdot 6 \cdot 8 \cdot 10 \cdot 12 \cdot 14} \cdot 2^{12} \\
& && \text{etc.,}
\end{aligned}$$

whence the first series will go over into this one

$$S = \frac{1}{2} - \sqrt{\frac{1}{4} - x} = x + x^2 + 2x^3 + 5x^4 + 14x^5 + 42x^6 + 132x^7 + \text{etc.},$$

which same series is found, if the surdic quantity $\sqrt{\frac{1}{4} - x}$ is expanded into a series and is subtracted from $\frac{1}{2}$.

§14 Let us put, that the transformation extends further, $x = y(1 + ny)^\nu$ and the propounded series

$$S = ax + bx^2 + cx^3 + dx^4 + ex^5 + \text{etc.}$$

will be transformed into the following

$$
\begin{aligned}
S = ay &+ \frac{\nu}{1}nay^2 + \frac{\nu(\nu-1)}{1\cdot 2}n^2ay^3 + \frac{\nu(\nu-1)(\nu-2)}{1\cdot 2\cdot 3}n^3ay^4 + \frac{\nu(\nu-1)(\nu-2)(\nu-3)}{1\cdot 2\cdot 3\cdot 4}n^4ay^5 \\
&+ b\,y^2 + \frac{2\nu}{1}nb\,y^3 + \frac{2\nu(2\nu-1)}{1\cdot 2}n^2b\,y^4 + \frac{2\nu(2\nu-1)(2\nu-2)}{1\cdot 2\cdot 3}n^3b\,y^5 \\
&+ c\,y^3 + \frac{3\nu}{1}nc\,y^4 + \frac{3\nu(3\nu-1)}{1\cdot 2}n^2c\,y^5 \\
&+ d\,y^4 + \frac{4\nu}{1}nd\,y^5 \\
&+ e\,y^5 \\
&\text{etc.}
\end{aligned}
$$

If therefore the sum of that series was known, one will at the same time also have the sum of this one and vice versa. Since n and ν can be taken ad libitum, hence from one summable series innumerable other summable ones can be found.

§15 One can also do transformations of such a kind that the sum of the found series becomes irrational in the following way.

Let this series be propounded

$$S = ax + bx^3 + cx^5 + dx^7 + ex^9 + fx^{11} + \text{etc.};$$

it will be

$$Sx = ax^2 + bx^4 + cx^6 + dx^8 + ex^{10} + fx^{12} + \text{etc.}$$

Now, put

$$x = \frac{y}{\sqrt{1-nyy}};$$

it will be $xx = \frac{y^2}{1-nyy}$ and the propounded series will be transformed into this one

$$\frac{Sy}{\sqrt{1-nyy}} = ay^2 + nay^4 + n^2ay^6 + n^3a\ y^8 + n^4a\ y^{10} + \text{etc.}$$
$$+ b\ y^4 + 2nby^6 + 3n^2by^8 + 4n^3by^{10} + \text{etc.}$$
$$+ c\ y^6 + 3nc\ y^8 + 6n^2cy^{10} + \text{etc.}$$
$$+ d\ y^8 + 4nd\ y^{10} + \text{etc.}$$
$$+ e\ y^{10} + \text{etc.}$$
$$\text{etc.}$$

Therefore, if the sum S was known from the first series, one will at the same time have the sum of the following series

$$\frac{S}{\sqrt{1-nyy}} = ay + (na+b)y^3 + (n^2a + 2nb + c)y^5 + (n^3a + 3n^2b + 3nc + d)y^7 + \text{etc.}$$

§16 If one takes $n = -1$, the coefficients of this series will be the continued differences of a from the series a, b, c, d etc.; but if the signs in the propounded series alternate, then having put $n = 1$ the coefficients will be these differences. Therefore, let Δa, $\Delta^2 a$, $\Delta^3 a$, $\Delta^4 a$ etc. denote the first, second, third etc. differences of a from the series of the numbers a, b, c, d, e, f etc. And if it was

$$S = ax + bx^3 + cx^5 + dx^7 + ex^9 + \text{etc.},$$

having put $x = \frac{y}{\sqrt{1+yy}}$ it will be

$$\frac{S}{\sqrt{1+yy}} = ay + \Delta a \cdot y^3 + \Delta^2 a \cdot y^5 + \Delta^3 a \cdot y^7 + \text{etc.}$$

But if it was

$$S = ax - bx^3 + cx^5 - dx^7 + ex^9 - \text{etc.}$$

and one puts $x = \frac{y}{\sqrt{1-yy}}$, it will be

$$\frac{S}{\sqrt{1-yy}} = ay - \Delta a \cdot y^3 + \Delta^2 a \cdot y^5 - \Delta^3 a \cdot y^7 + \text{etc.}$$

If therefore the series a, b, c, d, e etc. finally leads to constant differences, then both series can be summed explicitly; but this summation also following from the superior paragraphs.

§17 Let us put that the coefficients a, b, c, d etc. constitute this series

$$1, \quad \frac{1}{3}, \quad \frac{1}{5}, \quad \frac{1}{7}, \quad \frac{1}{9} \quad \text{etc.}$$

and it will be, as we already saw above [§11, II],

$$a = 1, \quad \Delta a = -\frac{2}{3}, \quad \Delta^2 a = \frac{2 \cdot 4}{3 \cdot 5}, \quad \Delta^3 a = -\frac{2 \cdot 4 \cdot 6}{3 \cdot 5 \cdot 7} \quad \text{etc.},$$

whence we will sum the following two series.

I. Let $S = x + \frac{1}{3}x^3 + \frac{1}{5}x^5 + \frac{1}{7}x^7 + \text{etc.}$; it will be $S = \frac{1}{2}\ln\frac{1+x}{1-x}$. Now, having put $x = \frac{y}{\sqrt{1+yy}}$, it will be

$$S = \frac{1}{2}\ln\frac{\sqrt{1+yy}+y}{\sqrt{1+yy}-y} = \ln(\sqrt{1+yy}+y),$$

whence it will be

$$\frac{\ln(\sqrt{1+yy}+y)}{\sqrt{1+yy}} = y - \frac{2}{3}y^3 + \frac{2 \cdot 4}{3 \cdot 5}y^5 - \frac{2 \cdot 4 \cdot 6}{3 \cdot 5 \cdot 7}y^7 + \text{etc.}$$

II. Let $S = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \text{etc.}$; it will be $S = \arctan x$. Now, having put $x = \frac{y}{\sqrt{1-yy}}$ it will be

$$S = \arctan\frac{y}{\sqrt{1-yy}} = \arcsin y = \arccos\sqrt{1-yy}.$$

Therefore, one will obtain this summation

$$\frac{\arcsin y}{\sqrt{1-yy}} = y + \frac{2}{3}y^3 + \frac{2 \cdot 4}{3 \cdot 5}y^5 + \frac{2 \cdot 4 \cdot 6}{3 \cdot 5 \cdot 7}y^7 + \text{etc.}$$

§18 One can also substitute transcendental functions of y for x and so can find other summations more difficult to find; but nevertheless, that the new series do not become too complex, one has to pick functions of such a kind, whose powers can easily be exhibited, as it is the case for the exponential quantities e^y. Therefore, having propounded this series

$$S = ax + bx^2 + cx^3 + dx^4 + ex^5 + fx^6 + \text{etc.}$$

put $x = e^{ny}y$ where e denotes the number whose hyperbolic logarithm is $= 1$; it will be $x^2 = e^{2ny}y^2$, $x^3 = e^{3ny}y^3$ etc. In general, it is indeed, as it is known,

$$e^z = 1 + z + \frac{z^2}{1 \cdot 2} + \frac{z^3}{1 \cdot 2 \cdot 3} + \frac{z^4}{1 \cdot 2 \cdot 3 \cdot 4} + \text{etc.}$$

Therefore, the propounded series will be transformed into this one

$$\begin{array}{rl} S = ay + 1nay^2 + \frac{1}{2}n^2ay^3 & + \frac{1}{6}n^3ay^4 + \frac{1}{24}n^4ay^5 + \text{etc.} \\ + b \quad y^2 + \frac{2}{1}nb\ y^3 & + \frac{4}{2}n^2by^4 + \frac{8}{6}n^3b\ \ y^5 + \text{etc.} \\ + c \qquad y^3 & + \frac{3}{1}nc\ \ y^4 + \frac{9}{2}n^2c\ \ y^5 + \text{etc.} \\ & + d \qquad y^4 + \frac{4}{1}nd \quad y^5 + \text{etc.} \\ & \qquad\quad + e \qquad\ \ y^5 + \text{etc.} \\ \text{etc.} & \end{array}$$

I. Let the geometric series be propounded $S = x + x^2 + x^3 + x^4 + x^5 + \text{etc.}$; it will be $S = \frac{x}{1-x}$. Now, put $n = -1$ that it is $x = e^{-y}y$ and $S = \frac{e^{-y}y}{1-e^{-y}y} = \frac{y}{e^y - y}$; one will find this sum

$$\frac{y}{e^y - y} = y - \frac{1}{2}y^3 - \frac{1}{6}y^4 + \frac{5}{24}y^5 + \frac{19}{120}y^6 - \text{etc.},$$

the law of which series is not recognized.

II. In the other series, let all terms except the first be $= 0$; it will be

$$b = -na, \quad c = \frac{3}{2}n^2a, \quad d = -\frac{8}{3}n^3a, \quad e = \frac{125}{24}n^4a, \quad f = -\frac{54}{5}n^5a \quad \text{etc.}$$

Because therefore the sum is $S = ay$ and $x = ye^{ny}$, it will be

$$y = x - nx^2 + \frac{3}{2}n^2x^3 - \frac{8}{3}n^3x^4 + \frac{125}{24}n^4x^5 - \frac{54}{5}n^5x^6 + \text{etc.}$$

Since in these series the law of progression is not manifest, the summations deduced from this substitution have hardly any use. But especially the transformations derived from the substitution $x = \frac{y}{1 \pm y}$, which not only yield extraordinary summations but also appropriate ways to approximate the sums of series, deserve to be mentioned. Therefore, having mentioned these things in advance, which were done without differential calculus, we want to proceed to show the use of this calculus in the doctrine of series itself.