LETTRE LXXXVII.

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Euler à Goldbach.

Sommaire. Réponse à la précédente.

Berlin d. 30. November 1745.

— — Ueber das problema catoptricum nehme die Freyheit Ew. meine Solution hiemit zu übersenden, deren analysis alle Umstände hinlänglich erläutern wird*). — Ew. Vorschlag, die Expression 3,1415926535 etc. auf eine bequeme Art vorzustellen, dass daraus zugleich die lex progressionis erhelle, läuft darauf hinaus, dass man eine bekannte Zahl m ausfindig machen soll, deren Cyphern in infinitum mit den obigen entweder einerley oder nur um 1 kleiner wären, denn solchergestalt würde der Rest $\pi - m$ durch eine solche Decimal-Fraction ausgedrückt werden, deren alle Figuren entweder 0 oder 1 seyn würden. Ich sehe aber noch keine Methode ein, wie man nur zur Erfindung der gemeldten Zahl m gelangen könnte. Ich wollte also vielmehr die Differenz $\pi - m$ als bekannt annehmen, als z. Exempel $\pi - m = 0{,}010010001000001$ etc. so würde

$$m = 3{,}131582652589 78,$$

und nun müsste man sehen, ob diese Zahl durch eine expressionem irrationalem finitam ausgedrückt werden könnte.

*) Voir ci-dessous.

Ich erinnere mich auch schon einmal einer gewissen leichten Operation Meldung gethan zu haben, wodurch man Zahlen bekommt, deren Werth vielleicht nullo modo in finitis ausgedrückt werden kann. Ich verfahre nehmlich wie in der ordinären Division, nur dass ich bey jeder Operation den divisorem um 1 vermehre, wie aus folgendem Exempel zu ersehen:

```
1|100000000000000000000|0464782743907639
2|10
  3|20
   4|20
    5|40
     6|50
      7|20
       8|60
        9|40
        10|40
        11|100
          12|10
          13|100
            14|90
             15|60
             16|150
                etc.
```

Wenn nun diese Zahl 0,464782743907639 etc. zur Peripherie des Zirkuls eine bekannte Verhältniss hätte, so hielte ich die Peripherie so gut als gefunden, indem dieselbe mit leichter Mühe auf so viel Figuren, als man immer verlangt, gefunden werden könnte. Man kann auch hierin auf unendlich vielerley Weise variiren und die divisores nach Belieben verändern.

Nach der arithmetica dyadica wird $\sqrt{2}$ folgendergestalt ausgedrückt gefunden $\sqrt{2} = 1,41421356236$, so operire man continuo duplando hinter der Verticallinie folgendergestalt:

1	41421356236
0	82842712472
1	65685424944
1	31370849888
0	62741699776
1	25483399552
0	50966799104
1	01933598208
0	03867196416
0	07734392832
0	15468785664

Die vor der Verticallinie herausgekommenen Zahlen 0 et 1 geben die gesuchte fractionem dyadicam, nehmlich

$$\sqrt{2} = 1,011010100000100111100110011001111111001,$$

worin sich aber keine lex wahrnehmen lässt.

Die von Ew. überschriebene series ist allerdings sehr merkwürdig. Dieselbe kann folgendergestalt generaler ausgedrückt werden: Es sey die tangens dieses Winkels $\frac{1}{n}90^0 = t$, so wird

$$1 - \frac{\pi}{2nt} = \frac{1}{1.2.3n^2}\cdot\frac{1}{2}\pi^2 + \frac{1}{1.2\ldots5n^4}\cdot\frac{1}{6}\pi^4 + \frac{1}{1.2\ldots7n^6}\cdot\frac{1}{6}\pi^6 + \frac{1}{1.2\ldots9n^8}\cdot\frac{3}{10}\pi^8 + \text{etc.}$$

Setzt man nun $n = 2$, so kommt Ew. series heraus.

Euler.

(Mémoire annexé à cette lettre.)

Solutio problematis in Actis Lipsiensibus A. 1745 propositi.

Circa datum focum C (Fig. 17) describere curvam $AEBF$, ut omnes radii ex C emissi post binas reflexiones in M et N factas, in ipsum punctum C revertantur.

I. *Lemma* 1. Determinare legem reflexionis, quam radii ex puncto C (Fig. 18) emissi ad curvam quamcunque EMm patiuntur.

Solutio. Consideretur radius incidens quicunque CM, et ducatur ad curvae punctum M tangens MT, in quam ex C perpendiculum demittatur CT, cui parallela MR erit normalis ad curvam. Sumto igitur angulo $RMO = CMR$, erit recta MO radius reflexus. Ponatur $CM = z$ et angulus $CMT = \varphi$, erit (posito sinu toto $= 1$) $CT = z \sin\varphi$ et $MT = z\cos\varphi$. Ac demisso ex T in CM perpendiculo TS, ob ang. $CTS = CMT = \varphi$ erit $CS = z\sin^2\varphi$ et $TS = z\sin\varphi\cos\varphi$. Jam ducatur radius proximus $Cm = z + dz$, eique conveniens reflexus mOo priorem radium reflexum MO secans in O, erit O punctum in caustica. Centro C describatur arculus MN, et cum in triangulo MNm ad N rectangulo sit $mN = dz$, et angulus $MmN = \varphi$, erit $mN = dz = Mm\cos\varphi$, ideoque $Mm = \frac{dz}{\cos\varphi}$ et $MN = \frac{dz\sin\varphi}{\cos\varphi}$, unde fit angulus $MCm = \frac{dz\sin\varphi}{z\cos\varphi}$. In puncto m ducatur pariter tangens mt ad eamque normalis mR, erit angulus $Cmt = \varphi + d\varphi$; at est $CmT = CMT - MCm = \varphi - \frac{dz\sin\varphi}{z\cos\varphi}$, unde fit $Tmt = MRm = d\varphi + \frac{dz\sin}{z\cos\varphi}$. Porro est $RMO = CMR = 90^\circ - \varphi$ et $RmO = 90^\circ - \varphi - d\varphi$. Quare ob $M\nu m = RMO + MRm = RMO + MOm$, fiet $MOm = RMO + MRm - RmO = 90^\circ - \varphi + d\varphi + \frac{dz\sin\varphi}{z\cos\varphi} - 90^\circ + \varphi + d\varphi$, seu $MOm = 2d\varphi + \frac{dz\sin\varphi}{z\cos\varphi}$. Centro O describatur arculus mn, et cum sint triangula MNm et mnM ob angulos ad N et n rectos, et $mMn = MmN$ aequalia et similia, erit $Mn = mN = dz$ et $mn = MN = \frac{dz\sin\varphi}{\cos\varphi}$. Unde habebitur quoque ang. $MOm = \frac{mn}{mO} = 2d\varphi + \frac{dz\sin\varphi}{z\cos\varphi}$, ex quo erit

$$mO = \frac{mn.z\cos\varphi}{2zd\varphi\cos\varphi + dz\sin\varphi} = \frac{zdz\sin\varphi}{2zd\varphi\cos\varphi + dz\sin\varphi}.$$

Sicque ob $mO = MO$ habemus radium reflexum

$$MO = \frac{zdz\sin\varphi}{2zd\varphi\cos\varphi + dz\sin\varphi}.$$ Q. E. I.

II. *Coroll.* 1. Cum sit angulus

$$MOm = 2d\varphi + \frac{dz\sin\varphi}{z\cos\varphi} = \frac{2zd\varphi\cos\varphi + dz\sin\varphi}{z\cos\varphi}$$

erit $MOm = \frac{2zd\varphi\sin\varphi\cos\varphi + dz\sin^2\varphi}{z\sin\varphi\cos\varphi}$. At hujus fractionis numerator est differentiale ipsius $z\sin^2\varphi = CS$, unde ob $z\sin\varphi\cos\varphi = TS$, erit angulus $MOm = \frac{d.CS}{TS}$.

III. *Coroll.* 2. Quodsi ergo vocemus $CS = r$ et $TS = s$, erit angulus $MOm = \frac{dr}{s}$. Tum vero erit $CT = \sqrt{(rr + ss)}$, $MT = \frac{s}{r}\sqrt{(rr + ss)}$ et $CM = z = \frac{rr+ss}{r}$, unde $dz = dr + \frac{2sds}{r} - \frac{ssdr}{rr}$. Hinc ob $\sin\varphi = \frac{r}{\sqrt{(rr+ss)}}$ et $\cos\varphi = \frac{s}{\sqrt{(rr+ss)}}$, erit $mn = \frac{dz\sin\varphi}{\cos\varphi} = \frac{rdr}{s} + 2ds - \frac{sdr}{r}$, atque $MO = \frac{mn}{MOm} = r + \frac{2sds}{dr} - \frac{ss}{r} = \frac{2sds}{dr} + \frac{rr-ss}{r}$.

IV. *Coroll.* 3. Ponatur radius reflexus $MO = w$, erit proximus $mo = w + dw$, et particula Oo erit elementum curvae causticae. Est vero $Oo = mo - nO = mo - MO + Mn = mo - MO + mN = dw + dz$, ideoque longitudo curvae causticae erit $= w + z \pm C = CM + MO \pm C$, uti constat.

V. *Coroll.* 4. Retentis autem denominatoribus $CS = r$ et $TS = s$, quarum relatione natura curvae EM definitur, erit $\sin MCT = \frac{s}{\sqrt{(rr+ss)}}$ et $\cos MCT = \frac{r}{\sqrt{(rr+ss)}}$. Quoniam vero est $CMO = 2CMR = 2MCT$, erit $\sin CMO = \frac{2rs}{rr+ss}$ et $\cos CMO = \frac{rr-ss}{rr+ss}$. Unde cum in triangulo CMO dentur latera CM et MO cum angulo intercepto CMO, tertium latus CO ejusque positio determinabitur.

VI. *Lemma* 2 Invenire relationem inter bina curvae quaesitae puncta M et m (Fig. 19) ad quae radius ex C reflexus eodem revertitur.

Solutio. Emittatur ex C radius CM, qui post primam reflexionem in m, hincque secunda reflexione iterum in C reflectatur. Manifestum est ejusmodi proprietatem reciprocam inter puncta M et m intercedere, ut radius quoque secundum directionem Cm emissus post binas reflexiones in m et M factas in C revertatur. Ducantur ergo ad M et m tangentes MT et mt, in quas ex C demittantur perpendicula CT et Ct, atque ex T et t porro perpendiculares TS et ts in radios CM et Cm. Jam ponatur ut ante $CS = r$, $TS = s$, atque $Cs = R$ et $ts = -S$, quia haec linea in partem oppositam cadit. Sitque O punctum in caustica. Erit ex ante inventis

$CT = \sqrt{(rr+ss)}$, $MT = \frac{s}{r}\sqrt{(rr+ss)}$ et $CM = \frac{rr+ss}{r}$	$Ct = \sqrt{(RR+SS)}$, $mt = -\frac{S}{R}\sqrt{(RR+SS)}$ et $Cm = \frac{RR+SS}{R}$
$\sin CMO = \frac{2rs}{rr+ss}$, $\cos CMO = \frac{rr-ss}{rr+ss}$	$\sin CmO = \frac{2RS}{RR+SS}$, $\cos CmO = \frac{RR-SS}{RR+SS}$
$MO = \frac{2sds}{dr} + \frac{rr-ss}{r}$	$mO = \frac{2SdS}{dR} + \frac{RR-SS}{R}$.

Ex C in Mm demittatur perpendiculum CV, eritque

$CV = 2s$, $MV = \frac{rr-ss}{r}$	$CV = -2S$, $mV = \frac{RR-SS}{R}$
$OV = MV - MO = -\frac{2sds}{dr}$	$OV = mO - mV = \frac{2SdS}{dR}$
$CO = \frac{2s\sqrt{(dr^2+ds^2)}}{dr}$;	$CO = -\frac{2S\sqrt{(dR^2+dS^2)}}{dR}$,
tang $COM = \frac{dr}{ds}$	tang $COm = -\frac{dR}{dS}$.

Ex quibus colligitur fore $CV = 2s = -2S$, ideoque $S = -s$, seu $ts = TS$. Deinde $OV = \frac{-2sds}{dr} = \frac{2SdS}{dR}$, ergo, ob $S = -s$ et $dS = -ds$, fit $-\frac{1}{dr} = \frac{1}{dR}$ atque $dR + dr = 0$, unde integrando oritur $R + r = 2a$; ita ut si ponatur $r = a + v$, fiat $R = a - v$. Cum igitur ex puncto M reperiatur punctum ipsi ex reflexione respondens m, si valor ipsius $TS = s$ statuatur negativus, hocque facto valor lineae $CS = r = a + v$ abeat in $Cs = R = a - v$, manifestum est quantitatem v ejusmodi fore functionem ipsius s, quae facto s negativo ipsa in sui negativam abeat, cujusmodi functiones equidem impares appellare soleo, quia potestates imparium exponentium ipsius s hac proprietate gaudent. Si igitur sumatur v hujusmodi functio impar ipsius s quaecunque, statuaturque $TS = s$ et $CS = a + v$, habebitur curva conditioni problematis satisfaciens. Q. E. I.

VII. *Coroll.* 1. Omnes igitur curvae problemati satisfacientes ita erunt comparatae, ut sumta pro v functione quacunque impari ipsius s, sit $TS = s$, $CS = a + v$, $CT = \sqrt{(ss + (a+v)^2)}$, $MT = \frac{s\sqrt{(ss+(a+v)^2)}}{a+v}$, $CM = a + v + \frac{ss}{a+v}$.

VIII. *Coroll.* 2. Positio autem radii reflexi Mm ita definietur ut sit $\sin CMO = \frac{2(a+v)s}{ss+(a+v)^2}$, $\cos CMO = \frac{(a+v)^2 - ss}{(a+v)^2 + ss}$, $MO = \frac{2sds}{dv} + a + v - \frac{ss}{a+v}$, ubi O est punctum in caustica, unde longitudo curvae causticae erit $= CM + MO \pm C$ $= 2(a+v) + \frac{2sds}{dv} \pm C$.

IX. *Coroll.* 3. Si porro ex C in radium reflexum Mm demittatur perpendiculum CV, erit $CV = 2s$, $MV = a + v - \frac{ss}{a+v}$, $OV = \frac{-2sds}{dv}$, $CO = \frac{2s\sqrt{(dv^2+ds^2)}}{dv}$ et tang $COM = \frac{dv}{ds}$.

X. *Coroll.* 4. Pro altero autem reflexionis puncto m erit $ts = -s$, $Cs = a - v$, $Ct = \sqrt{(ss + (a-v)^2)}$, $mt = \frac{-s\sqrt{(ss+(a-v)^2)}}{a-v}$ et $Cm = a - v + \frac{ss}{a-v}$, atque

$$mO = \frac{-2sds}{dv} + a - v - \frac{ss}{a-v},$$

unde fit radius reflexus totus $Mm = 2a - \frac{2ass}{aa-vv}$.

XI. *Problema.* Dato puncto C invenire omnes curvas AMB ita comparatas, ut radii ex C emissi post duplicem reflexionem in idem punctum C reflectantur.

Solutio. Consideretur radius quicunque CM (Fig. 20) ductaque tangente MT et ut ante rectis CT et TS, vocetur $TS = s$ et $CS = a + v$, habebiturque curva satisfaciens dummodo pro v capiatur functio impar ipsius s. In radio ergo reflexo $M(M)$, qui causticam in O tangat, erit

$$\sin CMO = \frac{2(a+v)s}{(a+v)^2+ss}, \quad \cos CMO = \frac{(a+v)^2 - ss}{(a+v)^2+ss},$$

$$CM = a + v + \frac{ss}{a+v}, \quad MO = \frac{2sds}{dv} + a + v - \frac{ss}{a+v},$$

$$CO = \frac{2s\sqrt{(dv^2+ds^2)}}{dv} \text{ atque tang } COM = \frac{dv}{ds}.$$

His praemissis sumatur recta quaecunque per C ducta, AB, pro axe, quae radium reflexum $M(M)$ in R secet, sitque angulus $CRM = \omega$. Cum igitur pro altero reflexionis puncto (M) iste angulus fiat $CR(M) = \omega - 180^\circ$, tam sinus quam cosinus anguli ω fieri debet negativus, si punctum M in (M) transferatur, hoc est si s fiat negativum. Ponatur igitur $\cos\omega = \frac{u}{c}$, erit $\sin\omega = \frac{\sqrt{(cc-uu)}}{c}$ et $d\omega = \frac{-du}{\sqrt{(cc-uu)}}$ debebitque u esse functio impar ipsius s, ut posito s negativo, abeat in $-u$: hocque casu quoque $\sqrt{(cc-uu)}$ ob signum radicale ambiguum induet valorem negativum. Ducatur radius reflexus proximus mOr, erit $Crm = \omega + d\omega$, ideoque

$d\omega = MOm$. At supra (§ III) invenimus angulum $MOm = \frac{dr}{s} = \frac{dv}{s}$ ob $r = a + v$, unde fiet $d\omega = \frac{dv}{s} = \frac{-du}{\sqrt{(cc-uu)}}$: erit ergo $s = \frac{-dv\sqrt{(cc-uu)}}{du}$. Quia igitur pro altero puncto reflexionis (M), v abit in $-v$, u in $-u$, et $\sqrt{(cc-uu)}$ in $-\sqrt{(cc-uu)}$, uti u erat functio impar ipsius s, ita nunc vicissim tam s quam v erunt functiones impares ipsius u. Jam in triangulo CRM, ob omnes angulos cum latere $CM = a + v + \frac{ss}{a+v} = a + v + \frac{dv^2(cc-uu)}{du^2(a+v)}$ datos, erit $\sin CRM : CM = \sin CMO : CR$ seu $CR = \frac{CM \sin CMO}{\sin CRM} = \frac{CV}{\sin CRM} = \frac{2cs}{\sqrt{(cc-uu)}} = \frac{-2cdv}{du}$, ob $s = \frac{-dv\sqrt{(cc-uu)}}{du}$. Sicque erit $CR = \frac{-2cdv}{du}$, quae pro puncto (M) eundem retinet valorem uti requiritur. Deinde erit $RV = \frac{-2udv}{du}$, hincque $MR = a + v - \frac{ss}{a+v} - \frac{2udv}{du} = a + v - \frac{2udv}{du} - \frac{dv^2(cc-uu)}{du^2(a+v)}$. Demittatur nunc ad axem ACB applicata MP, erit $MP = MR \sin\omega =$

$$\left(a + v - \frac{2udv}{du} - \frac{dv^2(cc-uu)}{du^2(a+v)}\right)\frac{\sqrt{(cc-uu)}}{c},$$

ubi ex signo radicali $\sqrt{(cc-uu)}$ patet axem ACB simul fore curvae diametrum orthogonalem. Tum vero erit

$$RP = MR\cos\omega = \frac{u(a+v)}{c} - \frac{2uudv}{cdu} - \frac{udv^2(cc-uu)}{cdu^2(a+v)}$$

et

$$CP = \frac{udv^2(cc-uu)}{cdu^2(a+v)} - \frac{2dv(cc-uu)}{cdu} - \frac{u(a+v)}{c}.$$

Positis ergo coordinatis orthogonalibus $CP = x$, $PM = y$, si pro v capiatur functio quaecunque impar ipsius u, omnes curvae problemati satisfacientes in sequentibus formulis continebuntur:

$$x = \frac{-u(a+v)}{c} - \frac{2dv(cc-uu)}{cdu} + \frac{udv^2(cc-uu)}{cdu^2(a+v)};$$

$$y = \left(a + v - \frac{2udv}{du} - \frac{dv^2(cc-uu)}{du^2(a+v)}\right)\frac{\sqrt{(cc-uu)}}{c}.$$

Unde fit $CM = a + v + \frac{dv^2(cc-uu)}{du^2(a+v)}$. Si ergo pro v capiatur functio algebraica ipsius u, curva quoque erit algebraica. Sicque tot, quot libuerit, curvas algebraicas exhibere licet. Q. E. I.

XII. *Coroll.* 1. Ad statum figurae expedit quantitatem v sumi negativam, quo facto erit per formulas hactenus inventas:

$$CP = x = \frac{-u(a-v)}{c} + \frac{2dv(cc-uu)}{cdu} + \frac{udv^2(cc-uu)}{c\,du^2(a-v)}$$

$$= \frac{-u(a-v)}{c}\left(1 - \frac{dv(cc-uu+c\sqrt{(cc-uu)})}{udu(a-v)}\right)$$

$$\left(1 - \frac{dv(cc-uu-c\sqrt{(cc-uu)})}{udu(a-v)}\right)$$

$$PM = y = \left(a - v + \frac{2udv}{du} - \frac{dv^2(cc-uu)}{du^2(a-v)}\right)\frac{\sqrt{(cc-uu)}}{c}$$

$$= \left(a - v - \frac{(c-u)dv}{du}\right)\left(a - v + \frac{(c+u)dv}{du}\right)\frac{\sqrt{(cc-uu)}}{c(a-v)}$$

$$CM = z = a - v + \frac{dv^2(cc-uu)}{du^2(a-v)}$$

factisque u et v itemque $\sqrt{(cc-uu)}$ negativis, hae formulae praebebunt alterum reflexionis punctum (M).

XIII. *Coroll.* 2. Reliquae autem lineae et anguli in figura expressi erunt $TS = s = \frac{dv\sqrt{(cc-uu)}}{du}$, $CS = r = a - v$, $CT = \sqrt{(rr+ss)}$ et $MT = \frac{s}{r}\sqrt{(rr+ss)}$, $CR = \frac{2cdv}{du}$, $MR = a - v + \frac{2udv}{du} - \frac{dv^2(cc-uu)}{du^2(a-v)}$ et $\cos CRM = \frac{u}{c}$, $\sin CRM = \frac{\sqrt{(cc-uu)}}{c}$. Porro $CV = \frac{2dv\sqrt{(cc-uu)}}{du}$, $RV = \frac{2udv}{du}$.

XIV. *Coroll.* 3. Caustica autem, in qua punctum O existit, ita definietur: Cum sit $MO = a - v - \frac{ss}{a-v} - \frac{2sds}{dv}$ et $MR = a - v + \frac{2udv}{du} - \frac{ss}{a-v}$, erit

$$RO = \frac{2sds}{dv} + \frac{2udv}{du} = \frac{2ds\sqrt{(cc-uu)}+2udv}{du}$$

ob $s = \frac{dv\sqrt{(cc-uu)}}{du}$: hincque $OQ = \frac{2ds(cc-uu)+2udv\sqrt{(cc-uu)}}{cdu}$ et $RQ = \frac{2uds\sqrt{(cc-uu)}+2uudv}{cdu}$, ideoque

$$CQ = \frac{2(cc-uu)dv - 2uds\sqrt{(cc-uu)}}{cdu}.$$

Tum vero longitudo curvae causticae

$$= 2(a-v) - \frac{2ds\sqrt{(cc-uu)}}{du} \pm C.$$

XV. *Coroll.* 4. Totus vero radius reflexus erit $M(M) = 2a - \frac{2ass}{aa-vv}$. Quare ob $s = \frac{dv\sqrt{(cc-uu)}}{du}$, erit

$$M(M) = 2a - \frac{2adv^2(cc-uu)}{du^2(aa-vv)}.$$

Proprietates ergo harum curvarum sunt sequentes:

XVI. Cum recta ACB simul sit curvae diameter, ita ut pars AMB aequalis sit et similis parti $A(M)B$, bini vertices A et B reperientur, faciendo $y = 0$, quod fit si vel $u = c$ vel $u = -c$. Quibus casibus, ob angulum CRM vel $= 0$ vel $= 180^0$, axis AB ad curvam erit normalis. Fiat ergo primo $u = c$ sitque $v = e$, erit $x = -a + e$, ideoque $AC = a - e$. Deinde sit $u = -c$, erit $v = -e$, atque $x = a + e$, ita ut pro altero vertice B sit $BC = a + e$ unde totus axis transversus erit $AB = 2a$.

XVII. Fieri interdum potest, ut applicata y aliis quoque casibus evanescat, scilicet si $a - v + \frac{2udv}{du} - \frac{dv^2(cc-uu)}{du^2(a-v)} = 0$, quod evenit si $a - v = -\frac{udv}{du} \pm \frac{cdv}{du}$. Hoc est si $\frac{dv}{du} = \frac{a-v}{\pm c - u}$. Hoc autem casu erit $x =$

$$\frac{-u(a-v)}{c} + \frac{2(\pm c + u)(a-v)}{c} + \frac{u(\pm c + u)(a-v)}{c(\pm c - u)} = \frac{2c(a-v)}{\pm c - u} = \frac{2cdv}{du}.$$

Quodsi ergo hujusmodi casus locum habet, erit simul $CP = CR$, quod quidem facile patet.

XVIII. Quantitas applicatae CE in ipso foco C innotescet ponendo $x = 0$. Fit autem $x = 0$ si

$$a - v = \frac{dv(cc - uu \pm c\sqrt{(cc-uu)})}{udu}.$$

Hoc autem valore substituto fiet $CE = y = \frac{2cdv\sqrt{(cc-uu)}}{udu}$, qui valor prodit, si $a - v = \frac{dv(\sqrt{(cc-uu)} \pm c)\sqrt{(cc-uu)}}{udu}$ seu si $\frac{dv\sqrt{(cc-uu)}}{udu} = \frac{a-v}{\sqrt{(cc-uu)} \pm c} = \frac{-(a-v)(\sqrt{(cc-uu)} \mp c)}{uu}$. Erit ergo quoque $CE = \frac{-2c(a-v)(\sqrt{(cc-uu)} \mp c)}{uu}$.

XIX. Si quaeretur locus, ubi axis reflexus $M(M)$ ad axem AB fit normalis, is reperietur ponendo angulum CRM rectum, seu $u = 0$. Hoc autem facto erit $x = \frac{2cdv}{du}$ et $y = a - v - \frac{ccdv^2}{du^2(a-v)}$. Quia vero v est functio impar ipsius u, posito $u = 0$, erit v vel $= 0$, vel $= \infty$.

XX. Denique ex formulis inventis maxima curvae applicata PM facile definiri poterit. Cum enim tangens in M tum sit axi AB parallela, triangulum CMR erit isosceles, ideoque $CM = MR$, hinc autem nascitur haec aequatio:

$$a - v + \frac{dv^2(cc-uu)}{du^2(a-v)} = a - v + \frac{2udv}{du} - \frac{dv^2(cc-uu)}{du^2(a-v)}$$

seu haec $\frac{udv}{du} = \frac{dv^2(cc-uu)}{du^2(a-v)}$. Hoc ergo evenit si vel $\frac{dv}{du} = 0$, vel $\frac{dv}{du} = \frac{u(a-v)}{cc-uu}$. Priori casu quo $\frac{dv}{du} = 0$, erit $x = \frac{-u(a-v)}{c}$ et $y = \frac{(a-v)\sqrt{(cc-uu)}}{c}$. Posteriori casu quo $\frac{dv}{du} = \frac{u(a-v)}{cc-uu}$, erit

$x = \frac{cu(a-v)}{cc-uu}$ et $y = \frac{c(a-v)}{\sqrt{(cc-uu)}}$ atque $CM = MR = \frac{cc(a-v)}{cc-uu}$ seu $x = \frac{cdv}{du}$, $y = \frac{cdv\sqrt{(cc-uu)}}{udu}$ et $CM = MR = \frac{ccdv}{udu}$.

XXI. *Exempl.* 1. Sit $v = u$; haec enim positio ob rationem $a : c$ indeterminatam aeque late patet ac $v = nu$; eritque $CR = \frac{2cdv}{du} = 2c$. Punctum ergo R, in quo radius reflexus $M(M)$ axem trajicit, est fixum, et caustica in punctum abit. Unde manifestum est curvam fore sectionem conicam circa focos C et R descriptam, cujus axis transversus sit $AB = 2a$ et distantia focorum $CR = 2c$. Lineae autem in figura expressae ita se habebunt: $TS = s = \sqrt{(cc-uu)}$, $CS = a - u$, $CT = \sqrt{(aa + cc - 2au)}$, $MT = \frac{\sqrt{(cc-uu)(aa+cc-2au)}}{a-u}$, $CM = a - u + \frac{cc-uu}{a-u} = \frac{aa-cc-2au}{a-u}$, $MR = a + u - \frac{cc+uu}{a-u} = \frac{aa-cc}{a-u}$, ideoque $CM + MR = 2a = AB$. Porro est $CV = 2\sqrt{(cc-uu)}$, $RV = 2u$ atque ob $\sin CRM = \frac{\sqrt{(cc-uu)}}{c}$ et $\cos CRM = \frac{u}{c}$, erit $PM = \frac{(aa-cc)\sqrt{(cc-uu)}}{c(a-u)}$, $PR = \frac{(aa-cc)u}{c(a-u)}$ et $CP = \frac{2acc-(aa+cc)u}{c(a-u)}$. Vertices sunt in A et B ut sit $AC = a - c$ et $BC = a + c$, an vero alibi quoque applicata y evanescat, indicat aequatio $a - u = \pm c - u$, unde nisi sit $a = \pm c$, quo casu fieret $CR = AB$, hoc evenire nequit. Si $CP = 0$, fit $u = \frac{2acc}{aa+cc}$, ideoque $CE = \frac{aa+cc}{ac}\sqrt{(cc-uu)} = \frac{aa-cc}{a}$. In puncto R fit radius reflexus $M(M)$ axi normalis. Applicata denique maxima habebitur si vel $\frac{dv}{du} = 1 = 0$, quod fieri nequit, vel si $1 = \frac{au-uu}{cc-uu}$, hoc est si $u = \frac{cc}{a}$, unde fit $x = c$, $y = \sqrt{(aa-cc)}$ et $CM = MR = a$.

XXII. *Exempl.* 2. Ponatur $v = \frac{u^3}{cc}$, erit $\frac{dv}{du} = \frac{3uu}{cc}$. Si igitur $\cos\ CRM = \frac{u}{c}$ et $\sin\ CRM = \frac{\sqrt{(cc-uu)}}{c}$, erit $CR = \frac{6uu}{c}$. Porro erit $TS = s = \frac{3uu}{cc}\sqrt{(cc-uu)}$, $CS = a - \frac{u^3}{cc}$ et
$$CM = a - \frac{u^3}{cc} + \frac{9u^4(cc-uu)}{ac^4 - ccu^3} = \frac{aac^4 - 2accu^3 + 9ccu^4 - 8u^6}{cc(acc-u^3)}$$
atque
$$MR = a + \frac{5u^3}{cc} - \frac{9u^4(cc-uu)}{ac^4 - ccu^3} = \frac{aac^4 + 4accu^3 - 9ccu^4 + 4u^6}{cc(acc-u^3)}$$
ideoque $CM + MR = \frac{2aac^4 + 2accu^3 - 4u^6}{cc(acc-u^3)} = \frac{2acc + 4u^3}{cc}$. Axis hujus curvae ut semper est $AB = 2a$: ad vertices autem inveniendos ponatur $u = c$, erit $v = e = c$, ideoque $AC = a - c$ et $BC = a + c$. Utrum autem alibi quoque applicata y evanescat, patebit si sit $\frac{3uu}{cc} = \frac{acc - u^3}{cc(\pm c - u)}$ seu $\pm 3cuu - 2u^3 = acc$. Quoties ergo haec aequatio radices habet reales ejusmodi ut sit $u < \pm c$, abscissae $x = \frac{6uu}{c}$ applicata respondebit evanescens. Applicata in foco C est $CE = \frac{6u}{c}\sqrt{(cc-uu)}$ existente $\frac{acc - u^3}{cc} = \frac{3u}{cc}\left(cc - uu \pm c\sqrt{(cc-uu)}\right)$ seu $acc - 3ccu + 2u^3 = \pm 3cu\sqrt{(cc-uu)}$ vel $4u^6 - 3ccu^4 + 4accu^3 - 6ac^4u + aac^4 = 0$. Radius vero reflexus $M(M)$ axem normaliter secabit si sit $u = 0$, quo casu fit $x = 0$ et $y = a$. Deinde cum applicata maxima sit ubi $\frac{dv}{du} = 0$, hoc est ubi $u = 0$; erit hoc casu $x = 0$ et $y = a$. Deinde vero quoque est maxima si $\frac{3uu}{cc} = \frac{u(acc-u^3)}{cc(cc-uu)}$, hoc est si $3ccuu - 2u^4 = accu$, unde fit vel $u = 0$, vel $2u^3 - 3ccu + acc = 0$. Caustica autem hujus curvae ita definietur: Cum sit $s = \frac{3uu}{cc}\sqrt{(cc-uu)}$, erit $\frac{ds}{du} = \frac{6u}{cc}\sqrt{(cc-uu)} - \frac{3u^3}{cc\sqrt{(cc-uu)}}$

$= \frac{6ccu - 9u^3}{cc\sqrt{(cc-uu)}}$; erit $RO = \frac{12ccu - 18u^3}{cc} + \frac{6u^3}{cc} = \frac{12u(cc-uu)}{cc}$,
$OQ = \frac{12u(cc-uu)\sqrt{(cc-uu)}}{c^3}$, $RQ = \frac{12uu(cc-uu)}{c^3}$, unde $CQ = \frac{-6ccuu + 12u^4}{c^3}$. Sit $CQ = p$, $QO = q$, erit
$$p = \frac{6uu(2uu-cc)}{c^3} \text{ et } q = \frac{12u(cc-uu)^{3:2}}{c^3}.$$
Sit angulus $CRM = \omega$, erit $\frac{u}{c} = \cos\omega$, $\frac{2uu-cc}{cc} = \cos 2\omega$, $\frac{\sqrt{(cc-uu)}}{c} = \sin\omega$, ideoque $p = 6c\cos^2\omega . \cos 2\omega$ et
$$q = 12c\cos\omega\sin^3\omega = 6c\sin^2\omega\sin 2\omega,$$
unde $\frac{q}{p} = \tang^2\omega\,\tang 2\omega = \frac{2\tang^3\omega}{1-\tang^2\omega}$. Vel cum sit
$$\cos^2\omega = \frac{1+\cos 2\omega}{2} \text{ et } \sin^2\omega = \frac{1-\cos 2\omega}{2},$$
erit $p = 3c(1+\cos 2\omega)\cos 2\omega$ et $q = 3c(1-\cos 2\omega)\sin 2\omega$. Erit ergo $\cos^2 2\omega + \cos 2\omega = \frac{p}{3c}$, ideoque
$$\cos 2\omega = -\frac{1}{2} \pm \sqrt{\left(\frac{1}{4} + \frac{p}{3c}\right)} \text{ et}$$
$$\cos^2 2\omega = \frac{1}{2} + \frac{p}{3c} \mp \sqrt{\left(\frac{1}{4} + \frac{p}{3c}\right)}$$
unde
$$\sin 2\omega = \sqrt{\left(\frac{1}{2} - \frac{p}{3c} \pm \sqrt{\left(\frac{1}{4} + \frac{p}{3c}\right)}\right)}.$$
Ergo prodibit
$$q = 3c\left(\frac{3}{2} \mp \sqrt{\left(\frac{1}{4} + \frac{p}{3c}\right)}\right)\sqrt{\left(\frac{1}{2} - \frac{p}{3c} \pm \sqrt{\left(\frac{1}{4} + \frac{p}{3c}\right)}\right)}.$$
Sit $\sqrt{\left(\frac{1}{4} + \frac{p}{3c}\right)} = t$ erit $\frac{p}{3c} = -\frac{1}{4} + tt$ et
$$q = 3c\left(\frac{3}{2} - t\right)\sqrt{\left(\frac{3}{4} + t - tt\right)},$$
unde
$$\frac{qq}{9cc} = \frac{27}{16} - \frac{9}{2}tt + 4t^3 - t^4 = \frac{1}{2} - \frac{5p}{3c} - \frac{pp}{9cc} + 4\left(\frac{1}{4} + \frac{p}{3c}\right)^{\frac{3}{2}},$$

quae aequatio ad rationalitatem perducta fit:

$$p^4 + 2ppqq + q^4 + 30cpqq - 18cp^3 - 9ccqq + 108ccpp - 216c^3p = 0.$$

Est ergo haec caustica linea quarti ordinis, quae ex aequatione

$$q = \frac{(9c \mp \sqrt{(9cc+12cp)})\sqrt{(3c - 2p \pm \sqrt{(9cc+12cp)})}}{2\sqrt{6c}},$$

non difficulter construetur.

Curva haec est tricuspidata triangulo aequilatero inscripta uti haec figura adjecta (Fig. 21) repraesentat, et curva problemati satisfaciens oritur, si filum huic curvae complicetur, alterque terminus in C figatur, sicque per evolutionem fili describetur.

LETTRE LXXXVIII.

GOLDBACH à EULER.

SOMMAIRE. Problème de la courbe catoptrique. Sur les nombres π et $\sqrt{2}$. Sommation d'une série.

St. Petersburg d. 28. December 1745.

Ew. sage ich für die mir übersandte ausführliche Solution des problematis in Act. Lips. propositi schuldigsten Dank. Ehe selbige noch ankam, hatte ich schon vor mich observiret, dass (Fig. 17) $CM + NR = 2(a+v) - \frac{2udv}{du}$ und $CN + NR = 2(a-v) + \frac{2udv}{du}$ (woraus denn folget, dass die drey latera trianguli $CM + MN + NC = 4a$) und dass $MR = \frac{cPM}{\sqrt{(cc-uu)}}$, folglich MR nur in dem einigen casu $= PM$, wenn $u = 0$; obzwar generaliter wahr ist, dass MR normalis ad axem wird, wenn nur $y = \frac{4aa - xx}{4a}$ (abstrahendo a valore ipsius x). Ich habe auch nicht gefunden, dass Ew. den casum deter-