Leonhard Euler
E190

Consideration of some series which are distinguished by special properties[a]

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Summary

In the preceding memoir[1] the author already dealt with a certain peculiar series in which the first term[b] is 0, the tenth 1, the hundredth 2, the thousandth 3, the ten-thousandth 4 and so on; thus, in general, to an index which is an arbitrary power of 10, there corresponds a term which is equal to the exponent of this power—if in fact it is an integer. Now since the logarithms are constructed on this principle, it might at first sight appear that each term of the series be the logarithm of the corresponding index. In the meantime, however, the author has shown that the ninth term of this series is[2] 0.89705 0585 and thus significantly smaller than the logarithm of the number 9. This is a remarkable example of a series which, while having infinitely many terms in common with the series of the logarithms, is not however

[a]This English translation is based on the memoir as printed in the *Opera Omnia* and translated into German by Martin Mattmüller, Euler Archiv, Basel. Footnotes in arabic numerals are those by the editor, Georg Faber, and footnotes in alphanumeric characters are those by the translator (W. G.).

[1]Memoir E189 of this volume, in particular, p. 467. G. F.

[b]By “first term” is meant here the value of the series for the argument (or “index”) 1, by “tenth term” the value of the series for the argument 10, etc.

[2]For this erroneous value, the number 0.89777 8587 should be substituted; see footnote on p. 519 [of this volume](p. 5 of the present translation W. G.). G. F.

identical with it. Series of this kind he has already investigated in detail in the above-mentioned memoir; here he now develops the said peculiar case in greater detail. It is contained in the following more general form in which to an arbitrary indeterminate index x there belongs the term[c]

$$= \frac{1-x}{1-a} + \frac{(1-x)(a-x)}{a-a^3} + \frac{(1-x)(a-x)(a^2-x)}{a^3-a^6} + \frac{(1-x)(a-x)(a^2-x)(a^3-x)}{a^6-a^{10}} + \text{etc.};$$

even though this expression is to be continued ad infinitum, it not only terminates when x is taken to be any rational power of a, but in fact becomes equal to the exponent of a. From this, the above-mentioned case obtains if one takes for a the number 10. By examining the more general form above, the author noticed: if the term corresponding to the index x is set $= s$, and the one corresponding to the index ax is set $= t$, one gets

$$1 + s - t = (1-x)\left(1-\frac{x}{a}\right)\left(1-\frac{x}{a^2}\right)\left(1-\frac{x}{a^3}\right)\left(1-\frac{x}{a^4}\right) \text{ etc.},$$

so that, when for x an arbitrary power of a is taken, evidently $t = 1 + s$, since one of the factors clearly vanishes. Now by transforming this form in different ways, he derives several other consequences not to be looked down on, through which, it may be said, the theory of series is considerably widened; for the quadrature of the circle, for example—denoting the ratio of the diameter to the circumference by $1 : \pi$—he finds the following rather elegant series

$$-\frac{9}{4}\pi + 8 = \frac{2\cdot 4}{5\cdot 5} + \frac{2\cdot 4\cdot 4\cdot 6}{5\cdot 5\cdot 7\cdot 7} + \frac{2\cdot 4\cdot 4\cdot 6\cdot 6\cdot 8}{5\cdot 5\cdot 7\cdot 7\cdot 9\cdot 9} + \text{ etc.}$$[d]

Subsequently, from the same general series he derives several other forms; for their development many remarkable computational artifices are to be

[c]Euler evidently encountered this series when he attempted to interpolate the logarithm to base a at the successive powers $1, a, a^2, a^3, \ldots$ of a. The series is in fact Newton's interpolation series for this interpolation problem. The case $a = 10$ was mentioned by Euler already about 20 years earlier in a letter to Daniel Bernoulli; see the translator's paper "On Euler's attempt to compute logarithms by interpolation: A commentary to his letter of February 16, 1734 to Daniel Bernoulli", *Journal of Computational and Applied Mathematics*, to appear.

[d]In the original (and also in the *Opera Omnia*), the minus sign on the left-hand side is missing; cf. footnote j.

observed, which, by having noted them on this occasion, may well be worth the effort also in other investigations.

1. The consideration of series which present themselves fortuitously, as it were, often furnishes artifices not to be looked down on, which can then be applied to good advantage in the whole theory of series. Now since the theory of series is most important for analysis, such observations definitely ought to be considered worth of being developed all-out. For this purpose I undertook to analyze the following series, which both because of the special properties attributable to it and because of the important applications which it leads us to, appears to deserve our full attention. The series looks as follows:

$$\frac{1-x}{1-a}+\frac{(1-x)(a-x)}{a-a^3}+\frac{(1-x)(a-x)(a^2-x)}{a^3-a^6} +\frac{(1-x)(a-x)(a^2-x)(a^3-x)}{a^6-a^{10}}+\text{etc.}$$

The formation of the numerators is evident by inspection: namely, they are formed by multiplying the members of the sequence

$$1-x,\quad a-x,\quad a^2-x,\quad a^3-x,\quad a^4-x,\quad a^5-x,\quad a^6-x,\quad \text{etc.}$$

The denominators are all made up of two parts: they are powers of a whose exponents are triangular numbers. The nth term[e] of the series presented thus is

$$\frac{(1-x)(a-x)(a^2-x)(a^3-x)\cdots(a^{n-1}-x)}{a^{n(n-1):2}-a^{n(n+1):2}}.$$

2. To begin with, it is clear that the series, when x is assumed to be any power of a, will terminate at some point in such a way that all subsequent terms become 0. If we thus let s be the sum of the series in question,

$$s=\frac{1-x}{1-a}+\frac{(1-x)(a-x)}{a-a^3}+\frac{(1-x)(a-x)(a^2-x)}{a^3-a^6} +\frac{(1-x)(a-x)(a^2-x)(a^3-x)}{a^6-a^{10}}+\text{etc.}$$

and first take $x=1$ or $x=a^0$, one gets $s=0$ since all terms vanish. Let further $x=a$, so that only the first term survives; then one gets $s=1$. If

[e]Here, "term" is to be understood in the modern sense of the word.

$x = a^2$, one obtains

$$s = \frac{1-a^2}{1-a} + \frac{(1-a^2)(a-a^2)}{a-a^3}$$

or $s = 2$. Putting $x = a^3$, one finds

$$s = \frac{1-a^3}{1-a} + \frac{(1-a^3)(a-a^3)}{a-a^3} + \frac{(1-a^3)(a-a^3)(a^2-a^3)}{a^3-a^6}\,.$$

The first of these terms yields $1 + a + a^2$, the second $1 - a^3$, and the third $1 - a - a^2 + a^3$; collecting them, one obtains $s = 3$.

3. If similarly one puts $x = a^4$, one finds, after doing the computation, that $s = 4$, and for $x = a^5$ one gets $s = 5$. It thus appears that almost certainly one can deduce by induction that whenever x is taken to be any power of a whose exponent is $= n$, then this very exponent will give the value of s. This induction, however, is valid only when n is a positive integer; for if it were valid also for an arbitrary fraction, then s would be equal to the logarithm of x, where a is taken to be the number whose logarithm is 1. If this were true, then for $a = 10$ the sum s of the series would always have to express the common logarithm of x, and one would have

$$\begin{aligned} s = -\frac{(1-x)}{9} - \frac{(1-x)(10-x)}{990} - \frac{(1-x)(10-x)(100-x)}{999\,000} \\ -\frac{(1-x)(10-x)(100-x)(1000-x)}{9\,999\,000\,000} - \text{etc.} = \ell x. \end{aligned}$$

From what follows it will become clear, however, that this equality does not hold if x is not a power of a with a positive integer exponent.

4. The fact that for $x = a^n$ one does not always have $s = n$ when n is not a positive integer is easily inferred from the case where $x = 0$. For in this case, if the above induction would really extend to all numbers, one would have to get $s = -\infty$, since the logarithm of 0 is always $-\infty$; but for $x = 0$ one obtains

$$s = \frac{1}{1-a} + \frac{1}{1-aa} + \frac{1}{1-a^3} + \frac{1}{1-a^4} + \frac{1}{1-a^5} + \text{etc.};$$

and even though this series cannot be summed, everyone will easily recognize that its sum must be finite and therefore cannot express the logarithm of 0.

Similarly, for $a = 10$, when x is not taken to be a power of 10, one will find by summation a value which most of the time deviates rather significantly from $\log x$. Indeed, for $a = 10$ let $x = 9$; then

$$s = \frac{8}{9} + \frac{8 \cdot 1}{990} + \frac{8 \cdot 1 \cdot 91}{999\,000} + \frac{8 \cdot 1 \cdot 91 \cdot 991}{9\,999\,000\,000} + \frac{8 \cdot 1 \cdot 91 \cdot 991 \cdot 9991}{999\,990\,000\,000\,000} + \text{etc.};$$

and when these terms are expressed in decimal fractions, one obtains[3]

$$\begin{array}{rcl}
s & = & 0.88888\,88888\,89 \\
 & & 0.00808\,08080\,81 \\
 & & 0.00072\,87287\,29 \\
 & & 0.00007\,21520\,15 \\
 & & 0.00000\,72080\,59 \\
 & & 0.00000\,07207\,35 \\
 & & 0.00000\,00720\,73 \\
 & & 0.00000\,00072\,07 \\
 & & 0.00000\,00007\,21 \\
 & & 0.00000\,00000\,72 \\
 & & 0.00000\,00000\,07 \\
\hline
s & = & 0.89777\,85865\,88,
\end{array}$$

and this value is significantly smaller than the logarithm of 9.

5. Our series thus has the following property: if one puts for x a rational power of a, the sum of the series becomes equal to the exponent of that power, i.e., for

$$x = a^0, \quad a^1, \quad a^2, \quad a^3, \quad a^4, \quad a^5, \quad a^6, \quad a^7, \quad a^8 \quad \text{etc.}$$

[3] In the original:

$$\begin{array}{rcl}
s & = & 0.88888\,88888\,88888\,88888 \\
 & & 0.00808\,08080\,80808\,08080 \\
 & & 0.00008\,00800\,80080\,08008 \\
 & & 0.00000\,08000\,80008\,00080 \\
 & & 0.00000\,00080\,00080\,00080 \\
 & & 0.00000\,00000\,80000\,08000 \\
 & & 0.00000\,00000\,00800\,00008 \\
 & & 0.00000\,00000\,00008\,00000 \\
 & & 0.00000\,00000\,00000\,08000 \\
 & & 0.00000\,00000\,00000\,00080 \\
\hline
s & = & 0.89705\,05852\,10673\,21224
\end{array}$$

Corrected by G. F.

one gets

$$s = 0,\quad 1,\quad 2,\quad 3,\quad 4,\quad 5,\quad 6,\quad 7,\quad 8 \ \text{etc.};$$

but even though this is the property of the logarithms, it still is only valid when the exponents of a are integers. Thus, if one visualizes a curve whose abscissae are $= s$ and whose ordinates are $= x$, this curve will intersect the logarithmic curve in innumerable points: namely as often as the abscissa is expressed by an integer, as often does the ordinate pass through a point of intersection. It is thus clear that the logarithmic curve is not determined, not even by infinitely many points, and this is of use also for all other curves. In this manner one thus can understand that an arbitrary series, even if all its terms corresponding to integer indices are given, can be interpolated in infinitely many different ways; this is a question that I will treat on another occasion in more detail.[4]

6. In order to arrive at a deeper understanding of our series, one can transform it into the following form:

$$s = \frac{1}{1-a}(1-x) + \frac{1}{1-a^2}(1-x)\left(1-\frac{x}{a}\right) + \frac{1}{1-a^3}(1-x)\left(1-\frac{x}{a}\right)\left(1-\frac{x}{a^2}\right) + \frac{1}{1-a^4}(1-x)\left(1-\frac{x}{a}\right)\left(1-\frac{x}{aa}\right)\left(1-\frac{x}{a^3}\right) + \text{etc.};$$

this form is simpler than the previous one since here no triangular numbers occur. If we now put ax in place of x and denote the sum of the series so obtained by t, one obtains

$$t = \frac{1}{1-a}(1-ax) + \frac{1}{1-a^2}(1-ax)(1-x) + \frac{1}{1-a^3}(1-ax)(1-x)\left(1-\frac{x}{a}\right) + \frac{1}{1-a^4}(1-ax)(1-x)\left(1-\frac{x}{a}\right)\left(1-\frac{x}{aa}\right) + \text{etc.}$$

Subtracting the first series from the second, one finds

$$t - s = x + \frac{x}{a}(1-x) + \frac{x}{aa}(1-x)\left(1-\frac{x}{a}\right) + \frac{x}{a^3}(1-x)\left(1-\frac{x}{a}\right)\left(1-\frac{x}{aa}\right) + \text{etc.};$$

subtract this series from 1, and since the rest is divisible by $1-x$, one obtains

$$1 + s - t = (1-x)\left(1 - \frac{x}{a} - \frac{x}{aa}\left(1-\frac{x}{a}\right) - \frac{x}{a^3}\left(1-\frac{x}{a}\right)\left(1-\frac{x}{aa}\right) - \text{etc.}\right).$$

[4]See Memoir E189, which was presented to the Academy on Sept. 21, 1750. G. F.

But here the second factor is further divisible by $1 - \frac{x}{a}$, so that

$$1 + s - t = (1 - x)\left(1 - \frac{x}{a}\right)\left(1 - \frac{x}{aa} - \frac{x}{a^3}\left(1 - \frac{x}{aa}\right) - \text{etc.}\right).$$

Here again one recognizes a factor $1 - \frac{x}{aa}$, and after factoring it out, there appears a factor $1 - \frac{x}{a^3}$, and so on; thus one finally obtains

$$1 + s - t = (1 - x)\left(1 - \frac{x}{a}\right)\left(1 - \frac{x}{a^2}\right)\left(1 - \frac{x}{a^3}\right)\left(1 - \frac{x}{a^4}\right)\left(1 - \frac{x}{a^5}\right) \text{ etc.}$$

7. It is thus clear that whenever x is taken to be an arbitrary power of a, a factor of this expression vanishes, so that

$$1 + s - t = 0 \quad \text{or} \quad t = 1 + s.$$

If for $x = a^n$, where n denotes a positive integer, the sum of the given series is $s = n$, then for $x = a^{n+1}$ the sum of the series

$$t = s + 1 = n + 1.$$

Since for $n = 0$ or $x = 1$ the sum of the series is $s = 0$, it thus follows that for $x = a$ the sum of the series $s = 1$; and from this it follows further that for $x = a^2$ one gets $s = 2$, and for $x = a^3$ one gets $s = 3$. What we have found before by induction is now generally clear: if $x = a^n$, where n denotes a positive integer, then one always has $s = n$. If, however, n is not a positive integer and s denotes the sum of the initially presented series for $x = a^n$, then for $x = a^{n+1}$ the sum of the series, which should be called t, will not be $= s + 1$; in fact, it is

$$t = 1 + s - (1 - a^n)(1 - a^{n-1})(1 - a^{n-2})(1 - a^{n-3})(1 - a^{n-4}) \text{ etc.}$$

In these cases, therefore, the value of the series evidently deviates from the nature of the logarithms.

8. In the same way as we have produced here, by multiplication of the values of x by a, from the value of s the value t, we can also, conversely, by division of the values of x by a, obtain from the value of t the value of s. Thus, in the original series, or in the one transformed into the form

$$s = \frac{1}{1 - a}(1 - x) + \frac{1}{1 - a^2}(1 - x)\left(1 - \frac{x}{a}\right) + \frac{1}{1 - a^3}(1 - x)\left(1 - \frac{x}{a}\right)\left(1 - \frac{x}{a^2}\right) + \text{etc.},$$

let us denote the sum of the series in the following cases as shown below:

$$\begin{array}{llll}
\text{for} & x = 1, & \text{there becomes} & s = A = 0, \\
& x = \frac{1}{a}, & & s = B, \\
& x = \frac{1}{a^2}, & & s = C, \\
& x = \frac{1}{a^3}, & & s = D, \\
& x = \frac{1}{a^4}, & & s = E
\end{array}$$

etc.

If one now puts $x = \frac{1}{a}$, one gets $s = B$ and $t = A = 0$, since t results from s by writing ax in place of x; from the above[f] one obtains

$$1 + B = \left(1 - \frac{1}{a}\right)\left(1 - \frac{1}{a^2}\right)\left(1 - \frac{1}{a^3}\right)\left(1 - \frac{1}{a^4}\right)\left(1 - \frac{1}{a^5}\right) \quad \text{etc.}$$

or

$$B = -1 + \left(1 - \frac{1}{a}\right)\left(1 - \frac{1}{a^2}\right)\left(1 - \frac{1}{a^3}\right)\left(1 - \frac{1}{a^4}\right)\left(1 - \frac{1}{a^5}\right) \quad \text{etc.;}$$

for $a = 10$ one thus gets

$$B = -0.10998\,99000\,01001.$$[5]

9. Let $x = \frac{1}{a^2}$; then $s = C$ and $t = B$, and therefore

$$1 + C - B = \left(1 - \frac{1}{a^2}\right)\left(1 - \frac{1}{a^3}\right)\left(1 - \frac{1}{a^4}\right)\left(1 - \frac{1}{a^5}\right) \quad \text{etc.;}$$

to this, one adds the previous $1 + B$ to obtain

$$2 + C = \left(2 - \frac{1}{a}\right)\left(1 - \frac{1}{a^2}\right)\left(1 - \frac{1}{a^3}\right)\left(1 - \frac{1}{a^4}\right)\left(1 - \frac{1}{a^5}\right) \quad \text{etc.}$$

and

$$C = -2 + \left(2 - \frac{1}{a}\right)\left(1 - \frac{1}{a^2}\right)\left(1 - \frac{1}{a^3}\right)\left(1 - \frac{1}{a^4}\right)\left(1 - \frac{1}{a^5}\right) \quad \text{etc.}$$

[f] That is, from the identity at the end of §6.

[5] The original has: $-0.10998\,99000\,00998$. Corrected by G. F.

Or, by eliminating the series[g],

$$1 + B = \left(1 - \frac{1}{a}\right)(1 + C - B)$$

or

$$C - 2B = \frac{1}{a}(1 + C - B).$$

If in the same manner one takes $x = \frac{1}{a^3}$, one gets $s = D$ and $t = C$; therefore,

$$1 + D - C = \left(1 - \frac{1}{a^3}\right)\left(1 - \frac{1}{a^4}\right)\left(1 - \frac{1}{a^5}\right)\left(1 - \frac{1}{a^6}\right) \quad \text{etc.},$$

and adding to it the previous series[h] yields

$$3 + D = \left(3 - \frac{1}{a} - \frac{2}{a^2} + \frac{1}{a^3}\right)\left(1 - \frac{1}{a^3}\right)\left(1 - \frac{1}{a^4}\right)\left(1 - \frac{1}{a^5}\right)\left(1 - \frac{1}{a^6}\right) \quad \text{etc.}$$

And since for $x = \frac{1}{a^4}$ one gets

$$1 + E - D = \left(1 - \frac{1}{a^4}\right)\left(1 - \frac{1}{a^5}\right)\left(1 - \frac{1}{a^6}\right)\left(1 - \frac{1}{a^7}\right) \quad \text{etc.},$$

there holds

$$4 + E = \left(4 - \frac{1}{a} - \frac{2}{a^2} - \frac{2}{a^3} + \frac{1}{a^4} + \frac{2}{a^5} - \frac{1}{a^6}\right) \times \left(1 - \frac{1}{a^4}\right)\left(1 - \frac{1}{a^5}\right)\left(1 - \frac{1}{a^6}\right)\left(1 - \frac{1}{a^6}\right) \quad \text{etc.};$$

and in this way one can continue as far as one wishes.

10. The relationship between the three values of the sum of the series s for three consecutive values of x can, however, also be represented by a finite expression. For the value x continue to denote the sum $= s$; if one takes ax in place of x, let the sum of the series $= t$; and if one takes aax in place of x, let the sum of the series $= u$. Since between t and s we have found the relationship

$$1 + s - t = (1 - x)\left(1 - \frac{x}{a}\right)\left(1 - \frac{x}{a^2}\right)\left(1 - \frac{x}{a^3}\right)\left(1 - \frac{x}{a^4}\right) \quad \text{etc.},$$

[g] Actually the infinite product for $1 + B$.
[h] The infinite product for $2 + C$.

there follows, if we write here ax for x, an analogous relationship between u and t:

$$1 + t - u = (1 - ax)(1 - x)\left(1 - \frac{x}{a}\right)\left(1 - \frac{x}{a^2}\right)\left(1 - \frac{x}{a^3}\right) \quad \text{etc.}$$

We therefore have

$$1 + t - u = (1 - ax)(1 + s - t)$$

or

$$u = 2t - s + ax(1 + s - t),$$

that is,

$$s = \frac{2t - u + ax(1 - t)}{1 - ax}.$$

For the values $A, B, C, D, \ldots$ defined above, there thus result the following relationships:

For $x = \frac{1}{a^2}$, one gets

$$A = 2B - C + \frac{1}{a}(1 + C - B)$$

or

$$C = \frac{1 + (2a - 1)B - aA}{a - 1} = B + \frac{1 + a(B - A)}{a - 1};$$

for $x = \frac{1}{a^3}$, one gets

$$D = C + \frac{1 + a^2(C - B)}{a^2 - 1};$$

for $x = \frac{1}{a^4}$, one gets

$$E = D + \frac{1 + a^3(D - C)}{a^3 - 1};$$

for $x = \frac{1}{a^5}$, one gets

$$F = E + \frac{1 + a^4(E - D)}{a^4 - 1}$$

etc.

These relationships can be expressed still more conveniently as follows:

$$C = 2B - A + \frac{1 + B - A}{a - 1},$$
$$D = 2C - B + \frac{1 + C - B}{a^2 - 1},$$

$$E = 2D - C + \frac{1 + D - C}{a^3 - 1},$$
$$F = 2E - D + \frac{1 + E - D}{a^4 - 1}$$

etc.

Now since $A = 0$, from this, once the value of the variable

$$B = -1 + \left(1 - \frac{1}{a}\right)\left(1 - \frac{1}{a^2}\right)\left(1 - \frac{1}{a^3}\right)\left(1 - \frac{1}{a^4}\right) \quad \text{etc.}$$

has been found, one can then assign uniquely the values to all subsequent variables $C, D, E, F, \dots$.

11. But since there holds $s = n$ when one puts $x = a^n$, where n denotes a positive integer, we obtain from our original series the following summable one:

$$n = \frac{1 - a^n}{1 - a} + \frac{(1 - a^n)(1 - a^{n-1})}{1 - a^2} + \frac{(1 - a^n)(1 - a^{n-1})(1 - a^{n-2})}{1 - a^3} + \text{etc.}$$

Thus, in this case, since $t = n + 1$, one gets[i]

$$1 = a^n + a^{n-1}(1 - a^n) + a^{n-2}(1 - a^n)(1 - a^{n-1}) + a^{n-3}(1 - a^n)(1 - a^{n-1})(1 - a^{n-2}) + \text{etc.},$$

and that this is true is evident if one moves all terms on the same side:

$$(1 - a^n)(1 - a^{n-1})(1 - a^{n-2})(1 - a^{n-3})(1 - a^{n-4}) \text{ etc.} = 0.$$

This motivates us to consider such formulae in a more general way. Let in fact

$$A, \quad B, \quad C, \quad D, \quad E, \quad F \quad \text{etc.}$$

[i] By subtracting from the preceding equation with n replaced by $n + 1$ the one in the present form.

be a sequence of arbitrary quantities, and let

$$(1-A)(1-B)(1-C)(1-D)(1-E) \text{ etc.} = S.$$

From this, one obtains

$$1 - A - B(1-A) - C(1-A)(1-B) - D(1-A)(1-B)(1-C) - \text{etc.} = S;$$

for this formula can be reduced quite easily to the previous one. We therefore obtain

$$A + B(1-A) + C(1-A)(1-B) + D(1-A)(1-B)(1-C) + \text{etc.} = 1 - S.$$ [6]

12. Thus, if now any of the quantities $A, B, C, \ldots$ becomes equal to 1, one gets $S = 0$, and there results a series whose sum is equal to 1. Take, for example, the sequence

A	B	C	D	E	F	etc.
$\frac{1}{2}$,	$\frac{2}{3}$,	$\frac{3}{4}$,	$\frac{4}{5}$,	$\frac{5}{6}$,	$\frac{6}{7}$	etc.;

since the fraction at infinitum is $= 1$, one gets $S = 0$, and there arises the following series:

$$1 = \frac{1}{2} + \frac{2}{2\cdot 3} + \frac{3}{2\cdot 3\cdot 4} + \frac{4}{2\cdot 3\cdot 4\cdot 5} + \frac{5}{2\cdot 3\cdot 4\cdot 5\cdot 6} + \text{etc.},$$

whose truth can also easily be verified; namely, it comes about as follows: Let

$$z = 1 + \frac{1}{1\cdot 2} + \frac{1}{1\cdot 2\cdot 3} + \frac{1}{1\cdot 2\cdot 3\cdot 4} + \frac{1}{1\cdot 2\cdot 3\cdot 4\cdot 5} + \text{etc.};$$

then one has

$$z - 1 = \frac{1}{1\cdot 2} + \frac{1}{1\cdot 2\cdot 3} + \frac{1}{2\cdot 3\cdot 4} + \frac{1}{2\cdot 3\cdot 4\cdot 5} + \frac{1}{2\cdot 3\cdot 4\cdot 5\cdot 6} + \text{etc.}$$

and by subtraction there follows

$$1 = \frac{1}{2} + \frac{2}{2\cdot 3} + \frac{3}{2\cdot 3\cdot 4} + \frac{4}{2\cdot 3\cdot 4\cdot 5} + \frac{5}{2\cdot 3\cdot 4\cdot 5\cdot 6} + \text{etc.}$$

[6] The original has: $A + \cdots + \text{etc.} = S + 1$. Several formulae of §13 also had to be corrected. G. F.

13. Let

$$A = \frac{1}{9}\,, \quad B = \frac{1}{25}\,, \quad C = \frac{1}{49}\,, \quad D = \frac{1}{81} \quad \text{etc.};$$

then one gets[7]

$$S = \frac{8}{9} \cdot \frac{24}{25} \cdot \frac{48}{49} \cdot \frac{80}{81} \cdot \frac{120}{121} \cdot \text{etc.} = \frac{\pi}{4};$$

here, π denotes the circumference of the circle with diameter 1. From this, one now arrives at the following series for the quadrature of the circle[j]:

$$-\frac{\pi}{4} + 1 = \frac{1}{9} + \frac{8}{9 \cdot 25} + \frac{8 \cdot 24}{9 \cdot 25 \cdot 49} + \frac{8 \cdot 24 \cdot 48}{9 \cdot 25 \cdot 49 \cdot 81} + \text{etc.}$$

or

$$-\frac{9}{4}\pi + 8 = \frac{2 \cdot 4}{5 \cdot 5} + \frac{2 \cdot 4 \cdot 4 \cdot 6}{5 \cdot 5 \cdot 7 \cdot 7} + \frac{2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot 8}{5 \cdot 5 \cdot 7 \cdot 7 \cdot 9 \cdot 9} + \text{etc.}$$

Now since there are infinitely many such products whose value S can be determined, one can in this way from each one of them derive an infinite series whose sum can be identified. This opens a wide area that allows us to find as many summable series as we wish.

14. I am returning, however, to the originally presented series

$$s = \frac{1}{1-a}\,(1-x) + \frac{1}{1-a^2}\,(1-x)\left(1 - \frac{x}{a}\right) + \frac{1}{1-a^3}\,(1-x)\left(1 - \frac{x}{a}\right)\left(1 - \frac{x}{a^2}\right) + \text{etc.},$$

with the intention of transforming it into another form in which the terms proceed in powers of x. This, in principle, could be done through expansion of the individual terms; but since in this way the individual coefficients turn out to be infinite series, it will be most convenient, for this purpose, to apply the formula found above,

$$u = 2t - s + ax(1 - t + s) \quad \text{or} \quad u - 2t + s = ax + ax(s - t),$$

where t is obtained from s by replacing x by ax, and likewise t becomes u if x is once more replaced by ax. Therefore, if we assume for the series to be sought

$$s = A + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 + \text{etc.},$$

[7] Compare footnotes 1, p. 3 and 2, p. 373 [reference to Wallis, *Arithmetica infinitorum*]. G. F.

[j] As corrected here (but not in the summary) by G. F.; see footnote 6.

one gets

$$t = A + Bax + Ca^2x^2 + Da^3x^3 + Ea^4x^4 + Fa^5x^5 + \text{etc.}$$

and

$$u = A + Ba^2x + Ca^4x^2 + Da^6x^3 + Ea^8x^4 + Fa^{10}x^5 + \text{etc.}$$

Now from this, one forms

$$u - 2t + s = B(1-a)^2x + C(1-aa)^2x^2 + D(1-a^3)^2x^3 + E(1-a^4)^2x^4 + \text{etc.},$$

$$ax(1+s-t) = ax + Ba(1-a)x^2 + Ca(1-aa)x^3 + Da(1-a^3)x^4 + \text{etc.}$$

From the equality of these series one concludes that

$$B = \frac{a}{(1-a)^2}\,, \quad C = \frac{Ba(1-a)}{(1-aa)^2}\,, \quad D = \frac{Ca(1-aa)}{(1-a^3)^2}\,, \quad E = \frac{Da(1-a^3)}{(1-a^4)^2} \quad \text{etc.}$$

15. Consequently, for the assumed coefficients one obtains the following values:

$$B = \frac{a}{(1-a)^2}\,,$$
$$C = \frac{a^2}{(1-a)(1-aa)^2}\,,$$
$$D = \frac{a^3}{(1-a)(1-aa)(1-a^3)^2}\,,$$
$$E = \frac{a^4}{(1-a)(1-aa)(1-a^3)(1-a^4)^2}\,,$$
$$F = \frac{a^5}{(1-a)(1-aa)(1-a^3)(1-a^4)(1-a^5)^2}$$

etc.

The first term A is not defined herewith; but since A is the value of s for $x = 0$, one clearly has

$$A = \frac{1}{1-a} + \frac{1}{1-a^2} + \frac{1}{1-a^3} + \frac{1}{1-a^4} + \frac{1}{1-a^5} + \text{etc.}$$

Through the determination of these values, the originally presented series

$$s = \frac{1}{1-a}(1-x) + \frac{1}{1-a^2}(1-x)\left(1-\frac{x}{a}\right) + \frac{1}{1-a^3}(1-x)\left(1-\frac{x}{a}\right)\left(1-\frac{x}{aa}\right) + \text{etc.}$$

is transformed into the form

$$
\begin{aligned}
s = \frac{1}{1-a} + \frac{1}{1-a^2} + \frac{1}{1-a^3} + \frac{1}{1-a^4} + \frac{1}{1-a^5} + \text{etc.} \\
+\frac{ax}{(1-a)^2} + \frac{a^2x^2}{(1-a)(1-aa)^2} + \frac{a^3x^3}{(1-a)(1-aa)(1-a^3)^2} \\
+\frac{a^4x^4}{(1-a)(1-a^2)(1-a^3)(1-a^4)^2} + \text{etc.}
\end{aligned}
$$

16. Since for $x = a^n$, where n denotes a positive integer, one has $s = n$, one obtains the following summation:

$$
\begin{aligned}
n + \frac{1}{a-1} + \frac{1}{a^2-1} + \frac{1}{a^3-1} + \frac{1}{a^4-1} + \frac{1}{a^5-1} + \text{etc.} \\
= \frac{a^{n+1}}{(a-1)^2} - \frac{a^{2n+2}}{(a-1)(aa-1)^2} + \frac{a^{3n+3}}{(a-1)(a^2-1)(a^3-1)^2} \\
-\frac{a^{4n+4}}{(a-1)(a^2-1)(a^3-1)(a^4-1)^2} + \text{etc.}
\end{aligned}
$$

Now if $n = 0$, one thus gets

$$
\begin{aligned}
\frac{1}{a-1} + \frac{1}{a^2-1} + \frac{1}{a^3-1} + \text{etc.} \\
= \frac{a}{(a-1)^2} - \frac{a^2}{(a-1)(aa-1)^2} + \frac{a^3}{(a-1)(a^2-1)(a^3-1)^2} - \text{etc.},
\end{aligned}
$$

and for $n = 1$ one gets

$$
\begin{aligned}
\frac{1}{a-1} + \frac{1}{a^2-1} + \frac{1}{a^3-1} + \text{etc.} \\
= \frac{a^2}{(a-1)^2} - \frac{a^4}{(a-1)(a^2-1)^2} + \frac{a^6}{(a-1)(a^2-1)(a^3-1)^2} - \text{etc.} - 1.
\end{aligned}
$$

In general, one has

$$
\begin{aligned}
\frac{1}{a-1} + \frac{1}{a^2-1} + \frac{1}{a^3-1} + \frac{1}{a^4-1} + \text{etc.} \\
= \frac{a^{n+1}}{(a-1)^2} - \frac{a^{2n+2}}{(a-1)(a^2-1)^2} + \frac{a^{3n+3}}{(a-1)(a^2-1)(a^3-1)^2} - \text{etc.} - n,
\end{aligned}
$$

where n denotes an arbitrary positive integer.

17. If one replaces n by $n-1$, one obtains

$$\begin{aligned}&\frac{1}{a-1}+\frac{1}{a^2-1}+\frac{1}{a^3-1}+\frac{1}{a^4-1}+\text{etc.}\\ =\frac{a^n}{(a-1)^2}-\frac{a^{2n}}{(a-1)(a^2-1)^2}&+\frac{a^{3n}}{(a-1)(a^2-1)(a^3-1)^2}-\text{etc.}-n+1;\end{aligned}$$

and subtracting from this the series above yields

$$\begin{aligned}1=\frac{a^n}{a-1}-\frac{a^{2n}}{(a-1)(a^2-1)}&+\frac{a^{3n}}{(a-1)(a^2-1)(a^3-1)}\\ &-\frac{a^{4n}}{(a-1)(a^2-1)(a^3-1)(a^4-1)}+\text{etc.}\end{aligned}$$

The sum of this series is thus always equal to 1, regardless of what value one assigns to the quantity a and also what positive integer one takes for n. In the case $n=1$, this summation is easily seen to be true: The fact that

$$1=\frac{a}{a-1}-\frac{a^2}{(a-1)(a^2-1)}+\frac{a^3}{(a-1)(a^2-1)(a^3-1)}-\text{etc.},$$

evidently follows by considering the series

$$z=1-\frac{1}{a-1}+\frac{1}{(a-1)(a^2-1)}-\frac{1}{(a-1)(a^2-1)(a^3-1)}+\text{etc.},$$

from which

$$\begin{aligned}1-z=\frac{1}{a-1}-\frac{1}{(a-1)(a^2-1)}&+\frac{1}{(a-1)(a^2-1)(a^3-1)}\\ &-\frac{1}{(a-1)(a^2-1)(a^3-1)(a^4-1)}+\text{etc.},\end{aligned}$$

and adding both yields

$$\begin{aligned}1=\frac{a}{a-1}-\frac{aa}{(a-1)(a^2-1)}&+\frac{a^3}{(a-1)(a^2-1)(a^3-1)}\\ &-\frac{a^4}{(a-1)(a^2-1)(a^3-1)(a^4-1)}+\text{etc.}\end{aligned}$$

18. But then, the validity of this series can be shown also for the remaining values of n as follows: If there holds

$$1 = \frac{a^n}{a-1} - \frac{a^{2n}}{(a-1)(a^2-1)} + \frac{a^{3n}}{(a-1)(a^2-1)(a^3-1)} - \text{etc.},$$

then—so I claim—one also has

$$1 = \frac{a^{n+1}}{a-1} - \frac{a^{2n+2}}{(a-1)(a^2-1)} + \frac{a^{3n+3}}{(a-1)(a^2-1)(a^3-1)} - \text{etc.}$$

For, since by assumption

$$1 = \frac{a^n}{a-1} - \frac{a^{2n}}{(a-1)(a^2-1)} + \frac{a^{3n}}{(a-1)(a^2-1)(a^3-1)} - \text{etc.},$$

one also has[k]

$$0 = a^n - \frac{a^{2n}}{a-1} + \frac{a^{3n}}{(a-1)(a^2-1)} - \text{etc.},$$

and these series added to each other yields[l]

$$1 = \frac{a^{n+1}}{a-1} - \frac{a^{2n+2}}{(a-1)(a^2-1)} + \frac{a^{3n+3}}{(a-1)(a^2-1)(a^3-1)} - \text{etc.}$$

Consequently, since the series

$$1 = \frac{a^n}{a-1} - \frac{a^{2n}}{(a-1)(a^2-1)} + \frac{a^{3n}}{(a-1)(a^2-1)(a^3-1)} - \text{etc.}$$

in the case $n = 1$ has been proved to be correct, it is also correct in the case $n = 2$, and thus further in the cases $n = 3$, $n = 4$, etc., so that the sum of this series will always be $= 1$ regardless of which positive integer one takes for n.

19. Now that I have, here with the help of the property

$$u - 2t + s = ax + ax(s - t)$$

[k]Multiply the preceding equation by a^n and rearrange.

[l]Compare also with §27.

established above, rearranged the initially presented series $s = \frac{1}{1-a}(1-x) +$ etc. in powers of x, it will be meaningful to derive the same transformation directly from the series s itself; because in this way, we will arrive at the summation of innumerable new series. One will thus have to develop the individual terms of the series s through multiplication; in order to do this as conveniently as possible, I consider an arbitrary term

$$\frac{1}{(1-a^m)}(1-x)\left(1-\frac{x}{a}\right)\left(1-\frac{x}{a^2}\right)\left(1-\frac{x}{a^3}\right)\cdots\left(1-\frac{x}{a^{m-1}}\right)$$

and put

$$P = (1-x)\left(1-\frac{x}{a}\right)\left(1-\frac{x}{a^2}\right)\left(1-\frac{x}{a^3}\right)\cdots\left(1-\frac{x}{a^{m-1}}\right).$$

Then one has

$$\ell P = \ell(1-x) + \ell\left(1-\frac{x}{a}\right) + \ell\left(1-\frac{x}{a^2}\right) + \cdots + \ell\left(1-\frac{x}{a^{m-1}}\right)$$

and by differentiation

$$\frac{dP}{P} = \frac{-dx}{1-x} - \frac{dx}{a-x} - \frac{dx}{aa-x} - \cdots - \frac{dx}{a^{m-1}-x}$$

or

$$\frac{dP}{P} = -dx\left\{\begin{array}{ccccccccccccc} & 1 & + & x & + & x^2 & + & x^3 & + & x^4 & + & x^5 & + \text{ etc.}\\ + & \frac{1}{a} & + & \frac{x}{a^2} & + & \frac{x^2}{a^3} & + & \frac{x^3}{a^4} & + & \frac{x^4}{a^5} & + & \frac{x^5}{a^6} & + \text{ etc.}\\ + & \frac{1}{a^2} & + & \frac{x}{a^4} & + & \frac{x^2}{a^6} & + & \frac{x^3}{a^8} & + & \frac{x^4}{a^{10}} & + & \frac{x^5}{a^{12}} & + \text{ etc.}\\ & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & \\ + & \frac{1}{a^{m-1}} & + & \frac{x}{a^{2m-2}} & + & \frac{x^2}{a^{3m-3}} & + & \frac{x^3}{a^{4m-4}} & + & \frac{x^4}{a^{5m-5}} & + & \frac{x^5}{a^{6m-6}} & + \text{ etc.}\end{array}\right\}.$$

Summation of the individual vertical series now yields

$$dP = -Pdx\left(\frac{a^m-1}{a^m-a^{m-1}} + \frac{a^{2m}-1}{a^{2m}-a^{2m-2}}x + \frac{a^{3m}-1}{a^{3m}-a^{3m-3}}x^2 + \frac{a^{4m}-1}{a^{4m}-a^{4m-4}}x^3 + \text{ etc.}\right).$$

20. Now assume for P the following series:

$$P = \alpha + \beta x + \gamma x^2 + \delta x^3 + \varepsilon x^4 + \text{ etc.}$$

Then

$$\frac{dP}{dx} = \beta + 2\gamma x + 3\delta x^2 + 4\varepsilon x^3 + 5\zeta x^4 + \text{ etc.}$$

Carrying out the substitution, one obtains

$$\begin{aligned}
&\beta + \frac{a^m - 1}{a^m - a^{m-1}}\,\alpha = 0,\\
&2\gamma + \frac{a^m - 1}{a^m - a^{m-1}}\,\beta + \frac{a^{2m} - 1}{a^{2m} - a^{2m-2}}\,\alpha = 0,\\
&3\delta + \frac{a^m - 1}{a^m - a^{m-1}}\,\gamma + \frac{a^{2m} - 1}{a^{2m} - a^{2m-2}}\,\beta + \frac{a^{3m} - 1}{a^{3m} - a^{3m-3}}\,\alpha = 0
\end{aligned}$$

etc.,

and since for $x = 0$ one has $P = 1$, one evidently has $\alpha = 1$. Thus, there holds

$$\beta = \frac{-a^m + 1}{a^m - a^{m-1}}$$

and

$$2\gamma - \frac{(a^m - 1)^2}{(a^m - a^{m-1})^2} + \frac{a^{2m} - 1}{a^{2m} - a^{2m-2}} = 0$$

or

$$2\gamma = \frac{a^m - 1}{a^m - a^{m-1}}\left(\frac{a^m - 1}{a^m - a^{m-1}} - \frac{a^m + 1}{a^m + a^{m-1}}\right) = \frac{2a^m(a^{m-1} - 1)(a^m - 1)}{(a^m - a^{m-1})(a^{2m} - a^{2m-2})}$$

and consequently

$$\gamma = \frac{(a^m - 1)(a^{m-1} - 1)}{(a^m - a^{m-1})(a^m - a^{m-2})}\,.$$

In a similar manner—though not without enormous labor—one can determine also the subsequent coefficients and see, eventually, that they can be expressed rather concisely.

21. In order to execute the determination of the coefficients more conveniently, I propose to apply a method used here already several times. Namely, in the series

$$P = \alpha + \beta x + \gamma x^2 + \delta x^3 + \varepsilon x^4 + \text{ etc.}$$

I put $\frac{x}{a}$ in place of x; let the sum of the series so obtained be $= Q$, that is,

$$Q = \alpha + \frac{\beta x}{a} + \frac{\gamma x^2}{a^2} + \frac{\delta x^3}{a^3} + \frac{\varepsilon x^4}{a^4} + \text{ etc.}$$

But since

$$P = (1-x)\left(1-\frac{x}{a}\right)\left(1-\frac{x}{aa}\right)\cdots\left(1-\frac{x}{a^{m-1}}\right),$$

one has

$$Q = \left(1-\frac{x}{a}\right)\left(1-\frac{x}{a^2}\right)\left(1-\frac{x}{a^3}\right)\cdots\left(1-\frac{x}{a^m}\right)$$

and therefore

$$P\left(1-\frac{x}{a^m}\right) = Q(1-x) \quad \text{or} \quad a^m P - Px - a^m Q + a^m Qx = 0.$$

Here, for P and Q, substitute the assumed series, so that

$$\left.\begin{array}{rrrrr} \alpha a^m & +\beta a^m x & +\gamma a^m x^2 & +\delta a^m x^3 & + \text{ etc.} \\ & -\alpha x & -\beta x^2 & -\gamma x^3 & - \text{ etc.} \\ -\alpha a^m & -\beta a^{m-1} x & -\gamma a^{m-2} x^2 & -\delta a^{m-3} x^3 & - \text{ etc.} \\ & +\alpha a^m x & +\beta a^{m-1} x^2 & +\gamma a^{m-2} x^3 & + \text{ etc.} \end{array}\right\} = 0.$$

By comparing corresponding terms, one obtains from this

$$\beta = \frac{-\alpha(a^m - 1)}{a^{m-1}(a-1)},$$
$$\gamma = \frac{-\beta(a^{m-1} - 1)}{a^{m-2}(aa-1)},$$
$$\delta = \frac{-\gamma(a^{m-2} - 1)}{a^{m-3}(a^3-1)},$$
$$\varepsilon = \frac{-\delta(a^{m-3} - 1)}{a^{m-4}(a^4-1)}$$

etc.

22. Since $\alpha = 1$, the coefficients thus look as follows:

$$\alpha = 1,$$
$$\beta = \frac{-(a^m - 1)}{a^{m-1}(a-1)},$$
$$\gamma = \frac{+(a^m - 1)(a^{m-1} - 1)}{a^{2m-3}(a-1)(aa-1)},$$
$$\delta = \frac{-(a^m - 1)(a^{m-1} - 1)(a^{m-2} - 1)}{a^{3m-6}(a-1)(aa-1)(a^3-1)},$$
$$\varepsilon = \frac{+(a^m - 1)(a^{m-1} - 1)(a^{m-2} - 1)(a^{m-3} - 1)}{a^{4m-10}(a-1)(a^2-1)(a^3-1)(a^4-1)}$$

etc.

Consequently, an arbitrary term

$$\frac{1}{1-a^m}(1-x)\left(1-\frac{x}{a}\right)\left(1-\frac{x}{aa}\right)\cdots\left(1-\frac{x}{a^{m-1}}\right)$$

of the series s, if expanded, yields this expression:

$$\frac{1}{1-a^m} - \frac{x}{a^{m-1}(1-a)} + \frac{(1-a^{m-1})x^2}{a^{2m-3}(1-a)(1-a^2)} - \frac{(1-a^{m-1})(1-a^{m-2})x^3}{a^{3m-6}(1-a)(1-a^2)(1-a^3)} + \text{etc.}$$

If now for m one takes in turn the numbers $1, 2, 3, 4$ etc., one obtains the following formulae for the terms of the series s:

First term : $= \dfrac{1}{1-a} - \dfrac{x}{1-a}$,

Second term : $= \dfrac{1}{1-a^2} - \dfrac{x}{a(1-a)} + \dfrac{(1-a)x^2}{a(1-a)(1-a^2)}$, Third term : $$= \frac{1}{1-a^3} - \frac{x}{a^2(1-a)} + \frac{(1-a^2)x^2}{a^3(1-a)(1-a^2)} - \frac{(1-a)(1-a^2)x^3}{a^3(1-a)(1-a^2)(1-a^3)},$$

Fourth term : $$= \frac{1}{1-a^4} - \frac{x}{a^3(1-a)} + \frac{(1-a^3)xx}{a^5(1-a)(1-a^2)} - \frac{(1-a^2)(1-a^3)x^3}{a^6(1-a)(1-a^2)(1-a^3)} + \frac{(1-a)(1-a^2)(1-a^3)x^4}{a^6(1-a)(1-a^2)(1-a^3)(1-a^4)}$$

etc.

23. If one now collects all these terms into a sum, one obtains a collection of infinitely many series which, taken together, are equal to the originally presented series. From

$$s = \frac{1}{1-a}(1-x) + \frac{1}{1-a^2}(1-x)\left(1-\frac{x}{a}\right) + \frac{1}{1-a^3}(1-x)\left(1-\frac{x}{a}\right)\left(1-\frac{x}{aa}\right) + \text{etc.}$$

one thus gets

$$\begin{aligned} s = {} & \frac{1}{1-a} + \frac{1}{1-a^2} + \frac{1}{1-a^3} + \frac{1}{1-a^4} + \frac{1}{1-a^5} + \text{etc.} \\ & - \frac{x}{1-a}\left(1 + \frac{1}{a} + \frac{1}{a^2} + \frac{1}{a^3} + \frac{1}{a^4} + \text{etc.}\right) \\ & + \frac{x^2}{a(1-a)(1-a^2)}\left(\frac{1-a}{1} + \frac{1-a^2}{a^2} + \frac{1-a^3}{a^4} + \frac{1-a^4}{a^6} + \text{etc.}\right) \\ & - \frac{x^3}{a^3(1-a)(1-a^2)(1-a^3)}\left(\frac{(1-a)(1-a^2)}{1} + \frac{(1-a^2)(1-a^3)}{a^3} + \frac{(1-a^3)(1-a^4)}{a^6} + \text{etc.}\right) \\ & + \frac{x^4}{a^6(1-a)(1-a^2)(1-a^3)(1-a^4)}\left(\frac{(1-a)(1-a^2)(1-a^3)}{1}\right. \end{aligned}$$

$$+\frac{(1-a^2)(1-a^3)(1-a^4)}{a^4} + \text{etc.}\Big)$$

etc.

Now since this series must agree with the one previously found[m], from the agreement of the individual of these series one finds their sums:

$$1+\frac{1}{a}+\frac{1}{a^2}+\frac{1}{a^3}+\frac{1}{a^4}+\text{etc.} = \frac{-a}{1-a},$$

$$\frac{1-a}{1}+\frac{1-a^2}{a^2}+\frac{1-a^3}{a^4}+\frac{1-a^4}{a^6}+\text{etc.} = \frac{+a^3}{1-aa},$$

$$\frac{(1-a)(1-a^2)}{1}+\frac{(1-a^2)(1-a^3)}{a^3}+\frac{(1-a^3)(1-a^4)}{a^6}+\text{etc.} = \frac{-a^6}{1-a^3},$$

$$\frac{(1-a)(1-a^2)(1-a^3)}{1}+\frac{(1-a^2)(1-a^3)(1-a^4)}{a^4}+\text{etc.} = \frac{+a^{10}}{1-a^4},$$

$$\frac{(1-a)(1-a^2)(1-a^3)(1-a^4)}{1}+\frac{(1-a^2)(1-a^3)(1-a^4)(1-a^5)}{a^5}+\text{etc.} = \frac{-a^{15}}{1-a^5}$$

24. These series can be transformed into the following form, in which the law of the sequence will be discernible still more clearly:

$$\frac{a}{a-1} = 1+\frac{1}{a}+\frac{1}{a^2}+\frac{1}{a^3}+\frac{1}{a^4}+\text{etc.},$$

$$\frac{a^2}{a^2-1} = \left(1-\frac{1}{a}\right)+\frac{1}{a}\left(1-\frac{1}{a^2}\right)+\frac{1}{a^2}\left(1-\frac{1}{a^3}\right)+\frac{1}{a^3}\left(1-\frac{1}{a^4}\right) +\frac{1}{a^4}\left(1-\frac{1}{a^5}\right)+\text{etc.},$$

$$\frac{a^3}{a^3-1} = \left(1-\frac{1}{a}\right)\left(1-\frac{1}{a^2}\right)+\frac{1}{a}\left(1-\frac{1}{a^2}\right)\left(1-\frac{1}{a^3}\right) +\frac{1}{a^2}\left(1-\frac{1}{a^3}\right)\left(1-\frac{1}{a^4}\right)+\text{etc.},$$

$$\frac{a^4}{a^4-1} = \left(1-\frac{1}{a}\right)\left(1-\frac{1}{a^2}\right)\left(1-\frac{1}{a^3}\right)+\frac{1}{a}\left(1-\frac{1}{a^2}\right)\left(1-\frac{1}{a^3}\right)\left(1-\frac{1}{a^4}\right)+\text{etc.},$$

$$\frac{a^5}{a^5-1} = \left(1-\frac{1}{a}\right)\left(1-\frac{1}{a^2}\right)\left(1-\frac{1}{a^3}\right)\left(1-\frac{1}{a^4}\right)$$

[m] In §15.

$$+\frac{1}{a}\left(1-\frac{1}{a^2}\right)\left(1-\frac{1}{a^3}\right)\left(1-\frac{1}{a^4}\right)\left(1-\frac{1}{a^5}\right)+\text{ etc.}$$

etc.

It can be seen from this that there holds generally

$$\frac{a^{m+1}}{a^{m+1}-1}=\frac{1}{1-\frac{1}{a^{m+1}}}=\left(1-\frac{1}{a}\right)\left(1-\frac{1}{a^2}\right)\cdots\left(1-\frac{1}{a^m}\right)$$
$$+\frac{1}{a}\left(1-\frac{1}{a^2}\right)\left(1-\frac{1}{a^3}\right)\cdots\left(1-\frac{1}{a^{m+1}}\right)+\frac{1}{a^2}\left(1-\frac{1}{a^3}\right)\left(1-\frac{1}{a^4}\right)\cdots\left(1-\frac{1}{a^{m+2}}\right)$$
$$+\frac{1}{a^3}\left(1-\frac{1}{a^4}\right)\left(1-\frac{1}{a^5}\right)\cdots\left(1-\frac{1}{a^{m+3}}\right)+\text{ etc.}$$

25. The sum of this series can also be investigated as follows: for brevity, let $\frac{1}{a}=b$ and let for the sum to be sought

$$z=(1-b)(1-b^2)\cdots(1-b^m)+b(1-b^2)(1-b^3)\cdots(1-b^{m+1})$$
$$+b^2(1-b^3)(1-b^4)\cdots(1-b^{m+2})+b^3(1-b^4)(1-b^5)\cdots(1-b^{m+3})+\text{ etc.}$$

Multiply both sides by $1-b^{m+1}$ to obtain

$$(1-b^{m+1})z=(1-b)(1-b^2)\cdots(1-b^m)(1-b^{m+1})+(1-b^2)(1-b^3)$$
$$\cdots(1-b^{m+1})(b-b^{m+2})+(1-b^3)(1-b^4)\cdots(1-b^{m+2})(b^2-b^{m+3})+\text{ etc.}$$

On the other hand, there holds

$$b-b^{m+2}=1-b^{m+2}-(1-b),$$
$$b^2-b^{m+3}=1-b^{m+3}-(1-bb),$$
$$b^3-b^{m+4}=1-b^{m+4}-(1-b^3)$$

etc.;

and substitution of these values for the last factors yields

$$(1-b^{m+1})z=(1-b)(1-b^2)\cdots(1-b^{m+1})+(1-b^2)(1-b^3)\cdots(1-b^{m+2})$$
$$-(1-b)(1-b^2)\cdots(1-b^{m+1})-(1-b^2)(1-b^3)\cdots(1-b^{m+2})$$

$$+(1-b^3)(1-b^4)\cdots(1-b^{m+3})+(1-b^4)(1-b^5)\cdots(1-b^{m+4})$$
$$-(1-b^3)(1-b^4)\cdots(1-b^{m+3})-(1-b^4)(1-b^5)\cdots(1-b^{m+4})$$
$$\text{etc.}$$

Thus, since all terms cancel, only the last one remains:

$$(1-b^{m+1})z=(1-b^{\infty})(1-b^{\infty+1})\cdots(1-b^{m+\infty});$$

it is clear, therefore, that for $b<1$, that is, $a>1$ as we assume, one gets $(1-b^{m+1})z=1$, and therefore

$$z=\frac{1}{1-b^{m+1}}=\frac{a^{m+1}}{a^{m+1}-1}\,,$$

as we had found.

26. From what was discussed in §21 one easily finds a series proceeding in powers of x which is equal to the product with infinitely many factors

$$P=(1-x)\left(1-\frac{x}{a}\right)\left(1-\frac{x}{a^2}\right)\left(1-\frac{x}{a^3}\right)\left(1-\frac{x}{a^4}\right)\text{ etc.}$$

Indeed, one puts

$$P=1-\alpha x+\beta x^2-\gamma x^3+\delta x^4-\varepsilon x^5+\text{ etc.},$$

writes ax for x, and denotes the resulting value $=Q$; then

$$Q=(1-ax)(1-x)\left(1-\frac{x}{a}\right)\left(1-\frac{x}{aa}\right)\left(1-\frac{x}{a^3}\right)\text{ etc.}=P-axP$$

and

$$Q=1-\alpha ax+\beta a^2x^2-\gamma a^3x^3+\delta a^4x^4-\varepsilon a^5x^5+\text{ etc.}$$

But

$$axP=ax-\alpha ax^2+\beta ax^3-\gamma ax^4+\delta ax^5-\text{ etc.},$$
$$-P=-1+\alpha x-\beta x^2+\gamma x^3-\delta x^4+\varepsilon x^5-\text{ etc.},$$

and therefore

$$\alpha=\frac{a}{a-1}\,,\quad \beta=\frac{\alpha a}{a^2-1}\,,\quad \gamma=\frac{\beta a}{a^3-1}\,,\quad \delta=\frac{\gamma a}{a^4-1}\quad\text{etc.}$$

For this reason the infinite product

$$P = (1-x)\left(1-\frac{x}{a}\right)\left(1-\frac{x}{aa}\right) \text{ etc.}$$

can be resolved into the following infinite series:

$$P = 1 - \frac{ax}{a-1} + \frac{a^2x^2}{(a-1)(a^2-1)} - \frac{a^3x^3}{(a-1)(a^2-1)(a^3-1)} + \frac{a^4x^4}{(a-1)(a^2-1)(a^3-1)(a^4-1)} - \text{ etc.}$$

27. If, therefore, this product P is set equal to 0, the infinite equation

$$0 = 1 - \frac{ax}{a-1} + \frac{a^2x^2}{(a-1)(a^2-1)} - \frac{a^3x^3}{(a-1)(a^2-1)(a^3-1)} + \text{ etc.}$$

will have only real roots x, and the values of x will be [equal to] the members of the geometric sequence

$$1,\quad a,\quad a^2,\quad a^3,\quad a^4,\quad a^5,\quad a^6,\quad a^7 \text{ etc.};$$

if one therefore puts $x = a^n$, where n denotes a positive integer, one gets

$$0 = 1 - \frac{a^{n+1}}{a-1} + \frac{a^{2n+2}}{(a-1)(a^2-1)} - \frac{a^{3n+3}}{(a-1)(a^2-1)(a^3-1)} + \text{ etc.},$$

the correctness of which has already been shown above in §18.

28. Above all, however, one series is remarkable, for which above (§16) inumerable others have already been found with the same value, namely

$$\frac{1}{a-1} + \frac{1}{a^2-1} + \frac{1}{a^3-1} + \frac{1}{a^4-1} + \frac{1}{a^5-1} + \text{ etc.};$$

its sum for $a > 1$, even though it is finite and can easily be determined by approximations, cannot be expressed neither in rational nor in irrational numbers. It appears therefore especially worth the effort that mathematicians investigate the nature of that transcendental quantity by which its sum is expressed.

29. I will now show how the sum of such series could be found approximately without effort, and in fact, I will consider this series in a somewhat more general setting. Let

$$s = \frac{1}{a-z} + \frac{1}{a^2-z} + \frac{1}{a^3-z} + \frac{1}{a^4-z} + \frac{1}{a^5-z} + \text{etc.}$$

Expand the individual terms in geometric series; one then gets

$$\begin{aligned} s = {} & \frac{1}{a} + \frac{1}{a^2} + \frac{1}{a^3} + \frac{1}{a^4} + \frac{1}{a^5} + \text{etc.} \\ +z & \left(\frac{1}{a^2} + \frac{1}{a^4} + \frac{1}{a^6} + \frac{1}{a^8} + \frac{1}{a^{10}} + \text{etc.}\right) \\ +z^2 & \left(\frac{1}{a^3} + \frac{1}{a^6} + \frac{1}{a^9} + \frac{1}{a^{12}} + \frac{1}{a^{15}} + \text{etc.}\right) \\ & \qquad \text{etc.} \end{aligned}$$

and these series, summed again, yield

$$s = \frac{1}{a-1} + \frac{z}{aa-1} + \frac{zz}{a^3-1} + \frac{z^3}{a^4-1} + \frac{z^4}{a^5-1} + \text{etc.}$$

When $z = 1$, these two series thus become one and the same, and the transformation makes no difference.

30. In order to sum this series, let us assume that n terms of the first form have in fact already been added up, and let us put for their sum $= A$, so that

$$A = \frac{1}{a-z} + \frac{1}{a^2-z} + \frac{1}{a^3-z} + \frac{1}{a^4-z} + \cdots + \frac{1}{a^n-z}$$

holds. The whole sum to be sought then becomes

$$s = A + \frac{1}{a^{n+1}-z} + \frac{1}{a^{n+2}-z} + \frac{1}{a^{n+3}-z} + \frac{1}{a^{n+4}-z} + \text{etc.}$$

Now expand the fractions in geometric series:

$$s = A + \frac{1}{a^{n+1}} + \frac{1}{a^{n+2}} + \frac{1}{a^{n+3}} + \frac{1}{a^{n+4}} + \text{etc.}$$

$$+z\left(\frac{1}{a^{2n+2}}+\frac{1}{a^{2n+4}}+\frac{1}{a^{2n+6}}+\frac{1}{a^{2n+8}}+\text{ etc.}\right)$$
$$+z^2\left(\frac{1}{a^{3n+3}}+\frac{1}{a^{3n+6}}+\frac{1}{a^{3n+9}}+\frac{1}{a^{3n+12}}+\text{ etc.}\right)$$

etc.,

and these series, summed again, yield

$$s = A + \frac{1}{a^n(a-1)} + \frac{z}{a^{2n}(aa-1)} + \frac{zz}{a^{3n}(a^3-1)} + \frac{z^3}{a^{4n}(a^4-1)} + \text{ etc.}$$

This series converges faster than the first, more so the larger the number n.

31. Let $a = 2$, hence

$$s = \frac{1}{2-z} + \frac{1}{4-z} + \frac{1}{8-z} + \frac{1}{16-z} + \text{ etc.}$$

If now

$$A = \frac{1}{2-z} + \frac{1}{4-z} + \frac{1}{8-z} + \cdots + \frac{1}{2^n - z},$$

one gets

$$s = A + \frac{1}{1\cdot 2^n} + \frac{z}{3\cdot 2^{2n}} + \frac{z^2}{7\cdot 2^{3n}} + \frac{z^3}{15\cdot 2^{4n}} + \frac{z^4}{31\cdot 2^{5n}} + \text{ etc.}$$

Let us now put $z = 1$ in order to thus seek the sum of the series

$$s = 1 + \frac{1}{3} + \frac{1}{7} + \frac{1}{15} + \frac{1}{31} + \frac{1}{63} + \text{ etc.}$$

Add, for example, the first four terms directly, so that $n = 4$; then one gets

1	=	1.00000 00000 00000
$\frac{1}{3}$	=	0.33333 33333 33333
$\frac{1}{7}$	=	0.14285 71428 57142
$\frac{1}{15}$	=	0.06666 66666 66666
A	=	1.54285 71428 57141.

Therefore, one has

$$s = A + \frac{1}{16\cdot 1} + \frac{1}{16^2\cdot 3} + \frac{1}{16^3\cdot 7} + \frac{1}{16^4\cdot 15} + \text{ etc.},$$

and these terms yield, in decimal fractions,

$$\begin{array}{rcl} & & 0.06383\,80095\,58149 \\ A & = & 1.54285\,71428\,57142 \\ \hline \text{hence } s & = & 1.60669\,51524\,15291. \end{array}$$

32. Incidentally, if the individual terms of the series[8]

$$s = \frac{1}{a-1} + \frac{1}{a^2-1} + \frac{1}{a^3-1} + \text{etc.}$$

are developed in geometric series and equal powers of a are collected, one arrives at the form

$$s = \frac{1}{a} + \frac{2}{a^2} + \frac{2}{a^3} + \frac{3}{a^4} + \frac{2}{a^5} + \frac{4}{a^6} + \frac{2}{a^7} + \frac{4}{a^8} + \frac{3}{a^9} + \text{etc.}$$

This series has the property, that the numerator of each fraction indicates how many divisors the exponent of a in the denominator has: thus the fraction $\frac{4}{a^6}$ has the numerator $= 4$, since the exponent 6 has four divisors $1, 2, 3, 6$. Therefore, if the exponent of a in the denominator is a prime number, then the numerator is always $= 2$; for numbers, however, that are not prime, it is greater than 2. Thus, as one easily sees, for $a = 10$ one gets

$$s = 0.12232\,42434\,26244\,52626\,44283\,44628.$$

[8]This series was investigated by J. H. Lambert (1728–1777) in a work entitled *Anlage zur Architektonic oder Theorie des Ersten und des Einfachen in der philosophischen und mathematischen Erkenntniß*, Riga 1771, Vol. 2, p. 507. G. F.