

conficitur  $\sin 3(\vartheta - \Phi) = \frac{\alpha(1-3tt) - 3t + t^3}{\sqrt{1 + \alpha x(1+tt)^2}}$ , ideoque:

$$(\sin 3(\vartheta - \Phi))^2 = \frac{(1(1-3tt) - 3t + t^3)^2}{(1 + \alpha x)^2 (1+tt)^2}$$

Hisq; valoribus substitutis nascitur:

$$\int p \partial v = -a^2 \int \frac{\partial t(1+tt)}{(3t-t^3)(\alpha(1-3tt) - 3t + t^3)^2}$$

$$\int Q \partial v = +a^2 \int \frac{t \partial t(1+tt)}{(3t-t^3)(\alpha(1-3tt) - 3t + t^3)^2}$$

Certo igitur affirmare licet, has formulas ab irrationalitate penitus liberari posse, etiam si mihi quidem nulla via pateat videatur hoc praeflandi; unde Geometris amplissimus campus aperit in suam sagacitatem exercendi.

§. 3c. Si loco formulae  $\int \frac{\partial z}{\sqrt{1+z^2}}$  assumissemus generalem  $\int \frac{\partial z}{\sqrt{1+z^2}}$ , eamque simili modo tractavissemus, perveniremus ad sequens theorema:

Integralia harum duarum formularum:

$$\int \frac{\partial \Phi \sin \Phi}{\sin n \Phi} \quad \text{et} \quad \int \frac{\partial \Phi \cos \Phi}{\sin n \Phi}$$

certe per logarithmos et arcus circulares exprimi possunt, ideoque dabitur certa substitutio, cujus ope haec formulae ad rationalitatem perducere possunt; unde haec observatio eo majorem attentionem meretur.

DIS-

DISQUISITIONES ANALYTICAE  
SUPER EVOLUTIONE POTESTATIS TRINOMIALIS

$$(1+x+xx)^n$$

AUCTORE  
L. E U L E R O.

Convēni exhibita die 17. Aug. 1775.

§. 1.

Cum olim in *Novorum Commentariorum Tomo XI*, sub titulo *observationum analyticarum*, istam potestatem trinomialem multo studio essem pertractatus, in tam egregia symptomata incidi, quae majore attentione Geometrarum non indigna videbantur. Hanc ob rem nuper hoc idem argumentum de novo tractare fuscipi, atque nonnullis artificis analyticis usus, multo plura insignia phaenomena se mihi obtulerunt, quorum expositionem Geometris non ingratalam fore confido.

§. 2. Incipio igitur ab ipsa evolutione hujus formulae:  $(1+x+xx)^n$ , quae pro singulis valoribus exponentis  $n$  sequentes praebet expressiones in tabula subjuncta representatas:

0	1
1	1 + x + xx
2	1 + 2x + 3xx + 2x <sup>3</sup> + x <sup>4</sup>
3	1 + 3x + 6xx + 7x <sup>3</sup> + 6x <sup>4</sup> + 3x <sup>5</sup> + x <sup>6</sup>
4	1 + 4x + 10xx + 16x <sup>3</sup> + 19x <sup>4</sup> + 16x <sup>5</sup> + 10x <sup>6</sup> + 4x <sup>7</sup> + x <sup>8</sup>
5	1 + 5x + 15xx + 30x <sup>3</sup> + 45x <sup>4</sup> + 51x <sup>5</sup> + 45x <sup>6</sup> + 30x <sup>7</sup> + 15x <sup>8</sup> + 5x <sup>9</sup> + x <sup>10</sup>
	etc.

K 2 Hic

Hic scilicet ex qualibet potestate facillime sequens deducitur; si enim pro quolibet valore exponentis  $n$ , quilibet coefficientis cum binis precedentibus in unam summam colligatur, obtinetur coefficientis pro potestate sequente exponentis  $n + 1$  subscribenda.

§. 3. Hanc tabulam aspicienti statim patet, in quolibet evolutione coefficientes terminorum usque ad medium, qui dignitatem  $x^2$  refert, crescere, inde autem iterum eodem ordine decrescere, usque ad ultimum terminum, qui est  $x^n$ . Deinde etiam haud difficulter perspicitur, pro potestate  $(1 + x + xx)^n$  in genere terminos initiales ita expressumiri:  $1 + nx + \frac{n(n+1)}{1 \cdot 2} x^2 + \frac{n(n-1)(n+1)}{1 \cdot 2 \cdot 3} x^3 + \frac{n(n-1)(n-2)(n+1)}{1 \cdot 2 \cdot 3 \cdot 4} x^4 + \dots$  etc.

His autem terminos ulterius proseguiri non attinet, quia in eorum coefficientibus nullus ordo deprehenditur.

§. 4. Hic autem imprimis ad coefficientem maximum seu medium reficito, quem pro potestate  $(1 + x + xx)^n$  in genere perpetuo statum  $= px^2$ ; tum vero terminos hunc sequentes ita repraesentabo:  $qx^{n+1}$ ;  $rx^{n+2}$ ;  $sx^{n+3}$ ;  $tx^{n+4}$ ; etc. unde termini medium precedentes erunt ordine retrogrado  $q'x^{n-1}$ ;  $r'x^{n-2}$ ;  $s'x^{n-3}$ ;  $t'x^{n-4}$ ; etc. Deinde vero pro potestate sequenti  $(1 + x + xx)^{n+1}$  eusdem litteras apicem sum notaturus; scilicet:  $p, q, r, s$ , etc. quas primo pro potestate deuno sequente  $(1 + x + xx)^{n+2}$  apicem duplitem designabo; pro sequentibus, apicem triplitem, quadruplitem, et ita porro.

§. 5. His praemisissis, in hac dissertatione ex facilius superioris tabulae potissimum terminos medios maximis coefficientibus affectos, sum contemplanturus, qui sunt

sunt:  $1, x, 3x^2, -x^3, 11x^4, 51x^5$  etc. qui junctim summi continentur seriem, cujus summam littera  $P$  indicabo, ita ut  $P = 1 + x + 3x^2 + 7x^3 + 19x^4 + 51x^5 + \dots + px^{n-1} + p'x^{n+1} + \dots$  etc.

§. 6. Praeterea vero, quemadmodum isti termini ex tabula superiore secundum diagonalem sunt desumpti, simili modo tales series formemus secundum diagonales ulteriores illi parallelas, quarum seriem summam pariter peculiaribus litteris denotemus sequenti modo:

$Q = x^2 + 2x^3 + 6x^4 + 16x^5 + 45x^6 + \dots + 9x^{n+1} + q'x^{n+2} + q''x^{n+3} + \dots$  etc.  
 $R = x^3 + 3x^4 + 11x^5 + 30x^6 + \dots + rx^{n+2} + r'x^{n+3} + r''x^{n+4} + \dots$  etc.  
 $S = x^4 + 4x^5 + 15x^6 + \dots + sx^{n+3} + s'x^{n+4} + s''x^{n+5} + \dots$  etc.  
 $T = x^5 + 5x^6 + \dots + tx^{n+4} + t'x^{n+5} + t''x^{n+6} + \dots$  etc.

His constitutis propositum mihi est primo in valores litterarum minuscularum  $p, q, r, s$ , etc. earumque derivatarum  $p', q', r', s'$  etc. inquirere, quo facto etiam valores litterarum majuscularum  $P, Q, R, S$ , etc. indagabo.

Investigatio litterarum  $p, q, r, s$ , etc.

§. 7. Cum  $p$  fit coefficientis potestatis  $x^n$  ex evolutione formulae  $(1 + x + xx)^n$  oriundae, istam formulam hoc modo repraesentemus:  $(x(1+x) + 1)^n$ , pro cuius evolutione utamur signandi modo jam aliquoties a me usitato, quo coefficientes similibus potestatis binomialis per hos characteres designare soleo:  $\binom{n}{1}, \binom{n}{2}, \binom{n}{3}, \binom{n}{4}$ , etc. ita ut sit

$$\begin{aligned} \binom{n}{1} &= n \\ \binom{n}{2} &= \frac{n(n-1)}{1 \cdot 2} \\ \binom{n}{3} &= \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} \\ \binom{n}{4} &= \frac{n(n-1)(n-2)(n-3)}{1 \cdot 2 \cdot 3 \cdot 4} \end{aligned}$$

$$\binom{n}{2} =$$

$$\binom{n}{2} = \frac{n \cdot (n-1) \cdot (n-2) \cdot (n-3) \cdot \dots \cdot (n-4)}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot \dots \cdot n}$$

$$\binom{n}{\lambda} = \frac{n \cdot (n-1) \cdot (n-2) \cdot (n-3) \cdot \dots \cdot (n-\lambda+1)}{1 \cdot 2 \cdot 3 \cdot 4 \cdot \dots \cdot \lambda}$$

Circa quos characteres hic annotasse iurabit, in genere semper esse  $\binom{n}{\lambda} = \binom{n}{n-\lambda}$ , quandoquidem hi coefficientes retro eundem ordinem servant, et quia coefficientes extreni sunt unitas, erit  $\binom{n}{0} = \binom{n}{n} = 1$ . Deinde, quia ex lege progressionis tam omnes termini primum antecedentes quam termini ultimum sequentes evanescent, erit ut sequitur:

$$\binom{n}{n-1} = \binom{n}{n+1} = 0$$

$$\binom{n}{n-2} = \binom{n}{n+2} = 0$$

$$\binom{n}{n-3} = \binom{n}{n+3} = 0$$

etc.

§. 8. His praemissis formula nostra  $(x(r+x)+1)^n$ , more solito tanquam binomium evoluta, dabit hanc feriem:

$$x^n (1+x)^n + \binom{n}{1} x^{n-1} (1+x)^{n-1} + \binom{n}{2} x^{n-2} (1+x)^{n-2} + \dots$$

ubi notetur esse in genere:

$$(1+x)^\lambda = 1 + \binom{\lambda}{1} x + \binom{\lambda}{2} x^2 + \binom{\lambda}{3} x^3 + \dots$$

Ex singulis igitur membris illius formae expostae deprimi debent termini potestatem  $x^n$  continentes, qui ipse qui conjunctum suum component terminum medium  $p x^n$ .

§. 9. Primum autem membrum  $x^n (1+x)^n$  tantum terminum hujus formae praebet  $x^n$ . Hic membro autem secundo hanc formam habebit terminus secundus, qui est  $\binom{n}{1} x^{n-1} x^n$ . Ex tertio membro potestas  $x^n$  oritur ex termi-

no

ante per  $\Phi$  expresso nanciscemur sequentem aequationem a fractionibus jam liberatam:

$$A(n-2a-\lambda-1) \binom{n}{n-2a-\lambda} + B(n-2a-\lambda)$$

no tertio, qui est  $\binom{n}{2} \binom{n-2}{2} x^n$ . Simili modo ex membro quarto deducitur  $\binom{n}{3} \binom{n-3}{2} x^n$ . Ex quinto oritur  $\binom{n}{4} \binom{n-4}{2} x^n$  et ita porro. Hinc igitur versus valor litterae  $p$  ita colligitur

$$p = 1 + \binom{n}{1} \binom{n-1}{2} + \binom{n}{2} \binom{n-2}{2} + \binom{n}{3} \binom{n-3}{2} + \binom{n}{4} \binom{n-4}{2} + \dots$$

§. 10. Simili modo ex eadem evolutione colligere licet coefficientes potestatis  $x^{n+1}$ , qui junctim summi dabunt valorem litterae  $q$ . Talis autem potestas ex primo membro orta erit  $\binom{n}{1} x^{n+1}$ . Ex secundo membro oritur  $\binom{n}{2} \binom{n-2}{2} x^{n+1}$ ; ex tertio membro  $\binom{n}{3} \binom{n-3}{2} x^{n+1}$ ; ex quarto  $\binom{n}{4} \binom{n-4}{2} x^{n+1}$ ; et ita porro, quocirca versus valor litterae  $q$  hoc modo exprimitur:

$$q = \binom{n}{1} + \binom{n}{2} \binom{n-2}{2} + \binom{n}{3} \binom{n-3}{2} + \dots$$

ubi ob analogiam primus terminus  $\binom{n}{1}$  ita representatus est intelligendus  $\binom{n}{0} \binom{n}{1}$ . Sic venim, cum quilibet terminus duobus conficit factoribus, priores factores constitunt hanc feriem:  $\binom{n}{0}$ ,  $\binom{n}{1}$ ,  $\binom{n}{2}$ ,  $\binom{n}{3}$ ,  $\binom{n}{4}$ , etc. posteriores vero istam  $\binom{n}{1}$ ,  $\binom{n-1}{2}$ ,  $\binom{n-2}{3}$ ,  $\binom{n-3}{4}$ , etc.

§. 11. Pari modo ex potestatibus  $x^{n+2}$ , quae ex singulis membris deducuntur, formabitur terminus  $r x^{n+2}$ ; at vero primum membrum pro hac potestate praebet  $1 \cdot \binom{n}{2} x^{n+2}$ , sive analogie gratia  $\binom{n}{0} \binom{n}{2} x^{n+2}$ . Ex membro secundo oritur eadem potestas  $\binom{n}{1} \binom{n-1}{2} x^{n+2}$ ; ex tertio membro  $\binom{n}{2} \binom{n-2}{2} x^{n+2}$ ; ex quarto  $\binom{n}{3} \binom{n-3}{2} x^{n+2}$  et ita porro; ex quibus ergo collectis nanciscimur valorem litterae  $r$  hoc modo expressum:

$$r = \binom{n}{0} \binom{n}{2} + \binom{n}{1} \binom{n-1}{2} + \binom{n}{2} \binom{n-2}{2} + \binom{n}{3} \binom{n-3}{2} + \dots$$

§. 12.

§. 37. Substituamus nunc loco litterarum  $A, B, C$  valores modo inventos, et aequatio inter has ternas litteras erit:

$$A(n-2a-\lambda-1) \binom{n}{n-2a-\lambda} + B(n-2a-\lambda)$$

§. 12. Superfluum foret eandem deductionem pro literis sequentibus apponere, quandoquidem jam satis perspicuum est fore:

$$s = \binom{n}{0} \binom{n}{1} + \binom{n}{1} \binom{n-2}{0} + \binom{n}{2} \binom{n-3}{0} + \text{etc.}$$

$$t = \binom{n}{1} \binom{n}{2} + \binom{n}{2} \binom{n-3}{1} + \binom{n}{3} \binom{n-4}{1} + \text{etc.}$$

$$u = \binom{n}{2} \binom{n}{3} + \binom{n}{3} \binom{n-4}{2} + \binom{n}{4} \binom{n-5}{2} + \text{etc.}$$

etc.

atque in genere si potestati  $x^{r+\lambda}$  tribuamus litteram  $z$ , erit  $z^r = \binom{n}{0} \binom{n}{1} + \binom{n}{1} \binom{n-2}{1} + \binom{n}{2} \binom{n-3}{1} + \text{etc.}$

§. 13. Manifestum hic est omnes terminos harum ferriem contineri in hac forma generali:  $\binom{n}{\alpha} \binom{n-\alpha}{\beta}$ , quam obfero femper huic esse aequalem;  $\binom{n}{\beta} \binom{n-\beta}{\alpha}$ , ita ut litterae  $\alpha$  et  $\beta$  permutationem patiantur. Cum enim facta evolutione fit:

$$\alpha = \frac{n \cdot (n-1) \cdot (n-2) \cdot \dots \cdot (n-\alpha+1)}{1 \cdot 2 \cdot 3 \cdot \dots \cdot \alpha}$$

$$\beta = \frac{(n-\alpha) \cdot (n-\alpha-1) \cdot (n-\alpha-2) \cdot \dots \cdot (n-\alpha-\beta+1)}{1 \cdot 2 \cdot 3 \cdot \dots \cdot \beta}$$

$$\text{et } \frac{n \cdot (n-1) \cdot (n-2) \cdot \dots \cdot (n-\alpha+1)}{1 \cdot 2 \cdot 3 \cdot \dots \cdot \alpha} \cdot \frac{(n-\alpha) \cdot (n-\alpha-1) \cdot (n-\alpha-2) \cdot \dots \cdot (n-\alpha-\beta+1)}{1 \cdot 2 \cdot 3 \cdot \dots \cdot \beta} = \frac{n \cdot (n-1) \cdot (n-2) \cdot \dots \cdot (n-\alpha-\beta+1)}{1 \cdot 2 \cdot 3 \cdot \dots \cdot (\alpha+\beta)}$$

facta multiplicatione erit  $\binom{n}{\alpha} \binom{n-\alpha}{\beta} = \binom{n}{\alpha+\beta} \binom{n-\alpha-\beta}{\alpha}$  ubi permutabilitas litterarum  $\alpha$  et  $\beta$  in oculo incurrit.

§. 14. Quodsi jam series ante inventae hoc modo immutentur, prima quidem pro  $p$  inventa nullam mutationem patitur, reliquae vero sequenti modo referentur:

$$q = \binom{n}{1} \binom{n-1}{0} + \binom{n}{2} \binom{n-2}{1} + \binom{n}{3} \binom{n-3}{2} + \text{etc.}$$

$$r = \binom{n}{2} \binom{n-2}{0} + \binom{n}{3} \binom{n-3}{1} + \binom{n}{4} \binom{n-4}{2} + \text{etc.}$$

$$s = \binom{n}{3} \binom{n-3}{0} + \binom{n}{4} \binom{n-4}{1} + \binom{n}{5} \binom{n-5}{2} + \binom{n}{6} \binom{n-6}{3} + \text{etc.}$$

$$z = \binom{n}{\alpha} \binom{n-\alpha}{0} + \binom{n}{\alpha+1} \binom{n-\alpha-1}{1} + \binom{n}{\alpha+2} \binom{n-\alpha-2}{2} + \text{etc.}$$

§. 15.

§. 15. Praeterea vero maxime memorabilis est haec conversio, qua est  $\binom{n}{\alpha} \binom{n-\alpha}{\beta} = \binom{n+\beta}{\alpha} \binom{n}{\alpha+\beta}$ . Cum enim sit

$$\binom{n+\beta}{\alpha} = \frac{n \cdot (n-1) \cdot (n-2) \cdot \dots \cdot (n-\alpha+1)}{1 \cdot 2 \cdot 3 \cdot \dots \cdot \alpha} \cdot \frac{(n-\alpha) \cdot (n-\alpha-1) \cdot (n-\alpha-2) \cdot \dots \cdot (n-\alpha-\beta+1)}{1 \cdot 2 \cdot 3 \cdot \dots \cdot \beta}$$

$$= \frac{n \cdot (n-1) \cdot (n-2) \cdot \dots \cdot (n-\alpha-\beta+1)}{1 \cdot 2 \cdot 3 \cdot \dots \cdot (\alpha+\beta)}$$

five

$$\binom{n}{\alpha+\beta} = \frac{n \cdot (n-1) \cdot (n-2) \cdot \dots \cdot (n-\alpha-\beta+1)}{1 \cdot 2 \cdot 3 \cdot \dots \cdot (\alpha+\beta)}$$

productum erit:

$$\binom{n+\beta}{\alpha} \binom{n}{\alpha+\beta} = \frac{n \cdot (n-1) \cdot (n-2) \cdot \dots \cdot (n-\alpha-\beta+1)}{1 \cdot 2 \cdot 3 \cdot \dots \cdot \alpha} \cdot \frac{n \cdot (n-1) \cdot (n-2) \cdot \dots \cdot (n-\alpha-\beta+1)}{1 \cdot 2 \cdot 3 \cdot \dots \cdot \beta}$$

in quam eandem formam resolvitur formula  $\binom{n}{\alpha} \binom{n-\alpha}{\beta}$ .

§. 16. Per hanc igitur transformationem superiores series sequenti modo exprimi poterunt:

$$p = \binom{n}{1} \binom{n}{0} + \binom{n}{2} \binom{n}{1} + \binom{n}{3} \binom{n}{2} + \binom{n}{4} \binom{n}{3} + \text{etc.}$$

$$q = \binom{n}{1} \binom{n}{2} + \binom{n}{2} \binom{n}{3} + \binom{n}{3} \binom{n}{4} + \text{etc.}$$

$$r = \binom{n}{2} \binom{n}{3} + \binom{n}{3} \binom{n}{4} + \binom{n}{4} \binom{n}{5} + \text{etc.}$$

$$s = \binom{n}{3} \binom{n}{4} + \binom{n}{4} \binom{n}{5} + \binom{n}{5} \binom{n}{6} + \text{etc.}$$

$$z = \binom{n}{\alpha} \binom{n}{\alpha} + \binom{n}{\alpha+1} \binom{n}{\alpha+1} + \binom{n}{\alpha+2} \binom{n}{\alpha+2} + \binom{n}{\alpha+3} \binom{n}{\alpha+3} + \text{etc.}$$

§. 17. Notari adhuc meretur alia transformatio, quae ad calculum numericum imprimis est accommodata. Cum enim ex prima forma sit  $z = \binom{n}{\alpha} + \binom{n}{\alpha+1} + \binom{n}{\alpha+2} + \text{etc.}$  quilibet terminus hujus seriei est  $\binom{n}{\alpha} \binom{n-\alpha}{\alpha}$ , qui dicatur = II, eoque facta evolutione:

$$II = \frac{n \cdot (n-1) \cdot (n-2) \cdot \dots \cdot (n-\alpha+1)}{1 \cdot 2 \cdot 3 \cdot \dots \cdot \alpha} \cdot \frac{(n-\alpha) \cdot (n-\alpha-1) \cdot (n-\alpha-2) \cdot \dots \cdot (n-\alpha-\alpha+1)}{1 \cdot 2 \cdot 3 \cdot \dots \cdot \alpha}$$

Quodsi jam hic loco  $\alpha$  scribamus  $\alpha + 1$ , ut oriatur terminus sequens, qui ergo erit  $\frac{n \cdot (n-1) \cdot (n-2) \cdot \dots \cdot (n-\alpha-1)}{1 \cdot 2 \cdot 3 \cdot \dots \cdot (\alpha+1)} \cdot \frac{(n-\alpha-1) \cdot (n-\alpha-2) \cdot \dots \cdot (n-\alpha-\alpha+1)}{1 \cdot 2 \cdot 3 \cdot \dots \cdot (\alpha+1)}$

hic

Nova Acta Acad. Imp. Scient. Tom. XIV.

hic per illum divisus praebet quotum  

$$\frac{(a-2z-\lambda)(a-2z-\lambda-1)}{(a+1)(\lambda+a+1)}$$
 Hinc ergo erit terminus sequens  

$$\Pi \cdot \frac{(a-2a-\lambda)(a-2z-\lambda-1)}{(a+1)(\lambda+a+1)}$$

§. 18. Quodsi ergo in hac serie, more Newtoniano, littera  $\Pi$  denotet quemlibet terminum praecedentem, sequens semper erit  $\Pi \cdot \frac{(a-2z-\lambda)(a-2z-\lambda-1)}{(a+1)(\lambda+a+1)}$ ; unde cum primus terminus sit  $(\frac{a}{\lambda})$ , ubi est  $a = c$ , § hic designetur per  $\Pi$ , erit terminus secundus  

$$\Pi \frac{(a-\lambda)(a-\lambda-1)}{(a+1)(\lambda+1)}$$
, qui si denuo vocetur  $\Pi$ , erit terminus tertius  

$$\Pi \frac{(a-\lambda)(a-\lambda-1)}{(a+1)(\lambda+1)}$$
, qui si denuo vocetur  $\Pi$ , erit terminus quartus  

$$\Pi \frac{(a-\lambda-1)(a-\lambda-2)}{(a+1)(\lambda+1)}$$
, et ita porro. Hoc modo nosstra series pro  $z$  hanc inducet formam:  

$$z = (\frac{a}{\lambda}) + \Pi \frac{(a-\lambda)(a-\lambda-1)}{(a+1)(\lambda+1)} + \Pi \frac{(a-\lambda-1)(a-\lambda-2)}{(a+1)(\lambda+1)} + \Pi \frac{(a-\lambda-2)(a-\lambda-3)}{(a+1)(\lambda+1)} + \text{etc.}$$
 ubi scilicet perpetua  $\Pi$  designat terminum praecedentem.

§. 19. Hinc igitur si loco  $\lambda$  successively scribamus valores 1, 2, 3, 4 etc. pro nostris litteris  $p, q, r, s$ , etc. sequentes nascemur series:  

$$p = 1 + \Pi \frac{(a-2)(a-3)}{1 \cdot 1} + \Pi \frac{(a-2)(a-3)}{2} + \Pi \frac{(a-1)(a-2)}{1 \cdot 2} + \Pi \frac{(a-1)(a-2)}{2} + \text{etc.}$$

$$q = (\frac{a}{2}) + \Pi \frac{(a-1)(a-2)}{1 \cdot 2} + \Pi \frac{(a-1)(a-2)}{2 \cdot 3} + \Pi \frac{(a-2)(a-3)}{3 \cdot 4} + \Pi \frac{(a-2)(a-3)}{4 \cdot 5} + \text{etc.}$$

$$r = (\frac{a}{3}) + \Pi \frac{(a-2)(a-3)}{1 \cdot 2} + \Pi \frac{(a-2)(a-3)}{2 \cdot 3} + \Pi \frac{(a-3)(a-4)}{3 \cdot 4} + \Pi \frac{(a-3)(a-4)}{4 \cdot 5} + \text{etc.}$$

$$s = (\frac{a}{4}) + \Pi \frac{(a-3)(a-4)}{1 \cdot 2} + \Pi \frac{(a-3)(a-4)}{2 \cdot 3} + \Pi \frac{(a-4)(a-5)}{3 \cdot 4} + \Pi \frac{(a-4)(a-5)}{4 \cdot 5} + \text{etc.}$$
 etc.

§. 20. Ista forma ad calculum numericum imprimis sunt accommodatae, quod pro sola littera  $p$  ostendisse sufficiet. Quaevis scilicet exempli gratia valorem ipsius  $p$  pro

pro casu  $n = 6$ , ac singulae ejus partes sequenti modo reperientur:

- I. = . = 1
- II. = 1.6.5 = 30
- III. = 30.4.3 = 90
- IV. = 90.2.1 = 20

ergo Summa =  $p = 141$

§. 21. Simili modo quaeramus valorem ipsius  $p$  pro casu  $n = 12$ , cujus singulae partes sequenti modo colligentur:

- I. = 1 . . . . . = 1
- II. = 1. 1.11 = 132
- III. = 132. 10.9 = 2970
- IV. = 2970. 8.7 = 18480
- V. = 18480. 4.3 = 34650
- VI. = 34650. 4.3 = 16632
- VII. = 16632. 2.1 = 924

ergo Summa =  $p = 73759$ .

§. 22. Mox autem trademus modum multo expeditiorem singulos terminos harum serierum ex binis praecedentibus eliciendi, unde facili calculo omnes valores pro litteris  $p, q, r$ , etc. pro singulis exponentibus  $n$  exhiberi poterunt, siquae omnes illos valores, quousque libuerit, continere licebit. Hanc autem relationem primo secunda pro numeris sub littera  $p$  contentis infutamus.

Investigatio relationis

inter ternos valores consecutivos  $p, p', p''$ .

§. 23. Cum sit  $p = 1 + \binom{n}{1} \binom{n-1}{1} + \binom{n}{2} \binom{n-2}{2} + \binom{n}{3} \binom{n-3}{3} + \text{etc.}$

hujus seriei conferemus terminum quemcumque  $\binom{n}{\alpha} \binom{n-\alpha}{\alpha}$ , quem vocemus =  $\Pi$ , ita ut facta evolutione sit

$$\Pi = \frac{1 \cdot 2 \cdot 3 \cdots (n-\alpha) \cdots 1}{1 \cdot 2 \cdot 3 \cdots \alpha \cdots 1} \cdot \frac{1 \cdot 2 \cdot 3 \cdots (n-\alpha) \cdots 1}{1 \cdot 2 \cdot 3 \cdots (n-\alpha) \cdots 1} = \frac{1 \cdot 2 \cdot 3 \cdots (n-\alpha) \cdots 1}{1 \cdot 2 \cdot 3 \cdots \alpha \cdots 1} \cdot \frac{1 \cdot 2 \cdot 3 \cdots (n-\alpha) \cdots 1}{1 \cdot 2 \cdot 3 \cdots (n-\alpha) \cdots 1}$$

terminum autem, qui hunc sequitur, designemus per  $\Phi$ , ut sit  $\Phi = \binom{n}{\alpha} \binom{n-\alpha}{\alpha}$ , id-  
eque facta evolutione  $\Phi = \frac{1 \cdot 2 \cdot 3 \cdots (n-\alpha) \cdots 1}{1 \cdot 2 \cdot 3 \cdots \alpha \cdots 1} \cdot \frac{1 \cdot 2 \cdot 3 \cdots (n-\alpha) \cdots 1}{1 \cdot 2 \cdot 3 \cdots (n-\alpha) \cdots 1}$ , hincque ergo habebitur:

$$\frac{\Phi}{\Pi} = \frac{1 \cdot 2 \cdot 3 \cdots (n-\alpha) \cdots 1}{1 \cdot 2 \cdot 3 \cdots \alpha \cdots 1} \cdot \frac{1 \cdot 2 \cdot 3 \cdots (n-\alpha) \cdots 1}{1 \cdot 2 \cdot 3 \cdots (n-\alpha) \cdots 1} \cdot \frac{1 \cdot 2 \cdot 3 \cdots (n-\alpha) \cdots 1}{1 \cdot 2 \cdot 3 \cdots (n-\alpha) \cdots 1} \cdot \frac{1 \cdot 2 \cdot 3 \cdots (n-\alpha) \cdots 1}{1 \cdot 2 \cdot 3 \cdots (n-\alpha) \cdots 1}$$

§. 24. Jam pro valoribus sequentibus  $p'$  et  $p''$  designemus valores ipsius  $\Phi$  respondentis per  $\Phi'$  et  $\Phi''$ , quibus quoniam oriuntur ex valore  $\Phi$ , si loco  $n$  scribatur  $n+1$  et  $n+2$ , erit facta evolutione

$$\Phi' = \frac{1 \cdot 2 \cdot 3 \cdots (n+1) \cdots 1}{1 \cdot 2 \cdot 3 \cdots \alpha \cdots 1} \cdot \frac{1 \cdot 2 \cdot 3 \cdots (n+1) \cdots 1}{1 \cdot 2 \cdot 3 \cdots (n+1) \cdots 1}$$

unde patet fore  $\Phi' : \Phi = \frac{1 \cdot 2 \cdot 3 \cdots (n+1) \cdots 1}{1 \cdot 2 \cdot 3 \cdots n \cdots 1} \cdot \frac{1 \cdot 2 \cdot 3 \cdots (n+1) \cdots 1}{1 \cdot 2 \cdot 3 \cdots (n+1) \cdots 1} \cdot \frac{1 \cdot 2 \cdot 3 \cdots n \cdots 1}{1 \cdot 2 \cdot 3 \cdots (n+1) \cdots 1}$

Simili modo, si hinc quoque loco  $n$  scribamus  $n+1$ , habebimus

$$\Phi'' = \frac{1 \cdot 2 \cdot 3 \cdots (n+2) \cdots 1}{1 \cdot 2 \cdot 3 \cdots \alpha \cdots 1} \cdot \frac{1 \cdot 2 \cdot 3 \cdots (n+2) \cdots 1}{1 \cdot 2 \cdot 3 \cdots (n+2) \cdots 1}$$

§. 25. Hinc jam formemus hanc expressionem:

$$A \Phi + \frac{B}{n-1} \Phi' + \frac{C}{(n-2) \cdot (n-1)} \Phi''$$

cujus ergo valor per ipsam litteram  $\Phi$  ita exprimitur:

$$\Phi \left( A + \frac{B}{n-2} \frac{1}{1} + \frac{C}{(n-2) \cdot (n-1)} \right)$$

ubi litteras  $A, B, C$  ita definiere conemur, ut forma ista aequalis evadat ipsi termino prae edenti  $\Pi$ . Evidens autem est, istas litteras, ut ad omnes terminos pariter pertinere queant, litteram  $\alpha$  involvere

gralia intra praescriptos terminos fuerint assumpta, scilicet ab  $x = a$  usque ad  $x = b$ , qui quidem termini nondum

vere non debere. Substituto igitur loco  $\Pi$  valore ante dato per  $\Phi$  expresso, nanciscemur sequentem aequationem per  $\Phi$  divisibilem:

$$A + \frac{B}{n-2} \frac{1}{1} + \frac{C}{(n-2) \cdot (n-1)} = \frac{1 \cdot 2 \cdot 3 \cdots (n-2) \cdots 1}{1 \cdot 2 \cdot 3 \cdots (n-2) \cdots 1} \cdot \frac{1 \cdot 2 \cdot 3 \cdots (n-2) \cdots 1}{1 \cdot 2 \cdot 3 \cdots (n-2) \cdots 1}$$

quae a fractionibus liberata evadit

$$A(n-2\alpha-1)(n-2\alpha) + B(n-2\alpha) + C = (\alpha+1)(\alpha+1)$$

§. 26. Cum in hac aequatione littera  $\alpha$  ad secundam dimensionem ascendat, ternae litterae  $A, B, C$  praeclie sufficiant, ut ex hac aequatione determinari queant. Primo igitur coaequemus utrinque terminos quadratum  $\alpha\alpha$  involventes, unde oriatur ista aequatio:  $4A\alpha\alpha = \alpha\alpha$ , idcirco  $A = \frac{1}{4}$ . Eodem modo coaequemus terminos ipsam litteram  $\alpha$  involventes, unde perducimur ad hanc aequationem:

$$2\alpha(1-2n)A - 2\alpha \cdot B = 2\alpha, \text{ unde sit } B = -\frac{(n-1) \cdot 1 - 2\alpha - 3}{4}$$

Denique termini ab  $\alpha$  immunes dant hanc aequationem:

$$(4n-1)A + nB + C = 1, \text{ unde reperitur } C = \frac{(n+1)^2}{4}$$

§. 27. His igitur valoribus inventis, pro singulis terminis semper erit  $A\Phi + \frac{B}{n-1} \Phi' + \frac{C}{(n-2) \cdot (n-1)} \Phi'' = \Pi$ . Quod si ergo hinc completemus hanc formulam:

$$A \cdot p + \frac{B}{n-1} p' + \frac{C}{(n-2) \cdot (n-1)} p''$$

ex primis terminis pro  $\Phi$  assumtis oriatur praecedens seriei  $p$  qui est =  $c$ ; ex secundis autem termini pro  $\Phi$  assumtis oriatur terminus primus, qui est  $1$ ; ex tertiis autem terminis conficitur terminus secundus, qui est  $\binom{n}{2} \binom{n-2}{2}$ ; ex quatuor terminis quartis pro  $\Phi$  assumtis conficitur tertius qui est  $\binom{n}{3} \binom{n-3}{3}$  et ita porro; sicque omnes tres series hoc modo collectae, producent hanc seriem:  $0 + 1 + \binom{n}{2} \binom{n-2}{2} + \binom{n}{3} \binom{n-3}{3} + \text{etc.}$  quae est ipsa series pro  $p$

data

tertio, casu quo  $x = -2$ , ex quibus ergo binos terminos illos  $a$  et  $b$  desumi oportet. Ita autem hos binos terminos

data. Hinc habebimus inter ternas literas p, p', p'' hanc aequationem:

$$A \cdot p + \frac{n}{n+1} p' + \frac{c}{(n+1)} p'' = p.$$

§. 28. Substituamus nunc loco litterarum A, B, C valores modo inventos, et nostra aequatio inter has ternas literas erit  $\frac{1}{4} p - \frac{(n+3)}{4(n+1)} p' + \frac{n}{4(n+1)} p'' = p$ , quae reducitur ad hanc:  $\frac{n-2}{n+1} p'' - \frac{(n+3)}{n+1} p' = 3p$ , unde fit  $p'' = p' + (\frac{n+3}{n+1})(p' + 3p)$ .

§. 29. Hinc igitur facile pro singulis valoribus exponentis  $n$  omnes numeri littera  $p$  designati definiti poterunt, dum quilibet ex duobus praecedentibus componitur. Ita sumto  $n = 0$ , erit  $p = 1$  et  $p' = 1$ , ideoque lectius  $p'' = 1 + \frac{1}{2}(1 + 3 \cdot 1)$ . Sumto  $n = 1$ , erit  $p = 1$ ,  $p' = 3$ , ideoque terminus quartus  $p'' = 5 + \frac{3}{2}(3 + 3 \cdot 1) = 7$ . Sumto  $n = 2$ , ob  $p = 3$  et  $p' = 7$ , erit terminus quintus  $p'' = 7 + \frac{1}{2}(7 + 3 \cdot 3) = 19$ . Si sumatur  $n = 3$ , ob  $p = 7$  et  $p' = 19$ , erit terminus sextus  $p'' = 19 + \frac{3}{2}(19 + 3 \cdot 7) = 51$ .

§. 30. Si hoc modo ulterius progrediamur, poterimus hanc progressionem continuare, quousque libuerit, ope formulae  $p' + \frac{1}{2}(p' + 3p) = p'$ , quae nobis suppediat sequentes determinationes:

- 51 +  $\frac{3}{2}(51 + 3 \cdot 19) = 141$
  - 141 +  $\frac{1}{2}(141 + 3 \cdot 51) = 393$
  - 393 +  $\frac{1}{2}(393 + 3 \cdot 141) = 1177$
  - 1177 +  $\frac{3}{2}(1177 + 3 \cdot 393) = 3139$
  - 3139 +  $\frac{1}{2}(3139 + 3 \cdot 1177) = 8953$
  - 8953 +  $\frac{3}{2}(8953 + 3 \cdot 3139) = 25653$
  - 25653 +  $\frac{1}{2}(25653 + 3 \cdot 8953) = 73789$ .
- etc.

§. 31.

§. 31. Simili methodo etiam relatio inter ternos terminos consecutivos pro sequentibus litteris q, r, s, etc. investigari poterit, quem laborem quo generaliter expediamus, inquiramus in relationem, inter ternos terminos z, z', et z'', quibus respondere assumimus litteram  $\lambda$ .

### Investigatio Relationis

inter ternos terminos consecutivos z, z', z''.

§. 32. Cum sit  $z = \binom{n}{\lambda} + \binom{n}{\lambda+1} + \binom{n}{\lambda+2} + \binom{n}{\lambda+3} + \dots$  etc. hujus seriei consideremus terminum quemcumque  $\Pi = \binom{n}{\lambda+1} + \binom{n}{\lambda+2} + \dots$  cuius valor evolutus est  $\Pi = \frac{n(n-1)(n-2) \dots (n-2\lambda+1)}{1 \cdot 2 \cdot 3 \dots \lambda}$   $\times \frac{1 \cdot 2 \cdot 3 \dots (n-2\lambda+1)}{(n+1)(n+2) \dots (n+1)}$   $\times \frac{(n-2\lambda+1) \dots (n-2\lambda+1)}{(n+1)(n+2) \dots (n+1)}$  unde fit  $\Phi = \frac{1 \cdot 2 \cdot 3 \dots (n-2\lambda+1)}{(n+1)(n+2) \dots (n+1)}$ . Hinc ergo colligimus  $\Phi = \frac{1 \cdot 2 \cdot 3 \dots (n-2\lambda+1)}{(n+1)(n+2) \dots (n+1)}$ , ideoque  $\Pi = \frac{(n+1)(n+2) \dots (n+1)}{(n-2\lambda+1) \dots (n-2\lambda+1)}$ .

§. 33. Jam pro valoribus sequentibus z' et z'' designemus valores ipsi  $\Phi$  respondentes per  $\Phi'$  et  $\Phi''$ , qui quoniam oriuntur ex valore  $\Phi$ , si loco  $n$  scribatur  $(n+1)$  et  $(n+2)$ , erit facta evolutione

$$\Phi' = \frac{(n+1) \cdot n \cdot (n-1) \cdot \dots \cdot (n-2\lambda+1)}{1 \cdot 2 \cdot 3 \cdot \dots \cdot \lambda} \times \frac{1 \cdot 2 \cdot 3 \cdot \dots \cdot (n-2\lambda+1)}{(n+1)(n+2) \dots (n+1)}$$

$$\Phi'' = \frac{(n+2) \cdot (n+1) \cdot n \cdot \dots \cdot (n-2\lambda+1)}{1 \cdot 2 \cdot 3 \cdot \dots \cdot \lambda} \times \frac{1 \cdot 2 \cdot 3 \cdot \dots \cdot (n-2\lambda+1)}{(n+1)(n+2) \dots (n+1)}$$

Eodemque modo erit

§. 34. Hinc profus ut supra formemus hanc expressionem:  $A\Phi + \frac{B}{n+1}\Phi' + \frac{C}{(n+2)(n+1)}\Phi''$ , cujus valor per  $\Phi$  ita exprimitur:  $\Phi(A + \frac{B}{n+1} + \frac{C}{(n+2)(n+1)})$ , ubi iterum litteras A, B, C ita definiti oportet, ut formula aequalis evadat ipsi termino praecedenti. It. Substituto, igitur loco,  $\Pi$  valore ante

ante per  $\Phi$  expresso nanciscemur sequentem aequationem a fractionibus jam liberatam:

$$A(n - 2x - \lambda - 1)(n - 2x - \lambda) + B(n - 2x - \lambda) + C = (x + 1)(x + \lambda + 1).$$

§. 35. Facta igitur evolutione et coequatis primo utrinque terminis  $xz$  involventibus prodiit haec aequatio pro determinatione litterae  $A$ :  $4xzA = x^2$ , ideoque  $A = \frac{x}{4}$ . Eodem modo si coequentur termini simplicem litteram  $x$  involventes, perducimur ad sequentem aequationem:

$$(4x\lambda - 4nz + 2x)A - 2x B = (\lambda + 2)x, \text{ unde colligitur } B = -\frac{2x - \lambda}{4}. \text{ Denique coequatis terminis ab } x \text{ liberis prodiit aequatio:}$$

$$C = \frac{(n-2)\lambda - n + \lambda + \lambda}{4} = \frac{(n-\lambda)(2n+3)}{4} + C = \lambda + 1, \text{ unde fit } C = \frac{(n+2)\lambda}{4} - \frac{\lambda}{4}.$$

§. 36. His igitur valoribus inventis, pro singulis terminis semper erit  $A\Phi + \frac{B}{n+1}\Phi' + \frac{C}{(n+2)(n+1)}\Phi'' = \text{II}$ . Quod si igitur hinc computemus istam formulam:

$Az + \frac{B}{n+1}z' + \frac{C}{(n+2)(n+1)}z''$ , ex primis terminis pro  $\Phi$  assumptis orietur praecedens seriei  $z$ , qui est  $o$ . Ex secundis autem terminis pro  $\Phi$  assumptis orietur terminus primus ( $\frac{x}{4}$ ); ex tertiis terminis conficitur terminus secundus ( $\frac{x}{4}$ ); ex quartis terminis pro  $\Phi$  assumptis conficitur tertius qui est ( $\frac{x}{4}$ ), et ita porro; quibus collectis oritur ipsa series pro  $z$  data

$$z = \left(\frac{x}{4}\right) + \left(\frac{x}{4}\right)\left(\frac{x-1}{n+1}\right) + \left(\frac{x}{4}\right)\left(\frac{x-2}{n+2}\right) + \left(\frac{x}{4}\right)\left(\frac{x-3}{n+3}\right) + \text{etc.}$$

Relatio igitur inter  $z, z', z''$  erit

$$Az + \frac{B}{n+1}z' + \frac{C}{(n+2)(n+1)}z'' = z.$$

§. 37.

§. 37. Substituamus nunc loco litterarum  $A, B, C$  valores modo inventos, et aequatio inter has ternas litteras est:

$$\frac{1}{4}z - \frac{(2n+3)}{4(n+1)}z' + \frac{(n+2)\lambda - n}{4(n+2)(n+1)}z'' = z,$$

quae reducitur ad hanc formam:

$$\frac{(n-2)}{(n+2)}\frac{-A\lambda}{n+1}z' = \frac{n+1}{n-1}z + 3z, \text{ unde colligitur } z'' = \frac{n+2}{(n+2)\lambda - n}((2n+3)z' + 3(n+1)z).$$

§. 38. Tribuamus nunc litterae  $\lambda$  successive valores  $0, 1, 2, 3, 4$ , etc. ac reperiemus sequentes relationes pro singulis litteris:

$$\begin{aligned} \frac{1}{(n-2)}\frac{-0^2}{(n+1)}p'' &= \frac{2n-3}{n-1}p' + 3p \\ \frac{(n+2)}{(n+2)}\frac{-1^2}{(n+1)}q'' &= \frac{2n-3}{n-1}q' + 3q \\ \frac{(n+2)^2}{(n-2)(n+1)}r'' &= \frac{2n-3}{n-1}r' + 3r \\ \frac{(n-2)^2}{(n+1)}s'' &= \frac{2n+3}{n+1}s' + 3s \\ &\text{etc.} \end{aligned}$$

§. 39. Cum igitur pro littera  $q$  habeamus hanc aequationem:  $q'' = \frac{2n-3}{n-1}q' + 3(2n+3)q$ , casu  $n=0$  erit  $q = 0$  et  $q' = 1$ , unde fit  $q'' = \frac{3}{2}(3 \cdot 1 + 3 \cdot 0) = 2$ . Nunc pro  $n=1$ , ob  $q = 1$  et  $q' = 2$ , erit  $q'' = \frac{3}{2}(5 \cdot 2 + 6 \cdot 1) = 6$ . Tum pro casu  $n=2$ , ob  $q = 2$  et  $q' = 6$ , erit  $q'' = \frac{3}{2}(7 \cdot 6 + 9 \cdot 2) = 16$ . Jam sumto  $n=3$ , ob  $q = 6$  et  $q' = 16$ , erit  $q'' = \frac{3}{2}(9 \cdot 16 + 12 \cdot 6) = 45$ . At casu  $n=4$ , ob  $q = 16$  et  $q' = 45$ , erit  $q'' = \frac{3}{2}(11 \cdot 45 + 15 \cdot 16) = 126$ .

§. 40. Hic autem calculus multo laboriosior et tediousior est quam praecedens pro valoribus litterae  $p$  expostus. Verum alia methodus multo facilior inde derivari poterit, qua omnes litteras  $q, r, s$  per solam litteram  $p$  cum suis

XI

deri-



derivatis  $p', p''$  determinare licebit; tum enim postquam series numerorum  $p$  jam satis longe fuerit computata, inde etiam valores litterarum  $q, r, s$ , etc. multo leviori labore colligi poterunt, id quod in sequenti articulo ostendemus:

Determinatio litterarum

$q, r, s, t$ , etc.

per solam primam  $p$  cum suis derivatis.

§. 41. Posito brevitalis gratia nostro trinomio  $1+x-rxx^{-X}$ , eius binas potestates  $X^n$  et  $X^{n+1}$  evolutas ita disponamus, ut pares potestates ipsius  $x$  sibi invicem subscriptae appareant, hoc modo:

$X^1 = 1 + nx + \dots + qx^{n-1} + px^n + qx^{n+1} + rx^{n+2} + sx^{n+3} + \text{etc.}$   
 $X^{n+1} = 1 + (n+1)x + \dots + rx^{n-1} + qx^{n+1} + q^2x^{n+2} + r^2x^{n+3} + \text{etc.}$   
 quo facto supra jam notavimus quemlibet coefficientem inferioris seriei aequari superiori cum binis precedentibus.

§. 42. Per hanc igitur legem sequentes nanciscemur aequalitates:

$$p' = q + p + q = 2q + p$$

$$q' = r + q + p$$

$$r' = s + r + q$$

etc.

unde colligimus sequentes determinationes

$$q = \frac{p^2 - p}{2}; r = q' - q - p; s' = r' - r - q; t' = s' - s - r; \text{ etc.}$$

§. 43. Manifestum est hic formulam  $p' - p$  exprimere incrementum quantitatis  $p$ , dum exponens  $n$  unitate auctur, quod cum per  $\Delta p$  exprimi solet, aequalitates inverse sequenti modo succinctius exhiberi poterunt:

$$q = \frac{1}{2}\Delta p, \text{ sive } 2q = \Delta p; 2r = 2\Delta q - 2p; 2s = 2\Delta r - 2q; \text{ etc.}$$

§. 44

§. 69. Quod si jam hanc aequationem per  $\partial\Phi \cos X\Phi$  multiplicemus et integremus, omnia haec integralia, ter-

§. 44. Characterem autem hoc differentiali  $\Delta$  in ulum vocato, cum sit  $2q = \Delta p$ , erit  $2\Delta q = \Delta\Delta p$  ideoque  $2r = \Delta\Delta p - 2p$ , hincque  $2\Delta r = \Delta^3 p - 2\Delta p$ , ex quo porro fit  $2s = \Delta^3 p - 3\Delta p$  ideoque

$$2\Delta s = \Delta^4 p - 3\Delta\Delta p, \text{ ergo } 2t = \Delta^4 p - 4\Delta\Delta p + 2p, \text{ ideoque}$$

$$2\Delta t = \Delta^5 p - 4\Delta^3 p + 2\Delta p. \text{ Hinc porro fit}$$

$$2u = \Delta^5 p - 5\Delta^3 p + 5\Delta p, \text{ ideoque}$$

$$2\Delta u = \Delta^6 p - 5\Delta^4 p + 5\Delta\Delta p, \text{ unde deducitur}$$

$$2v = \Delta^6 p - 6\Delta^4 p + 9\Delta\Delta p + 2p, \text{ et ita porro.}$$

§. 45. Quod si hos coefficientes numericos attentius consideremus, lex progressionis convenire deprehenditur cum serie (geometricis satis nota, unde pro valore  $z$ , cui ordinis index positus est  $\lambda$ , obtinebimus sequentem formam:

$$2z = \Delta^\lambda p - \lambda\Delta^{\lambda-2} p + \lambda(\lambda-1)\Delta^{\lambda-4} p - \lambda(\lambda-1)(\lambda-2)\Delta^{\lambda-6} p + \lambda(\lambda-1)(\lambda-2)(\lambda-3)\Delta^{\lambda-8} p - \lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)\Delta^{\lambda-10} p + \text{etc.}$$

quam seriem eo usque tantum continuari oportet, quamdiu indices ipsius  $\Delta$  non evadant negativi. Ita si sumamus  $\lambda = 6$ , quo casu sit  $z = v$ , ex hac lege generali utique proficit

$$2v = \Delta^6 p - 6\Delta^4 p + 9\Delta^2 p - 2p.$$

§. 46. Quo indoles hujus seriei clarius percipiat, meminisse oportet, hanc formam  $\frac{(x+Vxx-4)^n}{2^n} + \frac{(x-Vxx-4)^n}{2^n}$  in sequentem seriem resolvit:

$$x^2 - nx^{-2} + \frac{n(n-1)}{2}x^{-4} - \frac{n(n-1)(n-2)}{6}x^{-6} + \text{etc.}$$

Hoc igitur modo nostro scopo jam est satisfactum, cum omnes litteras  $q, r, s, t$ , etc. per solam primam  $p$  suasque derivatas  $p', p'', p'''$ , etc. expressas ellicuerimus.

§. 72. Cum jam series nostra sit  $1 + r + 2xr + x^2 + 19x^4 + \dots + nx^2 + r^2x^4 + p^2x^6 + \text{etc.}$

Determinatio quantitatis p per formulam finitam integram.

§. 47. Cum fit per tertiam formam supra expostam p = 1 + (2/3) x + (1/3) x^2 + (9/2) x^3 + etc. quilibet terminus in genere erit (2^n/n!) (x^n/n!), quem excipit iste sequens: (2^{n+1}/(n+1)!) (x^{n+1}/(n+1)!)

Cum igitur facta evolutione fit (2^n/n!) = (2^{n+1}/(n+1)!) - 1/n, similiq; modo (2^{n+1}/(n+1)!) = (2^{n+2}/(n+2)!) - 1/(n+1), haec posterior forma per priorem divisa dat quotum (2^{n+2}/(n+2)!) / (2^{n+1}/(n+1)!) = 2(n+1)/(n+2) = 2(n+1)/(n+2) x.

§. 48. Hac ergo reductione adhibita, sumto alpha = 1 erit (2/3) = 6/3 (2/3); sumto alpha = 2 erit (1/3) = 10/3 (2/3) = 10/3 \* 6/3 \* 2/3; si alpha = 3 fit (2/3) = 14/3 \* 6/3 \* 10/3 \* 6/3 \* 2/3; tum si alpha = 4 fiet (10/3) = 14/3 \* 6/3 \* 10/3 \* 6/3 \* 2/3 et ita porro. His igitur valoribus introductis erit per factores ordinarios numericos p = 1 + 2/3 (2/3) + 2^2/3^2 (2/3) + 2^3/3^3 (2/3) + 2^4/3^4 (2/3) + 2^5/3^5 (2/3) + etc.

§. 49. Nunc igitur videamus, quomodo formam finitam integram investigari oporteat, cujus integrale intratam terminos inclusum ad hanc ipsam seriem perducatur. Hunc in finem contemplari conveniet istam formulam (1+x)^n quippe cujus evolutio praebet hanc seriem: 1 + n x + (n/2) x^2 + (n/3) x^3 + etc., cujus termini alteri jam continent characteres nostros litterae n.

§. 50. Hanc igitur seriem in duas partes discerpamus, recundum terminos alternos, ac ponamus M = 1 + (n/2) x^2 + (n/4) x^4 + (n/6) x^6 + etc. N = (n/3) x + (n/5) x^3 + (n/7) x^5 + etc. ita

ita ut sit (1+x)^n = M + N. Nunc autem inquiramus, quomodo seriem priorem M per operationes analyticas tractari oporteat, ut ipsa series propolita, seu valor ipsius p, inde exoritur.

§. 41. Ad hoc efficiendum ducamus quantitatem M in certum quoddam differentiale dv cuiuspiam functionis ipsius x, atque sequentes integrationes ita determinemus, ut intra certos terminos, veluti ab x = a usque ad x = b, includantur, quas conditiones ita comparatas esse oportet, ut sequentibus conditionibus satisfiat:

- 1) fxx dv = 2/3 v
2) f x^4 dv = 2/6 v
3) f x^6 dv = 2/6 \* 10/3 v
4) f x^8 dv = 2/6 \* 10/3 \* 4 v
etc.

hoc enim modo integrale / M dv producet hanc seriem: v + 2/3 (2/3) v + 2^2/3^2 (2/3) v + 2^3/3^3 (2/3) v + etc. ita ut hoc modo id quod quaerimus nanciscamur p = / M dv

§. 52. Formularum igitur integralium, quas hic exposuimus, quaelibet ita a praecedente pendet, ut sit

fx dv = 2/3 dv
fx^4 dv = 2/6 fxx dv
fx^6 dv = 2/6 \* 10/3 f x^4 dv
fx^8 dv = 2/6 \* 10/3 \* 4 f x^6 dv
etc.

haec in genere effici debet ut fiat / x^{2m} dv = 4^{m-2}/m / x^{2m-2} dv. Haec scilicet reductio locum habere debet, postquam integralia

gratia intra praescriptos terminos fuerint assumpta, scilicet ab  $x = a$  usque ad  $x = b$ , qui quidem termini nondum sunt cogniti, sed ad ipsam conditionem praescriptam accommodari debent.

§. 53. Cum igitur pro his terminis integrationis esse debeat

$$m \int x^{2m} \partial v = (m-2) \int x^{2m-2} \partial v,$$

ponamus esse generatim

$$m \int x^{2m} \partial v = (4m-2) \int x^{2m-2} \partial v + \Pi x^{2m-1},$$

ubi scilicet  $\Pi$  ejusmodi sit functio, ut pars subnexa  $x^{2m-1}$  utroque termino tam  $x = a$  quam  $x = b$  in nihilum abeat. Haec jam aequatio differentiatam et per  $x^{2m-1}$  divisam dat

$$m x \partial v = (4m-2) \partial v + (3m-1) \Pi \partial x + x \partial \Pi,$$

quae aequatio subsistere debet pro omnibus numeris  $m$ .

§. 54. Hinc igitur ista aequatio in duas discerpi debet, quarum altera contineat solos terminos littera  $m$  affectos, altera vero reliquos, quae ergo duae aequationes erunt

$$x x \partial v = 4 \partial v + 2 \Pi \partial x \text{ et } 0 = -u \partial v - \Pi \partial x - x \partial \Pi.$$

Ex priori fit  $\partial v = \frac{2 \Pi \partial x}{x x - 4}$ ; ex altera vero fit  $\partial v = \frac{x \partial \Pi - \Pi \partial x}{x \Pi - \Pi \partial x}$ , qui ambo valores inter se coequati praebent hanc aequationem:

$$4 \Pi \partial x = (x x - 4) (x \partial \Pi - \Pi \partial x) = x^3 \partial \Pi - x x \Pi \partial x - 4 x \partial \Pi + 4 \Pi \partial x$$

hincque colligitur  $\frac{2 \Pi \partial x}{x x - 4} = \frac{x^3 \partial \Pi - x x \Pi \partial x - 4 x \partial \Pi + 4 \Pi \partial x}{x \Pi - \Pi \partial x}$ , unde integrando fit  $4 \Pi = 4 \int \frac{x x \partial x}{x \Pi - \Pi \partial x}$  ideoque  $\Pi = C \sqrt{x x - 4}$ , vel etiam  $\Pi = C \sqrt{4 - x x}$ , quo valore invento assequimur nostrum differentiale assumtum  $\partial v = \frac{2 C x \partial x}{\sqrt{4 - x x}}$ , unde fit  $v = 2 C A \sin \frac{x}{2}$ .

§. 55. Consideremus nunc formulam suffixam

$$\Pi x^{2n-1} = C x^{2n-1} \sqrt{4 - x x},$$

quam deprehendimus triplici modo in nihilum abire posse: primo scilicet quando  $x = 0$ , casu excepto quo  $m = 0$ ; secundo, casu quo  $x = 2$ ; ac tertio

tertio, casu quo  $x = -2$ , ex quibus ergo binos terminos illos  $a$  et  $b$  desumi oportet. Ita autem hos binos terminos eligi conveniet, ut etiam altera integrationis pars  $\int N \partial v$  commodè exprimat. Quia enim posuimus  $(1+x)^n = M+N$ , etiam ad integrale  $\int N \partial v$  est respiciendum, quod si penitus evanesceret, pro terminis integrationis sine dubio id esset commodissimum, tum enim foret  $\int (M+N) \partial v$ , five  $\int \partial v (1+x)^n = \int M \partial v$ , consequenter haberemus  $p = \frac{\int M \partial v}{0}$ .

§. 56. Supra autem posuimus

$$N = \left(\frac{1}{2}\right) x + \left(\frac{3}{2}\right) x^3 + \left(\frac{5}{2}\right) x^5 + \left(\frac{7}{2}\right) x^7 + \text{etc.}$$

unde conficitur  $\int N \partial v = \frac{1}{2} \int x \partial v + \frac{3}{2} \int x^3 \partial v + \left(\frac{5}{2}\right) \int x^5 \partial v + \text{etc.}$  ubi per easdem reductiones, quas pro littera  $M$  instituimus, quaelibet formula integralis ad praecedentem, ope reductionis  $\int x^{2m} \partial v = \frac{4m-2}{m} \int x^{2m-2} \partial v$  reduci potest. Summo enim  $m = \frac{3}{2}$  erit  $\int x^3 \partial v = \frac{5}{3} \int x \partial v$ . Summo  $m = \frac{5}{2}$  erit  $\int x^5 \partial v = \frac{16}{5} \int x \partial v$ . Summo  $n = \frac{7}{2}$  erit  $\int x^7 \partial v = \frac{24}{7} \int x \partial v$ , etc. unde patet, si modo  $\int x \partial v$  evanesceret etiam sequentia omnia esse evanitura.

§. 57. Quoniam igitur invenimus  $\partial v = \frac{2 C x \partial x}{\sqrt{4 - x x}}$ , erit  $x \partial v = \frac{2 C x \partial x}{\sqrt{(4 - x x)}}$ , hincque  $\int x \partial v = 2 C \sqrt{4 - x x}$ , quae expressio binis casibus vel  $x = +2$  vel  $x = -2$  evanescit. Quam obrem si terminos integrationis constituamus  $x = 2$ , et  $x = -2$ , non solum partes illae subnexae  $\Pi x^{2n-1}$ , verum etiam totus valor integralis  $\int N \partial v$  evanescet, atque adeo hoc casu quae sitio nostro perfecte satisfacimus, cum sit  $p = \frac{\int \partial v (1+x)^n}{0}$ .

§. 58. Cum igitur invenimus  $\partial v = \frac{2 C x \partial x}{\sqrt{(4 - x x)}}$ , ejus integrale, ita sumtum ut evanescat posito  $x = 2$ , erit  $v = 2$

$v = 2CA \sin \frac{x}{2} - 2C \frac{1}{2}$ , quare expressio reducitur ad hanc:  $v = -2CA \cos \frac{x}{2}$ ; unde hoc integrali usque ad alterum terminum  $x = -2$  extenso prodit  $v = -C\pi$ . His igitur valoribus substitutis erit formula quaesita  $p = -\frac{1}{2} \int_{-1}^1 \frac{1+x^n}{1-4-x^2} dx$ . Haec scilicet formula integralis. a termino  $x = 2$  usque ad terminum  $x = -2$  extensa, verum, necdebit valorem ipsius  $p$ .

§. 59. Quo hanc formulam concinuiorem reddamus, statuanus  $x = 2 \cos \Phi$ , ubi evidens est casu  $x = 1$  fieri angulum  $\Phi = c$ ; casu vero  $x = -1$  fieri  $\Phi = \pi$ , ita ut hoc angulo introducto integrale cuius debet a termino  $\Phi = 0$  usque ad  $\Phi = \pi$ ; tum vero erit  $\partial x = -2 \partial \sin \Phi$  et  $\sqrt{1-x^2} = 2 \sin \Phi$ , qua substitutione facta nasciscetur hanc aequationem:

$$p = + \frac{1}{2} \int_{\pi}^0 (1 + 2 \cos \Phi)^n \partial \Phi \left[ \begin{matrix} a & \partial & \equiv & 0 \\ ad & \equiv & \pi \end{matrix} \right].$$

Determinatio reliquarum litterarum per formulas finitas integrales.

§. 60. Hoc facile praestari poterit per relationes quas supra inter has litteras tradidimus. Primo scilicet habuimus  $2q = \Delta p = p' - p$ , ubi  $p'$  nascitur ex  $p$ , si loco  $n$  scribatur  $n+1$ . Quoniam igitur modo invenimus

$$p = \frac{1}{2} \int_{\pi}^0 (1 + 2 \cos \Phi)^n \partial \Phi, \text{ erit } p' = \frac{1}{2} \int_{\pi}^0 (1 + 2 \cos \Phi)^{n+1} \partial \Phi,$$

hincque ergo erit  $p' - p = \frac{1}{2} \int_{\pi}^0 \cos \Phi (1 + 2 \cos \Phi)^n \partial \Phi$ , quo valore substituto reperietur

$$q = \frac{1}{2} \int_{\pi}^0 \partial \Phi \cos \Phi (1 + 2 \cos \Phi)^n \left[ \begin{matrix} a & \partial & \equiv & 0 \\ ad & \partial & \equiv & \pi \end{matrix} \right]$$

hinc ergo porro erit

$$q = \frac{1}{2} \int_{\pi}^0 \partial \Phi \cos \Phi (1 + 2 \cos \Phi)^{n+1}.$$

§. 61.

§. 61. Supra autem vidimus esse  $r = q' - q - p$ , nunc vero erit  $q' - q = \frac{1}{2} \int_{\pi}^0 \partial \Phi \cos \Phi (1 + 2 \cos \Phi)^n$ . Hinc ergo si substituatue  $p$ , ob  $2 \cos \Phi^2 - 1 = \cos 2 \Phi$ , elicimus litteram

$$r = \frac{1}{2} \int_{\pi}^0 \partial \Phi \cos 2 \Phi (1 + 2 \cos \Phi)^n, \text{ unde iterum fit}$$

$$r = \frac{1}{2} \int_{\pi}^0 \partial \Phi \cos 2 \Phi (1 + 2 \cos \Phi)^{n+1}.$$

§. 62. Quoniam igitur supra invenimus  $s = r' - r - q$ , abstinens hic primo  $r' - r = \frac{1}{2} \int_{\pi}^0 \partial \Phi \cos \Phi \cos 2 \Phi (1 + 2 \cos \Phi)^n$ . Hinc ergo si substituatue  $q$ , ob  $2 \cos \Phi \cos 2 \Phi - \cos \Phi = \cos 3 \Phi$ , erit  $s = \frac{1}{2} \int_{\pi}^0 \partial \Phi \cos 3 \Phi (1 + 2 \cos \Phi)^n$ . Simili modo iam evidens est fore  $t = \frac{1}{2} \int_{\pi}^0 \partial \Phi \cos 4 \Phi (1 + 2 \cos \Phi)^n$ ; eodemque modo reperietur fore  $u = \frac{1}{2} \int_{\pi}^0 \partial \Phi \cos 5 \Phi (1 + 2 \cos \Phi)^n$ ; atque adeo in genere erit  $z = \frac{1}{2} \int_{\pi}^0 \partial \Phi \cos \lambda \Phi (1 + 2 \cos \Phi)^n$ .

§. 63. Quoniam Analysis, qua hic usi sumus, praestit est singularis et parum consueta, haud abs re erit veritatem harum formularum demonstratione analytica muniri, quam de singulis uno quasi labore sequenti modo insititue licebit. Inchoandum erit ab evolutione formulae  $(1 + 2 \cos \Phi)^n$ , quae perducit ad hanc seriem:

$$1 + \binom{n}{1} \cos \Phi + \binom{n}{2} 4 \cos^2 \Phi + \binom{n}{3} 8 \cos^3 \Phi + \binom{n}{4} 16 \cos^4 \Phi + \text{etc.}$$

Per notas autem angulorum reductiones confiat fore

$$\begin{aligned} 2 \cos \Phi &= 2 \cos \Phi \\ 4 \cos^2 \Phi &= 2 \cos 2 \Phi + 2 \\ 8 \cos^3 \Phi &= 2 \cos 3 \Phi + 6 \cos \Phi \\ 16 \cos^4 \Phi &= 2 \cos 4 \Phi + 8 \cos 2 \Phi + 6 \\ 32 \cos^5 \Phi &= 2 \cos 5 \Phi + 16 \cos 3 \Phi + 20 \cos \Phi \\ &\dots \dots \dots \\ 2^n \cos^n \Phi &= 2 \cos n \Phi + 2 \binom{n}{1} \cos^{n-2} \Phi + 2 \binom{n}{2} \cos^{n-4} \Phi \\ &\quad + 2 \binom{n}{3} \cos^{n-6} \Phi + \text{etc.} \end{aligned}$$

*Novae Acta Acad. Imp. Scient. Tom. XIV.*

N

ubi

nbi probe notandum est, quando terminus ultimus est abfolutus, tum tantum simpliciter capi debere; praeter ea vero etiam cosinus angulorum negativorum prorsus omitti debent.

§. 64. His igitur rite dispositis erit

$$(1 + z \cos \Phi)^n = (1 + \frac{z}{2})(\cos \Phi + 1 + \frac{z}{2})(\cos \Phi + 1 + z \cos \Phi) + (\frac{z}{2}(\cos \Phi + 1 + \frac{z}{2}) + (\frac{z}{2}(\cos \Phi + 1 + z \cos \Phi) + 1) \cos \Phi) - z(\frac{z}{2} \cos \Phi + 1 \cos \Phi + 1 \cos \Phi + 1 \cos \Phi + 1 \cos \Phi + 1 \cos \Phi) \text{ etc.}$$

unde sequentes integrationes sunt petendae.

§. 65. Incipiamus a prima littera  $p$ , ubi hanc seriem ducere in  $\partial \Phi$  et integrari oportet. Cum igitur in genere sit  $\int \partial \Phi \cos m \Phi = \frac{1}{m} \sin m \Phi$ , iste valor jam evanescit posito  $\Phi = c$ , pro altero integrationis termino  $\Phi = \pi$  manifesto evanescit, si quidem omnes numeri  $n$  sunt integri. Ad integrationem igitur soli termini absoluti relinquuntur, tum vero integrali rite sumto erit  $\int \partial \Phi = \pi$ , quo observato erit valorum integrale

$$\int \partial \Phi (1 + z \cos \Phi)^n = \pi + 2 \binom{n}{2} \pi + (\frac{n}{3}) \pi + 2c \binom{n}{6} \pi + \text{etc.}$$

Quod si hic forma generalis supra data consularur, hi coefficientes numerici revocentur ad formas  $(\frac{n}{2})$ ,  $(\frac{n}{3})$ , etc. prorsus uti veritas formulae postulat. Erit enim

$$p = \frac{1}{2} \int \partial \Phi (1 + z \cos \Phi)^n = 1 + \frac{1}{2} \binom{n}{2} + \binom{n}{3} + \binom{n}{6} + \text{etc.}$$

§. 66. Pergamus ad secundam litteram  $q$ , ubi superiorem seriem per  $\partial \Phi \cos \Phi$  multiplicavi et integrari oportet. Ad hoc observetur esse in genere

$$\int \partial \Phi \cos \Phi \cos m \Phi = \frac{1}{m} \sin(m+1)\Phi + \frac{1}{m} \sin(m-1)\Phi \text{ quae expressio posito } \Phi = \pi \text{ in nihilum abit solo casu excepto quo } m = 1, \text{ quippe quo fit } \int \partial \Phi \cos \Phi \cos \Phi = \frac{1}{2} \Phi = \frac{\pi}{2}.$$

Ex

Ex quo intelligitur, ex superiori serie alios terminos hic non in computum venire, nisi qui continent  $\cos \Phi$ , qui sunt:  $z(\frac{1}{2} \cos \Phi + \frac{1}{2} \cos \Phi + 2 \frac{z}{2} \cos \Phi + 2 \frac{z}{2} \cos \Phi + \text{etc.}$

Hi autem termini ducti in  $\partial \Phi \cos \Phi$  et integrati, ob

$$\int z \partial \Phi \cos \Phi^2 = \pi, \text{ dabunt, per } \pi \text{ divisi, ipsum valorem } q = (\frac{1}{2}) + (\frac{1}{2}) + (\frac{1}{2}) + (\frac{1}{2}) + \text{etc.}$$

§. 67. Pro littera  $r$  superior series multiplicari debet per  $\partial \Phi \cos \Phi$ . Cum igitur in genere sit

$$\cos \Phi \cos m \Phi = \frac{1}{2} \cos(m+2)\Phi + \frac{1}{2} \cos(m-2)\Phi, \text{ per } \partial \Phi \text{ multiplicando integrale pro termino } \Phi = \pi \text{ semper evanescit, excepto solo casu } m = 1, \text{ quippe quo fit } \int \partial \Phi \cos \Phi^2 = \frac{\pi}{2}.$$

Hic igitur ex serie superiori soli termini per  $\cos 2\Phi$  affecti in computum veniant, qui sunt

$$2 \cos 2\Phi (\binom{n}{2} + \frac{1}{2} \binom{n}{4} + \binom{n}{6} + \binom{n}{8} + \text{etc.})$$

Quia igitur  $\int \partial \Phi \cos 2\Phi^2 = \pi$ , omnibus terminis colligendis et per  $\pi$  dividendo reperitur

$$r = (\frac{n}{2}) + (\frac{1}{2}) \binom{n}{4} + (\frac{n}{6}) + \binom{n}{8} + \text{etc.}$$

§. 68. Quo haec clariora reddantur ac facilius ad valorem generalem  $z$  accommodari queant, evolutionem polynomii  $(1 + z \cos \Phi)^n$  statim secundum cosinus multiplicorum anguli  $\Phi$  disponamus hoc modo:

$$(1 + z \cos \Phi)^n = 1 + \frac{n}{1} \binom{n}{2} z + \frac{n}{2} \binom{n}{4} z^2 + \frac{n}{3} \binom{n}{6} z^3 + \frac{n}{4} \binom{n}{8} z^4 + \text{etc.} \\ + \cos \Phi (\frac{n}{1} \binom{n}{3} + (\frac{n}{5}) \binom{n}{7} + \frac{n}{2} \binom{n}{9} + (\frac{n}{11}) \binom{n}{13} + \text{etc.}) \\ + z \cos 2\Phi (\frac{n}{2} \binom{n}{4} + (\frac{n}{6}) \binom{n}{8} + (\frac{n}{10}) \binom{n}{14} + \text{etc.}) \\ + z^2 \cos 3\Phi (\frac{n}{3} \binom{n}{6} + (\frac{n}{9}) \binom{n}{12} + (\frac{n}{15}) \binom{n}{18} + \text{etc.}) \\ \dots \\ + z \cos 7\Phi (\frac{n}{7} \binom{n}{14} + (\frac{n}{21}) \binom{n}{21} + (\frac{n}{28}) \binom{n}{28} + \text{etc.})$$

§. 69.

§. 69 Quod si jam hanc aequationem per  $\partial\Phi \cos \lambda\Phi$  multiplicemus et integremus, omnia haec integralia, terminis praescriptis inclusa, evanescent, excepto membro  $2 \cos \lambda\Phi (\dots)$ , propterea quod productum  $2 \cos \lambda\Phi^2$  continet partem absolutam, unde per integrationem oritur  $\pi$ , ita ut sit  $\int \partial\Phi \cos \lambda\Phi (1 + 2 \cos \Phi)^n - \pi \left( \binom{n}{\lambda} + \binom{n}{\lambda-1} + \binom{n}{\lambda-2} + \dots + \binom{n}{\lambda-n} \right) + \text{etc.}$  qui valor per  $\pi$  divisus ipsam valorem ipsius  $2$  supra inventum praebet: unde veritas harum novarum expressionum luculenter est demonstrata.

§. Ceterum si fingulas series paragraphi penultimi vel leviter confideremus, deprehendimus eas ipsas litteris nostris  $p, q, r, s$ , etc. esse aequales, ita ut nunc sit  $(1 + 2 \cos \Phi)^n = p + q \cos \Phi + r \cos 2\Phi + s \cos 3\Phi + \dots + \text{etc.}$  ubi simul ratio est manifesta, cur litterae  $q, r, s$ , etc. duplicentur, quippe quae in hoc est posita, quod in evolutione formulae  $(1 + x + xx)^n$  littera  $p$  semel tantum in medio, reliquae vero litterae bis, a medio aequidistantes, occurrunt. Ex quo haec egregia afinitas inter illas binas potestates  $(1 + x + xx)^n$  et  $(1 + 2 \cos \Phi)^n$ , summa attentione digna est censenda.

Investigatio summae seriei

$$P = 1 + x + 3xx + 7x^3 + 19x^5 + \dots - px^r + p'x^{r+1} + \text{etc.}$$

§. 71. Quoniam hujus seriei terminus generalis est  $px^r$ , quoniam sequuntur  $p x^{r-1}$  et  $p x^{r-2}$ , inter has ternas quantitates  $p, p', p''$  invenimus supra hanc relationem:

$$(n+2)p'' = (n+5)p' + 3(n+1)p$$

quam hoc modo ad usum nostrum accommodatam referamus:

$$3(n+1)p + (n+1)p' + (n+2)p'' - (n+2)p'' = 0$$

§. 72.

§. 72. Cum jam series nostra sit  $1 + x + 3xx + 7x^3 + 19x^5 + \dots + px^r + p'x^{r+1} + p''x^{r+2} + \text{etc.}$  ejusmodi operationes instituamus, quibus relatio modo allata obviatur, id quod sequenti modo commodissime fiet:

$$\begin{aligned} \frac{3a \cdot 1x}{\partial x} &= 3 + 6x + 27xx + \dots + 3(n+1)px^r + \text{etc.} \\ \frac{\partial \cdot p}{\partial x} &= 1 + 6x + 21xx + \dots + (n+1)p'x^r + \text{etc.} \\ \frac{\partial^2 \cdot p^2}{\partial x^2} &= 1 + 2 + 9x + 28xx + \dots + (n+2)p''x^r + \text{etc.} \\ \frac{\partial^3 \cdot p^3}{\partial x^3} &= -\frac{1}{2} - 6 - 21x - 76xx + \dots - (n+2)p''x^r + \text{etc.} \end{aligned}$$

Colligantur jam hae quatuor series in unam summam atque obtinebimus sequentem aequationem:

$$\frac{\partial \partial \cdot p \cdot x}{\partial x} + \frac{\partial p}{\partial x} + \frac{\partial \cdot p^2}{\partial x^2} - \frac{\partial p}{\partial x} = 0,$$

quandoguidem omnes termini se mutuo destruant.

§. 73. Hoc ergo modo deducti sumus ad aequationem finitam differentialem primi gradus, quae per  $x \partial x$  multiplicata et in ordinem redacta ita se habebit:

$$P \partial x (3x+1) + \partial P (3xx + 2x - 1) = 0,$$

unde ergo sit  $\frac{\partial^2 P}{\partial x^2} = \frac{\partial x(1+3x)}{1-2x-3xx}$ , quae aequatio integrata praebet

$$P = -\frac{1}{2} l(1-2x-3xx) + C. \text{ consequenter } P = \frac{e}{\sqrt{1-x-3xx}}$$

Hic ad constantem  $C$  determinandam notetur tantum nostram seriem propolitam casu  $x = 0$  praebere  $P = 1$ , unde patet sumi debere  $C = 1$ , ita ut sit summa seriei  $P = \frac{1}{\sqrt{1-2x-3xx}}$ .

§. 74. Praeter ergo expectationem pertingimus ad summam algebraicam, quae expressio etiam ita est comparata, ut in seriem converfa ipsam nostram seriem reproducat, id quod ostendisse operae erit pretium. Cum igitur sit

$$P =$$

$$\int \frac{\partial \Phi \cos 3\Phi}{1 + b^2 - 2b \cos \Phi} = \frac{\pi b^3}{1 - b^2}$$

$P = (1 - 2x - 3xx)^{-1}$ , si hujus trinomi partes posteriores conjunctim spectentur, evolutio nobis dabit

$P = 1 + \frac{1}{2}(x + 3xx) + \frac{1}{4}(2x^2 + 3xx)^2 + \frac{1}{8}(x^3 + 3xx^2 + 3x^2 + \text{etc.})$  quam sufficiet ad potestatem tantum tertiam usque evolvisse. Hoc modo nascemur

$P = 1 + x + \frac{3}{2}xx + \frac{3}{2}x^3 + \text{etc.} = 1 + x + 3xx + 7x^3 + \text{etc.}$   
 $+ \frac{3}{2}xx + \frac{3}{2}x^3$   
 quae igitur perfecte congruit.

§. 75. At vero haec eadem summa adhuc alio modo investigari potest, ex formula scilicet integrali, quam pro valore litterae  $p$  invenimus:

$$p = \frac{1}{2} \int \Phi(1 + 2 \cos \Phi)^n [ad \Phi = 0].$$

Haec enitè formula, sumto  $n = 0$ , per  $x$  multiplicata, dat terminum secundum  $x$ ; casu porro  $n = 1$ , per  $xx$  multiplicata, dat terminum tertium  $3xx$ ; quo observato summa quaesita ita poterit representari:

$$P = \frac{1}{2} \int \partial \Phi (1 + x(1 + 2 \cos \Phi) + xx(1 + 2 \cos \Phi)^2 + x^3(1 + 2 \cos \Phi)^3 + \text{etc.})$$

ubi probe est observandum, in hac integratione quartitatem  $x$  tanquam constantem spectari, siquidem solus angulus  $\Phi$  est variabilis.

§. 76. Evidens autem est, seriem infinitam, in quam elementum  $\partial \Phi$  duci oportet, esse geometricam, cujus ergo summa erit  $\frac{1}{1 - \cos \Phi} = \frac{1}{1 - \frac{1-x}{1+x}} = \frac{1+x}{2-x}$ , sicque adeo pro  $P$  jam habemus hanc expressionem finitam:

$$P = \frac{1}{2} \int \frac{1}{1 - \frac{1-x}{2-x}} [ad \Phi = 0],$$

quae

quae aequatio ita poterit exhiberi:

$$P = \frac{1}{\pi(1-x)} \int \frac{\partial \Phi}{1 - \frac{x \cos \Phi}{1-x}} [ad \Phi = 0],$$

ubi jam brevitate gratia statuamus  $\frac{x^2}{1-x} = k$ , ut habeamus

$$P = \frac{1}{\pi(1-x)} \int \frac{\partial \Phi}{1 - k \cos \Phi}.$$

§. 77. Confat autem hujus formulae  $\frac{\partial \Phi}{1 - k \cos \Phi}$  integrale esse  $\frac{1}{1 + (1-k^2)} A \cos \frac{\cos \Phi + k}{1 - k \cos \Phi}$ ; unde si loco  $n$  scribamus  $-k$ , adiscimus pro nostro casu  $P = \frac{1}{\pi(1-x)(1-k^2)} A \cos \frac{\cos \Phi + k}{1 - k \cos \Phi}$ , ubi constantis additione non est opus, quia haec expressio casu  $\Phi = 0$  sponte evanescit. Faciamus igitur pro altero termino  $\Phi = \pi$ , unde sit  $\cos \Phi = -1$  et  $A \cos \frac{\cos \Phi + k}{1 - k \cos \Phi} = A \cos \frac{-1 + k}{1 - k}$ , ergo habebimus  $P = \frac{1}{\pi(1-x)(1-k^2)}$ , quae expressio ob  $k = \frac{x^2}{1-x}$  abit in hanc:  $P = \frac{1}{\pi(1-x)(1-3x^2)}$ , proflus ut ante.

§. 78. Cum sit  $1 - 3x = (1 - x)^2 - 4xx = (1 + x)(1 - 3x)$ , sequitur seriem nostram summendam duobus casibus fieri infinite magnam, scilicet altero casu quo  $x = -1$ , altero vero quo  $x = \frac{1}{3}$ . Tum vero nostra series habebit summam finitam, quando  $x$  continetur intra hos limites:  $-1$  et  $\frac{1}{3}$ ; sin autem  $x$  extra hos limites accipiat, tum summa semp. r erit imaginaria. Ita sumto  $x = \frac{1}{4}$ , habebitur haec summatio:

$$1 + \frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \frac{1}{4^4} + \frac{1}{4^5} + \frac{1}{4^6} + \text{etc.} = \frac{4}{3}.$$

Investigatio summae reliquarum serierum  $Q, R, S$ , etc. supra §. 6. expositarum.

§. 79. Incipiamus a serie  $Q$ , quae est  $Q = xx + 2x^3 + 6x^5 + \dots + qx^{r+1} + q'x^{r+2} + \text{etc.}$  cujus

ejus primus terminus  $rx$  ex potestate  $n = 1$  oritur; ubi perinde ac si seriem  $ab n = 0$  inchoare velimus, praefigi debet terminus  $or$ . Pro hac autem serie offendimus supra esse  $q = \frac{1}{2}p$   $p$ , unde hujus seriei summa ex serie prima  $p$  sequenti modo elici poterit.

§ 80. Cum sit  $P = 1 + x + 3rx + \dots + px^n + p'x^{n+1} + \text{etc.}$  erit  $Px = \dots + x + rx + \dots + px^{n+1} + \text{etc.}$  quae posterior series a priori subtrahita relinquit

$$P(1-x) = 1 + 2rx + \dots + (p'-p)x^{n+1} + \text{etc.}$$

Quare cum sit  $p' - p = 2q$ , erit

$P(1-x) = 1 + 2Q$ ; sicque innotescit hujus seriei summa, cum sit  $Q = \frac{P(1-x)-1}{2}$ . Modo ante autem vidimus esse  $P = \frac{1-x-2x^2-3x^3}{2r(1-2x-3x^2)}$  sicque habebimus  $Q = \frac{1-x-1x-2x-3x^2}{2r(1-2x-3x^2)}$ .

§ 81. Procedamus ad seriem  $R$ , quae ita se habebat:  $R = x^4 + 3x^5 + 10x^6 + \dots + rx^{n+2} + rx^{n+3} + \text{etc.}$  cujus primus terminus  $x^4$  ex potestate  $n = 2$  est ortus, unde praefixi concipiendi sunt bini termini  $ox^3 + ox^2$ , ad cujus summam inveniendam notetur esse  $r = q' - q - p$ . Hinc si sequentes operationes instituantur:

$$\begin{aligned} Q &= rx + 2x^2 + \dots + qx^{n+1} + qx^{n+2} + \text{etc.} \\ - Qx &= -x^3 \dots - qx^{n+2} - \text{etc.} \\ - Px^2 &= -rx - x^3 \dots - px^{n+2} - \text{etc.} \\ \text{inde conjunctim fiet} \end{aligned}$$

$$Q(1-x) - Pxx = (q' - q - p)x^{n+2} = R.$$

§ 82. Hoc igitur modo summam  $R$  determinavimus per binas series praecedentes  $Q$  et  $P$ , quae cum jam sint cognitae, etiam summam seriei  $R$  algebraice per certam functionem ipsius  $x$  expressam furus adepti, quae quomodo commode evolvi queat, deinceps ostendemus.

§ 83.

§ 83. Pro serie  $S$ , quae ita erat propofita:

$$S = x' + 4x^2 + 15x^3 + \dots + sx^{n+3} + s'x^{n+4} + \text{etc.}$$

ei tres termini evanescerentes praefigi sunt censendi, scilicet  $ox^3 + ox^4 + ox^5$ , siquidem a potestate  $n = 0$  incipere velimus. Supra autem invenimus esse  $s = r' - r - q$ , unde sequentes operationes instituamus:

$$\begin{aligned} R &= x^4 + 3x^5 + 10x^6 + \dots + rx^{n+2} + r'x^{n+3} + \text{etc.} \\ - Rx &= -x^5 - 3x^6 - \dots - rx^{n+3} - \text{etc.} \\ - Qrx &= -x^4 - 2x^5 - 6x^6 - \dots - qx^{n+3} - \text{etc.} \end{aligned}$$

quibus tribus seriibus collectis oritur haec series:

$$x^6 + \dots + sx^{n+3} + \text{etc.}$$

quae est ipsa series  $S$ . Quocirca summa hujus seriei per binas praecedentes  $Q$  et  $R$  ita determinatur ut sit  $S = R(1-x) - Qrx$ , cujus evolutio etiam satis simpliciter expediiri poterit, uti mox ostendetur.

§ 84. Eodem modo series  $T$  per binas praecedentes  $R$  et  $S$  definitur hoc modo:

$$\begin{aligned} S &= x^6 + 4x^7 + 15x^8 + \dots + sx^{n+3} + s'x^{n+4} + \text{etc.} \\ - Sx &= -x^7 - 4x^8 - \dots - sx^{n+4} - \text{etc.} \\ - Rxx &= -x^6 - 3x^7 - 10x^8 - \dots - rx^{n+4} - \text{etc.} \end{aligned}$$

Cum igitur  $s' - s - r = t$ , hae tres series collectae dabunt  $S(1-x) - Rxx = x^6 + \dots + tx^{n+4} + \text{etc.}$

$$\text{quae cum sit ipsa series } T, \text{ erit } T = S(1-x) - Rxx.$$

§ 85. Hinc igitur manifestum est figurulas harum serierum satis simpliciter per binas praecedentes determinari posse, atque adeo per legem penitus uniformem. Eas conjunctim ob oculos Nova Acta Acad. Imp. Scient. Tom. XLV. O pona-



ponamus:

$$Q \equiv \frac{p(1-x^{1-x})}{1-x^{1-x}}$$

$$R \equiv Q(1-x) - Pxx$$

$$S \equiv R(1-x) - Qxx$$

$$T \equiv S(1-x) - Rxx$$

$$U \equiv T(1-x) - Sxx$$

etc.

unde patet omnes has summas secundum feriem recurrentem procedere, cujus scilicet relationis est  $(1-x)^2, -xx$ . Verum mox patebit, hanc feriem adeo esse geometricam.

§. 86. Ad hoc ostendendum, cum facta evolutione fit

$$\frac{Q}{1-x} = \frac{1}{1-x} - \frac{1}{1-2x} + \frac{1}{1-3x} - \dots, \text{ ut habeamus } Q \equiv Pv; \text{ inde autem}$$

$$\frac{1-x}{1-x} - \frac{1}{1-2x} + \frac{1}{1-3x} - \dots = v, \text{ ut habeamus } Q \equiv Pv; \text{ inde autem}$$

substituta irrationalitate, cum fit  $\sqrt{1-x} = 1 - \frac{1}{2}xx - \frac{1}{8}x^3 - \dots$ , orietur haec aequatio:  $(1-x)^2 - 4xx \equiv (1-x)^2 - 4v(1-x) - 4vv$ , quae reducitur ad istam:  $v(1-x) - xx \equiv vv$ , quod probe notasse juvabit.

§. 87. Jam pro ferie R, si loco Q hunc valorem Pv substituamus, orietur haec aequatio:

$$R \equiv P(v(1-x) - xx), \text{ ideoque per relationem modo notatam}$$

$$R \equiv Pvv. \text{ Si jam porro loco } Q \text{ et } R \text{ valores inventos scribamus, nanciscemur simili modo:}$$

$$S \equiv Pv(v(1-x) - xx) \equiv Pvd^3$$

$$T \equiv Pvv(v(1-x) - xx) \equiv Pvd^4$$

$$U \equiv Pvd^3(v(1-x) - xx) \equiv Pvd^5$$

.....

$$Z \equiv Pvd^{\lambda-1}(v(1-x) - xx) \equiv Pvd^{\lambda}$$

§. 88.

§. 88. Quod n jam has determinationes ad formulas integrales, quas pro literis p, q, r, etc. invenimus, transferamus, quoniam invenimus  $z \equiv \frac{1}{\pi} \int \partial\Phi \cos \lambda\Phi (1 + z \cos \Phi)^n$ , si exponenti n successive valores tribuamus 0, 1, 2, 3, 4 etc. quia series Z a potestate  $x^\lambda$  incipere est censenda, formula differentialis  $\partial\Phi \cos \lambda\Phi$  per hanc feriem geometricam multiplicari debet:

$$(1 + z \cos \Phi)^0 x^\lambda + (1 + z \cos \Phi)^1 x^{\lambda+1} + (1 + z \cos \Phi)^2 x^{\lambda+2} + \text{etc.}$$

cujus summa est  $\frac{x^\lambda}{1-x-2x \cos \Phi}$ , qua ergo in calculum

introducenda summa quaesita Z ita exprimetur:

$$Z \equiv \frac{1}{\pi} \int \frac{x^\lambda \partial\Phi \cos \lambda\Phi}{1-x-2x \cos \Phi} \left[ a \Phi \equiv 0 \right], \text{ ubi quantitas } x \text{ est confians.}$$

§. 89. Quoniam igitur hic invenimus istam summam

$$\text{scil. } Z \equiv Pvd^\lambda = \frac{v^\lambda}{\sqrt{(1-2x-3xx)}}, \text{ existente}$$

$$v \equiv \frac{1-x}{1-x-\sqrt{(1-2x-3xx)}},$$

nunc huius ipsius formulae integralis valorem adeo algebraicum exhibere poterimus, quandoquidem nunc novimus esse.

$$\frac{1}{\pi} \int \frac{x^\lambda \partial\Phi \cos \lambda\Phi}{1-x-2x \cos \Phi} = \frac{v^\lambda}{\sqrt{(1-2x-3xx)}}$$

five multiplicando per  $\frac{\pi}{x^\lambda}$  habebimus

$$\int \frac{\partial\Phi \cos \lambda\Phi}{1-x-2x \cos \Phi} = \frac{\pi}{\sqrt{(1-2x-3xx)}} \left( \frac{v}{x} \right)^\lambda$$

§. 90. Quoniam haec integratio majori attentione digna videtur, eam in commodiorem formam transformamus, et quoniam  $x$  et  $y$  hic ut constantes spectantur, ponamus  $\frac{x}{y} = b$ , atque ob  $y = \frac{1-x}{1-x-3x^2}$ , erit

$2bx = 1-x-\sqrt{1-2x-3x^2}$ , quae aequatio, sublata irrationalitate, praebet:  $4b^2bx-4bx(1-x)+(1-x)^2=4x^2$ , quae reducitur ad hanc:  $4b^2x-b+bx=4x^2$ , unde ipsa quantitas  $x$  satis commode determinatur, cum fiat  $x = \frac{b}{2b^2-1}$ , ideoque  $1-x = \frac{2b^2-1-b}{2b^2-1}$ , hincque porro cum esset

$$\sqrt{1-2x-3x^2} = 1-x-2bx, \text{ erit nunc}$$

$$\sqrt{1-2x-3x^2} = \frac{1-b}{1+b-2b^2}$$

§. 91. Quod si ergo loco quantitatis  $x$  litteram  $b$  in nostrum calculum introducamus, integratio inventa ad hanc formam reducetur simpliciorum:

$$\int \frac{\partial \Phi \cos \lambda \Phi}{1-2b \cos \Phi + bb} \left[ \begin{matrix} a \Phi = 0 \\ \text{ad } \Phi = \pi \end{matrix} \right] = \frac{\pi b^{\lambda}}{1-bb}$$

cujus veritas ex calculis hactenus expeditis est deducta; verum etiam immediate et directe demonstrari potest, quo ipso praecedentia omnia eo magis corroborabuntur.

§. 92. Ad hoc igitur demonstrandum in subdium vocemus fati notam integrationem, qua est

$$\int \frac{\partial \Phi}{1+\beta \cos \Phi} = \frac{1}{1+\beta} A \cos \frac{\alpha \cos \Phi + \beta}{\alpha + \beta \cos \Phi}$$

Fiat nunc  $\alpha = 1+bb$  et  $\beta = -2b$  et habebimus

$$\int \frac{\partial \Phi}{1-2 \cos \Phi + bb} = \frac{1}{1-b^2} A \cos \frac{1-bb \cos \Phi}{1-2 \cos \Phi + bb}$$

quod integrale jam evanescit posito  $\Phi = c$ . Posito ergo pro altero termino  $\Phi = \pi$ , hoc integrale evadet  $\frac{\pi}{1-b^2}$ .

§. 93.

§. 93. Quoniam igitur pro nostris terminis integratio nis invenimus  $\int \frac{\partial \Phi}{1-2b \cos \Phi + bb} = \frac{\pi}{1-b^2}$ , atque manifeste est  $\int \partial \Phi = \pi$ , ideoque

$$\int \frac{\partial \Phi (1-\frac{c \cos \Phi}{b} + bh)}{1-2b \cos \Phi + bb} = \pi,$$

hanc formam in duas partes distribuendo habebimus

$$\pi = (1+bb) \int \frac{\partial \Phi}{1-2b \cos \Phi + bb} - 2b \int \frac{\frac{c \cos \Phi}{b} \partial \Phi}{1-2b \cos \Phi + bb},$$

unde colligimus

$$\int \frac{\frac{c \cos \Phi}{b} \partial \Phi}{1-2b \cos \Phi + bb} = \frac{\pi b}{1-bb^2}$$

§. 94. Quoniam pro nostris terminis integrationis in genere est  $\int \partial \Phi \cos i \Phi = 0$ , siquidem  $i$  fuerit numerus integer, multiplicemus hanc formulam supra et infra per  $x+bb = z b \cos \Phi$ , atque obtinebimus

$$\int \frac{\partial \Phi (x+bb \cos \Phi)}{1-bb} = \frac{b \cos i(i-1) \Phi - b \cos i(i+1) \Phi}{1-bb} = 0.$$

Haec forma jam in tres partes facta nobis dabit

$$(x+bb) \int \frac{\partial \Phi \cos i \Phi}{1-2b \cos \Phi + bb} = b \int \frac{\partial \Phi \cos i(i-1) \Phi}{1-2b \cos \Phi + bb} + b \int \frac{\partial \Phi \cos i(i+1) \Phi}{1-2b \cos \Phi + bb}$$

unde derivamus hanc reductionem generalem:

$$\int \frac{\partial \Phi \cos i(i-1) \Phi}{1-2b \cos \Phi + bb} = \frac{1+bb}{b} \int \frac{\partial \Phi \cos i \Phi}{1-2b \cos \Phi + bb} - \int \frac{\partial \Phi \cos i(i-1) \Phi}{1-2b \cos \Phi + bb}$$

cujus ope ex integralibus binis pro angulis  $i \Phi$  et  $(i-1) \Phi$  integrale pro angulo  $i+1) \Phi$  determinari potest, unde sequentem tabulam conficere licebit:

$$\int \frac{\partial \Phi}{1+bb} = \frac{\pi}{1-bb}$$

$$\int \frac{\partial \Phi \cos \Phi}{1+bb} = \frac{\pi b}{1-bb}$$

$$\int \frac{\partial \Phi \cos 2 \Phi}{1+bb} = \frac{\pi b^2}{1-bb}$$

$$\int \frac{\partial \Phi \cos 3 \Phi}{1+bb} = \frac{\pi b^3}{1-bb}$$

§. 95

$$\int \frac{\partial \Phi \cos 3\Phi}{1+bb-2b \cos \Phi} = \frac{\pi b^3}{1-bb}$$

$$\int \frac{\partial \Phi \cos 4\Phi}{1+bb-2b \cos \Phi} = \frac{\pi b^4}{1-bb}$$

$$\dots$$

$$\int \frac{\partial \Phi \cos \lambda \Phi}{1+bb-2b \cos \Phi} = \frac{\pi b \lambda}{1-bb}$$

Prorsus uti supra invenimus.

DE INNUMERIS  
CURVIS ALGEBRAICIS

QUARUM LONGITUDINEM  
PER ARCUS HYPERBOLICOS METIRI LICET.  
AUCTORE  
NICOLAO FUSS.

Conventui exhibita die 98. Jun. 1798.

§. 1.

In Tomo quinto novorum Actorum Academiae, pro anno 1787, dae reperiantur dissertationes viii immortalis Leonardi Euleri, super argumento plane novo et a nemine anlea tractato; prior inscripta est: *De innumeris curvis algebraicis, quarum longitudinem per arcus parabolicos metiri licet*; altera: *De innumeris curvis algebraicis, quarum longitudinem per arcus ellipticos metiri licet*. In calce posterioris dissertationis auctor d clarrat, se nullo adhuc modo vel unicam faltem curvam algebraicam eruere potuisse, cujus singulos arcus per arcus hyperbolicos metiri liceret. „Si talis quaestio“ inquit Eulerus loco citato „circa arcus hyperbolicos proponatur, facti cogor, nullo adhuc modo me vel unicam faltem curvam algebraicam eruere potuisse, cujus singuli arcus per formulam  $\int \sqrt{1+v^2}$  — quae arcum fittit Hyperbolae aequilaterae — exprimerentur. Sin autem aequationem generalem pro Hyperbola assumere velimus, pro qua est  $y=n\sqrt{1+vv}$ , elementum arcus inde nascitur  $\partial S = \frac{2n \sqrt{1+vv} \partial v}{\sqrt{1+vv}}$  „quae

DE