DE INTEGRATIONIBUS DIFFICILLIMIS

QUARUM INTEGRALIA TAMEN ALIUNDE EXHIBERI POSSUNT.

AUCTORE

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§. 1.

Cum hodie quidem nemo Geometrarum amplius dubitet, quin omnia imaginaria, undecunque originem trahant, ad hanc formam $A + B\sqrt{-1}$ reduci queant, quanquam haec veritas nondum satis firmis et claris rationibus est demonstrata: certum etiam erit, omnem formulam integralem $\int Z\partial z$, quaecunque Z fuerit functio ipsius z, si in ea loco z scribatur formula imaginaria $v(\cos\vartheta + \sqrt{-1}\sin\vartheta)$, resolvi posse in duas hujusmodi formulas integrales: $\int p\partial v + \sqrt{-1}.\int q\partial v$, ita ut litterae p et q sint functiones reales ipsius v.

§. 2. Hoc equidem nuper fusius ostendi circa formulas integrales $\int\frac{z^{m-1}\partial z}{1+z^n}$ et $\int\frac{z^{m-1}\partial z}{1-z^n}$, unde posito $z = v(\cos\vartheta + \sqrt{-1}\sin\vartheta)$ ortae sunt ejusmodi formulae integrales, quarum evolutio plura haud contemnenda calculi artificia requirebat. Ex quo intelligitur, si hujusmodi formulae magis fuerint complicatae, atque adeo quantitates irrationales in se involvant, tum formulas inde derivatas $\int p\partial v$ et $\int q\partial v$ ita prodituras esse perplexas, ut nemo facile earum

earum integrationem suscipere ausus fuerit, cum tamen, si formulae principalis $\int Z\partial z$ integrale fuerit cognitum, exinde valores formularum derivatarum haud difficulter deduci queant.

§. 3. Ad hoc clarius explicandum considerabo hic formulam simplicissimam $\int\frac{\partial z}{\sqrt{(1-zz)}}$, quae exprimit arcum circularem, cujus sinus $= z$. Quod si jam hic statuatur $z = v(\cos\vartheta + \sqrt{-1}\sin\vartheta)$, ita ut quaeri debeat forma finita, quae exhibeat arcum, cujus sinus est $v(\cos\vartheta + \sqrt{-1}\sin\vartheta)$, facile patet, facta hac substitutione resolutionem formulae integralis haud exiguas ambages postulare. Tum enim denominator induet hanc formam: $\sqrt{1 - vv(\cos 2\vartheta + \sqrt{-1}\sin 2\vartheta)}$; unde ante omnia imaginaria elidere oportet, quod quidem fieret si numerator et denominator multiplicarentur per formulam $\sqrt{(1 - vv(\cos 2\vartheta - \sqrt{-1}\sin 2\vartheta))}$; tum enim denominator prodiret realis $= \sqrt{1 - 2vv\cos 2\vartheta + v^4}$. At vero numerator fieret aeque intricatus, siquidem signum radicale etiamnunc involveret tam realia quam imaginaria, quae tamen a se invicem separari necesse est.

§. 4. Hanc ob rem ipsum denominatorem ante omnia in binas partes separatas, alteram realem alteram simpliciter imaginariam resolvi conveniet, id quod sequenti modo commodissime praestabitur. Introducatur quantitas s, ut sit $s = \sqrt{1 - 2vv\cos 2\vartheta + v^4}$, et quaeratur angulus ω, ut sit $\cos 2\omega = \frac{1 - vv\cos 2\vartheta}{s}$ et $\sin 2\omega = \frac{vv\sin 2\vartheta}{s}$; tum enim denominator noster induet hanc formam: $\sqrt{s\cos 2\omega - s\sqrt{-1}\sin 2\omega}$ quae forma jam sponte transit in hanc: $(\cos\omega - \sqrt{-1}\sin\omega)\sqrt{s}$.

§. 5. Nunc igitur fractionem nostram $\frac{\partial z}{\sqrt{(1-zz)}}$ multiplicemus supra et infra per $\cos\omega + \sqrt{-1}\sin\omega$, eaque abibit in hanc

hanc formam: $\frac{\partial z(\cos\omega + \sqrt{-1}\sin\omega)}{\sqrt{s}}$. Quare cum fit $\partial z = \partial v(\cos\vartheta + \sqrt{-1}\sin\vartheta)$, formula noſtra integranda erit $\frac{\partial v(\cos(\vartheta+\omega) + \sqrt{-1}\sin(\vartheta+\omega))}{\sqrt{s}}$, quae ergo jam ultro in duas partes requiſitas reſolvitur, quae ſunt $\int\frac{\partial v\cos(\vartheta+\omega)}{\sqrt{s}} + \sqrt{-1}\int\frac{\partial v\sin(\vartheta+\omega)}{\sqrt{s}}$, ubi eſt $s = \sqrt{1 - 2vv\cos 2\vartheta + v^4}$; tum vero angulum ω ita ſumi oportet, ut ſit tang $2\omega = \frac{vv\sin 2\vartheta}{1 - vv\cos 2\vartheta}$. Hic autem evidens eſt, ſi loco s et ω hos valores re ipſa ſubſtituere vellemus, has formulas tantopere fieri complicatas, ut vix ulla via pateat eas reſolvendi.

§. 6. Quo hoc clarius eluceat, retineamus primo quantitatem s in calculo, et cum ſit $\cos 2\omega = \frac{1 - vv\cos 2\vartheta}{s}$, hinc deducimus

$$\cos\omega = \sqrt{\frac{1+\cos 2\omega}{2}} = \sqrt{\frac{s + 1 - vv\cos 2\vartheta}{2s}} \text{ et}$$
$$\sin\omega = \sqrt{\frac{1-\cos 2\omega}{2}} = \sqrt{\frac{s - 1 + vv\cos 2\vartheta}{2s}}$$

unde pro formula reali integrali erit

$$\cos(\vartheta+\omega) = \cos\vartheta\sqrt{\frac{s+1-vv\cos 2\vartheta}{2s}} - \sin\vartheta\sqrt{\frac{s-1+vv\cos 2\vartheta}{2s}}.$$

Hinc jam formula realis $\int\frac{\partial v\cos(\vartheta+\omega)}{\sqrt{s}}$ reſolvetur in duas ſequentes:

$$\cos\vartheta\int\frac{\partial v}{s\sqrt{2}}\sqrt{(s+1-vv\cos 2\vartheta)} - \sin\vartheta\int\frac{\partial v}{s\sqrt{2}}\sqrt{(s-1+vv\cos 2\vartheta)}.$$

Quod ſi jam hic inſuper loco s ſuum valorem ſurrogare vellimus, unde fieret $s = \sqrt{(1 - 2vv\cos 2\vartheta + v^4)}$, vix credo quemquam fore, qui voluerit in formula tantopere intricata reſolvenda vires ſuas ſaltem tentare. Facta enim ſubſtitutione loco s hae duae formulae ſequenti modo prodibunt expreſſae:

$$\frac{\cos\vartheta}{\sqrt{2}}\int\frac{\partial v\sqrt{(\sqrt{1-2vv\cos 2\vartheta+v^4}+1-vv\cos 2\vartheta)}}{\sqrt{(1-2vv\cos 2\vartheta+v^4)}},$$
$$-\frac{\sin\vartheta}{\sqrt{2}}\int\frac{\partial v\sqrt{(\sqrt{1-2vv\cos 2\vartheta+v^4}-1+vv\cos 2\vartheta)}}{\sqrt{(1-2vv\cos 2\vartheta+v^4)}}.$$

§. 7.

§. 7. Simili modo pro parte imaginaria, ob $\sin(\vartheta+\omega) = \sin\vartheta\cos\omega + \cos\vartheta\sin\omega$, iſta pars componetur ex binis sequentibus formulis integralibus:

$$\frac{\sin\vartheta\sqrt{-1}}{\sqrt{2}}\int\frac{\partial v\sqrt{(\sqrt{1-2vv\cos 2\vartheta+v^4}+1+vv\cos 2\vartheta)}}{\sqrt{(1-2vv\cos 2\vartheta+v^4)}}$$
$$+\frac{\cos\vartheta\sqrt{-1}}{\sqrt{2}}\int\frac{\partial v\sqrt{(\sqrt{1-2vv\cos 2\vartheta+v^4}-1+vv\cos 2\vartheta)}}{\sqrt{(1-2vv\cos 2\vartheta+v^4)}}.$$

Unde etiam perspicuum eſt, totum laborem ad integrationem duarum tantum formularum integralium esse perductum, quae autem ita ſunt complicatae, ut vix quisquam laborem ſit suscepturus.

§. 8. Eo magis igitur erit mirandum, ſi haec ipsa integralia actu assignari poterunt. Cum enim iis junctim ſumtis exprimatur arcus circuli, cujus ſinus eſt $v(\cos\vartheta + \sqrt{-1}\sin\vartheta)$, ſi hunc arcum deſignemus per $x + y\sqrt{-1}$, ita ut x exhibeat integrale binarum illarum formularum realium, at $y\sqrt{-1}$ integrale binarum formularum imaginariarum, erit viciſſim $v(\cos\vartheta + \sqrt{-1}\sin\vartheta) = \sin(x + y\sqrt{-1}) = \sin x\cos y\sqrt{-1} + \cos x\sin y\sqrt{-1}$. Cum jam constet esse $\cos\Psi = \frac{1}{2}(e^{\Psi\sqrt{-1}} + e^{-\Psi\sqrt{-1}})$, poſito $\Psi = y\sqrt{-1}$, erit $\cos y\sqrt{-1} = \frac{1}{2}(e^{-y} + e^{+y})$. Deinde quia eſt $\sin\Psi = \frac{1}{2\sqrt{-1}}(e^{\Psi\sqrt{-1}} - e^{-\Psi\sqrt{-1}})$, erit $\sin y\sqrt{-1} = \frac{1}{2\sqrt{-1}}(e^{-y} - e^{+y})$.

§. 9. Subſtituantur igitur iſti valores, ac prodibit iſta aequatio:

$$v(\cos\vartheta + \sqrt{-1}\sin\vartheta) = \tfrac{1}{2}\sin x(e^{-y} + e^{+y}) + \frac{\cos x}{2\sqrt{-1}}(e^{-y} - e^{+y}),$$

ubi partes reales et imaginarias ſeorſim aequari oportet, unde duae ſequentes determinationes emergunt:

$$v\cos\vartheta = \tfrac{1}{2}\sin x(e^{-y} + e^{+y}) \text{ et } v\sin\vartheta = \tfrac{1}{2}\cos x(e^{y} - e^{-y}).$$

Neque jam adeo erit difficile hinc binas quantitates x et y determinare.

§. 10.

§. 10. Cum ex priore aequatione habeamus $\sin x = \frac{2v\cos\vartheta}{e^y + e^{-y}}$, ex altera vero $\cos x = \frac{2v\sin\vartheta}{e^y - e^{-y}}$, horum valorum quadrata invicem addita producent hanc aequationem: $1 = \frac{4vv\cos\vartheta^2}{(e^y + e^{-y})^2} + \frac{4vv\sin\vartheta^2}{(e^y - e^{-y})^2}$, unde ergo quantitatem y elicere oportet. Ad hoc autem notasse plurimum juvabit esse $(e^y + e^{-y})^2 = e^{2y} + e^{-2y} + 2$ et $(e^y - e^{-y})^2 = e^{2y} + e^{-2y} - 2$, unde si brevitatis gratia statuamus $e^{2y} + e^{-2y} = 2t$, aequatio inventa induet hanc formam: $1 = \frac{4vv\cos\vartheta^2}{2t+2} + \frac{4vv\sin\vartheta^2}{2t-2}$, unde resultat ista aequatio quadratica $tt - 1 = 2tvv - 2vv\cos 2\vartheta$.

§. 11. Hujus jam aequationis resolutio praebet $t = vv + \sqrt{v^4 - 2vv\cos 2\vartheta + 1} = vv + s$. Jam cum posuerimus $e^{2y} + e^{-2y} = 2t$, hinc elicitur $e^{2y} = t + \sqrt{tt - 1}$, ideoque $e^{-2y} = t - \sqrt{tt - 1}$. Quoniam igitur quantitatem t per v definivimus, Logarithmis sumendis erit $y = \frac{1}{2}l(t + \sqrt{tt - 1})$, quae ergo formula aequatur binis posterioribus formulis integralibus, imaginario $\sqrt{-1}$ omisso.

§. 12. Deinde vero, cum sit $e^y + e^{-y} = \sqrt{2t + 2}$ et $e^y - e^{-y} = \sqrt{2t - 2}$, pro quantitate x invenienda geminam habebimus aequationem, scil: $\sin x = \frac{2v\cos\vartheta}{\sqrt{2t+2}}$ et $\cos x = \frac{2v\sin\vartheta}{\sqrt{2t-2}}$. Sicque ipsa quantitas x erit $= A\sin\frac{2v\cos\vartheta}{\sqrt{2t+2}}$, atque hic ipse arcus circularis aequabitur summae binarum priorum formularum integralium realium.

§. 13. Postquam igitur posuerimus brevitatis gratia $t = vv + \sqrt{(1 - 2vv\cos 2\vartheta + v^4)}$, valores integralium supra inventorum ita se habebunt:

$$A\sin\frac{2v\cos\vartheta}{\sqrt{(2t+2)}} = \begin{cases} \frac{\cos\vartheta}{\sqrt{2}}\int\frac{\partial v\sqrt{(\sqrt{1 - 2vv\cos 2\vartheta + v^4} + 1 - vv\cos 2\vartheta)}}{\sqrt{(1 - 2vv\cos 2\vartheta + v^4)}} \\ -\frac{\sin\vartheta}{\sqrt{2}}\int\frac{\partial v\sqrt{(\sqrt{1 - 2vv\cos 2\vartheta + v^4} - 1 + vv\cos 2\vartheta)}}{\sqrt{(1 - 2vv\cos 2\vartheta + v^4)}}. \end{cases}$$

Similique modo erit

$$\tfrac{1}{2}l(t + \sqrt{tt - 1}) = \begin{cases} \frac{\sin\vartheta\sqrt{-1}}{\sqrt{2}}\int\frac{\partial v\sqrt{(\sqrt{1 - 2vv\cos 2\vartheta + v^4} + 1 - vv\cos 2\vartheta)}}{\sqrt{(1 - 2vv\cos 2\vartheta + v^4)}} \\ +\frac{\cos\vartheta\sqrt{-1}}{\sqrt{2}}\int\frac{\partial v\sqrt{(\sqrt{(1 - 2vv\cos 2\vartheta + v^4} - 1 + vv\cos 2\vartheta)}}{\sqrt{(1 - 2vv\cos 2\vartheta + v^4)}}. \end{cases}$$

Ubi notetur, loco $A\sin\frac{2v\cos\vartheta}{\sqrt{(2t+2)}}$ scribi posse $A\,tag\,\frac{\cos\vartheta\sqrt{(t-1)}}{\sin\vartheta\sqrt{(t+1)}}$.

§. 14. Hos autem valores integrales per t expressos penitius perscrutari operae erit pretium. Cum enim sit $t + 1 = 1 + vv + \sqrt{1 - 2vv\cos 2\vartheta + v^4}$, erit radicem extrahendo

$$\sqrt{t + 1} = \sqrt{\frac{1 + 2v\cos\vartheta + vv}{2}} + \sqrt{\frac{1 - 2v\cos\vartheta + vv}{2}}.$$

Simili modo cum sit $t - 1 = vv - 1 + \sqrt{1 - 2vv\cos 2\vartheta + v^4}$, erit $\sqrt{t - 1} = \sqrt{\frac{1 + 2v\sin\vartheta\sqrt{-1} + vv}{2}} + \sqrt{\frac{1 - 2v\sin\vartheta\sqrt{-1} + vv}{2}}$. His igitur valoribus substitutis pro priore integratione fiet $A\,tang\,\frac{\cos\vartheta\sqrt{(t-1)}}{\sin\vartheta\sqrt{(t+1)}} = A\,tang\,\frac{\cos\vartheta(\sqrt{1 + 2v\sin\vartheta\sqrt{-1} + vv} + \sqrt{1 - 2v\sin\vartheta\sqrt{-1} + vv})}{\sin\vartheta(\sqrt{1 + 2v\cos\vartheta + vv} + \sqrt{1 - 2v\cos\vartheta + vv})}$. Pro altera autem forma logarithmica, quoniam est $\frac{1}{2}l(t + \sqrt{tt - 1}) = l(\sqrt{\frac{t+1}{2}} + \sqrt{\frac{t-1}{2}})$, erit valor integralis posterior

$$\log\begin{cases} \frac{1}{2}\sqrt{1 + 2v\cos\vartheta + vv} + \frac{1}{2}\sqrt{1 - 2v\cos\vartheta + vv} \\ +\frac{1}{2}\sqrt{1 + 2v\sin\vartheta\sqrt{-1} + vv} + \frac{1}{2}\sqrt{1 - 2v\sin\vartheta\sqrt{-1} + vv}. \end{cases}$$

§. 15. Haud incommode autem ipsos hos valores integrales etiam per formulas integrales exprimere licet. Cum enim sit

$\partial.\,A\,tang\,\frac{\cos\vartheta\sqrt{(t-1)}}{\sin\vartheta\sqrt{(t+1)}} = \frac{\partial t\sin\vartheta\cos\vartheta}{(t - \cos 2\vartheta)\sqrt{(tt-1)}}$, prior integratio nunc erit $= \int\frac{\partial t\sin\vartheta\cos\vartheta}{(t - \cos 2\vartheta)\sqrt{(tt-1)}}$, scilicet binae

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binae formulae integrales priores aequabuntur huic unicae; binae posteriores vero, quae aequales erant $\frac{1}{2}l(t+\sqrt{tt-1})$, aequabuntur huic formulae integrali: $\frac{1}{2}\int\frac{\partial t}{\sqrt{(tt-1)}}$.

§. 16. Quantumvis autem hae formulae integrales difficiles videbantur, tamen, quia earum integralia constant, atque adeo per logarithmos et arcus circulares exprimi possunt, non amplius tantopere difficile erit in methodum inquirere haec ipsa integralia eruendi, id quod sequenti modo commodissime expediri posse videtur.

Integratio formularum:

$$\mathfrak{P}=\int\frac{\partial v\sqrt{(\sqrt{1-2vv\cos 2\vartheta+v^4}+1-vv\cos 2\vartheta)}}{\sqrt{1-2vv\cos 2\vartheta+v^4}},$$

$$\mathfrak{Q}=\int\frac{\partial v\sqrt{(\sqrt{1-2vv\cos 2\vartheta+v^4}-1+vv\cos 2\vartheta)}}{\sqrt{1-2vv\cos 2\vartheta+v^4}}.$$

§. 17. Hic ante omnia opus est formulam radicalem $\sqrt{1-2vv\cos 2\vartheta+v^4}$ ex calculo expellere, quod aptissime fiet loco v introducendo ipsam quantitatem t, quae erat $t=vv+\sqrt{1-2vv\cos 2\vartheta+v^4}$, unde elicitur $vv=\frac{1-tt}{2(\cos 2\vartheta-t)}$ atque hinc habebimus $\sqrt{1-2vv\cos 2\vartheta+v^4}=\frac{tt-2t\cos 2\vartheta+1}{2t-2\cos 2\vartheta}$. Deinde vero erit $1-vv\cos 2\vartheta=\frac{-tt\cos 2\vartheta+2t-\cos 2\vartheta}{2t-2\cos 2\vartheta}$.

§. 18. His jam valoribus substitutis nostrae formulae integrales sequentes induent formas:

$$\mathfrak{P}=\int\frac{\partial v(t+1)\sqrt{2(t-\cos 2\vartheta)(1-\cos 2\vartheta)}}{tt-2t\cos 2\vartheta+1}, \text{ et}$$

$$\mathfrak{Q}=\int\frac{\partial v(t-1)\sqrt{2(t-\cos 2\vartheta)(1+\cos 2\vartheta)}}{tt-2t\cos 2\vartheta+1},$$

quae ob $1-\cos 2\vartheta=2\sin\vartheta^2$ et $1+\cos 2\vartheta=2\cos\vartheta^2$ transibunt in has:

$$\mathfrak{P}=\int\frac{2\partial v(t+1)\sin\vartheta\sqrt{t-\cos 2\vartheta}}{tt-2t\cos 2\vartheta+1},$$

$$\mathfrak{Q}=\int\frac{2\partial v(t-1)\cos\vartheta\sqrt{t-\cos 2\vartheta}}{tt-2t\cos 2\vartheta+1}.$$

§. 19. Tantum igitur superest, ut loco ∂v valor debitus substituatur. Cum igitur sit $2vv=\frac{tt-1}{t-\cos 2\vartheta}$, erit differentiando

$$4v\partial v=\frac{\partial t(tt-2t\cos 2\vartheta+1)}{(t-\cos 2\vartheta)^2}, \text{ ideoque}$$

$$\partial v=\frac{(tt-2t\cos 2\vartheta+1)\partial t\sqrt{2}}{4\sqrt{(tt-1)}(t-\cos 2\vartheta)^{\frac{3}{2}}},$$

hocque valore introducto fiet

$$\mathfrak{P}=\frac{\sin\vartheta}{\sqrt{2}}\int\frac{\partial t\sqrt{t+1}}{\sqrt{t-1}\,(t-\cos 2\vartheta)}, \text{ et}$$

$$\mathfrak{Q}=\frac{\cos\vartheta}{\sqrt{2}}\int\frac{\partial t\sqrt{t-1}}{\sqrt{t+1}\,(t-\cos 2\vartheta)},$$

quarum formularum integratio nulla amplius laborat difficultate, quandoquidem facile ab omni irrationalitate liberari possunt.

§. 20. Tantum enim opus est poni $\sqrt{\frac{t+1}{t-1}}=u$, tum enim erit $t=\frac{uu+1}{uu-1}$, ideoque $t-\cos 2\vartheta=\frac{uu(1-\cos 2\vartheta)+1+\cos 2\vartheta}{uu-1}=\frac{2uu\sin\vartheta^2+2\cos\vartheta^2}{uu-1}$, tum vero erit $\partial t=-\frac{4u\partial u}{(uu-1)^2}$, quocirca erit $\frac{\partial t}{t-\cos 2\vartheta}=-\frac{2u\partial u}{(uu-1)(uu\sin\vartheta^2+\cos\vartheta^2)}$, ex quo ambae nostrae formulae ita prodibunt rationaliter expressae:

$$\mathfrak{P}=-\sin\vartheta\sqrt{2}\int\frac{uu\partial u}{(uu-1)(uu\sin\vartheta^2+\cos\vartheta^2)}$$

$$\mathfrak{Q}=-\cos\vartheta\sqrt{2}\int\frac{\partial u}{(uu-1)(uu\sin\vartheta^2+\cos\vartheta^2)},$$

quarum ergo integratio per regulas notissimas facile expeditur.

§. 21.

§. 21. Quoniam denominator duobus constat factoribus, pro priore formula statuamus

$$\frac{uu}{(uu-1)(uu\sin\vartheta^2+\cos\vartheta^2)} = \frac{F}{uu-1} + \frac{G}{uu\sin\vartheta^2+\cos\vartheta^2},$$

ac reperietur $F = \frac{uu}{uu\sin\vartheta^2+\cos\vartheta^2}$, posito $uu-1=0$, sicque erit $F=1$; tum vero reperitur $G=\frac{uu}{uu-1}$, posito $uu\sin\vartheta^2+\cos\vartheta^2=0$, sive $uu\sin\vartheta^2=-\cos\vartheta^2$, unde fit $G=\cos\vartheta^2$. Hinc ♄ in has duas formulas resolvitur:

$$♄ = -\sin\vartheta\sqrt{2}\int\frac{\partial u}{uu-1} - \sin\vartheta\sqrt{2}\int\frac{\partial u\cos\vartheta^2}{uu\sin\vartheta^2+\cos\vartheta^2}.$$

Est vero $\int\frac{\partial u}{uu-1} = \frac{1}{2}l\frac{u-1}{u+1}$ et $\int\frac{\partial u}{uu\sin\vartheta^2+\cos\vartheta^2} = \frac{1}{\sin\vartheta}$ A tang $\frac{u\sin\vartheta}{\cos\vartheta}$, quam obrem habebimus

$$♄ = -\frac{\sin\vartheta}{\sqrt{2}}\, l\frac{u-1}{u+1} - \sqrt{2}\cos\vartheta \text{ A tang } \frac{u\sin\vartheta}{\cos\vartheta}.$$

Quod si jam hic loco u scribamus valorem $\sqrt{\frac{t+1}{t-1}}$, erit

$$♄ = -\frac{\sin\vartheta}{\sqrt{2}}\, l\frac{\sqrt{t+1}-\sqrt{t-1}}{\sqrt{t+1}+\sqrt{t-1}} - \cos\vartheta^2\sqrt{2} \text{ A tang } \frac{\sin\vartheta\sqrt{t+1}}{\cos\vartheta\sqrt{t-1}},$$

cujus consensus cum integralibus supra exhibitis facile perspicitur.

§. 22. Simili modo pro ♃ statuamus

$$\frac{1}{(uu-1)(uu\sin\vartheta^2+\cos\vartheta^2)} = \frac{F}{uu-1} + \frac{G}{uu\sin\vartheta^2+\cos\vartheta^2},$$

eritque $F=\frac{1}{uu\sin\vartheta^2+\cos\vartheta^2}$, posito $uu-1=0$, sive $u=1$, unde ergo prodit $F=1$. Deinde erit $G=\frac{1}{uu-1}$, posito $uu=-\frac{\cos\vartheta^2}{\sin\vartheta^2}$, ideoque $G=-\sin\vartheta^2$; sicque formula pro ♃ inventa in has partes resolvitur:

$♃ = -\cos\vartheta\sqrt{2}\int\frac{\partial u}{uu-1} + \cos\vartheta\sqrt{2}\int\frac{\partial u\sin\vartheta^2}{uu\sin\vartheta^2+\cos\vartheta^2}$. Vidimus autem esse $\int\frac{\partial u}{uu-1} = \frac{1}{2}l\frac{u-1}{u+1}$, hic vero erit $\int\frac{\partial u\sin\vartheta^2}{uu\sin\vartheta^2+\cos\vartheta^2} = \sin\vartheta$ A tang $\frac{u\sin\vartheta}{\cos\vartheta}$, sicque habebimus

$$♃ = -\frac{\cos\vartheta}{\sqrt{2}}\, l\frac{u-1}{u+1} + \sin\vartheta\cos\vartheta\sqrt{2} \text{ A tang } \frac{u\sin\vartheta}{\cos\vartheta}.$$

Ac si hic iterum loco u scribamus valorem $\frac{\sqrt{t+1}}{\sqrt{t-1}}$, erit

$$♃ = -\frac{\cos\vartheta}{\sqrt{2}}\, l\frac{\sqrt{t+1}-\sqrt{t-1}}{\sqrt{t+1}+\sqrt{t-1}} + \sin\vartheta\cos\vartheta\sqrt{2} \text{ A tang } \frac{\sin\vartheta\sqrt{t+1}}{\cos\vartheta\sqrt{t-1}}.$$

§. 23.

§. 23. Haec autem resolutio ideo successit quod formula principalis proposita $\int\frac{\partial z}{\sqrt{(1-z^3)}}$ fuit in suo genere quasi simplicissima; unde facile intelligitur, si ejus loco aliae formulae difficiliores proponantur, tum resolutionem sine dubio multo magis futuram esse arduam, neque adeo expediri posse, nisi ipsa formula proposita per logarithmos et arcus circulares integrari queat. Sin autem hoc contigerit, quemadmodum evenit in hac formula: $\int\frac{\partial z}{(1\pm z^n)^{\frac{1}{n}}}$, sive adeo in hac: $\int\frac{z^{m-1}\partial z}{(1\pm z^n)^{\frac{m}{n}}}$, tum etiam posito $z=v(\cos\vartheta+\sqrt{-1}\sin\vartheta)$, certum erit, formulas integrales inde deductas, quantumvis fuerint perplexae, tamen semper etiam per logarithmos et arcus circulares resolvi posse, id quod unico exemplo ostendere conabimur.

Problema.

Si in formula integrali $\int\frac{\partial z}{\sqrt[3]{(1+z^3)}}$ ponatur $z=v(\cos\vartheta+\sqrt{-1}\sin\vartheta)$, unde haec formula resolvatur in has: $\int P\partial v+\sqrt{-1}\int Q\partial v$, ambas istas formulas integrales, quippe quae semper reales esse possunt, investigare.

Solutio.

§. 24. Hic ergo pro denominatore statim erit $1+z^3=1+v^3(\cos 3\vartheta+\sqrt{-1}\sin 3\vartheta)$, unde statuamus $\sqrt{1+2v^3\cos 3\vartheta+v^6}=s$, et quaeramus angulum 3ω, ut sit $\cos 3\omega=\frac{1+v^3\cos 3\vartheta}{s}$ et $\sin 3\omega=\frac{v^3\sin 3\vartheta}{s}$, quo facto erit $1+z^3=s(\cos 3\omega+\sqrt{-1}\sin 3\omega)$, ideoque $\sqrt[3]{(1+z^3)}=\sqrt[3]{s}.\cos\omega+\sqrt{-1}\sin\omega$, unde formula proposita erit

$$\int\frac{\partial z}{(\cos\omega+\sqrt{-1}\sin\omega)\sqrt[3]{s}} = \int\frac{\partial z(\cos\omega-\sqrt{-1}\sin\omega)}{\sqrt[3]{s}}.$$

§. 25.

§. 12. Superfluum foret eandem deductionem pro lit-

§. 15. Praeterea vero maxime memorabilis est haec

§. 25. Cum nunc fit $\partial z = \partial v(\cos\vartheta + \sqrt{-1}\sin\vartheta)$, formula refolvenda erit $\int\frac{\partial v(\cos(\vartheta-\omega)+\sqrt{-1}\sin(\vartheta-\omega))}{\sqrt[3]{s}}$; quamobrem pro refolutione quaefita erit

$$\int P\partial v = \int\frac{\partial v\cos(\vartheta-\omega)}{\sqrt[3]{s}} \quad \text{et} \quad \int Q\partial v = \int\frac{\partial v\sin(\vartheta-\omega)}{\sqrt[3]{s}},$$

quarum ergo formularum integralia inveftigari oportet. Evidens autem eft hinc angulum ω neutiquam commode per v exprimi pofse. Etsi enim tang $3\omega = \frac{v^3\sin 3\vartheta}{1+v^3\cos 3\vartheta}$, hic trifectione anguli opus foret, unde formulae noftrae plane inextricabiles prodirent.

§. 26. Maxime igitur memorabile eft, has ambas formulas integrales in quibus eft $s = \sqrt{(1+2v^3\cos\vartheta+v^6)}$ et tang $3\omega = \frac{v^3\sin 3\vartheta}{1+v^3\cos 3\vartheta}$, quas vix ac ne vix quidem per folam v referre liceret, nihilominus per logarithmos et arcus circulares integrari posse. Facile autem intelligitur per idoneam fubftitutionem loco v aliam variabilem idoneam in calculum introduci debere, cujus ope hae formulae fimpliciores reddi queant, id quod commodifsime fieri pofse videtur, si loco v angulus Φ introducatur, ita ut fit $\Phi = \vartheta - \omega$, unde ftatim oritur $\int P\partial v = \int\frac{\partial v\cos\Phi}{\sqrt[3]{s}}$ et $\int Q\partial v = \int\frac{\partial v\sin\Phi}{\sqrt[3]{s}}$, ubi ergo litteras v et s per Φ exprimi oportet.

§. 27. Cum fit tang $3\omega = \frac{v^3\sin 3\vartheta}{1+v^3\cos 3\vartheta}$, fi hunc angulum 3ω introducamus, erit $\text{tg}(3\vartheta-3\omega) = \frac{\text{tang}\,3\vartheta - \text{tang}\,3\omega}{1+\text{tang}\,3\vartheta\,\text{tang}\,3\omega}$, unde ob $3\vartheta - 3\omega = 3\Phi$ elicitur tang $3\Phi = \frac{\sin 3\vartheta}{v^3+\cos 3\vartheta}$, unde reperimus $v^3 + \cos 3\vartheta = \frac{\sin 3\vartheta}{\text{tang}\,3\Phi}$. Hinc fumtis quadratis erit

$$v^6 + 2v^3\cos 3\vartheta + \cos 3\vartheta^2 = \sin 3\vartheta^2\frac{\cos 3\Phi^2}{\sin 3\Phi^2}.$$

Addatur utrinque $\sin 3\vartheta^2$ eritque

$$v^6 + 2v^3\cos 3\vartheta + 1 = ss = \frac{\sin 3\vartheta^2}{\sin 3\Phi^2}, \text{ ideoque } s = \frac{\sin 3\vartheta}{\sin 3\Phi}.$$

Hacte-

Hactenus igitur noftrae formulae ad fequentes formas funt reductae:

$$\int P\partial v = \frac{1}{\sqrt[3]{\sin 3\vartheta}}\int\partial v\cos\Phi\sqrt[3]{\sin 3\Phi}$$
$$\int Q\partial v = \frac{1}{\sqrt[3]{\sin 3\vartheta}}\int\partial v\sin\Phi\sqrt[3]{\sin 3\Phi}.$$

§. 28. Cum denique fit $v^3 = \frac{\sin 3\vartheta}{\text{tang}\,3\Phi} - \cos 3\vartheta$, erit differentiando $3vv\partial v = -\frac{3\partial\Phi\sin 3\vartheta}{\sin 3\Phi^2}$, ideoque $\partial v = -\frac{\partial\Phi\sin 3\vartheta}{vv\sin 3\Phi^2}$. Cum igitur fit $v^3 = \frac{\sin 3(\vartheta-\Phi)}{\sin 3\Phi}$, erit $vv = \frac{(\sin 3(\vartheta-\Phi))^{\frac{2}{3}}}{\sin 3\Phi^{\frac{2}{3}}}$, unde fit $\partial v = -\frac{\partial\Phi\sin 3\vartheta}{(\sin 3(\vartheta-\Phi))^{\frac{2}{3}}\sin 3\Phi^{\frac{4}{3}}}$, ficque formulae noftrae, ad folam variabilem Φ reductae, erunt

$$\int P\partial v = -\sin 3\vartheta^{\frac{2}{3}}\int\frac{\partial\Phi\cos\Phi}{\sin 3\Phi(\sin 3(\vartheta-\Phi))^{\frac{2}{3}}}$$
$$\int Q\partial v = -\sin 3\vartheta^{\frac{2}{3}}\int\frac{\partial\Phi\sin\Phi}{\sin 3\Phi(\sin 3(\vartheta-\Phi))^{\frac{2}{3}}}$$

quarum formularum integratio haud exiguam dexteritatem in calculo angulorum poftulat.

§. 29. Ut calculum ad folitas quantitates revocemus, ftatuamus tang $\Phi = t$, ut fit

$$\sin\Phi = \frac{t}{\sqrt{1+tt}} \text{ et } \cos\Phi = \frac{1}{\sqrt{1+tt}}, \text{ unde fit}$$
$$\partial\Phi\cos\Phi = \frac{\partial t}{(1+tt)^{\frac{3}{2}}} \text{ et } \partial\Phi\sin\Phi = \frac{t\partial t}{(1+tt)^{\frac{3}{2}}}.$$

Praeterea vero habebitur tang $3\Phi = \frac{3t-t^3}{1-3tt}$, unde fit

$$\sin 3\Phi = \frac{3t-t^3}{(1+tt)^{\frac{3}{2}}} \text{ et } \cos 3\Phi = \frac{1-3tt}{(1+tt)^{\frac{3}{2}}}.$$ Hinc porro confi-

hic per illum divisus praebet quotum

$(n-2\alpha-\lambda)(n-2\alpha-\lambda-1)$ Hinc ergo erit terminus fequens

pro casu $n = 6$, ac fingulae ejus partes fequenti modo reconfiscitur $\sin 3(\vartheta - \Phi) = \frac{\alpha(1-3tt) - 3t + t^3}{\sqrt{1 + \alpha\alpha(1+tt)^3}}$, ideoque:

$$(\sin 3(\vartheta - \Phi))^{\frac{2}{3}} = \frac{(\alpha(1-3tt) - 3t + t^3)^{\frac{2}{3}}}{(1+\alpha\alpha)^{\frac{1}{3}}(1+tt)}.$$

Hisque valoribus substitutis nanciscemur.

$$\int P\partial v = -\alpha^{\frac{2}{3}} \int \frac{\partial t(1+tt)}{(3t-t^3)(\alpha(1-3tt)-3t+t^3)^{\frac{2}{3}}}$$

$$\int Q\partial v = +\alpha^{\frac{8}{3}} \int \frac{t\partial t(1+tt)}{(3t-t^3)(\alpha(1-3tt)-3t+t^3)^{\frac{2}{3}}}.$$

Certo igitur affirmare licet, has formulas ab irrationalitate penitus liberari posse, etiamsi mihi quidem nulla via patere videatur hoc praestandi; unde Geometris amplissimus campus aper[itur] ad suam sagacitatem exercendi.

§. 30. Si loco formulae $\int \frac{\partial z}{\sqrt[3]{1+z^3}}$ assumsissemus generalem $\int \frac{\partial z}{\sqrt[n]{(1+z^n)}}$, eamque simili modo tractavissemus, pervenissemus ad sequens theorema:

Integralia harum duarum formularum:

$$\int \frac{\partial\Phi \sin\Phi}{\sin n \ldots \sin(n\vartheta - n\Phi)^{\frac{n-1}{n}}} \text{ et } \int \frac{\partial\Phi \cos\Phi}{\sin n\Phi(\sin(n\vartheta - n\Phi))^{\frac{n-1}{n}}}$$

certe per logarithmos et arcus circulares exprimi possunt, ideoque dabitur certa substitutio, cujus ope hae formulae ad rationalitatem perduci possunt; unde haec observatio eo majorem attentionem meretur.

DIS-

DISQUISITIONES ANALYTICAE
SUPER EVOLUTIONE POTESTATIS TRINOMIALIS
$(1+x+xx)^n$.

AUCTORE
L. EULERO.

Conventui exhibita die 17. Aug. 1775.

§. 1.

Cum olim in *Novorum Commentariorum Tomo XI*, sub titulo *observationum analyticarum*, istam potestatem trinomialem multo studio essem perscrutatus, in tam egregia symptomata incidi, quae majore attentione Geometrarum non indigna videbantur. Hanc ob rem nuper hoc idem argumentum denuo tractare suscepi, atque nonnullis artificiis analyticis usus, multo plura insignia phaenomena se mihi obtulerunt, quorum expositionem Geometris non ingratam fore confido.

§. 2. Incipio igitur ab ipsa evolutione hujus formulae: $(1+x+xx)^n$, quae pro singulis valoribus exponentis n sequentes praebet expressiones in tabula subjuncta repraesentatas:

n	$(1+x+xx)^n$
0	1
1	$1+x+xx$
2	$1+2x+3xx+2x^3+x^4$
3	$1+3x+6xx+7x^3+6x^4+3x^5+x^6$
4	$1+4x+10xx+16x^3+19x^4+16x^5+10x^6+4x^7+x^8$
5	$1+5x+15xx+30x^3+45x^4+51x^5+45x^6+30x^7+15x^8+5x^9+x^{10}$.
etc.	etc.

Hic

Investigatio relationis