Solution of a geometric problem.

Leonhard Euler

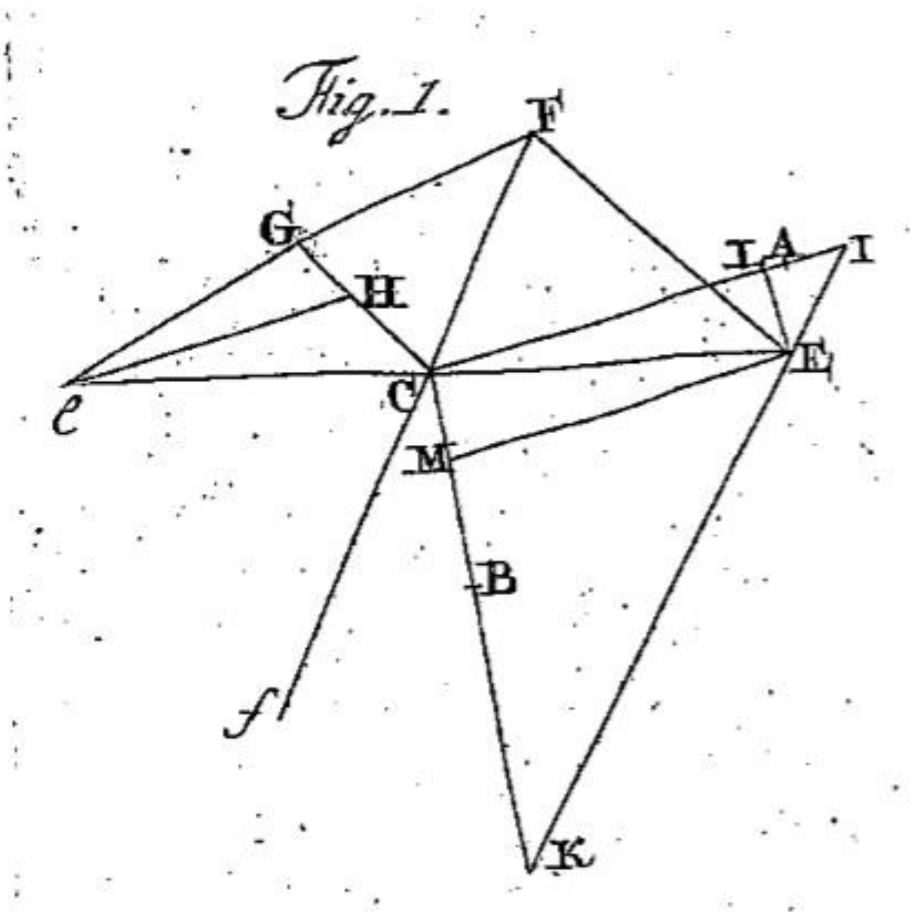
Translated by Pedro C. C. R. Pereira

Problem.

Given any conjugate diameters *Ee* and *Ff* of an ellipse in size and position, it's required to find the conjugate axes also in size and position.

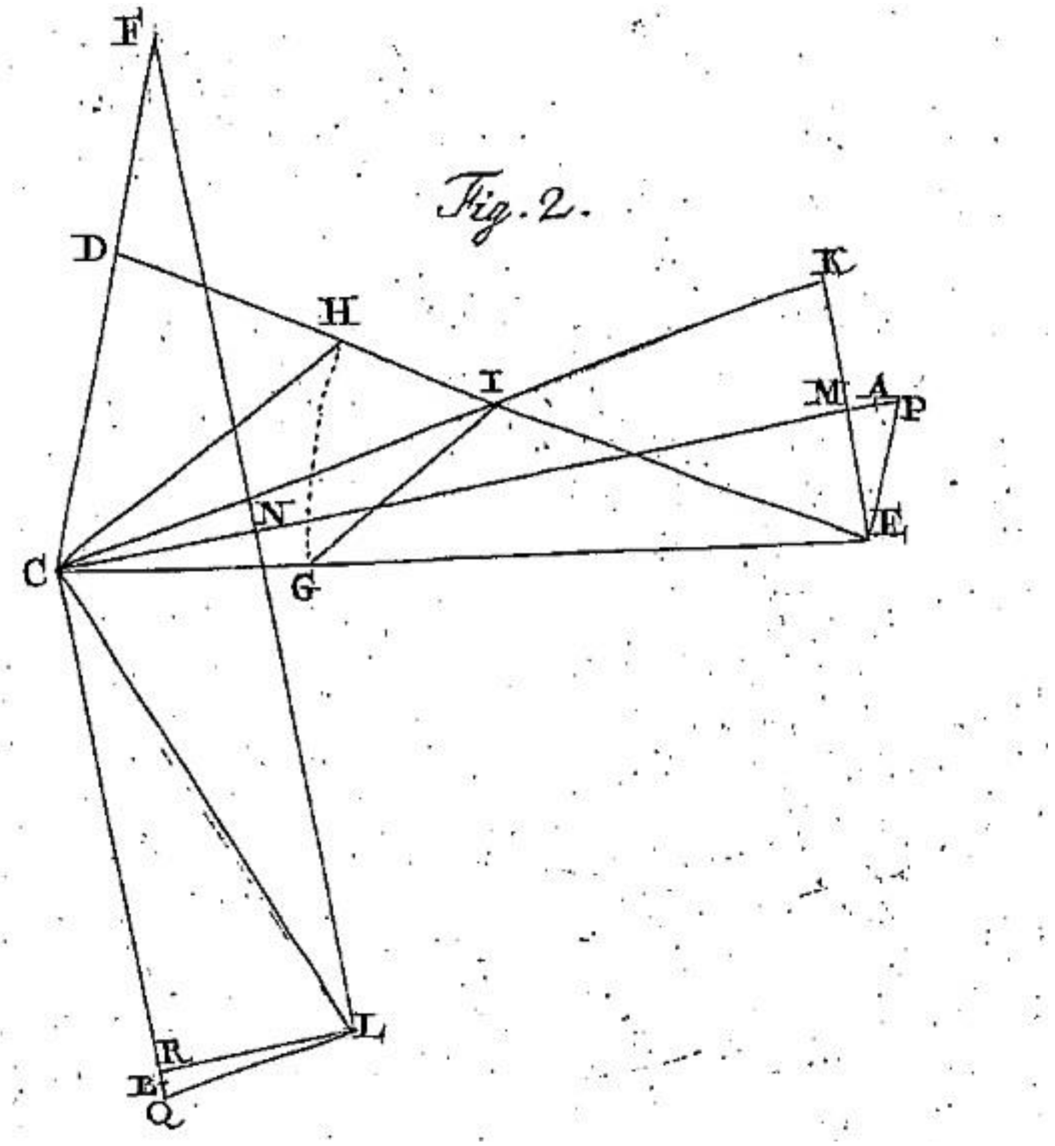
Construction.

Join the points *E* and *F*, and from the center *C* of the ellipse draw the line *CG* so that the angle *FCG* is equal to the angle *ECF*; then, draw *FG* so that the angle *CFG* is equal to the angle *CEF*; and so the triangle *CFG* is similar to the triangle *CEF.* Having joined *eG*, draw *eH,* bisecting the angle *CeG;* and, parallel to it, draw from the center *C* the line *CI,* and, perpendicular to *CI*, the line *CK*, so that the points *I,E* and *K* fall in a line parallel to the diameter *Ff.* From *E,* draw *EL* perpendicular to *CI*, and *EM* perpendicular to *CK*. Let *CA* be the mean proportional of *CL* and *CI*, and also *CB* the mean proportional of *CM* and *CK*. Then *CA* and *CB* are the principal semi axes, determined both in size and position.



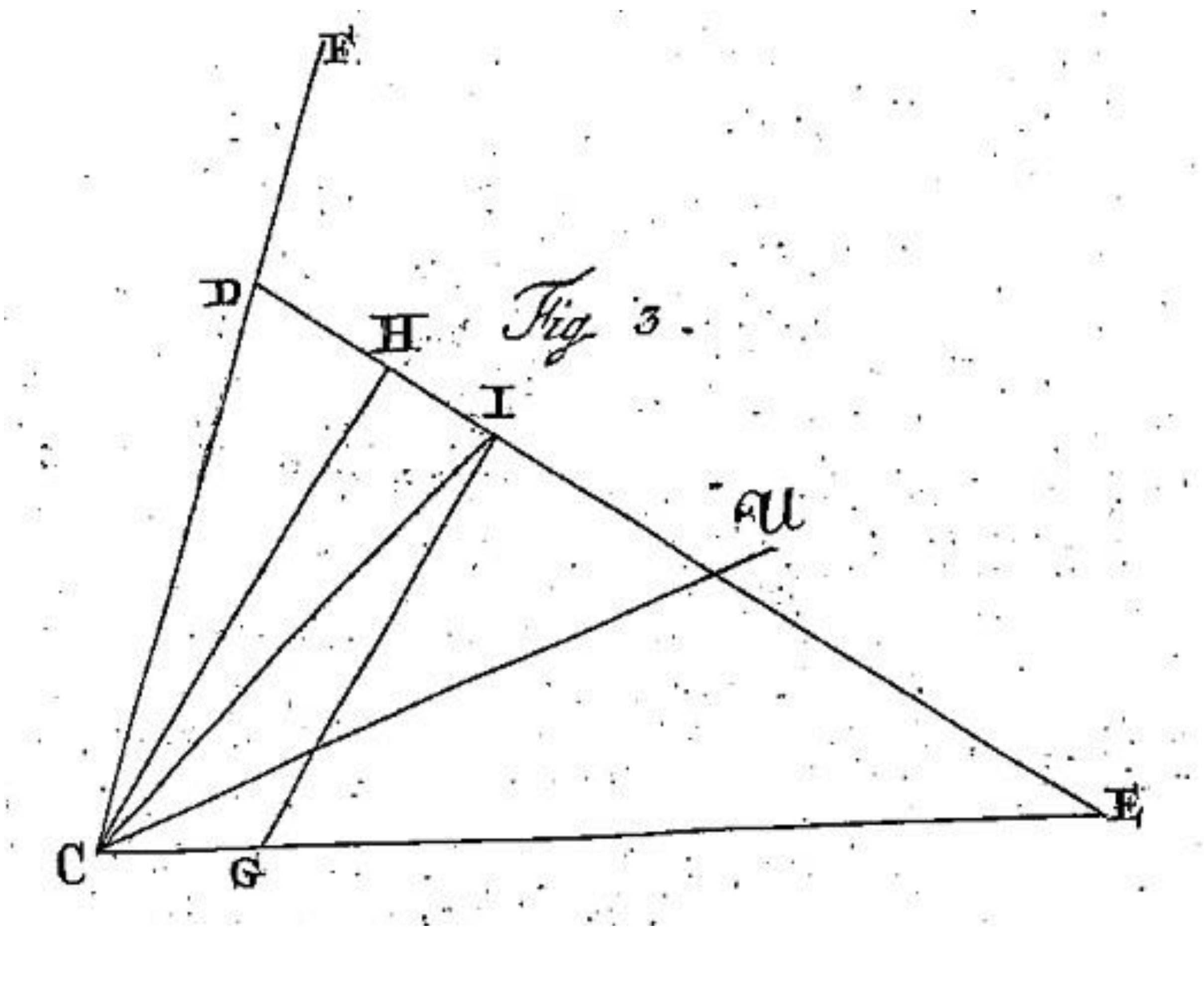
First alternative construction.

Let *CE* and *CF* be the conjugate semi diameters. Draw $ED = EC$, so that *CED* forms an isosceles triangle; and let $EG = EH = CF$. Join the points *C* and *H*, and, parallel to the line *CH*, draw the line *GI*, cutting *ED* in *I.* From *C,* draw the line *CIK,* equal to *CE* and passing through *I.* Join the points *E* and *K*, and let the line *EK* be bisected in *M*. Then the line *CM* will lie in the position of one of the axes, and its perpendicular *CR* will lie in the position of the other conjugate axis. From *F*, draw the line *FN*, perpendicular to *CM* and passing through it in *N*; extend the line *FN* to a new point *L* so that $NL = FN$. Then *EK* and *FL* will be ordinates to the axis *CM;* and, because of the above, the tangents at *E*, *F* and *L* are given. Indeed, the tangent *EP* at *E* is parallel to *CF*, and the tangent *LQ* at *L* parallel to *CK;* hence, the size of both semi axes are easily determined. Namely, the semi axis *CA* will be the mean proportional of *CM* and *CP*, and the semi axis *CB* will be the mean proportional of *CR* and *CQ*, *LR* being the line perpendicular to *CQ* and passing through *L*.

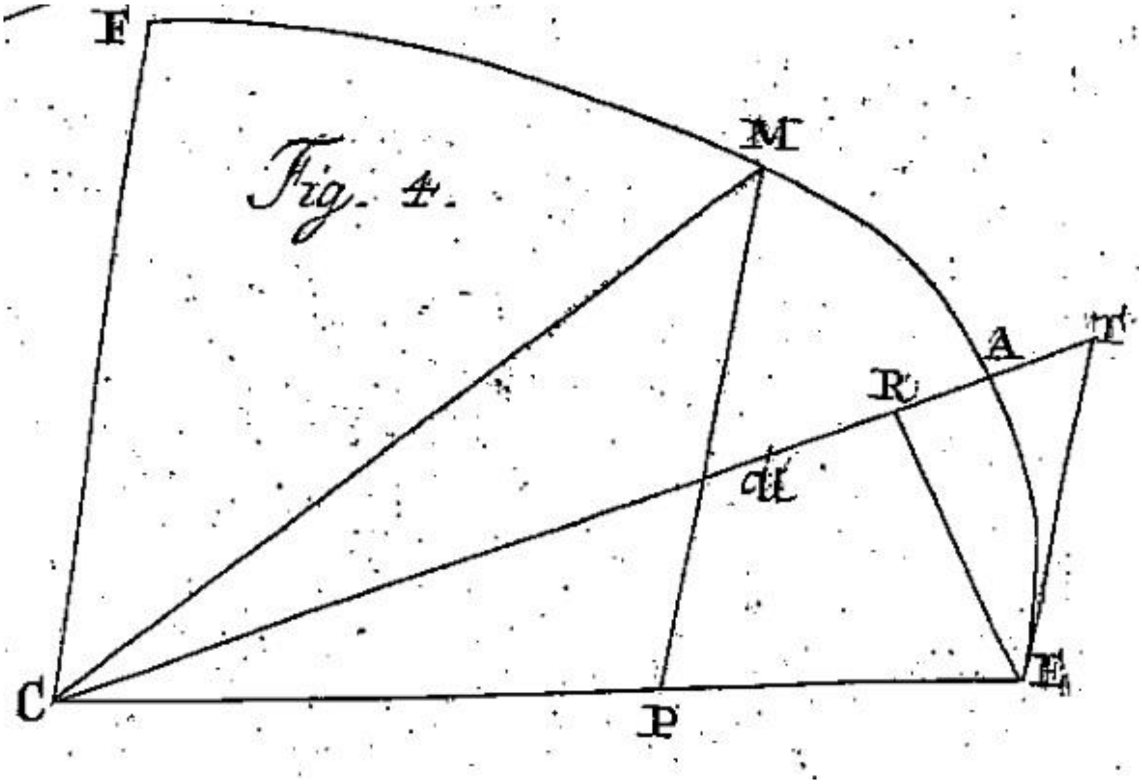


Second alternative construction.

Because the conjugate diameters cross each other in the centre at oblique angles, we can choose, from acute and obtuse, two conjugate semi diameters *CE* and *CF* that constitute an acute angle *ECF*. As before, draw $ED = EC$, and make $EG = EH = CF$, and *GI* parallel to *CH*. Join *CI* and draw the line *CU*, bisecting the angle *ECI,* and whose size is made to be the mean proportional of *CE* and *CI*. Then, the point *U* is one of the foci of the ellipse. Proceeding similarly, find the other focus. Then, given the foci and one point of the ellipse, it's easy to find its principal axes.



Demonstration of those constructions.



Let *ECF* be a quadrant of the ellipse, in which the conjugate semi diameters $CE = e$ and $CF = f$, forming an angle $ECF = \theta$, are given. Further, let *CA* be one of the semi axes whose position and size are required to be found. First, in order to find the position of this axis, let us make $angle\ ECA = \phi$, and let the angle *ACM* be equal to *ECA*, so that $angle\ ECM = 2\phi$. Because the points *E* and *M* are equidistant from the axis *CA*, they are also equidistant from the centre C, by which $CM = CE = e$. Draw *MP* parallel to *CF*, and it will be that $angle\ EPM = angle\ ECF = \theta$.

Now, besides knowing that its side *CM* is equal to *e,* we also know all the angles of the triangle *CPM*, namely $EPM = \theta$, $PCM = 2\phi$ and $CMP = \theta - 2\phi$. From trigonometry, we have:

$$\sin EPM : CM = \sin PCM : PM = \sin CMP : CP$$

Or, equivalently:

$$\sin \theta : e = \sin 2\phi : PM = \sin(\theta - 2\phi) : CP$$

Hence, we have:

$$PM = \frac{e \sin 2\phi}{\sin \theta} \text{ and } CP = \frac{e \sin (\theta\text{-}2\phi)}{\sin \theta}$$

From the nature of the ellipse:

$$PM^2 = \frac{CF^2}{CE^2}(CE^2 - CP^2)$$

Or:

$$CE^2 PM^2 + CF^2 CP^2 = CE^2 CF^2$$

In which we substitute the values of *PM* and *CP,* thus obtaining:

$$\frac{e^4 \sin^2 2\phi}{\sin^2 \theta} + \frac{eeff \sin^2(\theta\text{-}2\phi)}{\sin^2 \theta} = eeff$$

Which, divided by *ee* and multiplied by $\sin^2 \theta$, becomes:

$$ee \sin^2 2\phi + ff \sin^2 (\theta\text{-}2\phi) = ff \sin^2 \theta$$

But:

$$\sin(\theta\text{-}2\phi) = \sin\theta \cos 2\phi \text{-} \cos\theta \sin 2\phi$$

And, therefore:

$$\sin^2(\theta\text{-}2\phi) = \sin^2\theta \cos^2 2\,\phi \text{-} 2 \sin\theta \cos\theta \sin 2\phi \cos 2\phi + \cos^2\theta \sin^2 2\phi$$

From which, because $e \sin^2 2\,\phi = ff(\sin^2\theta\ - \sin^2(\theta - 2\,\phi))$, we have, on account of the fact that $1 - \cos^2 2\phi = \sin^2 2\phi$:

$$\sin^2\theta\text{-} \sin^2(\theta\text{-}2\ \phi) = \sin^2\theta \sin^2 2\phi + 2 \sin\theta \cos\theta \sin 2\phi \cos 2\phi \text{ - } \cos^2\theta \sin^2 2\phi$$

But, we know that $2 \sin\theta \cos\theta = \sin 2\theta$ and $\cos^2\theta - \sin^2\theta = \cos 2\theta$, so:

$$\sin^2\theta\text{-} \sin^2(\theta\text{-}2\ \phi) = \sin 2\theta \sin 2\phi \cos 2\phi \text{ - } \cos 2\theta \sin^2 2\phi$$

And so, we have:

$$ee \sin^2 2\ \phi = ff \sin 2\theta \sin 2\phi \cos 2\phi \text{-} ff \cos 2\theta \sin^2 2\phi$$

Which, divided by $\sin 2\phi$, will be:

$$ee \sin 2\phi = ff \sin 2\theta \cos 2\phi \text{-} ff \cos 2\theta \sin 2\phi$$

From which, at last, we have:

$$\frac{\sin 2\phi}{\cos 2\phi} = \tan 2\phi = \frac{ff \sin 2\theta}{ee + ff \cos 2\theta}$$

Therefore, once the angle $ECM = 2\phi$, whose tangent is $\frac{ff\ sin\ 2\theta}{ee+ff\ cos\ 2\theta}$, is found, if it's bisected by the line *CA*, this line will be in the position of one of the axes. The size of this axis is to be found as follows.

Draw from the point *E* the line *ER*, perpendicular to *CA*. Furthermore, draw the tangent *ET* at *E*, which is parallel to *CF,* being the point *T* the intersection of this tangent and the line *CA* extended. From the nature of the tangent, we know that *CA* is the mean proportional of *CR* and *CT.* Then, because $CE = e$ and $angle\ ECA = \phi$, we have $ER = e \sin\phi$ and $CR = e\cos\phi$. Moreover, since $CTE = \theta - \phi$, we shall have:

$$ET = \frac{\mathrm{e}\sin\phi}{\sin(\theta\text{-}\phi)}\text{ ; and } CT = \frac{\mathrm{e}\sin\theta}{\sin(\theta\text{-}\phi)}$$

And so, we have:

$$CA = e\sqrt{\frac{\sin\theta\cos\phi}{\sin(\theta\text{-}\phi)}}$$

The other semi axis is normal to *CA.* Let us call this semi axis *CB.* Since, from the nature of the ellipse, we have $RE^2 = \frac{CB^2}{CA^2}(CA^2 - CR^2)$; then, it'll be:

$$CB = \frac{CA.RE}{\sqrt{(CA^2 - CR^2)}}$$

Furthermore, we have:

$$CA \cdot RE = \mathrm{ee}\sin\phi\sqrt{\frac{\sin\theta\cos\phi}{\sin(\theta\text{-}\phi)}}$$

And:

$$\sqrt{(CA^2 - CR^2)} = \sqrt{\left(\frac{\mathrm{ee}\sin\theta\cos\phi}{\sin(\theta\text{-}\phi)}\text{-}\mathrm{ee}\cos^2\phi\right)}$$

Or, equivalently:

$$\sqrt{(CA^2 - CR^2)} = \mathrm{e}\sqrt{\frac{\cos\phi}{\sin(\theta\text{-}\phi)}(\sin\theta\text{-}\sin(\theta\text{-}\phi)\cos\phi)}$$

Moreover, because $\theta = \theta - \phi + \phi$, we have:

$$\sin\theta = \sin(\theta - \phi)\cos\phi + \cos(\theta - \phi)\sin\phi$$

From which, we shall have:

$$\sqrt{(CA^2 - CR^2)} = \mathrm{e}\sqrt{\frac{\cos(\theta\text{-}\phi)\sin\phi\cos\phi}{\sin(\theta\text{-}\phi)}}$$

Therefore, the semi axis CB is found:

$$CB = \mathrm{e} \sin \phi \sqrt{\frac{\sin \theta}{\cos(\theta\text{-}\phi) \sin \phi}} = \mathrm{e} \sqrt{\frac{\sin \theta \sin \phi}{\cos (\theta\text{-}\phi)}}$$

The same reasoning will find the same place, if we accommodate the figure so that the other semi diameter given $CF = f$ is used. In that case, we must permutate the quantities *e* and *f,* as well as the angles ϕ and $\theta - \phi$. Thus, we have:

$$CA = \mathrm{f} \sqrt{\frac{\sin \theta \cos(\theta\text{-}\phi)}{\sin \phi}} \text{ ; and } CB = \mathrm{f} \sqrt{\frac{\sin \theta \sin (\theta\text{-}\phi)}{\cos \phi}}$$

Let us put $CA = a$ and $CB = b$. We'll have:

$$\mathrm{a}=\mathrm{e} \sqrt{\frac{\sin \theta \cos \phi}{\sin(\theta\text{-}\phi)}} = \mathrm{f} \sqrt{\frac{\sin \theta \cos (\theta\text{-}\phi)}{\sin \phi}}$$

$$\mathrm{b}=\mathrm{e} \sqrt{\frac{\sin \theta \sin \phi}{\cos (\theta\text{-}\phi)}} = \mathrm{f} \sqrt{\frac{\sin \theta \sin (\theta\text{-}\phi)}{\cos \phi}}$$

Hence, we have $ab = ef \sin \theta$, by which it's shown the equality of all parallelograms described around the conjugate diameters. Then, once again we can determine the angle ϕ. In fact:

$$\mathrm{ee} \frac{\sin \theta \cos \phi}{\sin (\theta\text{-}\phi)} = \mathrm{ff} \frac{\sin \theta \cos (\theta\text{-}\phi)}{\sin \phi}$$

Or:

$$\mathrm{ee} \sin 2\phi = \mathrm{ff} \sin 2(\theta\text{-}\phi) = \mathrm{ff} \ (\sin 2\theta \cos 2\phi \text{-} \cos 2\theta \sin 2\phi)$$

From which, we have, as before:

$$\frac{\sin 2\phi}{\cos 2\phi} = \tan 2\phi = \frac{\mathrm{ff} \sin 2\theta}{\mathrm{ee}+\mathrm{ff} \cos 2\theta}$$

From the four formulas above, we also have:

$$\frac{\sin (\theta\text{-}\phi)}{\cos \phi} = \frac{\mathrm{ee} \sin \theta}{\mathrm{aa}} = \frac{\mathrm{bb}}{\mathrm{ff} \sin \theta} = \mathrm{A}$$

$$\frac{\cos(\theta\text{-}\phi)}{\sin \phi} = \frac{\mathrm{aa}}{\mathrm{ff} \sin \theta} = \frac{\mathrm{ee} \sin \theta}{\mathrm{bb}} = \mathrm{B}$$

And so:

$$\sin\theta - \frac{\cos\theta\sin\phi}{\cos\phi} = \mathrm{A}\ ;\ \text{and}\ \frac{\cos\theta\cos\phi}{\sin\phi} + \sin\theta = \mathrm{B}$$

Thus:

$$\frac{\cos\theta\sin\phi}{\cos\phi} = \sin\theta - \mathrm{A}\ ;\ \text{and}\ \frac{\cos\theta\cos\phi}{\sin\phi} = \mathrm{B} - \sin\theta$$

Multiplying these equations, we have:

$$\cos^2\theta = (\mathrm{A}+\mathrm{B})\sin\theta - \mathrm{AB} - \sin^2\theta$$

Or, equivalently:

$$1+\mathrm{AB} = (\mathrm{A}+\mathrm{B})\sin\theta$$

And, because of the values assigned to A and B above, we have:

$$\mathrm{AB} = \frac{\mathrm{ee}}{\mathrm{ff}}\ ;\ \text{and}\ (\mathrm{A}+\mathrm{B})\sin\theta = \frac{\mathrm{bb}}{\mathrm{ff}} + \frac{\mathrm{aa}}{\mathrm{ff}}$$

From which, we have $1 + \frac{ee}{ff} = \frac{aa+bb}{ff}$; and, therefore, we found the familiar propriety of an ellipse: $aa + bb = ee + ff$.

Furthermore, if we call one of the foci of the ellipse U, we'll have, as known:

$$CU = \sqrt{(\mathrm{aa}-\mathrm{bb})} = \sqrt[4]{(\mathrm{aa}-\mathrm{bb})^2}$$

But:

$$(\mathrm{aa}-\mathrm{bb})^2 = (\mathrm{aa}+\mathrm{bb})^2 - 4\mathrm{aabb}$$

Therefore, since $aa + bb = ee + ff$ and $ab = ef\sin\theta$, we have:

$$CU = \sqrt[4]{(\mathrm{e}^4+\mathrm{f}^4+2\mathrm{eeff}(1-2\sin^2\theta))}$$

But $1 - 2\sin^2\theta = \cos 2\theta$, and, hence, we have:

$$CU = \sqrt[4]{\mathrm{e}^4+\mathrm{f}^4+2\mathrm{eeff}\cos 2\theta}$$

Having found those, the reason for the given constructions will be made clear:

For the first construction (Fig. 1): From $angle\ ECF = angle\ FCG = \theta,$ we have $ECG = 2\theta$; and, because $EC{:}\,CF = CF{:}\,CG$, we shall have $CG = \frac{ff}{e}$. Furthermore, having drawn *eG*, we also have:

$$\tan CeG = \frac{CG \sin eCG}{Ce - CG \cos eCG}$$

But $Ce = \mathrm{e}$; $\sin eCG = \sin 2\theta$ and $\cos eCG = -\cos 2\theta$. Therefore, we have:

$$\tan CeG = \frac{\mathrm{ff} \sin 2\theta}{\mathrm{ee}+\mathrm{ff} \cos 2\theta}$$

Which is the expression found before for $\tan 2\phi$. Thus, we have $angle\ CeG = 2\phi$, and the angle *CeG* bisected by *eH* will be *CeH*, equal to $angle\ ECI = \phi$. Thereby, the line *CI* lies on the position of one of the axis, and its normal *CK* lies on the position of the other one. Then, line *IEK*, drawn through *E* and parallel to *CF*, is tangent to the ellipse at *E*. By which, if we draw the lines *EL* and *EM*, each perpendicular to one of the axes, then the semi axis *CA* will be the mean proportional of *CL* and *CI*, just as the semi axis *CB* will be the mean proportional of *CM* and *CK,* like the construction itself clearly states.

For the second construction (Fig. 2): From the fact that the triangle *CED* is isosceles and because $angle\ ECD = angle\ EDC = \theta$, we have $angle\ CED = 180° - 2\theta$. Then, from $EC = e$, $EG = EH = CF = f$, and since *GI* is parallel to *CH*, we have $EI = \frac{ff}{e}$. Furthermore, $\tan ECI = \frac{EI \sin CED}{CE - EI \cos CED}$, thus, because $\sin CED = \sin 2\theta$ and $\cos CED = -\cos 2\theta$, we have $\tan ECI = \frac{ff \sin 2\theta}{ee+ff \cos 2\theta}$, and, therefore, $angle\ ECI = 2\phi$. By which, knowing that $CK = CE = e$ and that *EK* is bisected in *M*, line *CM* will bisect the angle $ECK = 2\phi$, so that $angle\ ECM = \phi$; and so, *CMAP* will lie in the position of one of the axes. The length *CA* of its half is defined as before, and the determination of the other semi axis *CB* is manifest from the same construction.

For the third construction (Fig. 3): From the explanation of the precedent construction, it's understood that the line *CU,* bisecting the angle *ECI*, gives the position of the transverse axis. Therefore, because $CE = e$, $EI = \frac{ff}{e}$ and $CEI = 180° - 2\theta$; we have

$$CI = \sqrt{CE^2 + EI^2 - 2\, CE \cdot EI \cdot \cos(CEI)} = \sqrt{ee + \frac{f^4}{ee} + 2ff \cdot \cos(2\theta)}$$
$$= \frac{1}{e}\sqrt{e^4 + f^4 + 2eeff \cdot cos(2\theta)}$$

Hence, because *CU* is the mean proportional of *CE* and *CI*, we have:

$$CU = \sqrt{e \cdot \frac{1}{e}\sqrt{e^4 + f^4 + 2eeff \cdot cos(2\theta)}}$$

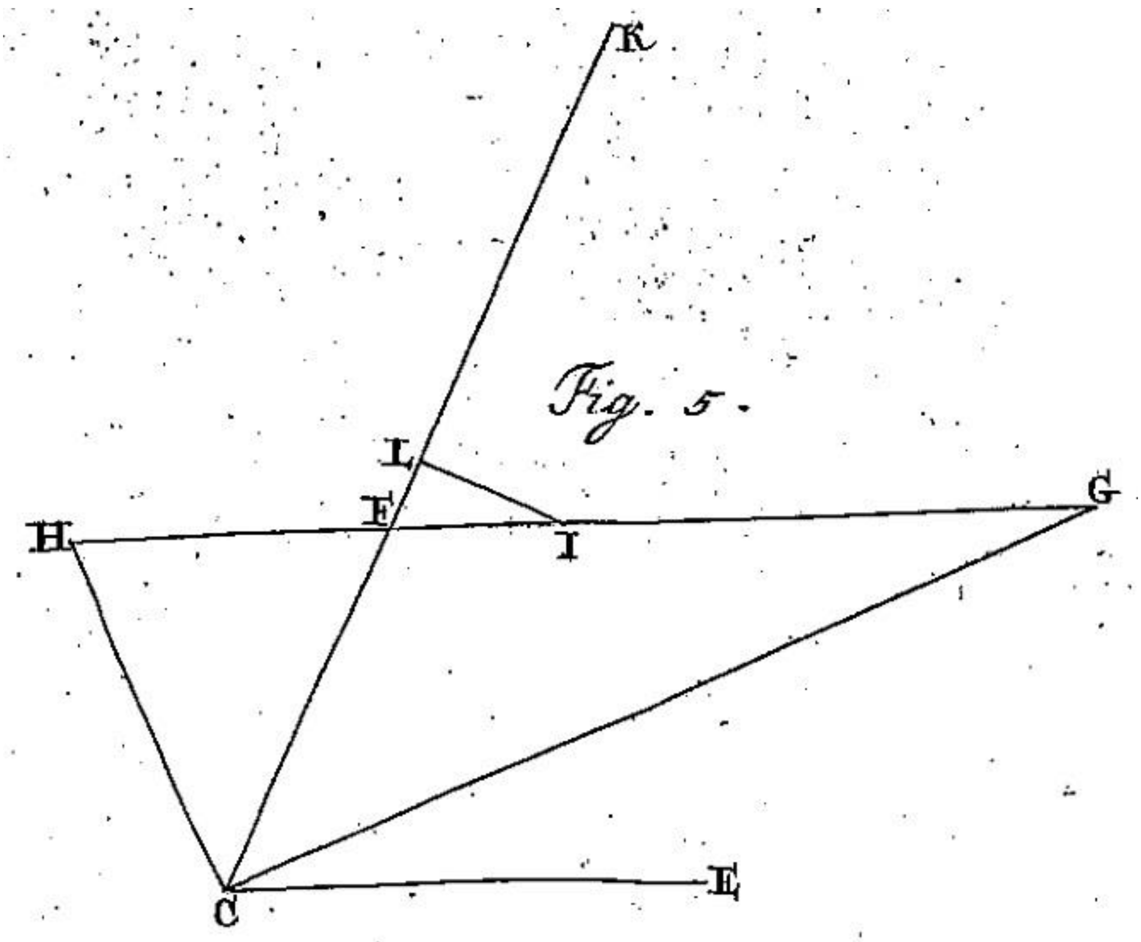
Or, equivalently:

$$CU = \sqrt[4]{e^4 + f^4 + 2eeff \cdot cos(2\theta)}$$

By this expression, it's manifest that *U* is one of the foci of the ellipse, from which the other focus becomes known.

Here, it's necessary only to assure that the line *CU* lies on the position of the transverse axis, and not the conjugate axis. However, it's easy to avoid making this mistake if we observe that the transverse axis is always within the acute angle that the conjugate diameters constitute.

It's true that these constructions are easily done, even though I must admit that the construction presented without demonstration by Pappus Alexandrinus is by far the best one. However, since he didn't add the demonstration, which gladly his commentator, Commandinus, is known to try to supply; here, I will annex not only Pappus' construction, but also its explanation.



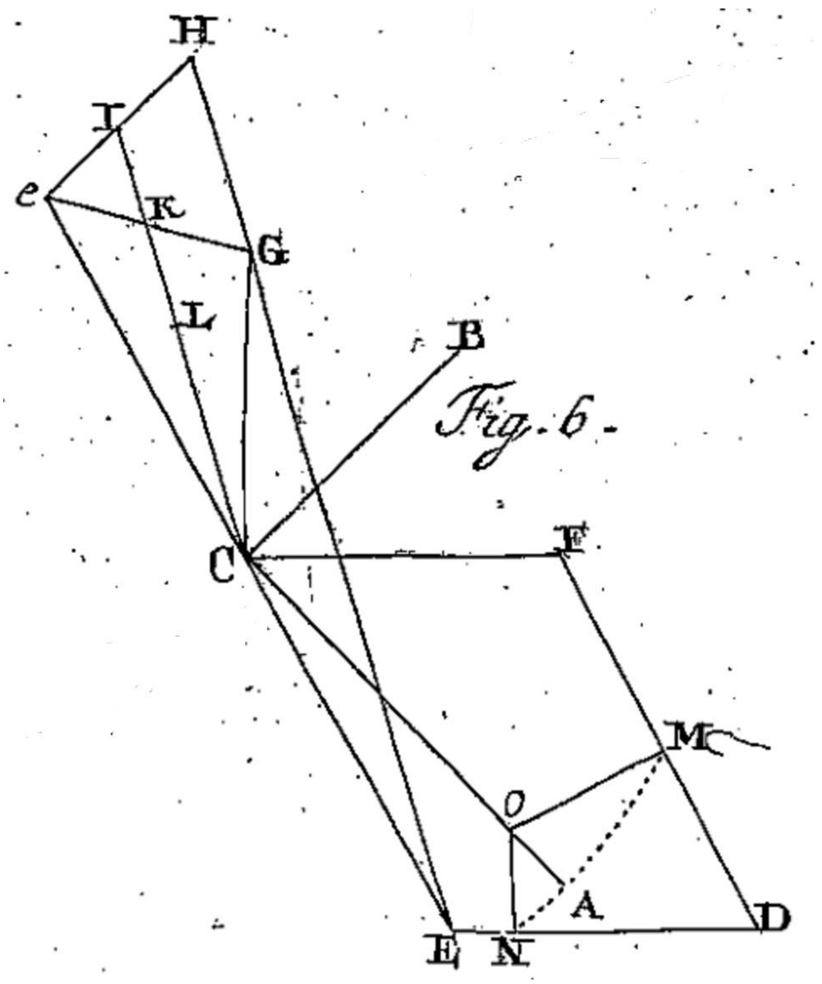
Thus, let *CE* and *CF* be the semi diameters of the given ellipse, and through *F* draw a line of indefinite length parallel to *CE.* This line is tangent to the ellipse at *F.* Let the principal axes lie on the lines *CG* and *CH*, then it's clear that, if we knew the points *G* and *H*, the position of the axes would be thence determined. We shall then direct our thought to the determination of those points.

Let us conceive a circle passing through the points *G, H* and the center of the ellipse, *C*. Since the angle *GCH* is a right angle, we know that the center of this circle shall lie on the line *GH.* Moreover, because of the nature of the tangents to the ellipse, and since the tangent at *F* is cut by the principal axes at the points *G* and *H,* we know that the rectangle *FG, FH* is equal to the square over *CE.* Therefore, we have two conditions, with which we can determine the aforementioned circle: Its center shall lie on the line *GH;* and the rectangle FG, FH shall be equal to the square over *CE.*

So, let the line *CF* be extended so that it meets the circle, still unknown, at the point *K*. Then, since $CF \cdot FK = FG \cdot FH$, we have $CF \cdot FK = CE^2$, and so *FK* is the "third proportional" to *CF* and *CE.* As both *CF* and *CE* are given, the point K can be found. This allows us to determine the circle, seeing that we know it should pass through the points *C* and *K*, and that its centre shall lie on the line *GH*. In order to that, let the line *CK* be bisected in *L*, and let us draw the line *LI*, perpendicular to *CK*, meeting *GH* at *I.* Then, *I* is the centre of the circle sought. Once we have determined the circle, we know the points G and H are its intersections with the tangent line at F. This is Pappus' construction.

Next, I'll present a construction simpler than all the ones showed hitherto, which will allow us to find not only the position of the axes, but also their quantity. Moreover, all the operations needed for this construction will be already included, so that not even the execution of a mean proportional will be required.

New Construction.



Let *CE* and *CF* be conjugate semi diameters, *ECF* being the angle between them, and let the parallelogram *CEDF* be completed. Extend the line *CE* to *e* so that $Ce = CE$, and draw $CG = CF$ perpendicular to *CF*. Join *EG* and *eG,* and extend *EG* to *H* so that $GH = Ge$. Also, join the points *e* and *H*, and bisect the line *eH* at *I*. Mark the point *L* at the line *CI* so that $KL = KI$. Then, from the centre *C*, use a compass to draw the arc of circle *MN*, of radius *CI,* being *M* and *N* the points of intersection of the arc and the lines *FD* and *ED.* Perpendicular to these lines, draw *MO* and *NO*, meeting at the point *O*. From the centre *C*, draw the line *COA*, passing through the point O and meeting the aforementioned arc of circle at *A*. Then, this line is the transverse semi axis of the ellipse, and the point *O* is one of its foci. Finally, draw the line *CB* perpendicular to *CA* so that $CB = CL$, and *CB* will be the conjugate semi axis.

Demonstration.

Let us put the conjugate semi diameters $CE = e$ and $CF = f$, and the angle $ECF = \theta$; By construction, we have $Ce = e$, $CG = f$, and $angle\ GCe = 90° - \theta$, so that $\cos GCe = \sin\theta$. Further, let us call the transverse semi axis $CA = a$, and the conjugate semi axis $CB = b$. As showed before, we have $aa + bb = ee + ff$ and $ab = ef \cdot \sin\theta$. Hence, we also have $aa + 2ab + bb = ee + ff + 2ef \cdot \sin\theta$ and $aa - 2ab + bb = ee + ff - 2ef \cdot \sin\theta$. Therefore, $a + b = \sqrt{ee + ff + 2ef \cdot \sin\theta}$ and $a - b = \sqrt{ee + ff - 2ef \cdot \sin\theta}$. Besides, because $CE = e$, $CG = f$ and $\cos GCe = \sin\theta$, we have, considering the triangle ECG, $EG = \sqrt{ee + ff + 2ef \cdot \sin\theta}$. Also, considering the triangle *eCG*, and because $Ce = e$, $CG = f$ and $\cos GCe = \sin\theta$, we have $Ge = \sqrt{ee + ff - 2ef \cdot \sin\theta}$. Thus:

$a + b = EG$; and $a - b = eG$.

Consequently, $2a = EG + eG$ and $2b = EG - eG$. Furthermore, since $GH = eG$, $2a = EH$; and bisecting *eH* at *I*, since *C* is the midpoint of the line *Ee*, we know that the line *CI* will be parallel to *EH*, and equal to half of this line. Thus, $a = CI$. Also, since $IK = \frac{1}{2}GH = \frac{1}{2}Ge$, and $KL = IK$, we have $CL = CI - 2IK = CI - GH = a - Ge$, and so $CL = b$. Therefore, we have established that *CI* is equal to the transverse semi axis and *CL* equal to the conjugate semi axis. Moreover, because of the nature of the ellipse we know that, since *ED* and *FD* are tangent to the ellipse, if from one of the foci *O* we draw the lines *ON* and *OM*, perpendicular to *ED* and *FD*, the distance from *M* or *N* to *C* is equal to the transverse semi axis. Thus, if the points *M* and *N* are taken so that their distance from the centre *C* is equal to the transverse semi axis *CI*, just as it's done in the construction, and from these points we draw the lines *MO* and *NO* perpendicular to the tangent lines, these perpendiculars will meet in the focus *O*. Thereby, one of the foci is found, and it lies on the transverse axis. If, then, we draw the line *CA*, passing through *O*, and extending to the arc of circle *MN*, not only this line is equal to the transverse semi axis, but also lies on its position. Hence, as *CA* is proved to be the transverse semi axis, if we draw the line *CB* perpendicular to it so that $CB = CL$, then *CB* will be the conjugate semi axis.