E752

DE INTEGRALIBUS QUIBUSDAM INVENTU DIFFICILLIMIS.

AUCTORE

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§. 1. Obtulerat se mihi jam dudum haec formula integralis $-\int \frac{\partial x\, lx}{\sqrt{1-xx}}$, cujus valorem, ab $x=0$ ad $x=1$ extensum, cognoscere optabam. Suspicabar enim, non sine ratione, eum partim per quadraturam circuli, partim per logarithmos exprimi: Verum omnes conatus istum valorem investigandi irriti fuere, atque semper in eiusmodi series infinitas incidi, quarum summam assignare nullo modo licebat. Primo enim evolvi formam radicalem in seriem more solito, ut haberem hanc formulam:

$$s = -\int \partial x\, lx \left(1+\tfrac{1}{2}x^2+\tfrac{1\cdot 3}{2\cdot 4}x^4+\tfrac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^6+\text{etc.}\right),$$

pro qua integranda cum sit:

$$\int -x^n\partial x\, lx = -\frac{x^{n+1}}{n+1}lx+\int\frac{x^n\partial x}{n+1} = -\frac{x^{n+1}}{n+1}lx+\frac{x^{n+1}}{(n+1)^2}.$$

Sumendo $x=1$ erit $\int -x^n\partial x\, lx = \frac{1}{(n+1)^2}$, et singulis terminis hoc modo integratis reperitur:

$$s = \frac{1}{1^2}+\frac{1}{2}\cdot\frac{1}{3^2}+\frac{1\cdot 3}{2\cdot 4}\cdot\frac{1}{5^2}+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{1}{7^2}+\text{etc.}$$

Haec autem series ita est comparata, ut ejus summatio nullo modo pateat.

§. 2. Conatus igitur eram factorem lx in seriem resolvere, cujus singuli termini perducerent ad formulas integrabiles, sed pluribus tentaminibus institutis res non successit, donec tandem nuper in idoneam resolutionem ipsius lx in seriem incidi, qua totum negotium feliciter expediri poterat. Scilicet cum sit $lx = \frac{1}{2} lxx$, hic loco xx scripsi $1 - (1 - xx)$. Hinc enim statim prodibat:

$$-lxx = \frac{1-xx}{1} + \frac{(1-xx)^2}{2} + \frac{(1-xx)^3}{3} + \text{etc.}$$

sicque formula proposita in hanc transformabatur:

$$s = \int \partial x \left(\frac{(1-xx)^{\frac{1}{2}}}{2} + \frac{(1-xx)^{\frac{3}{2}}}{4} + \frac{(1-xx)^{\frac{5}{2}}}{6} + \text{etc.}\right),$$

cujus omnes partes facile ad quadraturam circuli reducuntur.

§. 3. Quo hoc facilius praestari possit constituamus hanc reductionem:

$$\int \partial x (1 - xx)^n = Ax(1 - xx)^n + B \int \partial x (1 - xx)^{n-1},$$

unde differentiando et per $\partial x (1 - xx)^{n-1}$ dividendo oritur haec aequatio: $1 - xx = A(1 - xx) - 2nAx^2 + B$, unde fieri debet $A + B = 1$ et $A + 2nA = 1$. Hinc colligitur $A = \frac{1}{2n+1}$ et $B = \frac{2n}{2n+1}$, quocirca, sumto $x = 1$, habebitur ista reductio generalis:

$$\int \partial x (1 - xx)^n = \frac{2n}{2n+1} \int \partial x (1 - xx)^{n-1},$$

et loco n scribendo $\lambda + \frac{1}{2}$ erit:

$$\int \partial x (1 - xx)^{\lambda + \frac{1}{2}} = \frac{2\lambda + 1}{2\lambda + 2} \int \partial x (1 - xx)^{\lambda - \frac{1}{2}}.$$

§. 4. Cum nunc, integrationes semper ab $x = 0$ ad $x = 1$ extendendo, sit $\int \frac{\partial x}{\sqrt{1-xx}} = \frac{\pi}{2}$, reperietur per istam reductionem:

$$\int \partial x (1-xx)^{\frac{1}{2}} = \frac{1}{2} \cdot \frac{\pi}{2};$$
$$\int \partial x (1-xx)^{\frac{3}{2}} = \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2};$$
$$\int \partial x (1-xx)^{\frac{5}{2}} = \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2};$$
$$\int \partial x (1-xx)^{\frac{7}{2}} = \frac{7}{8} \cdot \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2};$$
etc.

His igitur valoribus ordine introductis nanciscemur sequentem seriem:

$$s = \frac{\pi}{2} \left(\frac{1}{2 \cdot 2} + \frac{1 \cdot 3}{2 \cdot 4 \cdot 4} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 6} + \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 8} + \text{etc.} \right),$$

quae series multo simplicior est ea quam supra attulimus; interim tamen insignem affinitatem tenet, atque adeo istae duae series inter se sunt aequales.

§. 5. Ut summam hujus seriei investigemus, consideremus hanc generaliorem:

$$v = \frac{t^2}{2 \cdot 2} + \frac{1 \cdot 3}{2 \cdot 4 \cdot 4} t^4 + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 6} t^6 + \text{etc.}$$

unde differentiando adipiscimur:

$$\frac{\partial v}{\partial t} = \frac{t}{2} + \frac{1 \cdot 3}{2 \cdot 4} t^3 + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} t^6 + \text{etc.}$$

cujus valor manifesto est $\frac{1}{t} \left(\frac{1}{\sqrt{1-tt}} - 1 \right)$. Hinc ergo fiet $v = \int \frac{\partial t}{t} \left(\frac{1}{\sqrt{(1-tt)}} - 1 \right)$, quo integrali invento poni debebit $t = 1$; ac tum erit summa quaesita $s = \frac{\pi}{2} v$.

§. 6. Hic primo irrationalitatem abigamus, ponendo $\sqrt{1-tt}=u$, ut sit $t=\sqrt{1-uu}$. Nunc autem integrationem extendi oportebit a termino $t=0$, hoc est $u=1$, usque ad $t=1$, hoc est $u=0$. Tum autem erit $\frac{\partial t}{t} = -\frac{u\partial u}{1-uu}$, ex quo conficitur:

$$v = -\int \frac{u\partial u}{1-uu}\left(\frac{1-u}{u}\right) = -\int \frac{\partial u}{1+u},$$

cujus integrale praebet $v = C - l(1+u)$, ubi constans C esse debet $l2$. Nunc igitur facto $u=0$ prodit $v=l2$ ideoque $s=\frac{1}{2}\pi l2$, qui ergo est valor formulae initio propositae, tantopere desideratus. Praeterea vero etiam series ante inventas nunc summare licet, scilicet series ex §. 1, quae erat

$$\frac{1}{1\cdot 1} + \frac{1}{2}\cdot\frac{1}{3^2} + \frac{1\cdot 3}{2\cdot 4}\cdot\frac{1}{5^2} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{1}{7^2} = \frac{1}{2}\pi l2,$$

tum vero

$$\frac{1}{2\cdot 2} + \frac{1\cdot 3}{2\cdot 4\cdot 4} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 6} + \text{etc.} = l2,$$

quae summationes per se satis abstrusae videri possunt. Hinc subjungimus sequens

Theorema.

Proposita formula integrali $\int \frac{-\partial x\, lx}{\sqrt{1-xx}}$ *ejus valor, a termino* $x=0$ *usque ad* $x=1$ *extensus, est* $=\frac{1}{2}\pi l2$.

§. 7. Si hanc formulam comparemus cum ista simpliciore: $\int \frac{-\partial x\, lx}{1-x}$, mirum utique erit, hanc non simili modo tractari posse, cum tamen aliunde constet, ejus valorem ab $x=0$ ad $x=1$ extensum, esse $=\frac{\pi\pi}{6}$. Cum enim sit:

$$\frac{1}{1-x}=1+x+x^2+x^3+x^4+\text{etc.} \text{ et}$$

$$\int -x^{n-1}\partial x\, lx \left[{ab\ x=0 \atop ad\ x=1}\right]=\frac{1}{nn},$$

inde orietur haec series:

$$\int\frac{-\partial x\, lx}{1-x}=1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+\text{etc.}$$

cujus summam olim primus inveni esse $=\frac{\pi\pi}{6}$, quem tamen valorem ex ipsa formula integrali nullo adhuc modo elicere potui. Hinc ergo sequitur ista proportio satis memorabilis:

$$\int\frac{-\partial x\, lx}{1-x}:\int\frac{-\partial x\, lx}{\sqrt{1-xx}}=\pi:3\,l\,2.$$

§. 8. His vestigiis insistenti, formulam integralem multo latius patentem mihi simili modo tractare licuit, quam operationem in sequente problemate explicabo:

Problema.

Proposita formula integrali $S=\int\frac{-x^{m-1}\partial x\, lx}{\sqrt[n]{(1-x^n)^m}}$, *ejus valorem, ab* $x=0$ *ad* $x=1$ *extensum, per expressionem finitam, tantum arcubus circularibus et logarithmis constantem, exhibere.*

Solutio.

§. 9. Hic ante omnia observasse juvabit hanc formulam: $\int\frac{x^{m-1}\partial x}{\sqrt[n]{(1-x^n)^m}}$ ab irrationalitate prorsus liberari posse, ponendo $\frac{x}{\sqrt[n]{(1-x^n)}} = t$. Hinc enim ista formula abit in hanc: $\int \frac{t^m \partial x}{x}$; tum autem erit $x^n = t^n (1 - x^n)$ ideoque $x^n = \frac{t^n}{1+t^n}$, sive in logarithmis:

$$n\,l\,x = n\,l\,t - l\,(1 + t^n),$$

unde differentiando prodit $\frac{\partial x}{x} = \frac{\partial t}{t(1+t^n)}$, ita ut per hanc substitutionem prodeat $\int \frac{t^{m-1} \partial t}{1+t^n}$, ubi, quia, sumto $x = 0$, fit etiam $t = 0$, at posito $x = 1$, fit $t = \infty$, hoc integrale a $t = 0$ usque ad $t = \infty$ est extendendum. Jam dudum autem demonstravi, istius formulae valorem hoc casu esse $\frac{\pi}{n \sin. \frac{m\pi}{n}}$.

Hinc ergo sequitur, etiam formulae integralis:

$$\int \frac{x^{m-1} \partial x}{\sqrt[n]{(1 - x^n)^m}},\ \text{ab } x = 0 \text{ ad } x = 1 \text{ extensae}$$

valorem esse $\frac{\pi}{n \sin. \frac{m\pi}{n}}$, cujus loco, brevitatis gratia, scribamus Δ.

§. 10. Hoc praenotato in ipsa formula proposita loco lx scribamus $\frac{1}{n} l x^n$, hujusque loco porro $\frac{1}{n} l\,(1 - (1 - x^n))$, sicque, facta evolutione habebimus:

$$-\,l\,x = \frac{1 - x^n}{n} + \frac{(1 - x^n)^2}{2n} + \frac{(1 - x^n)^3}{3n} + \text{etc.},$$

qua serie substituta, formula nostra integralis induet hanc formam:

$$S = \int x^{m-1}\partial x\left(\tfrac{1}{n}(1-x^n)^{1-\frac{m}{n}} + \tfrac{1}{2n}(1-x^n)^{2-\frac{m}{n}} + \tfrac{1}{3n}(1-x^n)^{3-\frac{m}{n}} + \text{etc.}\right)$$

cujus singula membra ad valorem ante introductum $\frac{\pi}{m \sin. \frac{m\pi}{n}}$ revocare licebit. Hunc enim in finem constituamus hanc reductionem generalem:

$$\int x^{m-1}\partial x\,(1-x^n)^{\lambda} = A\int x^{m-1}\partial x\,(1-x^n)^{\lambda-1} + Bx^m(1-x^n)^{\lambda},$$

factaque differentiatione ac divisione per

$$x^{m-1}\partial x\,(1-x^n)^{\lambda-1},$$

prodit haec aequatio:

$$1 - x^n = A + Bm(1-x^n) - \lambda n B x^n,$$

unde literae A et B ita definiuntur:

$$A = \frac{\lambda n}{m+\lambda n} \quad \text{et} \quad B = \frac{1}{m+\lambda n}.$$

Quamobrem, quia omnia haec integralia ab $x = 0$ ad $x = 1$ sunt extenden dn, habebimus hanc reductionem generalem:

$$\int x^{m-1}\partial x\,(1-x^n)^{\lambda} = \frac{\lambda n}{m+\lambda n}\int x^{m-1}\partial x\,(1-x^n)^{\lambda-1}.$$

§. 11. Hujus reductionis ope singulas partes evolvamus, ac pro prima parte erit $\lambda = 1 - \frac{m}{n}$ ideoque $\lambda n = n - m$, unde colligitur:

$$\int x^{m-1}\partial x\,(1-x^n)^{1-\frac{m}{n}} = \frac{n-m}{n}\Delta.$$

Pro secunda parte erit $\lambda = 2 - \frac{m}{n}$, sive $\lambda n = 2n - m$ hincque colligitur pars secunda:

$$\int x^{m-1}\partial x(1-x^n)^{2-\frac{m}{n}} = \frac{n-m}{n}\cdot\frac{2n-m}{2n}\cdot\Delta.$$

Pro tertia parte, ob $\lambda = 3-\frac{m}{n}$ et $\lambda n = 3n-m$, erit:

$$\int x^{m-1}\partial x(1-x^n)^{3-\frac{m}{n}} = \frac{n-m}{n}\cdot\frac{2n-m}{2n}\cdot\frac{3n-m}{3n}\cdot\Delta.$$

Eodemque modo erit pars quarta:

$$\int x^{m-1}\partial x(1-x^n)^{4-\frac{m}{n}} = \frac{n-m}{n}\cdot\frac{2n-m}{2n}\cdot\frac{3n-m}{3n}\cdot\frac{4n-m}{4n}\cdot\Delta,$$

et ita porro.

His igitur singulis partibus colligendis pro valore quaesito S hanc habebimus expressionem:

$$S = \Delta\left(\frac{n-m}{n\cdot n} + \frac{(n-m)(2n-m)}{n\cdot 2n\cdot 2n} + \frac{(n-m)(2n-m)(3n-m)}{n\cdot 2n\cdot 3n\cdot 3n} + \text{etc.}\right),$$

cujus ergo seriei summam investigari oportet.

§. 12. Hunc in finem consideremus istam seriem generaliorem:

$$T = \frac{n-m}{n\cdot n}t^n + \frac{(n-m)(2n-m)}{n\cdot 2n\cdot 2n}t^{2n} + \frac{(n-m)(2n-m)(2n-m)}{n\cdot 2n\cdot 3n\cdot 3n}t^{3n} + \text{etc.}$$

eritque differentiatione instituta et per t multiplicando:

$$\frac{t\partial T}{\partial t} = \frac{n-m}{n}t^n + \frac{(n-m)(2n-m)}{n\cdot 2n}t^{2n} + \frac{(n-m)(2n-m)(3n-m)}{n\cdot 2n\cdot 3n}t^{3n} + \text{etc.},$$

cujus seriei summa manifesto est $(1-t^n)^{-\frac{(n-m)}{n}} - 1$, unde ergo deducitur $\partial T = \frac{\partial t}{t}\left((1-t^n)^{-\frac{(n-m)}{n}} - 1\right)$, consequenter habebimus $T = \int\frac{\partial t}{t}\left((1-t^n)^{-\frac{(n-m)}{n}} - 1\right)$, quod integrale a termino $t=0$ usque ad $t=1$ extendi debet, quo facto erit noster valor quaesitus $S = \Delta T$.

§. 13. Nunc igitur jam tantum sumus lucrati, ut res deducta sit ad novam quidem formulam integralem sed nullos logarithmos involventem. Hanc vero formulam adeo ad rationalitatem perducere licebit statuendo $1 - t^n = u^n$; tum enim fiet $\frac{\partial t}{t} = - \frac{u^{n-1} \partial u}{1 - u^n}$, atque hinc nanciscemur

$$T = - \int \frac{(u^{m-1} - u^{n-1}) \partial u}{1 - u^n},$$

quod integrale, cum a termino $t = 0$ usque ad $t = 1$ extendi debebat nunc extendi, debet ab $u = 1$ usque ad $u = 0$.

Permutatis igitur terminis integrationis fiet:

$$S = \Delta \int \frac{(u^{m-1} - u^{n-1}) \partial u}{1 - u^n} \left[\begin{matrix} ab\ u = 0 \\ ad\ u = 1 \end{matrix} \right],$$

quod integrale jam certe per logarithmos et arcus circulares exprimere licet, sicque problemati proposito plane est satisfactum.

§. 14. Hujus formulae pars posterior integrationem sponte admittit, cum sit $\int \frac{u^{n-1} \partial u}{1 - u^n} = - \frac{1}{n} l (1 - u^n)$, qui valor jam evanescit facto $u = 0$, at vero pro altero termino prodit infinitus; pars prior integrata continet quoque tale membrum $- \frac{1}{n} l (1 - u)$, quod cum praecedente conjunctum dat $\frac{1}{n} l \frac{1 - u^n}{1 - u}$. Cum igitur sit $\frac{1 - u^n}{1 - u} = n$, ambo haec membra junctim sumta praebebunt $\frac{1}{n} l\, n$; omnes autem reliquae integralis partes habebunt finitam magnitudinem;

§. 15. Quanquam autem jam passim praecepta sunt tradita, integralia talium formularum inveniendi, haud inutile fore arbitror totam hanc integrationem ex primis principiis repetere; atque modo parumper discrepante tractare quam ergo investigationem, succinctius quam vulgo fieri solet, hic adjungam.

Problema.

Proposita formula integrali hac: $T = \int \frac{u^{m-1} - u^{n-1}}{1 - u^n} \partial u$, *ejus valorem, a termino* $u = 0$ *usque ad* $u = 1$ *extensum, investigare.*

Solutio:

§. 16. Modo notavimus, partis posterioris integrale esse $\frac{1}{n} l(1 - u^n)$, ejusque valorem infinitum, casu $u = 1$, a parte priore iterum destrui, unde solius partis prioris integrationem tradere sufficiet; hanc ob rem statuamus $U = \int \frac{u^{m-1} \partial u}{1 - u^n}$, ita ut sit $\partial U = \frac{u^{m-1} \partial u}{1 - u^n}$, ubi cum denominator manifesto habeat factorem $1 - u$, inde nascitur talis fractio partialis: $\frac{A \partial u}{1 - u}$, ubi erit $A = \frac{u^{m-1}(1 - u)}{1 - u^n}$, posito scilicet $u = 1$. Modo autem vidimus, fractionis $\frac{1 - u}{1 - u^n}$, valorem esse $\frac{1}{n}$, ita ut sit $A = \frac{1}{n}$, hincque orietur prima pars integralis $\frac{1}{n} \int \frac{\partial u}{1 - u} = - \frac{1}{n} l(1 - u)$ quae cum posteriore parte ipsius T conjuncta producit, ut vidimus, valorem $\frac{1}{n} l n$.

§. 17. Pro reliquis partibus hujus integralis inveniendis sit $1 - 2u \cos. \theta + uu$ factor quicunque denominatoris $1 - u^n$, quem ita comparatum esse oportet, ut, posito $uu = 2u \cos. \theta - 1$, etiam ipse denominator evanescat, ex qua conditione angulum θ determinare licebit. Hinc autem sequitur fore in genere $u^\lambda = 2u^{\lambda-1} \cos. \theta - u^{\lambda-2}$, ex qua forma intelligitur, potestates ipsius u seriem constituere recurrentem, cujus scala relationis est $2 \cos. \theta, -1$ atque hinc omnes potestates altiores ipsius u per solam primam et constantes difinire licebit. Evidens autem est, etiam quaevis multipla harum potestatum, veluti Auu, Au^3, Au^4, etc. secundum eandem scalam relationis $2 \text{coc.} \theta, -1$ progredi, ita ut ex binis quibuscunque sequens facile colligi queat.

§. 18. Observavi autem hanc progressionem fieri simplicissimam, sumto $A = \sin. \theta$, quo facto in subsidium vocamus hoc lemma notissimum:

$$\sin. (\lambda + 1) \theta = 2 \cos. \theta \sin. \lambda\theta - \sin (\lambda - 1) \theta,$$

hincque series harum potestatum sequenti modo adornabitur:

$$u \sin. \theta = u \sin. \theta;$$
$$u^2 \sin. \theta = u \sin. 2\theta - \sin. \theta;$$
$$u^3 \sin. \theta = u \sin. 3\theta - \sin. 2\theta;$$
$$u^4 \sin. \theta = u \sin. 4\theta - \sin 3\theta;$$
$$u^5 \sin. \theta = u \sin. 5\theta - \sin. 4\theta;$$

etc.

atque hinc in genere concludimus fore:

$$u^\lambda \sin.\theta = u \sin.\lambda\theta - \sin.(\lambda - 1)\theta.$$

§. 19. Cum nunc sit:

$$\sin.(\lambda - 1)\theta = \sin.\lambda\theta \cos.\theta - \cos.\lambda\theta \sin.\theta,$$

hinc fiet:

$$u^\lambda \sin.\theta = u \sin.\lambda\theta - \sin.\lambda\theta \cos.\theta + \cos.\lambda\theta \sin.\theta,$$

consequenter:

$$u^\lambda = \frac{(u - \cos.\theta)\sin.\lambda\theta}{\sin.\theta} + \cos.\lambda\theta,$$

quae formula ad sequentem usum optime est accomodata. Nunc ad ipsum angulum θ quaerendum sumamus $\lambda = n$, erit:

$$u^n = \frac{(u - \cos.\theta)\sin.n\theta}{\sin.\theta} + \cos.n\theta,$$

unde colligitur denominator:

$$1 - u^n = \frac{\sin.\theta(1 - \cos.n\theta) - (u - \cos.\theta)\sin.n\theta}{\sin.\theta},$$

qui cum debeat nihilo aequari, praebet has duas aequalitates: $\sin.n\theta = 0$ et $\cos.n\theta = 1$, unde patet fore $n\theta = i\pi$, ubi i est numerus integer sive par, sive impar, quia vero $\cos.n\theta$ debet esse $= 1$, evidens est pro i sumi debere numeros pares, ita ut valores pro angulo θ assumendi sint:

$$0, \frac{2\pi}{n}, \frac{4\pi}{n}, \frac{6\pi}{n}, \frac{8\pi}{n}, \text{ etc.}$$

quorum primus 0 dat factorem denominatoris $(1 - u)$, quem jam supra expedivimus.

§. 20. Denotet nunc θ quemcunque alium istorum valorum eritque haec formula $1 - 2u\cos.\theta + uu$ certe factor nostri denominatoris $1-u^n$, atque fractio $\frac{u^{m-1}}{1-u^n}$ resoluta certe continebit partem hujus formae: $\frac{N}{1-2u\cos.\theta+uu}$, cujus numerator N reperietur, uti alibi demonstravi, ex hac forma: $N=\frac{u^{m-1}(1-2u\cos.\theta+uu)}{1-u^n}$, posito scil. $uu-2u\cos.\theta+1=0$, quo ergo casu tam numerator quam denominator evanescet; unde ad valorem hujus fractionis $\frac{1-2u\cos\theta+uu}{1-u^n}$ inveniendum, differentialia loco numeratoris et denominatoris substituta dabunt $\frac{u-\cos.\theta}{-nu^{n-1}}$, quod, ob $u^n=1$, fit $\frac{u(u-\cos.\theta)}{-n}$, sicque numerator quaesitus N erit:

$$-\frac{u^m(u-\cos.\theta)}{n}=\frac{1}{n}(u^m\cos.\theta-u^{m+1}).$$

Supra autem invenimus:

$$u^\lambda=\frac{(u-\cos\theta)\sin.\lambda\theta}{\sin.\theta}+\cos.\lambda\theta,$$

quamobrem erit:

$$u^m\cos.\theta=\frac{(u-\cos.\theta)\cos.\theta\sin m\theta}{\sin.\theta}+\cos.\theta\cos.m\theta$$

$$-u^{m+1}=-\frac{(u-\cos.\theta)\sin.(m+1)\theta}{\sin.\theta}-\cos.(m+1)\theta,$$

hinc

$$N=\frac{1}{n}\left(\frac{(u-\cos.\theta)(\cos.\theta\sin.m\theta-\sin.(m+1)\theta)}{\sin.\theta}\right)+\cos.\theta\cos.m\theta-\cos.(m+1)\theta,$$

sive

$$N=\frac{1}{n}\left(-\frac{(u-\cos.\theta)\sin.\theta\cos.m\theta}{\sin.\theta}+\sin.m\theta\sin.\theta\right),\text{ sive}$$

$$N=\frac{1}{n}(\sin.\theta\sin.m\theta-(u-\cos.\theta)\cos.m\theta).$$

§. 21. Nostra igitur fractio $\frac{u^{m-1}}{1-u^n}$ hanc continebit fractionem partialem:

$$\frac{\sin.\theta\sin.m\theta-(u-\cos.\theta).\cos.m\theta}{n(1-2u\cos.\theta+uu)},$$

quam ergo, per ∂u multiplicatam, integrari oportet. Quia autem duabus constat partibus, earum postrema $\frac{1}{n}\int\frac{(u-\cos.\theta)\cos.m\theta\,\partial u}{1-2u\cos.\theta+uu}$ integrata dat $\frac{\cos.m\theta}{2n}l(1-2u\cos.\theta+uu)$; prior vero pars:

$$\frac{1}{n}\sin.m\theta\int\frac{\partial u\sin.\theta}{1-2u\cos.\theta+uu}=\frac{1}{n}\sin.m\theta\ \mathrm{A\,tag.}\frac{u\sin.\theta}{1-u\cos.\theta}.$$

Sicque totum integrale hujus partis erit:

$$=-\frac{\cos.m\theta}{2n}l(1-2u\cos.\theta+uu)+\frac{\sin.m\theta}{n}\mathrm{A\,tag.}\frac{u\sin.\theta}{1-u\cos.\theta}.$$

§. 22. Hoc integrale manifesto jam evanescit posito $u=0$. Superest igitur tantum ut loco u scribamus 1, quo facto pars logarithmica erit:

$$\frac{\cos.m\theta}{2n}l(2-2\cos.\theta)=\frac{\cos.m\theta}{2n}l4\sin.\tfrac{1}{2}\theta^2=\frac{\cos.m\theta}{n}l2\sin.\tfrac{1}{2}\theta.$$

Pars autem circularis erit:

$$\frac{\sin.m\theta}{n}\mathrm{A\,tag.}\frac{\sin.\theta}{1-\cos.\theta}=\frac{\sin.m\theta}{n}\mathrm{A\,tag.}\frac{\cos.\frac{1}{2}\theta}{\sin.\frac{1}{2}\theta}=\frac{(\pi-\theta)\sin.m\theta}{2n},$$

consequenter totum integrale ortum ex denominatoris factore $1-2u\cos.\theta+uu$ erit:

$$-\frac{\cos.m\theta}{n}l2\sin.\tfrac{1}{2}\theta+\frac{(\pi-\theta)}{2n}\sin.m\theta.$$

Quod si jam in hac formula loco θ successive substituantur valores supra assignati, qui erant $\frac{2\pi}{n}$, $\frac{4\pi}{n}$, $\frac{6\pi}{n}$ etc. et in genere $\frac{2i\pi}{n}$, summa omnium harum formularum dat verum valorem ipsius T, postquam scilicet addiderimus terminum $\frac{1}{n}ln$. Posito autem in genere $\theta=\frac{2i\pi}{n}$, integralis pars inde orta erit:

$$-\frac{1}{n}\cos.\frac{2mi\pi}{n}l2\sin.\frac{i\pi}{n}+\frac{(n-2i)\pi}{2nn}\sin.\frac{2mi\pi}{n},$$

ubi loco i scribi debent numeri 1, 2, 3, 4, etc., donec integrale fiat completum, quibus omnibus expeditis erit valor quaesitus T inventus.

Illustremus haec aliquot exemplis

Exemplum 1.

§. 23. Sit $n=2$, et quia m minus esse debet quam n, ne quantitas $\Delta=\frac{\pi}{n \sin. \frac{m\pi}{n}}$ fiat infinita, necessario erit $m=1$, ideoque $\Delta=\frac{\pi}{2}$. Tum vero erit $T=\frac{1}{2}l2$. Addatur igitur terminus $\frac{1}{2}l2$, prodibitque $T=l2$; sicque erit, ut ante invenimus, $S=\frac{\pi}{2}l2$, qui est valor formulae $\int\frac{-\partial x\, lx}{\sqrt{1-xx}}$.

Exemplum 2.

§. 24. Sit nunc $n=3$, eritque m vel 1 vel 2. Ex utroque autem valore prodit $\Delta=\frac{2\pi}{3\sqrt{3}}$. Tum vero sumi debet $\theta=\frac{2\pi}{3}$, quem valorem solum sumsisse sufficiet, ex quo:

$$T=-\tfrac{1}{3}\cos.\tfrac{2}{3}m\pi\, l\sqrt{3}+\tfrac{1}{18}\pi\sin.\tfrac{2}{3}m\pi,$$

ubi insuper addi debet $\frac{1}{3}l3$. Pro casu igitur $m=1$ erit

$$T=\tfrac{5}{12}l3+\tfrac{1}{12\sqrt{3}}\pi, \text{ hincque } S=\frac{\pi\pi}{3.18}+\frac{5\pi l3}{18\sqrt{3}},$$

qui est valor formulae integralis $\int\frac{-\partial x\, lx}{\sqrt[3]{1-x^3}}$. Pro altero casu, ubi $m=2$, erit ut ante $\Delta=\frac{2\pi}{3\sqrt{3}}$, at vero

$$T = \tfrac{5}{12} l 3 - \tfrac{\pi}{12\sqrt{3}}\pi, \quad \text{hinc } S = \tfrac{5\pi l 3}{18\sqrt{3}} - \tfrac{\pi\pi}{54},$$

qui ergo est valor hujus formulae integralis: $\int \frac{x\partial x l x}{\sqrt[3]{(1-x^3)^2}}$

§. 25. Loco plurium exemplorum formulam generalem tradamus pro numero quocunque n, sumendo $\theta = \frac{2i\pi}{n}$, donec fiat $2i > n$, quippe quos casus omnes rejici oportet. Tum igitur ex forma pro casu $\theta = \frac{2i\pi}{n}$ evoluta et loco i ordine scribendo 1, 2, 3, etc. reperiemus hunc valorem:

$$\begin{aligned} T = \tfrac{1}{n} l n &- \tfrac{1}{n}\cos.\tfrac{2m\pi}{n}\, l 2 \sin.\tfrac{\pi}{n} + \tfrac{(n-2)\pi}{2nn}\sin.\tfrac{2m\pi}{n} \\ &- \tfrac{1}{n}\cos.\tfrac{4m\pi}{n}\, l 2 \sin.\tfrac{2\pi}{n} + \tfrac{(n-4)\pi}{2nn}\sin.\tfrac{4m\pi}{n} \\ &- \tfrac{1}{n}\cos.\tfrac{6m\pi}{n}\, l 2 \sin.\tfrac{4\pi}{n} + \tfrac{(n-6)\pi}{2nn}\sin.\tfrac{6m\pi}{n} \\ &\text{etc.} \end{aligned}$$

Has scilicet partes eo usque continuari oportet, quamdiu fuerit $i < \frac{1}{2}n$, atque hoc valore invento habebitur pro problemate primo $S = \Delta T$, existente $\Delta = \frac{\pi}{n \sin.\frac{m\pi}{n}}$.

§. 26. Duplicis igitur generis termini in hac expressione occurrunt, quorum priores tantum logarithmos involvunt, posteriores autem quadraturam circuli π, atque hic commode usu venit, ut istae posteriores partes omnes in unicam formulam contrahi queant, quod, si etiam circa priores partes logarithmicas praestari posset, id pro invento maximi momenti esset habendum. Quod autem ad partes circulares attinet, earum contractionem sequenti problemate docebimus.

Problema.

Omnes partes circulares, ad quas in problemate praecedente sumus perducti in unam summam contrahere, sive, omisso factore communi $\frac{\pi}{2nn}$, *hanc seriem:*

$$(n-2)\sin.\frac{2m\pi}{n}+(n-4)\sin.\frac{4m\pi}{n}+\ldots\ldots(n-2i)\sin.\frac{2im\pi}{n}$$

summare quousque scil. 2i *non superat* n.

Solutio:

§. 27. Ponamus brevitatis gratia $\frac{m\pi}{n}=\phi$ atque series proposita sponte in has duas resolvitur:

$$n\sin.2\phi+n\sin.4\phi+n\sin.6\phi+\ldots\ldots+n\sin.2i\phi$$

$$2\sin.2\phi+4\sin.4\phi+6\sin.6\phi+\ldots\ldots+2i\sin.2i\phi$$

tum enim prior, demta posteriore, dabit valorem, quem quaerimus.

§. 28. Pro priore jam statuamus:

$$p=\sin.2\phi+\sin.4\phi+\sin.6\phi+\ldots\ldots+\sin.2i\phi$$

ac multiplicando per 2 sin. ϕ fiet:

$$\begin{aligned}2p\sin.\phi=\cos.\phi&-\cos.3\phi-\cos.5\phi-\cos.7\phi\\&+\cos.3\phi+\cos.5\phi+\cos.7\phi\\&-\cos.(2i-1)\phi-\cos.(2i+1)\phi\\&+\cos.(2i-1)\phi\end{aligned}$$

unde fit $p=\frac{\cos.\phi-\cos.(2i+1)\phi}{2\sin.\phi}$.

§. 29. Pro altera serie summanda consideremus primo hanc seriem:

$$q = \cos. 2\,\phi + \cos. 4\,\phi + \cos. 6\,\phi + \ldots\ldots + \cos. 2\,i\,\phi$$

cujus differentiale statim dat:

$$\frac{-\partial q}{\partial \phi} = 2 \sin. 2\,\phi + 4 \sin. 4\,\phi + 6 \sin. 6\,\phi + \ldots\ldots + 2\,i \sin. 2\,i\,\phi.$$

Jam vero reperiemus:

$$\begin{aligned} 2q \sin. \phi = & - \sin. \phi + \sin. 3\,\phi + \sin. 5\,\phi + \sin. 7\,\phi \\ & - \sin. 3\,\phi - \sin. 5\,\phi - \sin. 7\,\phi \\ & + \sin. (2\,i - 1)\,\phi + \sin. (2\,i + 1)\,\phi \\ & - \sin. (2\,i - 1)\,\phi \end{aligned}$$

sive

$$2\,q \sin. \phi = - \sin. \phi + \sin. (2\,i + 1)\,\phi,$$

ideoque

$$q = -\tfrac{1}{2} + \frac{\sin. (2\,i + 1)\,\phi}{2 \sin. \phi},$$

consequenter habebimus:

$$\frac{\partial q}{\partial \phi} = \frac{(2\,i + 1) \cos. (2\,i + 1)\,\phi}{2 \sin. \phi} - \frac{\sin. (2\,i + 1)\,\phi \cos. \phi}{2 \sin. \phi^2},$$

quibus valoribus inventis series prior, demta posteriore, hoc est $n\,p + \frac{\partial q}{\partial \phi}$, dabit valorem quaesitum, series vero in problemate proposita, ducta in $\frac{\pi}{2\,n\,n}$, dabit summam omnium partium circularium, quam quaerimus.

§. 30. Verum ad valores p et q inveniendos duos casus considerari convenit, prouti n fuerit vel numerus par, vel numerus impar. Sit igitur primo par, ponaturque $n = 2i$, ita ut i superare nequeat $\frac{m}{2}$, et quoniam posuimus $\Phi = \frac{m\pi}{n}$, erit nunc $\Phi = \frac{m\pi}{2i}$, unde deducimus:

$$p = \tfrac{1}{2}\cot.\tfrac{m\pi}{2i} - \tfrac{1}{2}\cos.m\pi\cot.\tfrac{m\pi}{2i} + \tfrac{1}{2}\sin.m\pi,$$

quae expressio, ob $\sin.m\pi = 0$, reducitur ad hanc:

$$p = \tfrac{1}{2}\cot.\tfrac{m\pi}{2i}(1 - \cos.m\pi);$$

ubi est $\cos.m\pi = \pm 1$, prouti m fuerit vel numerus par vel impar, atque priori casu erit $p = 0$, posteriori vero $p = \cot.\frac{m\pi}{2i}$.

§. 31. Porro autem hoc casu $n = 2i$ erit:

$$\frac{\partial q}{\partial \Phi} = i\cos.m\pi\cot.\tfrac{m\pi}{2i} - \tfrac{1}{2}(2i+1)\sin.m\pi - \tfrac{1}{2}\sin.m\pi\cot.\tfrac{m\pi^2}{2i},$$

quae expressio, ob $\sin.m\pi = 0$, abit in hanc:

$$\frac{\partial q}{\partial \Phi} = i\cos.m\pi\cot.\tfrac{m\pi}{2i}.$$

Quare cum summa quaesita sit $np + \frac{\partial q}{\partial \Phi}$, ea erit $i\cot.\frac{m\pi}{2i}$, consequenter, loco $2i$ restituendo n, erit summa $= \frac{1}{2}n\cot.\frac{m\pi}{n}$.

§. 32. Evolvamus nunc etiam alterum casum, quo n est numerus impar, et quoniam $2i+1$ superare non debet n, manifesto poni poterit $2i+1 = n$, quo facto statim habemus $p = \frac{1}{2}\cot.\frac{m\pi}{n} - \frac{\cos.m\pi}{2\sin.\frac{m\pi}{n}}$. Deinde vero

$$\frac{\partial q}{\partial \Phi} = \frac{n\cos.m\pi}{2\sin.\frac{m\pi}{n}} - \frac{\sin.m\pi\cot.\frac{m\pi}{n}}{2\sin.\frac{m\pi}{n}},$$

hincque ipsa summa quaesita:

$$np + \frac{\partial q}{\partial \phi} = \frac{n \cos. \frac{m\pi}{n}}{2 \sin. \frac{m\pi}{n}} = \tfrac{1}{2} n \cot. \frac{m\pi}{n}.$$

§. 33. Cum igitur, sive n sit numerus par sive impar, eadem summa prodeat, scilicet $\frac{1}{2} n \cot. \frac{m\pi}{n}$, haec per $\frac{\pi}{2nn}$ multiplicata dabit summam omnium partium circularium, quarum ergo summa erit $\frac{\pi}{4n} \cot. \frac{m\pi}{n}$; consequenter formula generalis supra pro T inuenta erit nunc:

$$\begin{aligned} T = \tfrac{1}{n} l n + \tfrac{\pi}{2n} \cot. \tfrac{m\pi}{n} &- \tfrac{1}{n} \cos. \tfrac{2m\pi}{n} \, l\, 2 \sin. \tfrac{\pi}{n} \\ &- \tfrac{1}{n} \cos. \tfrac{4m\pi}{n} \, l\, 2 \sin. \tfrac{2\pi}{n} \\ &- \tfrac{1}{n} \cos. \tfrac{6m\pi}{n} \, l\, 2 \sin. \tfrac{4\pi}{n} \\ &\text{etc.} \end{aligned}$$

quae expressio porro, ducta in $\frac{\pi}{n \sin. \frac{m\pi}{n}}$, dabit valorem expressionis in primo problemate tractatae, sicque nunc ex hac nova formula multo facilius erit exempla particularia, quotquot libuerit, evolvere.

§. 34. Circa formulam integralem, in problemate primo tractatam, casus prorsus singularis occurrit, quando $m = n$; tum enim fit $\Delta = \infty$. At vero habebitur $T = \int 0 \, \partial u$, sicque prodit $S = \Delta T = \infty . 0$, cujus ergo valor hoc modo plane non determinatur. Eum ergo immediate ex ipsa prima formula eruere convenit. Posito autem $m = n$ erit:

$$S = -\int \frac{x^{n-1}\partial x\, lx}{1-x^n},$$

quae formula per seriem ita evolvitur:

$$S = \int -x^{n-1}\partial x\, lx\,(1 + x^n + x^{2n} + x^{3n} + \text{etc.}),$$

quae, ab $x = 0$ ad $x = 1$ extensa, ob $\int -x^{\lambda-1}\partial x\, lx = \frac{1}{\lambda\lambda}$, statim ducit ad hanc seriem:

$$S = \frac{1}{nn}(1 + \tfrac{1}{4} + \tfrac{1}{9} + \tfrac{1}{16} + \tfrac{1}{25} + \text{etc.}),$$

cujus seriei summa cum sit $\frac{\pi\pi}{6}$, erit $S = \frac{\pi\pi}{6nn}$. At vero nulla via patet, hunc valorem ex praecedente solutione derivandi.

Problema.

Proposita formula integrali hac: $V = \int -\partial v\,(1-v)^{\theta-1}\, lv$, *ejus valorem, a termino* $v = 0$ *ad* $v = 1$ *extensum, per expressionem finitam repraesentare.*

Solutio:

§. 35. Cum sit $lv = l(1-(1-v))$ erit per seriem:

$$-lv = \frac{1-v}{1} + \frac{(1-v)^2}{2} + \frac{(1-v)^3}{3} + \text{etc.}$$

sicque erit:

$$V = \int \partial v\left(\frac{(1-v)^\theta}{1} + \frac{(1-v)^{\theta+1}}{2} + \frac{(1-v)^{\theta+2}}{3} + \text{etc.}\right).$$

Quare cum in genere sit $\int \partial v\,(1-v)^\lambda = -\frac{(1-v)^{\lambda+1}}{\lambda+1} + C$, hoc ut evanescat, posito $v = 0$, fieri debet $C = \frac{1}{\lambda+1}$. Facto nunc $v = 1$, erit pro nostro casu $\int \partial v\,(1-v)^\lambda = \frac{1}{\lambda+1}$.

Quamobrem habebimus:

$$V = \frac{1}{1(\theta+1)} + \frac{1}{2(\theta+2)} + \frac{1}{3(\theta+3)} + \text{etc.}$$

§. 36. Cum nunc sit $\frac{1}{1(\theta+1)} = \frac{1}{\theta}\left(1 - \frac{1}{\theta+1}\right)$, et in genere $\frac{1}{\alpha(\theta+\alpha)} = \frac{1}{\theta}\left(\frac{1}{\alpha} - \frac{1}{\theta+\alpha}\right)$, series nostra in duas partes resolvitur. Erit enim:

$$V = \frac{1}{\theta}\left\{\begin{array}{l} 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \text{etc.} \\ -\frac{1}{\theta+1} - \frac{1}{\theta+2} - \frac{1}{\theta+3} - \frac{1}{\theta+4} - \frac{1}{\theta+5} - \text{etc.} \end{array}\right\}$$

quae quo facilius ad formulas integrales redigi queant, prior ita repraesentetur:

$$p = \frac{y}{1} + \frac{y^2}{2} + \frac{y^3}{3} + \frac{y^4}{4} + \text{etc.}$$

altera vero hoc modo:

$$q = \frac{y^{\theta+1}}{\theta+1} + \frac{y^{\theta+2}}{\theta+2} + \frac{y^{\theta+3}}{\theta+3} + \text{etc.}$$

quippe quae casu $y = 1$ in nostras series abeunt. Inde vero prodit:

$$\frac{\partial p}{\partial y} = 1 + y + y^2 + y^3 + \text{etc.} = \frac{1}{1-y},$$
$$\frac{\partial q}{\partial y} = y^\theta + y^{\theta+1} + y^{\theta+2} + \text{etc.} = \frac{y^\theta}{1-y}.$$

Hinc igitur habebimus $\partial p - \partial q = \frac{\partial y(1-y^\theta)}{1-y}$ consequenter erit $p - q = \int\frac{\partial y(1-y^\theta)}{1-y}$. Formulam istam integralem ab $y = 0$ ad $y = 1$ extendendo, valor noster quaesitus erit $V = \frac{1}{\theta}\int\frac{\partial y(1-y^\theta)}{1-y}$, qui utique semper per logarithmos et arcus circulares assignari poterit.

Corollarium 1.

§. 37. Quo has formulas ad majorem affinitatem cum ante tractata reducamus, ponamus primo $v = x^n$, ita ut ipsa formula proposita nunc sit:

$$V = nn\int -x^{n-1}\partial x(1-x^n)^{\theta-1}lx.$$

Tum vero in formula, ad quam sumus perducti, statuamus simili modo $y = u^n$, fietque $V = \frac{n}{\theta}\int\frac{u^{n-1}\partial u(1-u^{n\theta})}{1-u^n}$, cujus formulae jam denominator et alterum membrum cum forma supra inventa $T = \int\frac{(u^{m-1}-u^{n-1})\partial u}{1-u^n}$ congruit. Quare ut paritas perfecta reddatur, statuamus $n\theta + n = m$, ideoque $\theta = \frac{m-n}{n}$, sicque forma proposita fiet:

$$V = nn\int -x^{n-1}\partial x(1-x^n)^{\frac{m-2n}{n}}lx, \text{ sive}$$

$$V = nn\int\frac{-x^{n-1}\partial x\, lx}{\sqrt[n]{(1-x^n)^{2n-m}}}.$$

Tum vero idem valor etiam ita exprimetur:

$$V = \frac{nn}{m-n}\int\frac{u^{n-1}-u^{m-1}}{1-u^n}\partial u,$$

hoc est $V = \frac{nn}{n-m}T$. Hinc ergo, cum ex problemate primo sit $S = \Delta T$, nunc ambae formulae S et V ita a se invicem pendent ut sit $V = \frac{nn}{n-m}\frac{S}{\Delta}$.

Scholion.

§. 38. Haec reductio ad similitudinem adhuc alio modo peragi potest, statuendo $n\theta = m$, sive $\theta = \frac{m}{n}$, atque nunc formula proposita erit:

$$V = nn\int -x^{n-1}\partial x\, lx\, (1-x^n)^{\frac{m-n}{n}} \text{ sive}$$

$$V = nn\int \frac{-x^{n-1}\partial x\, lx}{\sqrt[n]{(1-x^n)^{n-m}}};$$

tum vero formula inde derivata erit:

$$V = \tfrac{nn}{m}\int\left(\frac{u^{n-1}-u^{m+n-1}}{1-u^n}\right)\partial u.$$

Cum autem sit $u^{m+n-1} = u^{m-1}(1-(1-u^n))$, haec formula transformabitur in hanc:

$$\tfrac{nn}{m}\int u^{m-1}\partial u + \tfrac{nn}{m}\int \frac{u^{n-1}-u^{m-1}}{1-u^n}\partial u,$$

ideoque $V = \frac{nn}{mm} - \frac{nn}{m}T$, quam obrem istae duae formulae integrales:

$$S = \int\frac{-x^{m-1}\partial x\, lx}{\sqrt[n]{(1-x^n)^m}} \quad \text{et} \quad \frac{V}{nn} = \int\frac{-x^{n-1}\partial x\, lx}{\sqrt[n]{(1-x^n)^{n-m}}}$$

ita inter se cohaerent, ut, ob $T = \frac{S}{\Delta}$, sit $V = \frac{nn}{mm} - \frac{nn}{m}\frac{S}{\Delta}$, unde, quia formula posterior tanquam simplicior ipsius S spectari potest, valore ipsius V invento erit $S = \Delta\left(\frac{1}{m} - \frac{m}{nn}V\right)$. Haec autem reductio longe est praeferenda illi, quam ante invenimus, quippe quae laborabat hoc defectu, quod fractio differentialis ibi integranda erat spuria, cum in numeratore occurrat potestas $u^{n\theta+n-1}$, quae utique altior est quam potestas denominatoris u^n; quocirca nunc demum integrale pro quantitate T ante evolutum hic usurpari poterit.