DE UNCIIS POTESTATUM BINOMII EARUMQUE INTERPOLATIONE.

AUCTORE

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§. 1. [illegible] Evolutionem potestatis $(1+x)^n$ sequenti modo per [illegible] characteres repraesentemus:

$$(1+x)^n = 1 + \left(\frac{n}{1}\right) x + \left(\frac{n}{2}\right) x^2 + \left(\frac{n}{3}\right) x^3 + \text{etc.}$$

[illegible] characteres uncinulis inclusi: $\left(\frac{n}{1}\right)$, $\left(\frac{n}{2}\right)$, $\left(\frac{n}{3}\right)$, etc. uncias [illegible] ferunt. Erit ergo $\left(\frac{n}{1}\right) = n$; $\left(\frac{n}{2}\right) = \frac{n}{1} \cdot \frac{n-1}{2}$; $\left(\frac{n}{3}\right) = \frac{n}{1} \cdot \frac{n-1}{2} \cdot \frac{n-2}{3}$; etc. Hinc in genere erit:

$$\left(\frac{n}{q}\right) = \frac{n}{1} \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \frac{n-3}{4} \cdot \ldots \cdot \frac{n-q+1}{q}$$

quae ergo evolutio nullam habet difficultatem, quoties q fuerit numerus integer positivus. Totum igitur negotium eo redit, ut etiam valores hujus characteris generalis $\left(\frac{n}{q}\right)$ explorentur, quando pro q numeri vel fracti vel etiam negativi accipiuntur. Ceterum pro casu $q = 0$ per se manifestum est fore $\left(\frac{n}{0}\right) = 1$, siquidem hinc primus terminus potestatis evolutae prodire debet.

§. 2. Cum ipsa evolutio potestatis $(1+x)^n$ alias potestates ipsius x non involvat, nisi quarum exponentes sint numeri integri positivi, ea revera nullam interpolationem admittit. Interim tamen si hanc formam $\left(\frac{n}{q}\right)$ ut certam functionem numerorum n et q spectemus, ita ut, si q consideretur ut abscissa cujusdam curvae cujus applicata sit $\left(\frac{n}{q}\right)$, nullum est dubium, quin talis curva quandam le-

gem continuitatis sit habitura, quam ergo hic investigare consti[...]
Principia autem interpolationis ex serie hypergeometrica *Wallisii* [...]
2, 6, 24, 120, 720, etc. repetere conveniet, quandoquidem ev[...]
lutio nostrorum characterum insigni affinitate cum hac serie [...]
praedita.

§. 3. Quoniam quilibet terminus seriei hypergeometricae h[...] producto involvitur, $1 . 2 . 3 . 4 \ldots\ldots m$ ejus loco brevitatis gra[...] scribamus $\Phi : m$, siquidem ista forma tanquam certa functio ips[...] m spectari potest, cujus adeo interpolationem jam pridem docui, [...]que demonstravi esse $\Phi : \frac{1}{2} = \sqrt{\pi}$ et $\Phi : -\frac{1}{2} = \sqrt{\pi}$, denotante [...] peripheriam circuli radio 1 descripti. At si aliae fractiones, vel $\frac{1}{3}$, $\frac{1}{4}$, etc. sumantur, valores continuo altiores quantitates transce[...]dentes requirunt; quamobrem, si nostros characteres ad hujusm[...] formulas $\Phi : m$ revocaverimus, interpolatio nulla amplius laborat d[...]ficultate.

Problema.

§. 4. *Valorem characteris $\left(\frac{n}{q}\right)$ ad terminos progressionis h[...]pergeometricae revocare.*

Solutio.

Cum sit $\left(\frac{n}{q}\right) = \frac{n(n-1)(n-2)\ldots\ldots(n-q+1)}{1 . 2 . 3 \ldots\ldots q}$, at vero ex p[...]gressione hypergeometrica sit $\Phi : n = n(n-1)(n-2)\ldots$ [...] ea ita referri potest

$$\Phi : n = n(n-1)(n-2)\ldots(n-q+1)\times(n-q)(n-q-1)\ldots$$

unde patet, numeratorem nostrae fractionis esse $\frac{\Phi : n}{\Phi : (n-q)}$; qua[...]obrem, cum denominator sponte sit $\Phi : q$, valor nostri charact[...] $\left(\frac{n}{q}\right)$ erit $\frac{\Phi : n}{\Phi : q \times \Phi : (n-q)}$.

Corollarium.

§. 5. Quod si ergo loco n scribamus $a+b$ et a loco q, hab[...]mus istam aequationem: $\left(\frac{a+b}{a}\right) = \frac{\Phi : (a+b)}{\Phi : a \times \Phi : b}$, in qua formula litt[...]

[illegible] permutationem admittunt, unde concluditur, semper fore [illegible], hincque etiam $(\frac{n}{q}) = (\frac{n}{n-q})$, unde deduci possunt sequentia theoremata notatu maxime digna.

Theorema 1.

Quicunque numeri pro a, b *et* n *accipiantur, semper haec aequatio locum habebit:* $(\frac{n}{a})(\frac{n-a}{b}) = (\frac{n}{b})(\frac{n-b}{a})$.

Demonstratio.

Loco n scribatur $a+b+c$, et cum sit per superiorem reductionem $(\frac{a+b+c}{a}) = \frac{\Phi:(a+b+c)}{\Phi:a \times \Phi:(b+c)}$ atque $(\frac{b+c}{b}) = \frac{\Phi:(b+c)}{\Phi:b \times \Phi:c}$ productum fiet $(\frac{a+b+c}{a})(\frac{b+c}{b}) = \frac{\Phi:(a+b+c)}{\Phi:a \times \Phi:b \times \Phi:c}$; unde patet litteras $a, b, c,$ pro lubitu inter se permutari posse. Hinc pro $a+b+c$ restituto n erit $(\frac{n}{a})(\frac{n-a}{b}) = (\frac{n}{b})(\frac{n-b}{a})$; utraque enim pars aequalis est huic formae $\frac{\Phi:n}{\Phi:a \times \Phi:b \times \Phi:(n-a-b)}$.

Theorema 2.

Productum ex ternis characteribus $(\frac{n}{a})(\frac{n-a}{b})(\frac{n-a-b}{c})$ *semper eundem valorem retinet, utcunque litterae* a, b, c, *inter se permutentur.*

Demonstratio.

Per reductionem enim ad seriem hypergeometricam habebimus $(\frac{n}{a}) = \frac{\Phi:n}{\Phi:a \times \Phi:(n-a)}$; $(\frac{n-a}{b}) = \frac{\Phi:(n-a)}{\Phi:b \times \Phi:(n-a-b)}$; $(\frac{n-a-b}{c}) = \frac{\Phi:(n-a-b)}{\Phi:c \times \Phi:(n-a-b-c)}$; unde productum propositum reducetur ad hanc formam:

$$\frac{\Phi:n}{\Phi:a \times \Phi:b \times \Phi:c \times \Phi:(n-a-b-c)},$$

quae expressio manifesto eundem retinet valorem, utcunque litterae a, b, c inter se permutentur; quod cum pluribus modis fieri possit, etiam plura hujusmodi producta inter se aequalia exhiberi poterunt.

Corollarium.

§. 8. Hoc modo ulterius progredi licet atque demonstr poterit istud produstum: $(\frac{n}{a})(\frac{n-a}{b})(\frac{n-a-b}{c})(\frac{n-a-b-c}{d})$ perpet eundem valorem retinere, utcunque litterae a, b, c, d, permutentu Ejus enim valor semper erit

$$\frac{\Phi : n}{\Phi : a \times \Phi : b \times \Phi : c \times \Phi : d \times \Phi : (n-a-b-c-d)}.$$

Theorema 3.

§. 9. *Hoc productum:* $(\frac{a}{b})(\frac{b}{a})$ *semper aequale est huic ch racteri:* $(\frac{0}{a-b})$.

Demonstratio.

Cum enim sit per reductionem ad numeros hypergeometric $(\frac{a}{b}) = \frac{\Phi : a}{\Phi : b \times \Phi : (a-b)}$ et $(\frac{b}{a}) = \frac{\Phi : b}{\Phi : a \times \Phi : (b-a)}$, manifesto e $(\frac{a}{b})(\frac{b}{a}) = \frac{1}{\Phi : (a-b) \times \Phi : (b-a)}$. Tum vero simili modo erit:

$$(\frac{0}{a-b}) = \frac{\Phi : 0}{\Phi : (a-b) \times \Phi : (b-a)} = \frac{1}{\Phi : (a-b) \times \Phi : (b-a)},$$

ob $\Phi : 0 = 1$, unde sequitur $(\frac{a}{b})(\frac{b}{a}) = (\frac{0}{a-b})$; hincque patet h productum semper nihilo aequari, quoties $a-b$ est numerus intege

Scholion.

§. 10. His praemissis sit $(\frac{P}{Q})$ forma generalis omnium h jus generis functionum, quas hic evolvere constitui, ubi P et Q d notent numeros quoscunque, sive integros sive fractos, sive negat vos sive positivos, ita ut in hac formula infinities-infinita multitud casuum contineatur, atque jam notavimus, quoties denominator Q fuerit numerus integer positivus, evolutionem revera semper insti posse; unde has formas: $(\frac{P}{1})$ pro cognitis habebimus, earumque o reliquos casus ad majorem simplicitatem reducere conabimur. S

Sequenti autem theoremate numerus omnium casuum ad semissem redigitur.

Theorema 4.

Omnes casus hujus formae: $\left(\frac{P}{Q}\right)$, *facillime reducuntur ad casus quibus est Q major quam* $\frac{1}{2}P$.

Demonstratio.

Ponatur enim $Q = \frac{1}{2}P - s$ et cum sit in genere $\left(\frac{a}{b}\right) = \left(\frac{a}{a-b}\right)$, erit $\left(\frac{P}{\frac{1}{2}P - s}\right) = \left(\frac{P}{\frac{1}{2}P + s}\right)$, sicque omnes casus, quibus Q superatur ab $\frac{1}{2}$P, prorsus congruunt cum iis, quibus superat $\frac{1}{2}$P.

Corollarium.

§ 12. Si ergo concipiatur curva, cujus abscissae x respondeat applicata $y = \left(\frac{a}{x}\right)$, tum applicata abscissae $x = \frac{1}{2}a$ simul erit diameter curvae, quandoquidem binis abscissis $x = \frac{1}{2}a + t$ et $x = \frac{1}{2}a - t$ aequales respondent applicatae; unde sufficiet alteram tantum medietatem curvae determinasse.

Scholion.

§ 13. Cum igitur hoc modo omnes casus in formula $\left(\frac{P}{Q}\right)$ contenti ad semissem redigantur, in sequentibus ostendam, quomodo intra multo arctiores limites compingi queat. Si scilicet litterae m et n denotent numeros integros positivos, haec formula generalis: $\left(\frac{p+m}{q+n}\right)$ semper reduci potest ad hanc formam: $M.\left(\frac{p}{q}\right)$, ubi valor factoris M absolute assignari potest. Hoc igitur modo forma nostra generalis $\left(\frac{P}{Q}\right)$ semper redigi poterit ad talem: $\left(\frac{p}{q}\right)$, in qua numeri p et q intra limites 0 et 1 subsistant. Quin etiam redigi possent intra limites 0 et -1. Huic igitur reductioni inservient sequentia problemata, quorum solutiones his lemmatibus innituntur.

Lemma 1.

§. 14. Cum sit $\left(\frac{p+m}{m}\right) = \frac{\Phi:(p+m)}{\Phi:m \times \Phi:p}$, erit

$$\Phi:(p+m) = \Phi:m \times \Phi:p \times \left(\frac{p+m}{m}\right),$$

cujus characteris valor, ob m numerum integrum positivum, sempe absolute assignari poterit. Eodem igitur modo erit:

$$\Phi:(q+n) = \Phi:n \times \Phi:q \times \left(\frac{q+n}{n}\right).$$

Lemma 2.

§. 15. Cum sit $\left(\frac{p}{m}\right) = \frac{\Phi:p}{\Phi:m \times \Phi:(p+m)}$, concluditur fore

$$\Phi:(p-m) = \frac{\Phi:p}{\Phi:m} : \left(\frac{p}{m}\right).\quad \text{Eodem modo erit}$$

$$\Phi:(q-n) = \frac{\Phi:q}{\Phi:n} : \left(\frac{q}{n}\right).$$

Problema I.

§. 16. *Hanc formulam:* $\left(\frac{p+m}{q}\right)$, *ubi* m *denotat numerum in-tegrum positivum, reducere ad hanc simpliciorem:* $\left(\frac{p}{q}\right)$.

Solutio.

Per reductionem nostram generalem ad numeros hypergeom tricos erit $\left(\frac{p+m}{q}\right) = \frac{\Phi:(p+m)}{\Phi:q \times \Phi:(p-q+m)}$. Quod si jam hic lemmate primo loco $\Phi:(p+m)$ et $\Phi:(p-q+m)$ valores su stituamus, prodibit $\left(\frac{p+m}{q}\right) = \frac{\Phi:p}{\Phi:q \times \Phi:(p-q)} \times \frac{\left(\frac{p+m}{m}\right)}{\left(\frac{p-q+m}{m}\right)}$. Cu igitur sit $\frac{\Phi:p}{\Phi:q \times \Phi:(p-q)} = \left(\frac{p}{q}\right)$, habebimus

$$\left(\frac{p+m}{q}\right) = \frac{\left(\frac{p+m}{m}\right)}{\left(\frac{p-q+m}{m}\right)} \times \left(\frac{p}{q}\right).$$

Problema II.

§. 17. *Hanc formulam:* $\left(\frac{p-m}{q}\right)$, *ubi* m *sit numerus integer positivus, reducere ad formam simpliciorem* $\left(\frac{p}{q}\right)$.

Solutio.

Reductio nostra statim praebet hanc aequationem:

$$\left(\frac{p-m}{q}\right) = \frac{\Phi:(p-m)}{\Phi:q \times \Phi:(p-q-m)}.$$

Hic jam loco $\Phi:(p-m)$ et $\Phi:(p-q-m)$ valores ex lemmate secundo substituantur, ac reperietur sequens expressio:

$$\left(\frac{p-m}{q}\right) = \frac{\Phi:p}{\Phi:q \times \Phi:(p-q)} \times \frac{\left(\frac{p-q}{m}\right)}{\left(\frac{p}{m}\right)},$$

Cum sit $\frac{\Phi:p}{\Phi:q \times \Phi:(p-q)} = \left(\frac{p}{q}\right)$, hanc habebimus formam:

$$\left(\frac{p-m}{q}\right) = \frac{\left(\frac{p-q}{m}\right)}{\left(\frac{p}{m}\right)} \times \left(\frac{p}{q}\right).$$

Problema III.

§. 18. *Hanc formulam:* $\left(\frac{p}{q+n}\right)$, *ubi* n *denotet numerum integrum positivum, reducere ad simpliciorem* $\left(\frac{p}{q}\right)$.

Solutio.

Reductio nostra hic praebet $\left(\frac{p}{q+n}\right) = \frac{\Phi:p}{\Phi:(q+n) \times \Phi:(p-q-n)}$. Jam ex lemmate primo loco $\Phi:(q+n)$, ex secundo vero loco $\Phi:(p-q-n)$, valores substituantur, prodibitque:

$$\left(\frac{p}{q+n}\right) = \frac{\Phi:p}{\Phi:q \times \Phi:(p-q)} \times \frac{\left(\frac{p-q}{n}\right)}{\left(\frac{q+n}{n}\right)} = \frac{\left(\frac{p-q}{n}\right)}{\left(\frac{q+n}{n}\right)} \times \left(\frac{p}{q}\right).$$

Problema IV.

§. 19. *Hanc formulam:* $\left(\frac{p}{q-n}\right)$, *ubi* n *denotet numerum integrum positivum, ad formam simpliciorem* $\left(\frac{p}{q}\right)$ *reducere.*

Solutio.

Per reductionem ad numeros hypergeometricos erit:

$$\binom{p}{q-n} = \frac{\Phi : p}{\Phi : (q-n) \times \Phi : (p-q+n)}.$$

Quod si jam loco $\Phi : (q-n)$ ex lemmate secundo, at lo $\Phi : (p-q+n)$ ex lemmate primo valores substituantur, resulta expressio:

$$\binom{p}{q-n} = \frac{\Phi : p}{\Phi : q \times \Phi : (p-q)} \times \frac{\binom{q}{n}}{\binom{p-q+n}{n}} = \frac{\binom{q}{n}}{\binom{p-q+n}{n}} \times \binom{p}{q}.$$

Problema V.

§. 20. *Si fuerit* $\binom{P}{Q} = \binom{p+m}{q+n}$, *ejus valorem ad hanc fo mam reducere:* $\binom{p}{q} M$, *ubi M absolute assignare lice inde quod* m *et* n *sint numeri integri positivi.*

Solutio.

Ex problemate I. invenimus $\binom{p+m}{q} = \frac{\binom{p+m}{m}}{\binom{p-q+m}{m}} \times \binom{p}{q}$. Qu si jam hic loco q ubique scribamus $q+n$, erit

$$\binom{p+m}{q+n} = \frac{\binom{p+m}{m}}{\binom{p-q-n+m}{m}} \times \binom{p}{q+n}.$$

Hic loco $\binom{p}{q+n}$ valorem ex problemate III. substituamus, quo facto fi

$$\binom{p+m}{q+n} = \frac{\binom{p+m}{m} \times \binom{p-q}{n}}{\binom{p-q-n+m}{m} \times \binom{q+n}{n}} \times \binom{p}{q},$$

ubi igitur erit

$$M = \frac{\binom{p+m}{m} \times \binom{p-q}{n}}{\binom{p-q-n+m}{m} \times \binom{q+n}{n}},$$

cujus valorem, ob m et n numeros integros positivos, semper a solute assignare licebit.

Problema VI.

§. 21. *Si fuerit* $\left(\frac{p}{Q}\right) = \left(\frac{p+m}{q-n}\right)$, *ejus valorem reducere ad formam* $M\left(\frac{p}{q}\right)$.

Solutio.

Ex problemate primo cum sit $\left(\frac{p+m}{q}\right) = \frac{\left(\frac{p+m}{m}\right)}{\left(\frac{p-q+m}{m}\right)} \times \left(\frac{p}{q}\right)$, hic ubique loco q scribatur $q-n$, ut prodeat

$$\left(\frac{p+m}{q-n}\right) = \frac{\left(\frac{p+m}{m}\right)}{\left(\frac{p-q+n+m}{m}\right)} \times \left(\frac{p}{q-n}\right),$$

Atque hic loco $\left(\frac{p}{q+n}\right)$ valor ex problemate IV substituatur, quo facto pro formula nostra hanc impetrabimus expressionem:

$$\left(\frac{p+m}{q-n}\right) = \frac{\left(\frac{p+m}{m}\right) \times \left(\frac{q}{n}\right)}{\left(\frac{p-q+n+m}{m}\right) \times \left(\frac{p-q+n}{n}\right)} \times \left(\frac{p}{q}\right)$$

Problema VII.

§. 22. *Si fuerit* $\left(\frac{p}{Q}\right) = \left(\frac{p-m}{q+n}\right)$, *ejus valorem reducere ad formam* $M\left(\frac{p}{q}\right)$.

Solutio.

In Problemate II invenimus $\left(\frac{p-m}{q}\right) = \frac{\left(\frac{p-q}{m}\right)}{\left(\frac{p}{m}\right)} \times \left(\frac{p}{q}\right)$, ubi si loco q scribamus $q+n$ orietur forma proposita

$$\left(\frac{p-m}{q+n}\right) = \frac{\left(\frac{p-q-n}{m}\right)}{\left(\frac{p}{m}\right)} \times \left(\frac{p}{q+n}\right).$$

Hinc si ex problemate III loco $\left(\frac{p}{q+n}\right)$ valor substituatur, orietur expréssio $\left(\frac{p-m}{q+n}\right) = \frac{\left(\frac{p-q-n}{m}\right) \times \left(\frac{p-q}{n}\right)}{\left(\frac{p}{m}\right) \times \left(\frac{q+n}{n}\right)} \times \left(\frac{p}{q}\right)$.

Problema VIII.

§. 23. *Si fuerit* $\left(\frac{P}{Q}\right) = \left(\frac{p-m}{q-n}\right)$, *ejus valorem ad formam si*
plicem $\left(\frac{p}{q}\right) M$ *reducere.*

Solutio.

Sumatur iterum ex problemate secundo expressio

$$\left(\frac{p-m}{q}\right) = \frac{\left(\frac{p-q}{m}\right)}{\left(\frac{p}{m}\right)} \times \left(\frac{p}{q}\right),$$

in eaque loco q scribatur $q-n$, ut oriatur forma proposita, quae e

$$\left(\frac{p-m}{q-n}\right) = \frac{\left(\frac{p-q+n}{m}\right)}{\left(\frac{p}{m}\right)} \times \left(\frac{p}{q-n}\right),$$

unde, substituendo loco characteris $\left(\frac{p}{q-n}\right)$ ejus valorem problem
IV inventum, prodibit $\left(\frac{p-m}{q-n}\right) = \frac{\left(\frac{p-q+n}{m}\right)\left(\frac{p}{q}\right)}{\left(\frac{p}{m}\right) = \left(\frac{p-q+n}{n}\right)} \times \left(\frac{p}{q}\right)$.

Corollarium.

§. 24. Quoties igitur denominator Q fuerit numerus integ
sive positivus sive negativus, tum loco q semper statui poterit
et quia $\left(\frac{p}{o}\right) = 1$, valor talis formulae $\left(\frac{P}{Q}\right)$ per nostras reductio
semper absolute assignari potest, quia in omnibus characteribus d
nominatores sunt vel m vel n, ideoque numeri integri. Tantum
tur superest, ut eos casus investigemus, quibus Q est quaepi
fractio sive positiva sive negativa, quae revocari possit ad $\left(\frac{p}{q}\right)$
ubi q erit fractio simplicissima ejusdem generis et quidem unit
minor; quamobrem totum negotium eo redit, ut valor hujus formu

$\left(\frac{p}{q}\right)$ indagetur, quando q est fractio. Pro his igitur casibus valorem formulae $\left(\frac{p}{q}\right)$ per formulam quandam integralem exprimemus.

Problema.

§. 25. *Valorem formulae $\left(\frac{p}{q}\right)$ per formulam integralem exprimere.*

Solutio.

Hunc in finem consideremus hanc formulam: $\int x^{q-1}\partial x\,(1-x)^n$, cujus valor, ab $x=0$ ad $x=1$ extensus, desiquetur per Δ, qui cum sit certa functio ipsius q, puta $f:q$, loco q hic scribamus $q+1$ et $\Delta' = f:(q+1)$, erit $\Delta-\Delta' = \int x^{q-1}\partial x\,(1-x)^{n+1}$; hocque modo ex quovis casu numeri n reperietur valor ipsius Δ pro casu $n+1$. Incipiamus a casu $n=0$ et valores ipsius Δ pro sequentibus numeris n ita se habebunt:

n	Δ
0	$\frac{1}{q}$
1	$\frac{1}{q(q+1)}$
2	$\frac{1\cdot 2}{q(q+1)(q+2)}$
3	$\frac{1\cdot 2\cdot 3}{q(q+1)(q+2)(q+3)}$.

Hinc jam manifestum est fore in genere:

$$\Delta = \frac{1}{q} \times \frac{1\cdot 2\cdot 3\cdot 4\ldots\ldots n}{(q+1)(q+2)(q+3)\ldots(q+n)}.$$

Cum nunc sit $\left(\frac{q+n}{n}\right) = \frac{(q+n)(q+n-1)\ldots\ldots(q+1)}{1\cdot 2\cdot 3\ldots\ldots n}$, evidens est fore $\Delta = \frac{1}{q}:\left(\frac{q+n}{n}\right)$, unde vicissim erit $\left(\frac{q+n}{n}\right) = \frac{1}{q\Delta}$. Sit nunc $q+n=p$, sive $n=p-q$, ut fiat $\left(\frac{p}{n}\right) = \frac{p}{p-q} = \left(\frac{p}{q}\right)$. Et cum jam sit $\Delta = \int x^{q-1}\partial x\,(1-x)^{p-q}$, concludimus fore $\left(\frac{p}{q}\right) = \frac{1}{q\int x^{q-1}\partial x\,(1-x)^{p-q}}$, ita ut valor hujus formulae integralis, ab $x = 0$ ad $x = 1$ extensus, perducat ad valorem characteris (

Corollarium.

§. 26. Quaecunque ergo fractiones loco p et q substituan[tur] semper curva algebraica exhiberi potest, a cujus quadratura, ea[que] definita, scilicet quando $x = 1$, valor formulae $(\frac{p}{q})$ pendeat.

Scholion.

§. 27. Analysis, qua hic usi sumus, videtur quidem tan[tum] locum habere casibus quibus n est numerus integer positivus, ne[c] ergo ad casus, quibus $p - q$ est fractio, applicari posse. Ver[um] ipsum principium continuitatis applicationem ad numeros fractos [sa]tis confirmare videtur; interim tamen juvabit consensus cum verit[ate] in casu aliunde cognito ostendisse. Consideretur ergo haec form[ula] $(\frac{1}{\frac{1}{2}})$, ubi $p = 1$ et $q = \frac{1}{2}$, eritque per reductionem generale[m] $(\frac{1}{\frac{1}{2}}) = \frac{\Phi : 1}{\Phi : \frac{1}{2} \times \Phi : \frac{1}{2}}$, quae expressio, ob $\Phi : 1 = 1$ et $\Phi : \frac{1}{2} = \frac{1}{2}\sqrt{}$ evadit $\frac{4}{\pi}$. Nunc igitur videamus num ista expressio conveniat c[um] $\frac{1}{\frac{1}{2}\int\frac{\partial x}{\sqrt{x}}(1 - x)^{\frac{1}{2}}}$. At vero iste denominator, posito $x = yy$, abit

$$\int \partial y \sqrt{1 - yy} = \int \frac{\partial y}{\sqrt{1 - yy}} - \int \frac{yy\partial y}{\sqrt{1 - yy}}.$$

Constat autem, his integralibus ab $y = 0$ ad $y = 1$ extensis, e[sse] $\int \frac{\partial y}{\sqrt{1 - yy}} = \frac{\pi}{2}$ et $\int \frac{yy\partial y}{\sqrt{1 - yy}} = \frac{\pi}{4}$, ita ut differentia sit $\frac{\pi}{4}$, ideo[que] valor hic inventus $\frac{4}{\pi}$ egregie convenit cum praecedente.

Scholion 2.

§. 28. Quod autem ad formulam integralem $\int x^{q-1}\partial x(1 - x)$ attinet, ex analysi patet, ejus valorem, ab $x = 0$ ad $x = 1$ ex[ten]sum, finitum fieri non posse, nisi sit $q > 0$, simulque $p - q >$

Quoniam vero in nostra potestate est istos numeros p et q, ad quos formulam generalem $\left(\frac{P}{Q}\right)$ reduximus, intra limites 0 et 1 redigere, formula integralis inventa semper ad omnes plane casus transferri poterit. Ceterum jam manifestum est, casibus quibus Q est numerus integer, sive positivus sive negativus, evolutionem actu institui posse, hocque etiam succedet casibus quibus P — Q est numerus integer, unde usus nostrae formulae integralis erit amplissimus casibus quibus neque Q neque P — Q sunt integri. Casus maxime memorabilis hic occurrit, quando P est numerus integer sive positivus sive negativus; tum enim, quaecunque fractio pro Q accipiatur, valor hujus expressionis $\left(\frac{P}{Q}\right)$ per peripheriam circuli assignari poterit.

Problema.

§. 20. *Valorem formulae $\left(\frac{P}{Q}\right)$, quoties P fuerit numerus integer sive positivus sive negativus, ad quadraturam circuli reducere.*

Solutio.

Quando P est numerus integer sive positivus sive negativus, ista forma semper reduci poterit ad hanc: $\left(\frac{0}{q}\right)$, ita ut $p = 0$; sicque per formulam integralem erit $\left(\frac{0}{q}\right) = \frac{1}{q \int x^{q-1} \partial x (1-x)^{-q}}$; quamobrem hanc formulam integralem accuratius evolvamus, quae reducta ad hanc formam: $\int \frac{\partial x}{x} \left(\frac{x}{1-x}\right)^q$, posito $\frac{x}{1-x} = z$, sive $x = \frac{z}{1+z}$, a $z = 0$ usque ad $z = \infty$ extendi debet. Ob $\frac{\partial x}{x} = \frac{\partial z}{z(1+z)}$ vero formula transmutatur in hanc: $\int \frac{z^{q-1} \partial z}{1+z}$. At vero olim ostendimus hujus formulae integralis $\int \frac{z^{m-1} \partial z}{(1+z)^n}$ valorem, a $z = 0$ ad $z = \infty$ extensum, esse $\frac{\pi}{n \sin. \frac{m\pi}{n}}$. Nostro igitur casu erit $m = q$ et $n = 1$, unde nostrum integrale erit $\frac{\pi}{\sin. q\pi}$, quo substituto habebimu

$$\left(\frac{0}{q}\right) = \frac{1}{\frac{q\pi}{\sin. q\pi}} = \frac{\sin. q\pi}{q\pi}.$$

Corollarium.

§. 30. Quoties q fuerit numerus integer sive positivus siv negativus formula illa, ob $\sin. q\pi = 0$, semper in nihilum abit, sol casu excepto $q = 0$. Sumto autem q quasi infinite parvo ob $\sin. q\pi = q\pi$ erit utique $\left(\frac{0}{q}\right) = 1$, quemadmodum rei natura postula

Corollarium.

§. 31. Cum per reductionem nostram generalem sit:

$$\left(\frac{0}{q}\right) = \frac{\Phi : 0}{\Phi : q \times \Phi : -q},$$

ob $\Phi : 0 = 1$ erit $\Phi : q \times \Phi : -q = \frac{q\pi}{\sin. q\pi}$, ita ut, quicunque va lores ipsi q tribuantur, tam valores $\Phi : q$ quam $\Phi : -q$ ad quan titates transcendentes superiorum generum referantur; interim tame eorum productum per quadraturam circuli exprimetur.

Scholion.

§. 32. Cum sit $\left(\frac{p}{q}\right) = \frac{1}{q\int x^{q-1}\partial x(1-x)^{p-q}}$, siquidem ho integrale ab $x = 0$ ad $x = 1$ extendatur, si istos valores in theo rematibus supra allatis circa relationem formularum $\left(\frac{p}{q}\right)$ substitua mus, sequentia nanciscemur theoremata, pro relatione formularu integralium, quae maxime videntur memorabilia.

Theorema.

§. 33. *Si sequentia integralia ab* $x = 0$ *ad* $x = 1$ *extendantu semper haec aequalitas subsistet:*

[illegible] $\partial x\,(1-x)^{n-a}\times\int x^{b-1}\,\partial x\,(1-x)^{n-a-b}$

[illegible] $\partial x\,(1-x)^{n-b}\times\int x^{a-1}\,\partial x\,(1-x)^{n-b-a}$.

Corollarium.

[illegible] Si in talibus formulis exponens ipsius x evanescat, [illegible] $\int\partial x(1-x)^p$, ejus valor absolute assignari potest, erit [illegible] At si exponens ipsius $1-x$ evanescat, ut habeamus [illegible] ejus valor manifesto erit $\frac{1}{p+1}$: sin autem formula integra- [illegible] $\partial x\,(1-x)^{-q}$, ejus valor, ut vidimus, erit [illegible] unde plures relationes notatu dignae oriuntur. Ceterum hic [illegible] exponentes ipsius x et $1-x$ inter se permutari [illegible] ut semper sit $\int x^p\,\partial x\,(1-x)^q=\int x^q\,\partial x\,(1-x)^p$.

Theorema.

§. 35. *Si omnia integralia ab* $x=0$ *ad* $x=1$ *extendantur, productum ex his tribus formulis integralibus:*

$$\int x^{a-1}\,\partial x\,(1-x)^{n-a};\quad \int x^{b-1}\,\partial x\,(1-x)^{n-a-b};$$
$$\int x^{c-1}\,\partial x\,(1-x)^{n-a-b-c}$$

semper eundem valorem retinebit, quomodocunque litterae a, b, c, *inter se permutentur.*

Theorema.

§. 36. *Si omnia integralia ab* $x=0$ *ad* $x=1$ *extendantur, productum ex his quatuor formulis integralibus semper eundem valorem retinebit, quomodocunque litterae* a, b, c, d *inter se permutantur scilicet:*

$$\int x^{a-1}\,\partial x\,(1-x)^{n-a}\times\int x^{b-1}\,\partial x\,(1-x)^{n-a-b}$$
$$\int x^{c-1}\,\partial x\,(1-x)^{n-a-b-c}\times\int x^{d-1}\,\partial x\,(1-x)^{n-a-b-c-d}.$$

Corollarium.

§. 36. Hic evidens est numerum talium formularum integr- lium continuo ulterius augeri posse unde numerus variationum, qu- in singulis productis locum habere possunt, in infinitum excresc- ubi quidem observo, casum simplicissimum Theorematis primi pr- sus convenire cum iis quae olim de relatione inter diversas for- las integrales proposueram.

Scholion.

§. 37. Omnia illa integralia in hac forma generali con- nentur: $\int x^p \partial x (1-x)^q$, quam constat plurimis modis in alias fo- mas transmutari posse, dum scilicet binos exponentes p et q quo- numero integro sive augere sive minuere licet, atque inter has d- versas formas sine dubio simplicissima est ea, in qua isti exponentes i- tra limites 0 et -1 deprimuntur, quam transformationem per s- quentes reductiones commodissime institui posse facile patet:

$$\int x^p \partial x (1-x)^q = \frac{p}{p+q+1} \int x^{p-1} \partial x (1-x)^q,$$

$$\int x^p \partial x (1-x)^q = \frac{p+q+2}{p+1} \int x^{p+1} \partial x (1-x)^q,$$

$$\int x^p \partial x (1-x)^q = \frac{q}{p+q+1} \int x^p \partial x (1-x)^{q-1},$$

$$\int x^p \partial x (1-x)^q = \frac{p+q+2}{q+1} \int x^p \partial x (1-x)^{q+1}.$$

Saepenumero etiam haec reductio, qua binae praecedentium sim- instituuntur, insignem usum praestat:

$$p \int x^{p-1} \partial x (1-x)^q = q \int x^p \partial x (1-x)^{q-1}.$$

Problema.

§. 38. *Describere lineam curvam, cujus abscissae* x *respond- applicata* $y = \left(\frac{m}{x}\right)$, *ubi* m *denotet numerum integr- positivum.*

Solutio.

Hic primo investigentur applicatae, quando abscissae x numeris integris tribuuntur, easque immediate ex forma $y = \left(\frac{m}{x}\right)$ facile definire licet, cum sit $\left(\frac{m}{0}\right) = 1$; $\left(\frac{m}{1}\right) = m$; $\left(\frac{m}{2}\right) = \frac{m(m-1)}{1 \cdot 2}$; etc, donec perveniatur ad $x = m$, ubi iterum est $\left(\frac{m}{m}\right) = 1$. Praeter hos enim casus omnes applicatae, quae respondent valoribus negativis ipsius x, quin etiam majoribus quam m, evanescunt. At vero jam observavimus hanc curvam semper praeditam esse diametro, quem praebet applicata abscissae $x = \frac{1}{2} m$ respondens, unde sufficiet casus tantum evolvere, quibus $x > \frac{1}{2} m$.

At si abscissae x valores fractos tribuamus, necesse est primum formulam $\left(\frac{m}{x}\right)$ ad hanc reducere: $\left(\frac{0}{x}\right)$, quippe cujus valorem ostendimus esse $\frac{\sin. \pi x}{\pi x}$, id quod facillime praestatur ope reductionis supra allatae, qua ostendimus esse $\left(\frac{p+m}{q}\right) = \frac{\left(\frac{p+m}{m}\right)}{\left(\frac{p-q+m}{m}\right)} \times \left(\frac{p}{q}\right)$.

Nunc igitur fiat $p = 0$ et $q = x$ atque colligitur

$$\left(\frac{m}{x}\right) = \frac{\left(\frac{0}{x}\right)}{\left(\frac{m-x}{m}\right)} = \frac{\sin. \pi x}{\pi x} : \left(\frac{m-x}{m}\right).$$

Ad formulam evolvendam unicum intervallum abscissae $= 1$ percurrisse sufficiet, quem in finem statuamus $x = n + q$, ita ut q sit fractio unitate minor, existente n numero integro quovis, eritque $\sin. \pi x = \pm \sin. \pi q$, ubi signum $+$ valebit si n sit numerus par, $-$ vero si impar. Hoc observato habebimus

$$y = \pm \frac{\sin. q\pi}{\pi(q+n)} : \left(\frac{m-n-q}{m}\right),$$

ex qua formula jam omnes valores intermedii facile assignari poterunt, sicque tota curva erit descripta.

Corollarium.

§. 39. Hic evidens est istius curvae maximam applicatam semper respondere abscissae $x = \frac{1}{2}m$, quae simul erit curvae dia- meter, cujus determinatio pro casibus, quibus m est numerus pa nulla laborat difficultate; at si m sit numerus impar, ista maxim applicata a quadratura circuli pendebit, quam in sequente proble- mate investigemus.

Problema.

§. 40. *Investigare maximam applicatam curvae modo ante de- scriptae, qua abscissae* x *respondeat applicata* $y = \left(\frac{m}{x}\right)$.

Solutio.

Designemus hanc maximam applicatam littera M, ita u $M = \left(\frac{m}{\frac{1}{2}m}\right)$, atque hic duos casus evolvi oportebit, prouti m fuer vel numerus par vel impar. Sit igitur primo $m = 2i$, erit $M = \left(\frac{2i}{i}\right)$ cujus valorem jam dudum constat reduci ad hanc expressionem:

$$\frac{2 \cdot 6 \cdot 10 \cdot 14 \ldots\ldots (4i-2)}{1 \cdot 2 \cdot 3 \cdot 4 \ldots\ldots i}.$$

Hinc enim patet, pro casu $i = 1$ fore $M = 2$. Si $i = 2$, er $M = 6$; si $i = 3$, erit $M = 20$ et ita porro.

At si m fuerit numerus impar, ponatur $m = 2i + 1$, eritqu $M = \left(\frac{2i+1}{i+\frac{1}{2}}\right)$, qui valor, si ad numeros hypergeometricos reduca- tur, fiet $M = \frac{\Phi:(2i+1)}{(\Phi:(i+\frac{1}{2}))^2}$, ubi est $\Phi:(2i+1) = 1 \cdot 2 \cdot 3 \cdot 4 \ldots (2i+1$ At cum sit $\Phi:\frac{1}{2} = \frac{1}{2}\sqrt{\pi}$, hincque porro $\Phi:(1+\frac{1}{2}) = \frac{1 \cdot 3}{2 \cdot 2} \cdot \sqrt{\pi}$ $\Phi:(2+\frac{1}{2}) = \frac{1 \cdot 3 \cdot 5}{2 \cdot 2 \cdot 2} \cdot \sqrt{\pi}$, ideoque in genere

$$\Phi:(i+\tfrac{1}{2}) = \frac{1 \cdot 3 \cdot 5 \ldots (2i+1)}{2 \cdot 2 \cdot 2 \ldots\ldots 2} \cdot \sqrt{\pi}, \text{ erit}$$

$$\frac{\Phi:(2i+1)}{\Phi:(i+\frac{1}{2})} = \frac{2 \cdot 4 \cdot 6 \cdot 8 \ldots 2i \times 2 \cdot 2 \cdot 2 \ldots\ldots 2}{\sqrt{\pi}}, \text{ sive}$$

$$\frac{\Phi:(2i+1)}{\Phi:(i+\frac{1}{2})} = \frac{2}{\sqrt{\pi}} \times 4 \cdot 8 \cdot 12 \cdot 16 \ldots 4i,$$

quae expressio denuo per $\Phi:(i+\frac{1}{2})$ divisa subministrat istam:

$$\frac{(\Phi:i)^2(2i+1)}{(\Phi:(i+\frac{1}{2}))^2} = \frac{4}{\pi} \times \frac{8 \cdot 16 \cdot 24 \cdot 32 \ldots 8i}{3 \cdot 5 \cdot 7 \cdot 9 \ldots (2i+1)}.$$

Ita pro casu $m = 1$ erit $i = 0$, et $M = \frac{4}{\pi}$;

pro casu $m = 3$ erit $i = 1$ et $M = \frac{8}{3} \cdot \frac{4}{\pi} = \frac{32}{3\pi}$;

pro casu $m = 5$ erit $i = 2$ et $M = \frac{8 \cdot 16}{3 \cdot 5} \cdot \frac{4}{\pi} = \frac{512}{15\pi}$;

et ita porro.

Problema.

§ 41. *Describere curvam, cujus abscissis* x *respondeant applicatae* $\left(\frac{-m}{x}\right)$, *denotante* m *numerum quemcunque integrum positivum.*

Solutio.

Ex ipsa hac formula $y = \left(\frac{-m}{x}\right)$ sine difficultate eliciuntur applicatae pro omnibus abscissis per numeros integros expressis; erit enim $\left(\frac{-m}{0}\right) = 1$; $\left(\frac{-m}{1}\right) = -m$; $\left(\frac{-m}{2}\right) = \frac{m(m+1)}{1 \cdot 2}$; et ita porro, quae ergo applicatae signis alternantibus in infinitum progrediuntur. Pro applicatis praecedentibus notetur esse $\left(\frac{-m}{-m}\right) = 1$; $\left(\frac{-m}{-m-1}\right) = -m$; et cet. At vero inter abscissas $x = 0$ et $x = -m$ applicatae intermediae abscissis $-1, -2, -3, \ldots -(m+1)$ respondentes omnes nihilo erunt aequales. Si abscissae x valores fracti tribuantur, formulam $\left(\frac{-m}{x}\right)$ iterum reduci convenit ad formulam $\left(\frac{0}{x}\right)$. Supra autem invenimus esse

$$\left(\frac{p-m}{q}\right) = \frac{\left(\frac{p-q}{m}\right)}{\left(\frac{p}{m}\right)} \times \left(\frac{p}{q}\right).$$

Quod si jam hic faciamus $p = 0$ et $q = x$, erit

$$\left(\frac{-m}{x}\right) = \frac{\left(\frac{-x}{m}\right)}{\left(\frac{0}{m}\right)} \times \left(\frac{0}{x}\right) = \frac{\left(\frac{-x}{m}\right)}{\left(\frac{0}{m}\right)} \cdot \frac{\sin.\pi x}{\pi x}.$$

Quia igitur formula $\left(\frac{o}{m}\right)$ semper evanescit, numerator vero, ob ex-
clusos jam numeros integros pro x, nunquam evanescere potest, evi-
dens est istam applicatam y semper esse infinitam, qui est cas
prorsus singularis curvae infinitas habentis applicatas finitas, int
quas intermediae omnes evadant infinite magnae, cujusmodi cas
mihi quidem adhuc nondum occurrit, quem ergo attentione Geometr
rum haud indignum esse arbitror.