An Example of the Solution of Differential Equations without Separation of Variables[1]

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§1. I believe that the separation of variables in differential equations is so carefully sought because a solution of the equation follows directly from that discovery, which is evident to one practiced enough in these matters. Moreover the integration of differential equations, if indeed it succeeds, is begun best by separating variables. Though certainly innumerable equations have been given, whose integrals can be found without such a separation – the Celebrated Johann Bernoulli exhibited a method of this type in our Comm. Tom. I page 167[2] – yet all of these equations have been arranged in such a way, so that either the separation of variables is obvious by itself, or that at least the separation may be derived from that integration. It is indeed likely that the computation of solutions that Analysts have found up until now are all of this type of equation, that, even if the variables can be separated in no other way, a separation still arises from that solution. For this reason, I have believed until now that no solvable differential equation could be produced whose separation would elude all men.

§2. Recently however while engaged in the rectifying of an ellipse, I unexpectedly came upon a differential equation by which I was able to solve the rectification of the ellipse, yet a separation of variables could not be found, not even from the method of solution. In fact the equation I obtained was $dy + \frac{y^2 dx}{x} = \frac{x\ dx}{x^2-1}$ which closely resembles the Riccati equation, and as it happens it is as difficult to separate as $dy + y^2\ dx = x^2\ dx$. This case seemed exceedingly paradoxical to me at first; but after studying the solution more carefully I realized easily not only that a separation could not be deduced from it, but also, that if a separation were to succeed by another method, far greater absurdities would follow. One might find a comparison of the perimeters of different ellipses, which, it surely seems to me, would overturn all of analysis. This solution moreover is extremely easy, it is completed indeed by the elongation of infinite ellipses having one of two axes in common, and for this reason it must be substantially preferred to the usual way of solving quadratures.

§3. I will therefore relate the entire matter, just as I arrived at it [**Note:** See Figure 1]. Let ACB be a quarter of an ellipse, whose center is C, and whose semi-axes are AC and BC. Let $AC = a$ and $BC = b$ and from A draw an infinite tangent AT, and to this from the center C draw a secant CT cutting off an arc $AM = s$, and call $AT = t$. After dropping a perpendicular from M onto AC and calling $CP = x$, then from the nature of the ellipse $PM = \frac{b\sqrt{a^2-x^2}}{a}$; and $tx = b\sqrt{a^2-x^2}$ or $x = \frac{ab}{\sqrt{bb+tt}}$ will be obtained from the proportion $CP{:}PM = CA{:}AT$. To the arc AM is added an element Mm, and mp and Ct are drawn nearest respectively to MP and CT; then for Mm,

[1]*Comm. Acad. Sci. Imp. Petropol.* **6**, 1732/3, pp. 168-174. [E28]

[2]Bernoulli, Johann. On Integrations of Differential Equations, wherein is Related an Example of Integration without a Prior Separation of Variables. *Comm. Acad. Sci. Imp. Petropol.* **1**, 1726, pp. 167-184.

$ds = \frac{-dx\sqrt{a^4-(a^2-b^2)x^2}}{a\sqrt{a^2-x^2}}$ and $Tt = dt$ will result. However because $x = \frac{ab}{\sqrt{b^2+t^2}}$; $dx = \frac{-abtdt}{(b^2+t^2)^{3/2}}$ will result, as will $\sqrt{a^2-x^2} = \frac{at}{\sqrt{b^2+t^2}}$, and $\sqrt{a^4-(a^2-b^2)x^2} = \frac{a\sqrt{b^4+a^2tt}}{\sqrt{b^2+t^2}}$. From these it is found that $ds = \frac{bdt\sqrt{b^4+a^2tt}}{(bb+tt)^{3/2}}$. To find the integral of this expression through series I set $a^2 = (n+1)b^2$, which will give $ds = \frac{b^2dt\sqrt{(b^2+t^2)+nt^2}}{(b^2+t^2)^{3/2}}$, and the irrational expression in the numerator may be considered a binomial, one of whose terms is b^2+t^2, and whose other simple term is nt^2. I now expand $\sqrt{(b^2+t^2)+nt^2}$ by the standard method into this series

$$(b^2+t^2)^{1/2} + \frac{Ant^2}{(b^2+t^2)^{1/2}} + \frac{Bn^2t^4}{(b^2+t^2)^{3/2}} + \frac{Cn^3t^6}{(b^2+t^2)^{5/2}} + \text{etc.}$$

in which for the sake of brevity $A = \frac{1}{2}$, $B = \frac{-1\cdot 1}{2\cdot 4}$, $C = \frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 6}$, $D = \frac{-1\cdot 1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 8}$ etc. Therefore

$$ds = \frac{b^2dt}{b^2+t^2} + \frac{Ab^2nt^2dt}{(b^2+t^2)^2} + \frac{Bb^2n^2t^4dt}{(b^2+t^2)^3} + \frac{Cb^2n^3t^6dt}{(b^2+t^2)^4} + \text{etc.}$$

will result and the entire elliptic arc s will be the integral of this series.

§4. It should be noted here that it is possible for the integration of these separate terms to be reduced to the first term $\int \frac{bb\,dt}{bb+tt}$, which gives the arc of a circle of radius b whose tangent is t. For this reason I will integrate the individual terms as follows:

$$\begin{aligned}
\int \frac{b^2t^2\,dt}{(b^2+t^2)^2} &= \frac{1}{2}\int \frac{bb\,dt}{bb+tt} - \frac{1}{2}\frac{b^2t}{bb+tt}; \\
\int \frac{b^2t^4\,dt}{(b^2+t^2)^3} &= \frac{1}{2}\cdot\frac{3}{4}\int \frac{b^2\,dt}{bb+tt} - \frac{1}{2}\cdot\frac{3}{4}\frac{b^2t}{bb+tt} - \frac{1}{4}\frac{b^2t^3}{(bb+tt)^2}; \\
\int \frac{b^2t^6\,dt}{(b^2+t^2)^4} &= \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\int \frac{b^2\,dt}{bb+tt} - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{b^2t}{bb+tt} - \frac{1\cdot 5}{4\cdot 6}\frac{b^2t^5}{(bb+tt)^2} - \frac{1}{6}\frac{b^2t^5}{(bb+tt)^3},
\end{aligned}$$

from which the rule for the integrals of the remaining terms is now apparent enough.

§5. If the quarter part AMB of the perimeter of the ellipse is required, one ought to make t infinite, and by doing this all of the algebraic terms in the above integrals will vanish. After having set $t = \infty$, the circular arc $\int \frac{bb\,dt}{bb+tt}$ will give a quarter part of the periphery of a circle, whose radius is b or BC, which we will denote by the letter e. Therefore $\int \frac{b^2\,dt}{bb+tt} = e$, $\int \frac{b^2t^2\,dt}{(bb+tt)^2} = \frac{1\cdot e}{2}$, $\int \frac{b^2t^4\,dt}{(bb+tt)^3} = \frac{1\cdot 3\cdot e}{2\cdot 4}$, $\int \frac{b^2t^6\,dt}{(bb+tt)^4} = \frac{1\cdot 3\cdot 5\cdot e}{2\cdot 4\cdot 6}$ etc. will result. Thus the quarter part of the perimeter of the ellipse will turn out to be

$$AMB = e\left(1 + \frac{1An}{2} + \frac{1\cdot 3}{2\cdot 4}Bn^2 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}Cn^3 + \frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8}Dn^4 + \text{etc.}\right).$$

And after substituting the due values in place of A, B, C, D, etc.

$$AMB = e\left(1 + \frac{1\cdot n}{2\cdot 2} - \frac{1\cdot 1\cdot 3n^2}{2\cdot 2\cdot 4\cdot 4} + \frac{1\cdot 1\cdot 3\cdot 3\cdot 5n^3}{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6} - \frac{1\cdot 1\cdot 3\cdot 3\cdot 5\cdot 5\cdot 7n^4}{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6\cdot 8\cdot 8} + \text{etc.}\right)$$

will result.

§6. If n or $\frac{a^2-b^2}{b^2}$ which is equivalent to it is very small, in which case the ellipse is quite close to a circle, this series converges rapidly; therefore in this case the perimeter of the ellipse is found easily. When indeed n is as small as possible a quantity, or $a = b + \omega$, denoting by ω the quantity that is as small as possible, $n = \frac{2\omega}{b}$ will result, and $AMB = e\left(1 + \frac{\omega}{2b}\right)$ as near as possible. When indeed $a = 0$, the point A lies upon C, and $AMB = BC = b$ results; in this case indeed $n = -1$, therefore $\frac{b}{e} = 1 - \frac{1}{2\cdot 2} - \frac{1\cdot 1\cdot 3}{2\cdot 2\cdot 4\cdot 4} - \frac{1\cdot 1\cdot 3\cdot 3\cdot 5}{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6} - \text{etc.}$ will be obtained. The sum of this series therefore expresses the ratio of the radius to a quarter part of the periphery of a circle.

§7. Therefore whatever value the letter n in the series found in §5 may have, the sum of the series can always be found by means of the rectification of an ellipse, whose major axis is to its minor axis as $\sqrt{n+1}$ is to 1. Since this is the case, I also used my method of reducing summations of series to the solutions of equations, which I have presented recently[3], so that I might investigate the equations upon whose solutions the summations of a given series depends. Now in order that this method may be able to be applied more easily, I set $n = -x^2$, and the series that must be summed will be $1 - \frac{1\cdot x^2}{2\cdot 2} - \frac{1\cdot 1\cdot 3\cdot x^4}{2\cdot 2\cdot 4\cdot 4} - \frac{1\cdot 1\cdot 3\cdot 3\cdot 5\cdot x^6}{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6} - \text{etc.}$, the sum of which I will call s. Thus by differentiating $\frac{ds}{dx} = -\frac{1\cdot x}{2} - \frac{1\cdot 1\cdot 3\cdot x^3}{2\cdot 2\cdot 4} - \frac{1\cdot 1\cdot 3\cdot 3\cdot 5\cdot x^5}{2\cdot 2\cdot 4\cdot 4\cdot 6} - \text{etc.}$ will result. Now in turn I multiply by x, and I take differentials while holding dx constant, and $\frac{d.xds}{dx^2} = -1\cdot x - \frac{1\cdot 1\cdot 3\cdot x^3}{2\cdot 2} - \frac{1\cdot 1\cdot 3\cdot 3\cdot 5\cdot x^5}{2\cdot 2\cdot 4\cdot 4} - \text{etc.}$ will result. I further divide everywhere by x, and the opposite side I multiply by dx, and I take integrals, and $\int \frac{d.xds}{xdx} = -x - \frac{1\cdot 1\cdot x^3}{2\cdot 2} - \frac{1\cdot 1\cdot 3\cdot 3\cdot x^5}{2\cdot 2\cdot 4\cdot 4} - \text{etc.}$ will result. Finally I multiply again by dx, I divide indeed by x^3, and I take integrals, and $\int \frac{1}{x^3}\int \frac{d.xds}{x} = \frac{1}{x} - \frac{1\cdot x}{2\cdot 2} - \frac{1\cdot 1\cdot\cdot 3\cdot x^3}{2\cdot 2\cdot 4\cdot 4} - \frac{1\cdot 1\cdot 3\cdot 3\cdot 5\cdot x^5}{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6} - \text{etc.}$ will result. This very series is the original one divided by x: its sum is therefore $\frac{s}{x}$. For this reason we have the equation $\int \frac{1}{x^3}\int \frac{d.xds}{x} = \frac{s}{x}$, which after differentials are taken changes into $x^2ds - sxdx = \int \frac{d.xds}{x}$. This in turn may be differentiated and $x^2dds + xdxds - sdx^2 = \frac{d.xds}{x} = dds + \frac{dxds}{x}$ will appear. The solution of this equation depends upon the summation of the proposed series, which since it may be obtained through the rectification of an ellipse, a solution of the equation will also be given.

§8. Since in that equation s has one dimension, it can be reduced by my method introduced in Tom. III. Comm.[4] to a differential equation simply by making the substitution $s = c^{\int pdx}$, where c denotes the number whose log. is 1. Having done this, $ds = c^{\int pdx}pdx$ and $dds = c^{\int pdx}(dpdx + ppdx^2)$ will result, and also the given equation will be transformed into $x^2dp + x^2p^2dx + pxdx - dx = dp + ppdx + \frac{pdx}{x}$, which being divided by $xx - 1$ is changed into $dp + ppdx + \frac{pdx}{x} = \frac{dx}{xx-1}$. To make this simpler, I put $p = \frac{y}{x}$, and this will produce $dy + \frac{yydx}{x} = \frac{xdx}{xx-1}$. How this is able to be separated I do not know, nor does consideration of a construction lead there.

§9. Now in order that the construction of this equation may be deduced from the preceding, I set the semi-axis AC, which before I denoted by the letter a, equal to r, as it ought to be considered as a variable; and I set a quarter part of the perimeter of the corresponding ellipse equal to q; then $-xx = n = \frac{r^2-b^2}{b^2}$ and $x = \frac{\sqrt{b^2-r^2}}{b}$ will result. Further as $q = es$, and truly $s = c^{\int pdx} = c^{\int \frac{ydx}{x}}$, wherefore $q = ec^{\int \frac{ydx}{x}}$, and $lq - le = \int \frac{y\,dx}{x}$, and even $y = \frac{xdq}{qdx} = \frac{(r^2-b^2)dq}{qrdr}$ will be had. However

[3] *Comm. Acad. Sci. Imp. Petropol.* **6**,1732/3, pp. 68-97. [E25]

[4] *Comm. Acad. Sci. Imp. Petropol.* **3**, 1728, pp. 124-137. [E10]

lest irrationals be produced, when r is larger than b, I put the value $-n$ back in the place of xx, and $\frac{dx}{x} = \frac{dn}{2n}$ and $\frac{xdx}{xx-1} = \frac{dn}{2(n+1)}$ will result. After substituting these, the equation $2dy + \frac{y^2dn}{n} = \frac{dn}{n+1}$ will be had, which will be constructed by taking $n = \frac{r^2-b^2}{b^2}$ and $y = \frac{(r^2-b^2)dq}{qrdr}$ [**Note:** The text reads dy for dq], or now having found n, $y = \frac{2ndq}{qdn}$. Following from here the construction is developed [**Note:** See Figure 2.]: after the quarter ellipse BCA has been drawn, whose center is at C and whose semi-axis BC is a constant set equal to 1, here I put 1 in the place of b, as homogeneity may be observed quite readily. Therefore the semi-axis AC will be equal to r. From A is erected a normal AD equal to the elliptic arc AB. The point B will be on some curve BD, whose construction by this method is clear. Therefore $AD = q$ will result. Let F be the focus of the ellipse, then $CF = \sqrt{r^2-1}$. To BF a normal FP may be drawn, and $CP = r^2 - 1 = n$ [**Note:** The text reads EP for CP] will result. Here it may be noted, when $AC < BC$ and the focus F lies upon BC, that the value of n becomes negative, and that one ought to take B to the other side of the point C. [**Note:** This direction is unclear. In the case where $AC < BC$, several changes to Figure 2 must be made to construct the given differential equation.] Next a tangent DT to the curve BD at D is drawn, and $AT = \frac{qdr}{dq}$ will result. After AP has been drawn, from T a normal line TG is drawn [**Note:** to AP] intersecting AP – extended if necessary – at O, and meeting an extended DA at G. Because of the similar triangles PCA and TAG, $AG = \frac{rqdr}{(r^2-1)dq}$. After setting $CI = CB = 1$, CH may be taken equal to AG itself. Having drawn HI, to it may be erected a perpendicular IK, and $CK = \frac{(r^2-1)dq}{rqdr} = y$ will result. Let PM be equal to this CK, and M will be on the desired curve BM. This curve certainly has the property that after calling $CP = n$ and $PM = y$, $2dy + \frac{y^2dn}{n} = \frac{dn}{n+1}$.

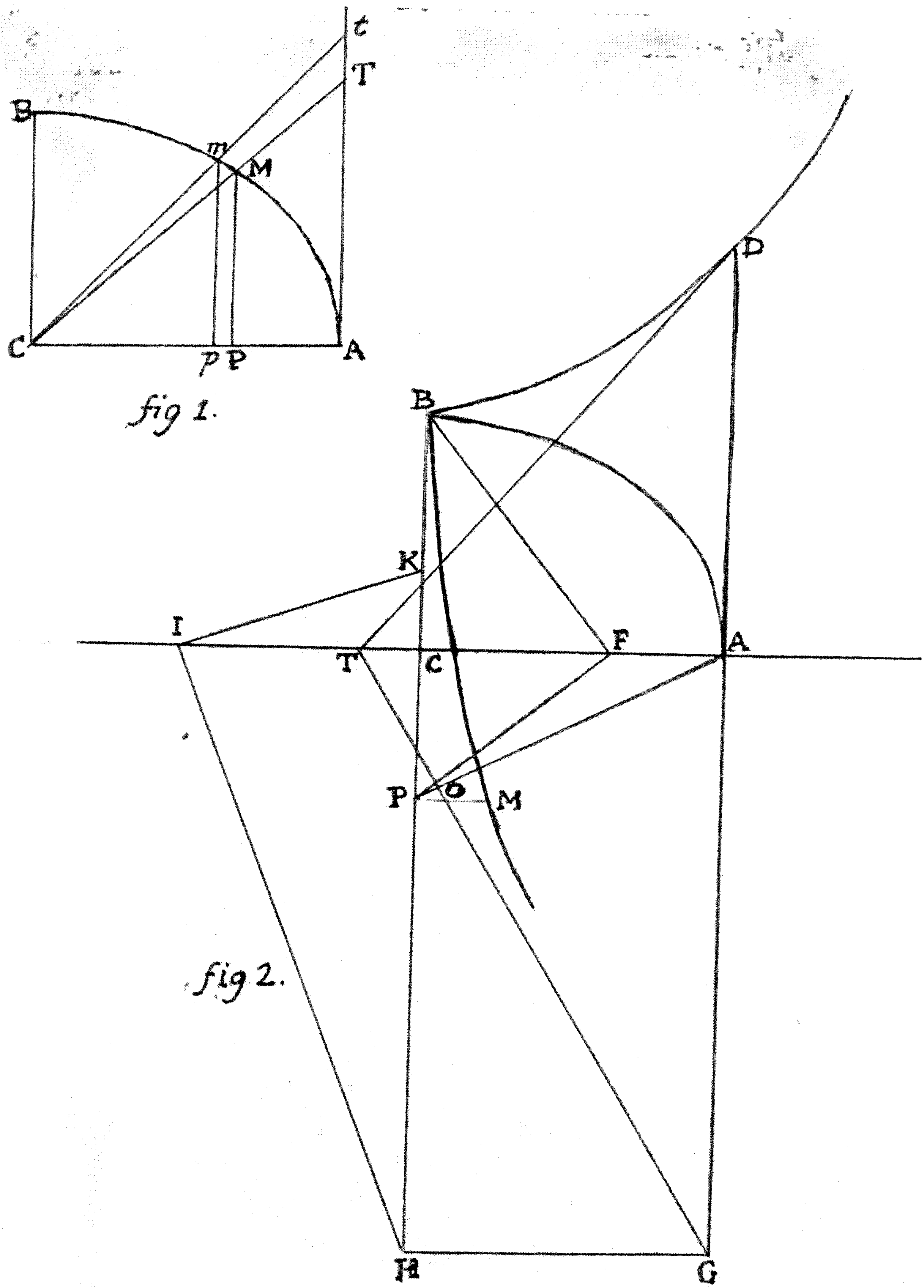

fig 1.

fig 2.