Synopsis of Leonhard Euler's 1749 paper:

METHOD FOR DETERMINING THE LONGITUDE OF PLACES BY OBSERVING OCCULTATIONS OF FIXED STARS BY THE MOON*

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1. History of the problem

Navigation has been important in traveling as well as safety. For example, in 1707, a British fleet made an error on their longitude findings that sent them in the wrong direction where four of the five ships were destroyed and nearly two thousand soldiers lost their lives. This problem may have been avoided if the sailors had an accurate method of finding longitude.

In 1714, the British Parliament passed the Longitude Act, which offered £20,000 to anyone who could calculate longitude to within a half degree. Since one degree of longitude can be as much as 69 miles, even a fraction of a degree comes out to a significant distance. Therefore, the required method would need to be very accurate.

In the late 1720s, a self-educated British watchmaker named John Harrison set out to construct a sea clock which could keep accurate enough time to calculate longitude. Harrison succeeded in constructing a sea clock in 1735 which was deserving of the £20,000 prize. However, he was very critical of his machine even though everyone else, including the royal board of longitude, was very impressed.

The sextant was then invented by a couple of inventors during the same time period. A sextant had incorporated a longer measuring arc as well as a telescope. These additional pieces enabled the device to measure distances during the day from the moon to the sun and from the moon to the stars at night. With a star chart and a sextant a sailor now could measure all sorts of lunar distances. A sailor makes an observation at a set time and found the same observation through the charts in London at a different time; the sailor is now able to find how many degrees they are away from London.

At this time, Leonhard Euler set out to predict the motions of the moon and from these particular motions he constructed lunar tables. [9] These tables were found through the studies of Isaac Newton who introduced the theory of gravity into astronomy.

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2. The body of the paper

I & II. These first two sections allow Euler to set up for the observations. He says, at some known location, in this case Paris; it will be α hours after noon and the moon's right ascension will be ζ when the occultation is observed. We can see this in Figure 1 where Paris is denoted by P_1. To determine the longitude of the unknown location, which he specifies should be west of Paris, the same occultation needs to be viewed with time and right ascension recorded. At that unknown location, which we say is X_2, he lets β be the hours past noon and η denotes the moon's right ascension when the occultation is observed (see Figure 2).

III. Since these two observations would occur within a few hours of each other, it is necessary that α and ζ be predetermined. Mr. Euler explains that the hourly movement of the moon's right ascension can be deduced from "the Astronomical Tables."[1] These lunar tables give information about the occultation's occurrence in Paris and can be compared to what is observed in the unknown location. These tables were quite complex, including predictions of the moon's orbit and how it would be viewed from Paris at every hour on any day to come. Euler remarks that these tables are not completely accurate; however, he says that they are very close to the truth. He assigns γ to be the hourly movement of the moon's right ascension at this time.

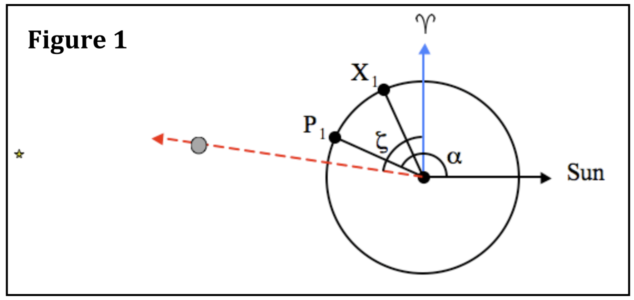

Figure 1

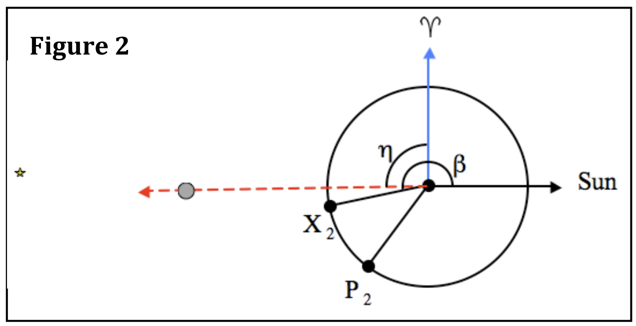

Figure 2

IV. Here, another variable is defined. Euler denotes the difference in longitude between Paris and the unknown location by the letter z (see Figure 3). He then states that it will be $\beta + z$ hours after noon in Paris, when the angle of the moon's right ascension is η.

V. In this section Euler first reminds the reader that at α hours the right ascension of the moon was said to be ζ. Then he makes the connection that during $\beta + z - \alpha$ hours the right ascension of the moon has changed by $\eta - \zeta$. We will refer to $\beta + z - \alpha$ hours as T, this value represents the total elapsed time between observations. P_1 denotes Paris's original location; after T hours Paris has moved to P_2. Similarly, X_1 and X_2 represent the positions of the unknown location (Figure 3). Euler goes on to say that T (time) multiplied by the earlier defined γ (speed) must equal the moon's total change in right ascension.

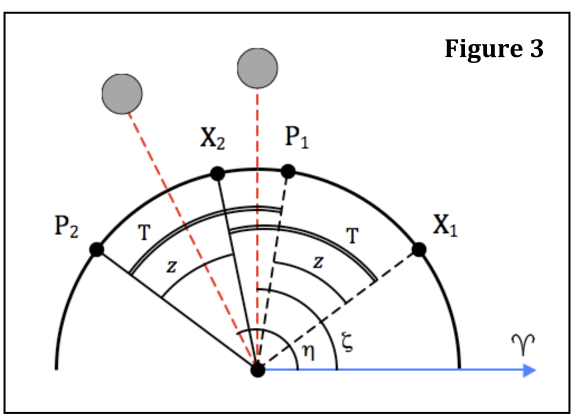

Figure 3

[1] This presumably refers to Euler's "Tabulæ Astronomicæ Solis et Lunæ" [2]

So, one can say that $(\beta + z - \alpha)\,\gamma = \eta - \zeta$. This equation can be reordered so that one may find the change in longitude $z = \alpha - \beta + (\eta - \zeta)/\gamma$ hours.

VI. Now that Euler has set up the equation to find the difference in longitudes, a few precautionary statements are made. Although Euler had assumed that the unknown location was west of Paris, he says that if the value of z is found to be negative then it is clear that this location is actually east of Paris. In the equation used, if α and β were swapped and η and ζ were swapped, then the opposite sign would result because of the symmetry of the equation. Therefore, the connection between east and west and positive and negative values makes logical sense.

VII. In this last statement, Euler says that if the unknown location is too far from the one that had been used to calculate the tables, in this case Paris, then the difference cannot be measured. Next, he apparently refers to "paragraph six" of his previous article [3], and says that it provides an alternate step that is needed when the two locations are too far from each other. Euler says that by repeating all the same steps and research, a new table could be developed that is appropriate for the unknown location.

3. Appraisal of method

Although all of these observations and calculations can be quite accurate, there are some pitfalls that cannot be avoided. In the 1700s, celestial navigation was dependent on the human eye; there were no computers to help gain this knowledge, just the eye and a few simple tools. The sextant is a tool that was commonly used to find longitude and it is one thing that could cause error. It could be calibrated poorly, placement of the eye could be off, or an error could be made in calculation. Some of these problems could be corrected; however, human error is hard to escape. [1]

Euler's theory is completely dependent on astronomical tables to provide at least half of the information needed, and he states that they are not completely accurate. These tables predicted the change in the sun and moon's orbits, in his lunar tables [2] Euler states,

> Although it is claimed of several lunar tables that they are based on this theory, I dare to assert that the calculations to which this theory leads are so intricate, that such tables must be considered to differ greatly from the theory. Nor do I claim that I have included in these tables all the inequalities of motion which the theory implies. [10]

There are many gravitational factors that alter a planet's orbit derived from Newton's theory of gravity. Since the sun, moon, and earth are not always equidistant the amount of gravitational pull is also a variable. The table's predictions are difficult to calculate because of the irregular change that is caused by the gravity. Also, based on the complexity of these equations and the calculations that are done, the results should be expected to have some error.

By setting up an example one can see how a small error in measurement can greatly affect the longitude that is found. One can calculate a correct z value from the equation Euler offers. Then alter γ by say 1°, 0.5°, and 0.1° and see how much a small change to γ will affect z. Below is an example where α is set to be 10 hours past noon, β to be 11 hours, ζ is 75°, and η is observed to be 88°. Suppose the speed of the moon's right ascension is said to be 8.2°/hour, when this value is used the true z value comes out to 0.585365 hours. In order to receive the prizes discussed, the proposed technique needed to be within half a degree, $^2/_3$ of a degree, or just within one degree of the actual longitude. If we make an error of ε in measurement of γ the calculated longitude would be, $z' = \alpha - \beta + \frac{\eta - \zeta}{\gamma + \varepsilon}$. Then, $\varepsilon' = z - z' = (\eta - \zeta)\left(\frac{1}{\gamma} - \frac{1}{\gamma + \varepsilon}\right)$, where ε' represents the actual error in the results. In the table below this error is given in longitudinal hours and degrees.

ε	z	z'	ε'	
1°/hour	0.585365 hrs.	0.413043 hrs.	0.172322 hrs.	2.584830°
0.5°/hour	0.585365 hrs.	0.494253 hrs.	0.091112 hrs.	1.366695°
0.1°/hour	0.585365 hrs.	0.566265 hrs.	0.019100 hrs.	0.286513°

In order to win the top prize, ε could be at most 0.176113°/hour.

4. Comparison with other methods

The measurement of the moon has been successful in the past, but the surroundings have to be just right. If the sky is cloudy then the astronomical method of finding longitude fails because the moon and stars cannot be seen. At times this method is impractical because it requires one to know how to calculate longitude once a measurement has been taken. When one takes the measurement, the reference point cannot differ too much from the proposed location. Also, time must be kept at least on a daily basis, if not more frequently. The benefit to using the astronomical method is that no complicated machines are necessary, such as the chronometer. The chronometer method requires a tool that is very difficult to construct. It took Harrison a great deal of time to build this product and there still could have been problems with its operation. Even though the chronometer method does not need perfect weather, numerous chronometers are needed on board. This may have been a problem back in the 1700s because they were not mass-produced and the chronometer's size proved to be an issue. The process to mass-produce these chronometers was impractical and inconvenient.

5. Epilogue

Euler's work on the longitude problem did not end with this paper. In *Letters to a German Princess*, published some 20 years later, Euler describes a total of six methods that could be used to calculate longitude. While he gives several different astronomical methods, he also mentions chronometers and compasses as providing plausible solutions to the longitude problem. [6] Near the end of his life, Euler wrote another paper on longitude, in which he describes a process now known as the "lunar distance method." [8] Additionally, Euler wrote a second book on lunar motion, namely his *Theoria motuum lunae* [7], which was published in 1772.

In the end, however, it was Euler's first lunar theory that achieved the most recognition. The astronomer Tobias Mayer used Euler's *Theoria motus lunae* [5] to develop his method for calculating longitude, which earned him (posthumously) a £3,000 prize from the Board of Longitude. Euler was awarded £300 for his contributions to Mayer's success.

From a historical perspective, Harrison's chronometers were more successful than the astronomical methods. Eventually, the problems of design and production were overcome, thus permitting their widespread use. Since the astronomical methods required difficult calculations, the mass-production of chronometers (which required no such calculations) was sufficient to make these methods obsolete.

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[4] ——— "Methode de determiner la longitude des lieux par l'observation d'occultations des étoiles fixes par la lune," *Mémoires de l'académie royale des sciences et belles-lettres de Berlin* **3** (1747) 1749, pp. 178-179. [E115]

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