ON TRANSCENDENTAL PROGRESSIONS THAT IS, THOSE WHOSE GENERAL TERMS CANNOT BE GIVEN ALGEBRAICALLY

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SUMMARY

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The theory of progressions of numbers, not merely because of its beauty, but also because of its great utility, deserves the attention of Geometers. The most important problem in this subject is to exhibit, for any given progression, its general term, i.e., that general formula or expression from which any desired term of the progression could be found separately and immediately, apart from the other terms. Now there are some progressions, the general terms of which either cannot be exhibited at all, or at least not by any algebraic expression. Hence the author considers progressions of this type, the general terms of which cannot be given by any algebraic formula. Thus, e.g., one might consider the progression

$$1,\ 2,\ 6,\ 24,\ 120 \text{ etc.},$$

whose general term cannot be exhibited algebraically, yet, remarkably, it happens that some of its intermediate terms depend on the quadrature of a circle, while it is possible to express others algebraically. From this the author concludes that the general term of this progression is a formula such that, although to be sure it is not algebraic, nevertheless in certain cases it becomes algebraic, while in others it presupposes the quadrature of a circle. Now since there are many differential formulas occurring in analysis whose integrals, in general, cannot be determined, but which nevertheless in certain cases are easy to find, it occurred to the author that formulas of this kind could be taken as the general terms of progressions. Taking as given, therefore, formulas of this type, he shows how any desired terms of a progression could be found from them.

1. Recently, prompted by those results concerning series which the celebrated GOLDBACH has communicated to the Society, I was looking for a general expression which would give all the terms of the progression

$$1 + 1\cdot 2 + 1\cdot 2\cdot 3 + 1\cdot 2\cdot 3\cdot 4 + \text{etc.}$$

Supposing that, if continued to infinity, this progression would ultimately behave like a geometric progression, I came upon the following expression

$$\frac{1\cdot 2^n}{1+n}\cdot\frac{2^{1-n}\cdot 3^n}{2+n}\cdot\frac{3^{1-n}\cdot 4^n}{3+n}\cdot\frac{4^{1-n}\cdot 5^n}{4+n}\cdot\text{etc.},$$

which represents the term of order n of the stated progression. This expression, indeed, never breaks off, whether n is a whole number or a fraction, but only gives approximations for any desired term, except in the cases $n = 0$ and $n = 1$, in which it just becomes 1. Let $n = 2$; we will have

$$\frac{2\cdot 2}{1\cdot 3}\cdot\frac{3\cdot 3}{2\cdot 4}\cdot\frac{4\cdot 4}{3\cdot 5}\cdot\frac{5\cdot 5}{4\cdot 6}\cdot\text{etc.} = \text{the second term } 2.$$

If $n = 3$, we will have

$$\frac{2\cdot 2\cdot 2}{1\cdot 1\cdot 4}\cdot\frac{3\cdot 3\cdot 3}{2\cdot 2\cdot 5}\cdot\frac{4\cdot 4\cdot 4}{3\cdot 3\cdot 6}\cdot\frac{5\cdot 5\cdot 5}{4\cdot 4\cdot 7}\cdot\text{etc.} = \text{the third term } 6.$$

2. However, although this expression would not appear to be of any use for computing the terms of the series, nevertheless it is wonderfully suitable for interpolating terms whose indices are fractional numbers. I have decided, however, not to expound the matter in this way, since more convenient methods for this purpose will present themselves below. I will just consider that general term to the extent that will be needed for what follows. I sought the term of index $n = \frac{1}{2}$, that is, the term which lies equally between the first, 1, and the preceding, which is also 1. But setting $n = \frac{1}{2}$, I got the series

$$\sqrt{\frac{2 \cdot 4}{3 \cdot 3} \cdot \frac{4 \cdot 6}{5 \cdot 5} \cdot \frac{6 \cdot 8}{7 \cdot 7} \cdot \frac{8 \cdot 10}{9 \cdot 9}} \cdot \text{etc.},$$

which expresses the required term. But this series immediately appeared to me to be similar to that one for the area of a circle which I recalled having seen in the works of WALLIS. For WALLIS found that the circle was to the square of its diameter as

$$2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot 8 \cdot 8 \cdot 10 \cdot \text{etc.} \quad \text{to} \quad 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdot 7 \cdot 9 \cdot 9 \cdot \text{etc.}$$

If therefore the diameter $= 1$, the area of the circle will be

$$= \frac{2 \cdot 4}{3 \cdot 3} \cdot \frac{4 \cdot 6}{5 \cdot 5} \cdot \frac{6 \cdot 8}{7 \cdot 7} \cdot \text{etc.}$$

From this correspondence with my series, therefore, we can conclude that the term of index $\frac{1}{2}$ is equal to the square root of the circle whose diameter $= 1$.

3. I had previously supposed that the general term of the series 1, 2, 6, 24, etc. could be given, if not algebraically, at least exponentially. But after I had seen that some intermediate terms depended on the quadrature of a circle, I recognized that neither algebraic nor exponential quantities were suitable for expressing it. For the general term of that progression must thus include not only algebraic quantities but also those depending on the quadrature of a circle, and perhaps even on other quadratures; thus it could not be represented either by any algebraic formula, or by an exponential.

4. Since I knew, however, of formulas involving differential quantities, which indeed in some cases could be integrated, and then produce algebraic quantities, in other cases however could not, and then represent quantities depending on the quadratures of curves, it occurred to me that perhaps there were formulas of this kind, suitable for expressing the general terms of the aforementioned progression, as well as of other similar ones. Indeed, progressions whose general terms cannot be expressed algebraically, I call *transcendental*; just as everything in Geometry which surpasses the powers of ordinary Algebra is commonly called transcendental.

5. I therefore asked myself in what way differential formulas would be best suited to express the general terms of progressions. Now a general term is a formula involving not only constant quantities, but also some other non-constant quantity, say n, which gives the order or index of the terms; thus, if the third term is wanted, one should put 3 in place of n. But a differential formula must contain some variable quantity. It would not make sense, of course, to take n for this quantity; n is not the variable of integration, but after the formula has been integrated, or after its integration has been indicated, then n should serve to express the formation of the progression. Thus, a differential formula has to contain some variable quantity x, which however after integration must be set equal to some other quantity related to the progression; and this is how we get the particular term of index n.

6. In order to make this more clearly understood, I let $\int p\,dx$ be the general term of a progression, which is to be derived from it in the following way; here p stands for any function of x and of constants, among which n itself must also occur. Suppose $p\,dx$ integrated, taking the constant of integration so that when $x = 0$, the whole integral vanishes; then let x be taken equal to some known quantity. Only quantities belonging to the progression will remain in the resulting integral, and it will express the term of index $= n$. Thus the integral determined in this way will properly be the general term. To be sure, if this can be done, then there is no need for the differential formula, for the progression formed in this way will have an algebraic general term; in other cases it may happen that the integration cannot be done unless particular numbers are substituted for n.

7. I took, therefore, several differential formulas of this kind, which could not be integrated unless one substituted a positive whole number in place of n, so that the principal terms of the series would become algebraic, and in this way I generated the progressions. Their general terms will thus be evident, and it will be possible to determine the quadrature on which all their intermediate terms would depend. Here, to be sure, I do not propose to run through several formulas of this type, but I will pick a particular one which is sufficiently general as to make very clear how all those progressions can be dealt with in which a typical term is composed of a number of factors depending on the index; all these factors are fractions whose numerators and denominators proceed in some arithmetic progression, such as

$$\frac{2}{3} + \frac{2 \cdot 4}{3 \cdot 5} + \frac{2 \cdot 4 \cdot 6}{3 \cdot 5 \cdot 7} + \frac{2 \cdot 4 \cdot 6 \cdot 8}{3 \cdot 5 \cdot 7 \cdot 9} + \text{etc.}$$

8. Let the formula

$$\int x^e \, dx \, (1-x)^n$$

stand for the general term, and let it be integrated so that it becomes $= 0$ if $x = 0$; and then, by setting $x = 1$, let it give the term of order n of the resulting progression. Let us see, therefore, what kind of progression it produces. We have

$$(1-x)^n = 1 - \frac{n}{1}x + \frac{n(n-1)}{1 \cdot 2}x^2 - \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}x^3 + \text{etc.}$$

and therefore

$$x^e \, dx \, (1-x)^n = x^e \, dx - \frac{n}{1}x^{e+1} \, dx + \frac{n(n-1)}{1 \cdot 2}x^{e+2} \, dx - \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}x^{e+3} \, dx + \text{etc.}$$

Consequently,

$$\int x^e \, dx \, (1-x)^n = \frac{x^{e+1}}{e+1} - \frac{nx^{e+2}}{1 \cdot (e+2)} + \frac{n(n-1)x^{e+3}}{1 \cdot 2 \cdot (e+3)} - \frac{n(n-1)(n-2)x^{e+4}}{1 \cdot 2 \cdot 3 \cdot (e+4)} + \text{etc.}$$

Setting $x = 1$, since the addition of a constant is not necessary, we will have

$$\frac{1}{e+1} - \frac{n}{1 \cdot (e+2)} + \frac{n(n-1)}{1 \cdot 2 \cdot (e+3)} - \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3 \cdot (e+4)} + \text{etc.},$$

the general term of the series which was to be found. Thus, if $n = 0$, it will produce the term $= \frac{1}{e+1}$; if $n = 1$, the term $= \frac{1}{(e+1)(e+2)}$; if $n = 2$, the term $\frac{1 \cdot 2}{(e+1)(e+2)(e+3)}$; if $n = 3$, it gives the term $= \frac{1 \cdot 2 \cdot 3}{(e+1)(e+2)(e+3)(e+4)}$; the rule which these terms follow is obvious.

9. Thus I arrived at the progression

$$\frac{1}{(e+1)(e+2)} + \frac{1 \cdot 2}{(e+1)(e+2)(e+3)} + \frac{1 \cdot 2 \cdot 3}{(e+1)(e+2)(e+3)(e+4)} + \text{etc.},$$

whose general term is

$$\int x^e \, dx \, (1-x)^n.$$

The term of order n of this progression will then be of the form

$$\frac{1 \cdot 2 \cdot 3 \cdot 4 \cdots n}{(e+1)(e+2) \cdots (e+n+1)}.$$

This form enables us, at any rate, to find the terms of integral index, but if the indices are not integral, then the corresponding terms cannot be obtained from it. To get those, it is better to use the series

$$\frac{1}{e+1} - \frac{n}{1 \cdot (e+2)} + \frac{n(n-1)}{1 \cdot 2 \cdot (e+3)} - \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3 \cdot (e+4)} + \text{etc.}$$

If $\int x^e\,dx\,(1-x)^n$ is multiplied by $e+n+1$, we will have a progression whose term of order n has the form

$$\frac{1\cdot 2\cdot 3\cdots n}{(e+1)(e+2)\cdots(e+n)},$$

of which, therefore, the actual general term will be

$$(e+n+1)\int x^e\,dx\,(1-x)^n.$$

Here it should be noted that the progression will always become algebraic when e is replaced by a positive [whole] number. Let, e. g., $e=2$; the n^{th} term of the progression will be

$$\frac{1\cdot 2\cdot 3\cdots n}{3\cdot 4\cdot 5\cdots(n+2)} \quad \text{or} \quad \frac{1\cdot 2}{(n+1)(n+2)}.$$

We can also write down the corresponding general term, which will be

$$(n+3)\int xx\,dx\,(1-x)^n.$$

But the integral of this is

$$\left(C-\frac{(1-x)^{n+1}}{n+1}+\frac{2(1-x)^{n+2}}{n+2}-\frac{(1-x)^{n+3}}{n+3}\right)(n+3);$$

setting this $=0$, when $x=0$, we get

$$C=\frac{1}{n+1}-\frac{2}{n+2}+\frac{1}{n+3}.$$

Let $x=1$; the general term will be

$$\frac{n+3}{n+1}-\frac{2(n+3)}{n+2}+1=\frac{2}{(n+1)(n+2)}.$$

10. Therefore, in order that we might obtain a transcendental progression, let e be equal to a fraction $\frac{f}{g}$. The term of order n of the progression will be

$$=\frac{1\cdot 2\cdot 3\cdots n}{(f+g)(f+2g)(f+3g)\cdots(f+ng)}g^n$$

that is

$$\frac{g\cdot 2g\cdot 3g\cdots ng}{(f+g)(f+2g)(f+3g)\cdots(f+ng)}.$$

And the general term will be

$$=\frac{f+(n+1)g}{g}\int x^{\frac{f}{g}}\,dx\,(1-x)^n.$$

Dividing by g^n, the progression becomes

$$\frac{1}{f+g}+\frac{1\cdot 2}{(f+g)(f+2g)}+\frac{1\cdot 2\cdot 3}{(f+g)(f+2g)(f+3g)}+\text{etc.},$$

of which the term of order n is

$$\frac{1\cdot 2\cdot 3\cdots n}{(f+g)(f+2g)\cdots(f+ng)}.$$

Therefore the general term of the progression will be

$$= \frac{f+(n+1)g}{g^{n+1}} \int x^{\frac{f}{g}}\, dx\, (1-x)^n.$$

Here, if the fraction $\frac{f}{g}$ is not equal to a whole number, that is if the ratio of f to g is not that of a multiple, then the progression will be transcendental and the intermediate terms will depend on quadratures.

11. Let me put an example in evidence, in order to bring the use of the general term more clearly before the eye. In the first progression of the preceding paragraph, let $f = 1$, $g = 2$; the term of order n will be

$$= \frac{2 \cdot 4 \cdot 6 \cdot 8 \cdots 2n}{3 \cdot 5 \cdot 7 \cdot 9 \cdots (2n+1)},$$

so the progression itself will be

$$\frac{2}{3} + \frac{2 \cdot 4}{3 \cdot 5} + \frac{2 \cdot 4 \cdot 6}{3 \cdot 5 \cdot 7} + \text{etc.},$$

of which the general term will therefore be

$$\frac{2n+3}{2} \int dx\, (1-x)^n \sqrt{x}.$$

Let the term of index $\frac{1}{2}$ be required; therefore $n = \frac{1}{2}$, and we will have the required term

$$= 2 \int dx \sqrt{(x - xx)}.$$

Since the element of integration is that of the area of a circle, it is clear that the required term is the area of a circle of diameter $= 1$.

Next consider the series

$$1 + \frac{r}{1} + \frac{r(r-1)}{1 \cdot 2} + \frac{r(r-1)(r-2)}{1 \cdot 2 \cdot 3} + \text{etc.},$$

which is the series of coefficients of a binomial raised to the power r. The term of order n is therefore

$$\frac{r(r-1)(r-2)\cdots(r-n+2)}{1 \cdot 2 \cdot 3 \cdots (n-1)}.$$

In the preceding § we had this term

$$\frac{1 \cdot 2 \cdot 3 \cdots n}{(f+g)(f+2g)\cdots(f+ng)}.$$

In order to compare this with the previous one, it is necessary to invert, whence we get

$$\frac{(f+g)(f+2g)\cdots(f+ng)}{1 \cdot 2 \cdots n};$$

multiplying by $\frac{n}{f+ng}$ this will be

$$\frac{(f+g)(f+2g)\cdots(f+(n-1)g)}{1 \cdot 2 \cdots (n-1)};$$

we should therefore take $f + g = r$ and $f + 2g = r - 1$, whence $g = -1$ and $f = r + 1$. The general term

$$\frac{f+(n+1)g}{g^{n+1}} \int x^{\frac{f}{g}}\, dx\, (1-x)^n$$

is to be treated in the same way. For the proposed progression

$$1 + \frac{r}{1} + \frac{r(r-1)}{1 \cdot 2} + \text{etc.}$$

this general term

$$\frac{n(-1)^{n+1}}{(r-n)(r-n+1)\int x^{-r-1}dx\,(1-x)^n}$$

will result. Let $r = 2$; the general term of this progression

$$1,\ 2,\ 1,\ 0,\ 0,\ 0\ \text{ etc.}$$

will be

$$\frac{n(-1)^{n+1}}{(2-n)(3-n)\int x^{-3}dx\,(1-x)^n}.$$

Here, however, it is necessary to point out that this case, and others in which $e+1$ becomes a negative number, cannot be deduced from the general case, since in these cases the integral does not become $= 0$, when $x = 0$. In fact, for these integrals

$$\int x^e\,dx\,(1-x)^n$$

a peculiar method of integration is appropriate; for after integrating we should add an infinite constant. But when $e+1$ is a positive number, as I assumed in §8, adding a constant is not necessary. In the case, however, of the progression whose term of order n was the following

$$\frac{r(r-1)(r-2)\cdots(r-n+2)}{1 \cdot 2 \cdot 3 \cdots (n-1)},$$

it is possible to change that form of the term of exponent n into

$$\frac{r(r-1)\cdots 1}{\big(1 \cdot 2 \cdot 3 \cdots (n-1)\big)\big(1 \cdot 2 \cdots (r-n+1)\big)}.$$

But from §14 we have

$$r(r-1)\cdots 1 = \int dx\,(-lx)^r$$

and

$$1 \cdot 2 \cdot 3 \cdots (n-1) = \int dx\,(-lx)^{n-1}$$

and

$$1 \cdot 2 \cdots (r-n+1) = \int dx\,(-lx)^{r-n+1}.$$

Whence for the progression under discussion

$$1 + \frac{r}{1} + \frac{r(r-1)}{1 \cdot 2} + \frac{r(r-1)(r-2)}{1 \cdot 2 \cdot 3} + \text{etc.}$$

we will get this general term

$$\frac{\int dx\,(-lx)^r}{\int dx\,(-lx)^{n-1}\int dx\,(-lx)^{r-n+1}}.$$

If $r = 2$, the general term will be

$$\frac{2}{\int dx\,(-lx)^{n-1}\int dx\,(-lx)^{3-n}},$$

to which corresponds the progression

$$1,\ 2,\ 1,\ 0,\ 0,\ 0\ \text{ etc.;}$$

and if we ask for the term of index $\frac{3}{2}$, it will be

$$\frac{2}{\int dx\,(-lx)^{\frac{1}{2}}\int dx\,(-lx)^{\frac{3}{2}}}.$$

Now let the area of a circle of diameter $= 1$ be called A, so that

$$\int dx\,(-lx)^{\frac{1}{2}} = \sqrt{A} \quad\text{and}\quad \int dx\,(-lx)^{\frac{3}{2}} = \tfrac{3}{2}\sqrt{A};$$

the term falling halfway between the two first terms of the progression 1, 2, 1, 0, 0, 0, etc. will be of the form $\frac{4}{3A}$, which is approximately $\frac{5}{3}$.

12. I now go on to the progression which I mentioned at the beginning,

$$1 + 1\cdot 2 + 1\cdot 2\cdot 3 + \text{etc.},$$

and in which the term of order n is $1\cdot 2\cdot 3\cdot 4\cdots n$. This sequence belongs to the kind we have been considering, but the general term must be derived in a peculiar way. Recall, now, that if the term of order n is

$$\frac{1\cdot 2\cdot 3\cdots n}{(f+g)(f+2g)\cdots(f+ng)},$$

I found the corresponding general term; setting $f = 1$ and $g = 0$, this becomes $1\cdot 2\cdot 3\cdots n$, and we now require its general term; therefore in the general term

$$\frac{f+(n+1)g}{g^{n+1}}\int x^{\frac{f}{g}}\,dx\,(1-x)^n$$

let these values be substituted for f and g; the required general term will be

$$\int \frac{x^{\frac{1}{0}}\,dx\,(1-x)^n}{0^{n+1}}.$$

And what, indeed, the value of this expression might be, I investigate in the following way.

13. From the rules for setting up general terms of this type, we see that x can be replaced by other functions of x, provided that they are $= 0$ when $x = 0$ and $= 1$ when $x = 1$. For if functions of this kind are substituted for x, the general term will be just as good as before. Therefore put $x^{\frac{g}{f+g}}$ in place of x, and consequently $\frac{g}{f+g}x^{\frac{-f}{g+f}}\,dx$ in place of dx, so that we will have

$$\frac{f+(n+1)g}{g^{n+1}}\int \frac{g}{f+g}\,dx\,\left(1-x^{\frac{g}{f+g}}\right)^n.$$

Now let $f = 1$ and $g = 0$; we will have

$$\int \frac{dx\,(1-x^0)^n}{0^n}.$$

Since, however, $x^0 = 1$, we have here the case in which the numerator and denominator vanish, $(1-x^0)^n$ and 0^n. By applying a known rule, therefore, let us determine the value of the fraction $\frac{1-x^0}{0}$. The value we are seeking will be that assumed by the fraction $\frac{1-x^z}{z}$ when z vanishes; therefore differentiate the numerator and denominator, taking only z as variable; we will have $\frac{-x^z\,dz\,lx}{dz}$ or $-x^z\,lx$; if now we let $z = 0$, we get $-lx$. Thus

$$\frac{1-x^0}{0} = -lx.$$

14. Since therefore

$$\frac{1-x^0}{0}=-lx,$$

it follows that

$$\frac{(1-x^0)^n}{0^n}=(-lx)^n$$

and consequently the required general term $\int \frac{dx\,(1-x^0)^n}{0^n}$ has been transformed into $\int dx\,(-lx)^n$. The value of this can be determined by quadratures. Consequently the general term of the sequence

$$1,\ 2,\ 6,\ 24,\ 120,\ 720\ \text{ etc.}$$

is

$$\int dx\,(-lx)^n,$$

which is to be computed in the way which was explained previously. It can also be seen that this is the general term of the proposed sequence, in that it correctly reproduces the terms whose indices are positive whole numbers. For example, let $n=3$; we get

$$\int dx\,(-lx)^3=\int -dx\,(lx)^3=-x(lx)^3+3x(lx)^2-6x\,lx+6x;$$

here it is not necessary to add a constant, since everything vanishes when $x=0$; therefore put $x=1$; since $l1=0$, all the terms involving logarithms will vanish and what will remain is 6, which is the third term.

15. It is true that this method of obtaining the terms of that series is much too tedious, at least for those terms whose indices are whole numbers, which at any rate can be obtained more easily just by continuing the progression. Nevertheless, it is very useful for getting the terms of fractional index, especially since, up to this time, not even the most tedious method has been able to do this. If we set $x=\frac{1}{2}$,[1)] the corresponding term will be $=\int dx\,\sqrt{-lx}$, the value of which is given by quadratures. But at the beginning [§11] I claimed that this term is equal to the square root of a circle whose diameter is 1. Up to this point, indeed, it is not permissible to draw that conclusion, because of the shortcomings of analysis; below, however, will appear a method by which those intermediate terms can be reduced to the quadratures of algebraic curves. By putting these things together, it may be that no little improvement of analysis will result.

16. The general term of the progression whose term of order n is given by

$$\frac{1\cdot 2\cdot 3\cdots n}{(f+g)(f+2g)(f+3g)\cdots(f+ng)},$$

is, according to §10,

$$\frac{f+(n+1)g}{g^{n+1}}\int x^{\frac{f}{g}}\,dx\,(1-x)^n.$$

But if the term of order n is

$$1\cdot 2\cdot 3\cdots n,$$

then the general term is

$$\int dx\,(-lx)^n.$$

If this formula is substituted in place of $1\cdot 2\cdot 3\cdots n$, we will have

$$\frac{\int dx\,(-lx)^n}{(f+g)(f+2g)\cdots(f+ng)}=\frac{f+(n+1)g}{g^{n+1}}\int x^{\frac{f}{g}}\,dx\,(1-x)^n.$$

[1)] Evidently a slip for ‘$n=\frac{1}{2}$’. *Tr.*

From this it follows that

$$(f+g)(f+2g)\cdots(f+ng)=\frac{g^{n+1}\int dx\,(-lx)^n}{(f+(n+1)g)\int x^{\frac{f}{g}}dx\,(1-x)^n}.$$

This expression is therefore the general term of the general progression

$$f+g,\ (f+g)(f+2g),\ (f+g)(f+2g)(f+3g)\ \text{ etc.}$$

In this way, therefore, by means of the general term, all the terms of whatever index are defined in any progression. This will follow below from the reduction of $\int dx\,(-lx)^n$ to well-known quadratures of algebraic curves, and will also be of use here.

17. Let $f+g=1$ and $f+2g=3$; so that $g=2$ and $f=-1$. From this will arise the particular progression

$$1,\ 1\cdot 3,\ 1\cdot 3\cdot 5,\ 1\cdot 3\cdot 5\cdot 7\ \text{ etc.}$$

Therefore its general term is

$$\frac{2^{n+1}\int dx\,(-lx)^n}{(2n+1)\int x^{-\frac{1}{2}}dx\,(1-x)^n}.$$

Although the exponent of x here is negative, nevertheless the difficulty which was mentioned above does not occur here, since it is less than unity. Take $n=\frac{1}{2}$, in order to get the term of order $\frac{1}{2}$; it will be

$$=\frac{2^{\frac{3}{2}}\int dx\,\sqrt{-lx}}{2\int x^{-\frac{1}{2}}dx\,\sqrt{(1-x)}}=\frac{\sqrt{2}\cdot\int dx\,\sqrt{-lx}}{\displaystyle\int\frac{dx-x\,dx}{\sqrt{(x-xx)}}}.$$

In §15, however, it was stated that $\int dx\,\sqrt{-lx}$ gives the square root of a circle whose diameter $=1$; let the circumference of this circle be p; the area will be $=\frac{1}{4}p$, so that $\int dx\,\sqrt{-lx}$ gives $\frac{1}{2}\sqrt{p}$. Next

$$\int\frac{dx-x\,dx}{\sqrt{(x-xx)}}=\int\frac{dx}{2\sqrt{(x-xx)}}+\sqrt{(x-xx)};$$

but $\int\frac{dx}{2\sqrt{(x-xx)}}$ gives the arc of a circle whose versed sine is x. Therefore setting $x=1$ we will get $\frac{1}{2}p$. It follows that the desired term will be

$$=\sqrt{\frac{2}{p}}.$$

18. Since the general term of the progression whose term of order n is given by

$$(f+g)(f+2g)\cdots(f+ng)$$

is, by §16,

$$\frac{g^{n+1}\int dx\,(-lx)^n}{(f+(n+1)g)\int x^{\frac{f}{g}}dx\,(1-x)^n},$$

similarly, if the term of order n is

$$(h+k)(h+2k)\cdots(h+nk),$$

the general term will be

$$\frac{k^{n+1}\int dx\,(-lx)^n}{(h+(n+1)k)\int x^{\frac{h}{k}}dx\,(1-x)^n}.$$

Let the previous progression be divided by this one, that is to say, the first term by the first, the second by the second, and so forth; the result will be a new progression, whose term of order n will be

$$\frac{(f+g)(f+2g)\cdots(f+ng)}{(h+k)(h+2k)\cdots(h+nk)}.$$

And the general term of this progression, composed of those two, will be

$$\frac{g^{n+1}(h+(n+1)k)\int x^{\frac{h}{k}}\,dx\,(1-x)^n}{k^{n+1}(f+(n+1)g)\int x^{\frac{f}{g}}\,dx\,(1-x)^n}.$$

This no longer contains the logarithmic integral $\int dx\,(-lx)^n$.

19. In all the general terms of this kind, it must be particularly stressed that not only may constant numbers be substituted for f, g, h, k, but that they can also assume values which depend on n in any way. For in the integration these letters, just like n, are all to be treated as constants. Let the term of order n be

$$(f+g)(f+2g)\cdots(f+ng);$$

take $g=1$, but $f=\frac{nn-n}{2}$. Since the corresponding progression is

$$f+g,\ (f+g)(f+2g),\ (f+g)(f+2g)(f+3g)\ \text{ etc.},$$

put everywhere 1 in place of g; the result will be

$$f+1,\ (f+1)(f+2),\ (f+1)(f+2)(f+3)\ \text{ etc.}$$

But in place of f is to be written in the first term 0, in the second 1, in the third 3, in the fourth 6, and so forth; thus we get the progression

$$1,\ 2\cdot 3,\ 4\cdot 5\cdot 6,\ 7\cdot 8\cdot 9\cdot 10\ \text{ etc.},$$

of which therefore the general term is

$$\frac{2\int dx\,(-lx)^n}{(nn+n+2)\int x^{\frac{nn-n}{2}}dx\,(1-x)^n}=\frac{2\int dx\,(-lx)^n}{(nn+n+2)\int dx\,\left(x^{\frac{n-1}{2}}-x^{\frac{n+1}{2}}\right)^n}.$$

20. I come now to those progressions by means of which I found that handy method of determining the intermediate terms of the progression

$$1,\ 2,\ 6,\ 24,\ 120\ \text{ etc.}$$

Indeed the method applies more generally than just to that progression only, since the general term

$$\int dx\,(-lx)^n$$

occurs in the general terms of infinitely many other progressions. I take the general term

$$\frac{f+(n+1)g}{g^{n+1}}\int x^{\frac{f}{g}}\,dx\,(1-x)^n,$$

to which corresponds this term of order n

$$\frac{1\cdot 2\cdot 3\cdots n}{(f+g)(f+2g)(f+3g)\cdots(f+ng)}.$$

Here I put $f = n$, $g = 1$; the general term will be

$$(2n+1)\int x^n\,dx\,(1-x)^n \quad \text{or} \quad (2n+1)\int dx\,(x-xx)^n$$

and its form of order n will be

$$\frac{1\cdot 2\cdot 3\cdots n}{(n+1)(n+2)(n+3)\cdots 2n}.$$

So the corresponding progression is

$$\frac{1}{2},\ \frac{1\cdot 2}{3\cdot 4},\ \frac{1\cdot 2\cdot 3}{4\cdot 5\cdot 6}\ \text{ etc.}$$

or

$$\frac{1\cdot 1}{1\cdot 2},\ \frac{1\cdot 2\cdot 1\cdot 2}{1\cdot 2\cdot 3\cdot 4},\ \frac{1\cdot 2\cdot 3\cdot 1\cdot 2\cdot 3}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}\ \text{ etc.}$$

In this, the numerators are the squares of the progression 1, 2, 6, 24 etc., while between two consecutive denominators it is easy to interpolate an equidistant value. Let A be the term of index $\frac{1}{2}$ in the progression 1, 2, 6, 24 etc.; then the term of order $\frac{1}{2}$ of the above progression will be $= \frac{AA}{1}$.

21. In the general term

$$(2n+1)\int x^n\,dx\,(1-x)^n$$

let $n = \frac{1}{2}$; the term of this exponent will be

$$= 2\int dx\,\sqrt{(x-xx)} = \frac{AA}{1},$$

whence

$$A = \sqrt{1\cdot 2\int dx\,\sqrt{(x-xx)}}$$

= the term of the progression 1, 2, 6, 24 etc., whose index is $\frac{1}{2}$, which therefore, as is clear from this, is the square root of the circle of diameter 1. Now let A stand for the term of this progression of order $\frac{3}{2}$; the corresponding term in the given progression will be

$$= \frac{AA}{1\cdot 2\cdot 3} = 4\int dx\,(x-xx)^{\frac{3}{2}},$$

therefore

$$A = \sqrt{1\cdot 2\cdot 3\cdot 4\int dx\,(x-xx)^{\frac{3}{2}}}.$$

In a similar way we can find that the term of order $\frac{5}{2}$

$$= \sqrt{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\int dx\,(x-xx)^{\frac{5}{2}}}.$$

From this I conclude generally that the term of order $\frac{p}{2}$ will be

$$= \sqrt{1\cdot 2\cdot 3\cdot 4\cdots (p+1)\int dx\,(x-xx)^{\frac{p}{2}}}.$$

In this way are found all the terms of the progression 1, 2, 6, 24 etc., whose indices are fractions in which the denominator is 2.

22. Next, in the general term

$$\frac{f+(n+1)g}{g^{n+1}}\int x^{\frac{f}{g}}\,dx\,(1-x)^n$$

I put $f = 2n$, keeping $g = 1$; this will give the general term

$$(3n+1)\int dx\,(xx - x^3)^n$$

of the progression

$$\frac{1}{3},\ \frac{1\cdot 2}{5\cdot 6},\ \frac{1\cdot 2\cdot 3}{7\cdot 8\cdot 9}\ \text{ etc.}$$

Let this be multiplied by the preceding $(2n+1)\int dx\,(x-xx)^n$; the result is

$$(2n+1)(3n+1)\int dx\,(x-xx)^n \int dx\,(xx-x^3)^n.$$

This will give the progression

$$\frac{1\cdot 1\cdot 1}{1\cdot 2\cdot 3},\ \frac{1\cdot 2\cdot 1\cdot 2\cdot 1\cdot 2}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}\ \text{etc.},$$

where the numerators are the cubes of the corresponding terms of the progression 1, 2, 6, 24 etc. Let A be the term of this progression of order $\frac{1}{3}$; the corresponding term of the previous one will be

$$\frac{A^3}{1} = 2\left(\frac{2}{3}+1\right)\int dx\,(x-xx)^{\frac{1}{3}} \int dx\,(xx-x^3)^{\frac{1}{3}},$$

therefore the term of order $\frac{1}{3}$ is

$$\sqrt[3]{1\cdot 2\cdot \frac{5}{3}\int dx\,(x-xx)^{\frac{1}{3}} \int dx\,(xx-x^3)^{\frac{1}{3}}};$$

similarly the term of order $\frac{2}{3}$ is

$$\sqrt[3]{1\cdot 2\cdot 3\cdot \frac{7}{3}\int dx\,(x-xx)^{\frac{2}{3}} \int dx\,(xx-x^3)^{\frac{2}{3}}}.$$

And the term of order $\frac{4}{3}$ is

$$\sqrt[3]{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot \frac{11}{3}\int dx\,(x-xx)^{\frac{4}{3}} \int dx\,(xx-x^3)^{\frac{4}{3}}}$$

and in general the term of order $\frac{p}{3}$ is[1)]

$$\sqrt{1\cdot 2\cdots p\cdot \frac{2p+3}{3}\cdot (p+1)\int dx\,(x-xx)^{\frac{p}{3}} \int dx\,(xx-x^3)^{\frac{p}{3}}}.$$

23. If we wish to proceed further by letting $f = 3n$, it will be necessary to multiply the general term

$$(4n+1)\int dx\,(x^3-x^4)^n$$

by the preceding, whence we will have

$$(2n+1)(3n+1)(4n+1)\int dx\,(x-xx)^n \int dx\,(x^2-x^3)^n \int dx\,(x^3-x^4)^n,$$

[1)] The square root should of course be a cube root. *Tr.*

corresponding to the series

$$\frac{1\cdot 1\cdot 1\cdot 1}{1\cdot 2\cdot 3\cdot 4},\ \frac{1\cdot 2\cdot 1\cdot 2\cdot 1\cdot 2\cdot 1\cdot 2}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8}\quad \text{etc.}$$

From this will be defined the terms of the progression 1, 2, 6, 24 etc., whose indices are fractions having denominator 4. Indeed, the term whose index is $\frac{p}{4}$ will be found to be

$$= \sqrt[4]{1\cdot 2\cdot 3\cdots p\left(\frac{2p}{4}+1\right)\left(\frac{3p}{4}+1\right)(p+1)}$$
$$\times \int dx\,(x-xx)^{\frac{p}{4}} \int dx\,(xx-x^3)^{\frac{p}{4}} \int dx\,(x^3-x^4)^{\frac{p}{4}}.$$

Hence we can conclude in general that the term of order $\frac{p}{q}$ is

$$= \sqrt[q]{1\cdot 2\cdot 3\cdots p\left(\frac{2p}{q}+1\right)\left(\frac{3p}{q}+1\right)\left(\frac{4p}{q}+1\right)\cdots(p+1)}$$
$$\times \int dx\,(x-xx)^{\frac{p}{q}} \int dx\,(xx-x^3)^{\frac{p}{q}} \int dx\,(x^3-x^4)^{\frac{p}{q}} \cdots \int dx\,(x^{q-1}-x^q)^{\frac{p}{q}}.$$

From this formula, therefore, the terms of any fractional index whatever can be found by quadratures of algebraic curves; however, to do this it is necessary to have $1\cdot 2\cdot 3\cdots p$, the term whose index is the numerator of the proposed fraction.

24. One could proceed further in the same manner to more complex progressions, by taking more complex general terms, but I will not pursue these things any further. It is possible, indeed, to multiply the integral signs, so that the general term would be

$$\int q\,dx \int p\,dx;$$

clearly the integral of $p\,dx$ must be multiplied by $q\,dx$, and the result once again integrated, so that the term of the series will be given, finally, by setting $x=1$. In order, however, to determine each of these integrals, it is necessary to add a constant, chosen so that when $x=0$ the integral will also be $=0$.

General terms which contain several integral signs, such as

$$\int r\,dx \int q\,dx \int p\,dx,$$

are to be treated similarly. But still the functions to be taken in place of p, q, r etc., should always be such that, whenever n is a positive whole number, the terms which are produced are at least algebraic.

25. Let the general term be

$$\int \frac{dx}{x} \int x^e\,dx\,(1-x)^n;$$

converted into a series, this gives

$$\frac{x^{e+1}}{(e+1)^2} - \frac{nx^{e+2}}{1\cdot(e+2)^2} + \frac{n(n-1)x^{e+3}}{1\cdot 2\cdot(e+3)^2} - \text{etc.}$$

Setting $x=1$, we will have the term of order n

$$\frac{1}{(e+1)^2} - \frac{n}{1\cdot(e+2)^2} + \frac{n(n-1)}{1\cdot 2\cdot(e+3)^2} - \text{etc.}$$

of this series. In fact, the progression itself, starting from the term of index 0, will be

$$\frac{1}{(e+1)^2},\ \frac{(e+2)^2-(e+1)^2}{(e+2)^2(e+1)^2},\ \frac{(e+3)^2(e+2)^2-2(e+3)^2(e+1)^2+(e+2)^2(e+1)^2}{(e+3)^2(e+2)^2(e+1)^2},$$
$$\frac{(e+4)^2(e+3)^2(e+2)^2-3(e+4)^2(e+3)^2(e+1)^2+3(e+4)^2(e+2)^2(e+1)^2-(e+3)^2(e+2)^2(e+1)^2}{(e+4)^2(e+3)^2(e+2)^2(e+1)^2}$$

etc.

The rule of this progression is evident, and does not require explanation. Let $e = 0$; then we get

$$\int dx\,(1-x)^n = \frac{1-(1-x)^{n+1}}{n+1};$$

therefore the general term is

$$\int \frac{dx - dx\,(1-x)^{n+1}}{(n+1)x},$$

and the corresponding progression will be

$$\frac{1}{1},\ \frac{4-1}{4\cdot 1},\ \frac{9\cdot 4 - 2\cdot 9\cdot 1 + 4\cdot 1}{9\cdot 4\cdot 1},\ \frac{16\cdot 9\cdot 4 - 3\cdot 16\cdot 9\cdot 1 + 3\cdot 16\cdot 4\cdot 1 - 9\cdot 4\cdot 1}{16\cdot 9\cdot 4\cdot 1}\quad \text{etc.}$$

The differences of these constitute the progression

$$\frac{-1}{4\cdot 1},\ \frac{-9+4}{9\cdot 4\cdot 1},\ \frac{-16\cdot 9 + 2\cdot 16\cdot 4 - 9\cdot 4}{16\cdot 9\cdot 4\cdot 1}\quad \text{etc.}$$

26. In this essay I have thus accomplished what I primarily intended to do, namely, to find the general terms of all progressions whose individual terms are formed of factors in arithmetic progression, and in which the number of the factors depends in any way on the indices of the terms. And although in every case studied here the number of factors was equal to the index, nevertheless, if some other form of dependence is of interest, there is no difficulty in the matter. The index has been denoted by the letter n; if now someone requires that the number of factors should be $\frac{nn+n}{2}$, it is not necessary to do anything other than everywhere to replace n by $\frac{nn+n}{2}$.

27. To round off this discussion, let me add something which certainly is more curious than useful. It is known that $d^n x$ denotes the differential of x of order n, and if p denotes any function of x and dx is taken to be constant, then $d^n p$ is homogeneous with dx^n; but whenever n is a positive whole number, the ratio of $d^n p$ to dx^n can be expressed algebraically; for example, if $n = 2$ and $p = x^3$, then $d^2(x^3)$ is to dx^2 as $6x$ to 1. We now ask, if n is a fractional number, what the value of that ratio should be. It is easy to understand the difficulty in this case; for if n is a positive whole number, d^n can be found by continued differentiation; such an approach is not available if n is a fractional number. But it will nevertheless be possible to disentangle the matter by using interpolation in progressions, which I have discussed in this essay.

28. Let it be required to determine the ratio between $d^n(z^e)$ and dz^n, dz being taken as constant; that is to say, it is required to find the value of the fraction $\frac{d^n(z^e)}{dz^n}$. Let us see first what its values are if n is a whole number, in order to be able afterwards to extend our reasoning to other values. If $n = 1$, its value will be

$$ez^{e-1} = \frac{1\cdot 2\cdot 3\cdots e}{1\cdot 2\cdot 3\cdots (e-1)} z^{e-1};$$

I express e in this way, in order to make it easier to relate the following results to this case. If $n = 2$, the value will be

$$e(e-1)z^{e-2} = \frac{1\cdot 2\cdot 3\cdots e}{1\cdot 2\cdot 3\cdots (e-2)} z^{e-2}.$$

If $n = 3$, we will have

$$e(e-1)(e-2)z^{e-3} = \frac{1\cdot 2\cdot 3\cdots e}{1\cdot 2\cdot 3\cdots (e-3)} z^{e-3}.$$

From this I infer in general that, whatever n may be, it will always be the case that

$$\frac{d^n(z^e)}{dz^n} = \frac{1\cdot 2\cdot 3\cdots e}{1\cdot 2\cdot 3\cdots (e-n)} z^{e-n}.$$

But from §14,

$$1\cdot 2\cdot 3\cdots e = \int dx\,(-lx)^e \quad \text{and} \quad 1\cdot 2\cdot 3\cdots (e-n) = \int dx\,(-lx)^{e-n}.$$

Thus we have

$$\frac{d^n(z^e)}{dz^n} = z^{e-n}\frac{\int dx\,(-lx)^e}{\int dx\,(-lx)^{e-n}}$$

or

$$d^n(z^e) = z^{e-n}dz^n\,\frac{\int dx\,(-lx)^e}{\int dx\,(-lx)^{e-n}}.$$

Here dz is taken as constant and $\int dx\,(-lx)^n$ and $\int dx\,(-lx)^{e-n}$ must be integrated in the way which was explained above, following which it is necessary to set $x = 1$.

29. It is not necessary to show the correctness of this proceeding; it will be clear if any positive whole number is substituted in place of n. Let us inquire, however, what $d^{\frac{1}{2}}z$ should be, if dz is constant. We will have therefore $e = 1$ and $n = \frac{1}{2}$. Thus we will have

$$d^{\frac{1}{2}}z = \frac{\int dx\,(-lx)}{\int dx\,\sqrt{-lx}}\sqrt{z}\,dz.$$

But

$$\int dx\,(-lx) = 1$$

and letting A be the area of a circle, whose diameter is 1, we will have

$$\int dx\,\sqrt{-lx} = \sqrt{A},$$

whence

$$d^{\frac{1}{2}}z = \sqrt{\frac{z\,dz}{A}}.$$

Suppose, therefore, that we are given the equation

$$yd^{\frac{1}{2}}z = z\sqrt{dy}$$

of a certain curve, where dz is taken to be constant, and let us ask, what is this curve? But since $d^{\frac{1}{2}}z = \sqrt{\frac{z\,dz}{A}}$, the equation will go over into

$$y\sqrt{\frac{z\,dz}{A}} = z\sqrt{dy},$$

which squared gives

$$\frac{yy\,dz}{A} = z\,dy;$$

whence one finds

$$\frac{1}{A}\,lz = c - \frac{1}{y}$$

or

$$y\,lz = cAy - A,$$

which is the equation of the required curve.