

DISQVISITIO VLTERIOR
S V P E R S E R I E B V S
 SECUNDVM MVLTIPLA CVIVSDAM ANGULI PRO-
 GREDIENTIBVS.

Autore
L. EVLE RO

Conventui exhib. die 26 Maii 1777.

§. I.

Contemplabor hic denuo eiusmodi functiones cuiuspiam anguli Φ , quas in series, quarum termini cosinus angularum multiplorum ipsius Φ continent, evolvere liceat. Scilicet si Φ denotet talem functionem anguli Φ , quae per evolutionem huiusmodi seriei oriatur:

$\Phi = A + B \cos \Phi + C \cos 2\Phi + D \cos 3\Phi + E \cos 4\Phi + \dots$
 manifestum est talem resolutionem semper succedere, quando eadem fundio Φ per solutionem communem in talem series converti potest:

$\Phi = \alpha + \beta \cos \Phi + \gamma \cos^2 \Phi + \delta \cos^3 \Phi + \varepsilon \cos^4 \Phi + \dots$
 propterea quod omnes potestates cosinuum in cosinus multiplorum eiusdem anguli resolvi possunt, id quod in potestatis finium non succedit, quoniam tantum potestates pares in cosinus multiplorum resolvuntur, potestates vero impares ad finus multiplorum perduntur. Quia vero omne

finit

sinus facilissime ad cosinus revocantur, ea quae hic sum tradituras, pariter quoque ad sinus pertinere sunt censenda.

§. 2. Nisi autem functio proposita Φ fuerit rationalis et satis simplex, seriei quae ex eius evolutione nascitur:

$A + B \cos \Phi + C \cos^2 \Phi + D \cos^3 \Phi + E \cos^4 \Phi + \text{etc.}$
singuli termini ita deprehenduntur comparati, ut eorum valores non nisi per quantitates maxime transcendentes exhiberi queant. Veluti si functio proposita fuerit

$$\Phi = (1 - n \cos \Phi)^{-\frac{3}{2}},$$

a cuius evolutione propemodum universa theoria Astronomiae pendet, seriei inde oriundae primus terminus A per hanc seriem exprimi invenitur:

$$1 + \frac{3 \cdot 5}{4 \cdot 4} n n + \frac{3 \cdot 5}{4 \cdot 4} \cdot \frac{7 \cdot 9}{8 \cdot 8} n^4 + \frac{3 \cdot 5}{4 \cdot 4} \cdot \frac{7 \cdot 9}{8 \cdot 8} \cdot \frac{11 \cdot 13}{12 \cdot 12} \cdot n^6 + \text{etc.}$$

cuius summatio omnia artificia analytica hactenus inventa eludit; hinc olim plurimum in hoc elaboravi, ut eius summationem ad resolutionem aequationis differentialis reducere, unde deinceps haec investigatio ad genera quantitatum transcendentium, sive ad quadraturas curvarum magis cognitas, deduci posset; verum etiam in hoc labore operam meam nequicquam consumsi. Nuper autem se mihi obtulit idea, quae me ad formulas integrales satis concinnas manuduxit, quibus non solum primus huius seriei terminus A, sed adeo omnes termini, satis commode exprimi possunt, quas in sequenti theoremate sum complexurus.

Theorema generale.

§. 3. Si functio Φ anguli Φ ita fuerit comparata, ut in talem seriem resolvi se patiatur:

P 2

$\Phi =$

$$\Phi = A + B \cos \Phi + C \cos 2\Phi + D \cos 3\Phi \\ + E \cos 4\Phi + \text{etc.}$$

tum singulae quantitates **A**, **B**, **C**, **D**, **E**, etc. per sequentes formulas integrales determinantur, siquidem in singulis integratio a termino $\Phi = 0$, usque ad terminum $\Phi = \pi$ extendatur, denotante π semiperipheriam circuli cuius radius = 1.

1. $A = \frac{1}{\pi} \int \Phi \partial \Phi$.
 2. $B = \frac{2}{\pi} \int \Phi \partial \Phi \cos \Phi$.
 3. $C = \frac{2}{\pi} \int \Phi \partial \Phi \cos 2\Phi$.
 4. $D = \frac{2}{\pi} \int \Phi \partial \Phi \cos 3\Phi$.
 5. $E = \frac{2}{\pi} \int \Phi \partial \Phi \cos 4\Phi$.
- etc. etc.

ubi notetur primum coefficientem esse $\frac{1}{\pi}$ dum sequentes omnes sunt $\frac{2}{\pi}$.

§. 4. Hic primum observasse iuvabit omnes has formulas integrales facillime per quadraturas curvarum satis simplicium repraesentari posse. Si enim super axe rectilineo Tab. I. A B abscissae A P aequales capiantur arcibus, qui angulos Φ metiuntur, ita ut sit $A P = \Phi$, tum vero super hoc axe construatur curva E M F, cuius applicatae P M referant functionem propositam Φ , tum formula $\int \Phi \partial \Phi$ exprimet aream A E M P, cuius initium in A statuitur, ubi $\Phi = 0$. Quodsi iam punctum P usque ad B moveatur, ut fiat $A B = \pi$, tum area A E F B per $\frac{1}{\pi}$ multiplicata statim praebet primum terminum A seriei quam quaerimus. Simili modo secundus terminus B, simulque omnes sequentes, con-

construi poterunt, si curva E M F ita describatur, ut pro secundo termino B capiatur applicata P M $= \Phi \cos \Phi$; pro tertio vero P M $= \Phi \cos^2 \Phi$; pro quarto P M $= \Phi \cos^3 \Phi$, et ita porro; tum enim tota area A E M F in $\frac{2}{\pi}$ duda has ipsas quantitates B, C, D, etc. exhibebit.

§. 5. Quoniam hoc modo abscissae A P arcibus circularibus aequales sunt capienda, istae curvae descriptae pro algebraicis haberi nequeunt; interim tamen harum curvarum loco algebraicae substitui poterunt, ita ut omnes nostrae quantitates adeo per quadraturas curvarum algebraicarum exhiberi queant; tantum enim ponatur $\cos \Phi = x$, et cum fundio Φ spectari possit tanquam fundio ipsius Φ , erit nunc Φ fundio algebraica ipsius x . Cum autem hinc fiat $\partial \Phi = \frac{-\partial x}{\sqrt{1-x^2}}$, pro prima quantitate habebimus:

$$A = -\frac{1}{\pi} \int \frac{\Phi \partial x}{\sqrt{1-x^2}},$$

unde constructio ita erit instituenda, ut singulis abscissis A P $= x$ respondeant applicatae P M $= \frac{\Phi}{\sqrt{1-x^2}}$, ita ut iam futura sit area A E M P $= \int \frac{\Phi \partial x}{\sqrt{1-x^2}}$, quam autem nunc a termino $x = 1$ usque ad terminum $x = -1$ extendi oportet. Hic ergo abscissas a punto fixo medio C capi conveniet, statuique C P $= x$ et P M $= \frac{\Phi}{\sqrt{1-x^2}}$; tum enim, si fuerit C A $= 1$ et C B $= -1$, area A E F B, toti bafi A B imminens, proposito satisfaciet, ita ut omes istae determinationes per quadraturas linearum curvarum expediri queant.

§. 6. His praenotatis adgrediamur demonstrationem nostri theorematis, ac primo manifestum est, si i denotet numerum

rum integrum quemcunque, integrale

$$\int \partial \Phi \cos. i \Phi = \frac{1}{i} \sin. i \Phi,$$

annihilari tam posito $\Phi = 0$, quam posito $\Phi = \pi$, quod ergo pro omnibus numeris integris i valebit, solo casu $i = 0$ excepto, quippe quo prodit $\int \partial \Phi \cos. i \Phi = \pi$. Hoc observato, quoniam per hypothesin est

$$\Phi = A + B \cos. \Phi + C \cos. 2\Phi + D \cos. 3\Phi + \text{etc.}$$

erit $\int \Phi \partial \Phi = A \pi$, integralibus scilicet a $\Phi = 0$ usque ad $\Phi = \pi$ extensis. Hinc igitur iam evisa est pars prima nostrae theorematis, qua est $A = \frac{1}{\pi} \int \Phi \partial \Phi$.

§. 7. Pro reliquis partibus consideremus formulam differentialem $\partial \Phi \cos. i \Phi \cos. \lambda \Phi$, quae in simplices cosinus resoluta dat

$$\frac{1}{2} \partial \Phi [\cos. (i - \lambda) \Phi + \cos. (i + \lambda) \Phi],$$

unde eius integrale erit

$$\int \partial \Phi \cos. i \Phi \cos. \lambda \Phi = \frac{\sin. (i - \lambda) \Phi}{2(i - \lambda)} + \frac{\sin. (i + \lambda) \Phi}{2(i + \lambda)},$$

quod integrale utique evanescit, tam sumto $\Phi = 0$ quam sumto $\Phi = \pi$, ob i et λ numeros integros; si modo unicum casum excipiamus, quo $\lambda = i$, quippe quo casu reperitur

$$\int \partial \Phi \cos. i \Phi^2 = \frac{1}{2} \Phi + \frac{1}{4i} \sin. 2i \Phi,$$

qui valor sumto $\Phi = \pi$ abit in $\frac{1}{2}\pi$.

§. 7. Cum igitur, integrationem a $\Phi = 0$ usque ad $\Phi = \pi$ extendendo, semper fit $\int \partial \Phi \cos. i \Phi \cos. \lambda \Phi = 0$, solo casu excepto $\lambda = i$, quippe quo casu integrale erit $= \frac{\pi}{2}$, ex aequatione

$$\Phi =$$

$\Phi = A + B \cos \phi + C \cos 2\phi + D \cos 3\phi + E \cos 4\phi + \text{etc.}$
pro parte secunda theorematis nostri reperiemus

$$\int \Phi \partial \phi \cos \phi = \frac{1}{2} \pi B,$$

propterea quod ex omnibus reliquis formulis nihil oritur;
hinc vicissim concluditur fore

$$B = \frac{2}{\pi} \int \Phi \partial \phi \cos \phi,$$

si quidem integrale a $\phi = 0$ ad $\phi = \pi$ extendatur.

§. 8. Simili modo pro parte tertia reperiemus

$$\int \Phi \partial \phi \cos 2\phi = \frac{1}{2} \pi C,$$

ideoque vicissim habebimus:

$$C = \frac{2}{\pi} \int \Phi \partial \phi \cos 2\phi.$$

Pari modo pro partibus sequentibus prodibit

$$D = \frac{2}{\pi} \int \Phi \partial \phi \cos 3\phi.$$

$$E = \frac{2}{\pi} \int \Phi \partial \phi \cos 4\phi,$$

sicque porro in infinitum: hocque ergo modo veritas nostri theorematis perfide est demonstrata.

§. 9. Postquam veritatem nostri theorematis extra omne dubium collocavimus, haud difficile erit, pro quovis casu, quo functio proposita Φ per seriem datur, cuius singuli termini secundum potestates ipsius $\cos \phi$ progrediuntur, alteram seriem, quam intendimus

$$A + B \cos \phi + C \cos 2\phi + D \cos 3\phi + \text{etc.}$$

formare atque dilucide ostendere, quemadmodum singuli eius termini $A, B, C, D, \text{ etc.}$ exprimantur. Quoniam vero singulae

gulae litterae latinae maiores ab omnibus litteris graecis, sequentibus in infinitum pendent, ne ex ordine istarum litterarum confusio oriatur, loco litterarum graecarum sequentes characteres introducamus:

$$\Phi = (\circ) + (1) \text{ cof. } \Phi + (2) \text{ cof. } \Phi^2 + (3) \text{ cof. } \Phi^3 \\ + (4) \text{ cof. } \Phi^4 + \text{etc.}$$

et iam quaestio huc redit, quomodo singulae litterae latinae A, B, C, D, etc, ex ipsis characteribus (1); (2); (3); etc. definiri debeant.

§. 10. Incipiamus a primi littera A, cuius evolutio postulat sequens Lemma:

Lemma.

Si integralia a $\Phi = \circ$ usque ad $\Phi = \pi$, extendantur, semper erit

$$\int \partial \Phi \text{ cof. } \Phi^\lambda = \frac{\lambda - 1}{\lambda} \int \partial \Phi \text{ cof. } \Phi^{\lambda - 2}.$$

Ad hoc demonstrandum ponatur

$$\int \partial \Phi \text{ cof. } \Phi^\lambda = f \text{ fin. } \Phi \text{ cof. } \Phi^{\lambda - 1} + g \int \partial \Phi \text{ cof. } \Phi^{\lambda - 2},$$

et summis differentialibus erit

$$\text{cof. } \Phi^\lambda = f \text{ cof. } \Phi^\lambda - f(\lambda - 1) \text{ fin. } \Phi^2 \text{ cof. } \Phi^{\lambda - 2} + g \text{ cof. } \Phi^{\lambda - 2},$$

quae aequatio, ob fin. $\Phi^2 = 1 - \text{cof. } \Phi^2$, induet hanc formam:

$$\text{cof. } \Phi^\lambda = \lambda f \text{ cof. } \Phi^\lambda - f(\lambda - 1) \text{ cof. } \Phi^{\lambda - 2} + g \text{ cof. } \Phi^{\lambda - 2},$$

unde primo fit $g = f(\lambda - 1)$ et $f = \frac{1}{\lambda}$, ideoque $g = \frac{\lambda - 1}{\lambda}$.

Sicque in genere habemus hanc reductionem:

$$\int \partial \Phi \text{ cof. } \Phi^\lambda = \frac{1}{\lambda} \text{ fin. } \Phi \text{ cof. } \Phi^{\lambda - 1} + \frac{\lambda - 1}{\lambda} \int \partial \Phi \text{ cof. } \Phi^{\lambda - 2},$$

quod integrale ita capi debet, ut posito $\Phi = \circ$ evanescat.

Quam-

Quamobrem si statuamus $\phi = \pi$, unde fit $\sin \phi = 0$, casu Lemmatis habebimus:

$$\int \partial \phi \cos \phi^\lambda = \frac{\lambda - 1}{\lambda} \int \partial \phi \cos \phi^{\lambda - 2}.$$

§. 11. Quoniam igitur a casibus simplicissimis incipiendo habemus:

$$\text{I. } \int \partial \phi \cos \phi^0 = \pi.$$

$$\text{II. } \int \partial \phi \cos \phi^1 = 0,$$

hinc omnes sequentes formulas assignare possumus:

$$\text{III. } \int \partial \phi \cos \phi^2 = \frac{1}{2} \pi.$$

$$\text{IV. } \int \partial \phi \cos \phi^3 = 0.$$

$$\text{V. } \int \partial \phi \cos \phi^4 = \frac{1}{2} \cdot \frac{3}{4} \pi.$$

$$\text{VI. } \int \partial \phi \cos \phi^5 = 0.$$

$$\text{VII. } \int \partial \phi \cos \phi^6 = \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \pi.$$

$$\text{VIII. } \int \partial \phi \cos \phi^7 = 0.$$

$$\text{IX. } \int \partial \phi \cos \phi^8 = \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \frac{7}{8} \pi.$$

$$\text{X. } \int \partial \phi \cos \phi^9 = 0.$$

etc.

etc.

§. 12. Quia igitur supra invenimus esse

$$\pi A = \int \Phi \partial \phi, \text{ ob}$$

$$\Phi = (0) + (1) \cos \phi + (2) \cos \phi^2 + (3) \cos \phi^3 + \text{etc.}$$

integrationes modo assignatae praebent

$$\begin{aligned} \pi A = & (0) \pi + (2) \frac{1}{2} \pi + (4) \frac{1}{2} \cdot \frac{3}{4} \pi + (6) \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \pi \\ & + (8) \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \frac{7}{8} \pi + \text{etc.} \end{aligned}$$

divisione ergo per π facta nanciscimur hanc determinationem:

$$\begin{aligned} A = & (0) + \frac{1}{2}(2) + \frac{1}{2} \cdot \frac{3}{4}(4) + \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6}(6) \\ & + \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \frac{7}{8}(8) + \text{etc.} \end{aligned}$$

five elegantius

Nova Acta Acad. Imp. Scient. Tom. XI.

Q A =

$A = (0) + \frac{2}{4}(2) \frac{4 \cdot 3}{4 \cdot 8}(4) + \frac{6 \cdot 5 \cdot 4}{4 \cdot 8 \cdot 12}(6) + \frac{8 \cdot 7 \cdot 6 \cdot 5}{4 \cdot 8 \cdot 12 \cdot 16}(8) + \text{etc.}$
 quae est eadem series, quam olim per satis longas ambages sum adeptus.

§. 13. Ope eiusdem Lemmatis etiam secundâ litterâ B definiri poterit. Quia enim invenimus $\frac{1}{2} \pi B = \int \partial \Phi \cos \Phi$, si loco Φ seriem cognitam substituamus, integrationes Lemmatis nobis dabuntur:

$$\begin{aligned} \frac{1}{2} \pi B &= 1 + \frac{1}{2} \pi (1) + \frac{1}{2} \cdot \frac{3}{4} \pi (3) + \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \pi (5) \\ &\quad + \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \frac{7}{8} \pi (7) + \text{etc.} \end{aligned}$$

undè per π dividendo erit

$$\frac{1}{2} B = \frac{1}{2}, (1) + \frac{1}{2} \cdot \frac{3}{4}, (3) + \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6}, (5) + \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \frac{7}{8}, (7) + \text{etc.}$$

$$B = (1) + \frac{3}{4}, (3) + \frac{5 \cdot 4}{4 \cdot 8}, (5) + \frac{7 \cdot 6 \cdot 5}{4 \cdot 8 \cdot 12}, (7) + \frac{9 \cdot 8 \cdot 7 \cdot 6}{4 \cdot 8 \cdot 12 \cdot 16}, (9) + \text{etc.}$$

§. 14. Pro tertia litterâ C peculiari Lemmate opus erit, quo est:

$\int \partial \Phi \cos^2 \Phi \cos \Phi^\lambda = \frac{\lambda(\lambda-1)}{\lambda \lambda - 4} \int \partial \Phi \cos^2 \Phi \cos \Phi^{\lambda-2}$
 siquidem integralia a $\Phi = 0$ usque $\Phi = \pi$ extendantur
 Ad hoc demonstrandum ponamus in genere esse:

$$\begin{aligned} \int \partial \Phi \cos^2 \Phi \cos \Phi^\lambda &= f \sin^2 \Phi \cos \Phi^\lambda + g \cos^2 \Phi \sin \Phi \cos \Phi^\lambda \\ &\quad + h \int \partial \Phi \cos^2 \Phi \cos \Phi^{\lambda-2}, \end{aligned}$$

undè differentiatio præbet hanc aequationem:

$$\begin{aligned} \cos^2 \Phi \cos \Phi^\lambda &= 2 f \cos^2 \Phi \cos \Phi^\lambda + g \cos^2 \Phi \cos \Phi^\lambda \\ &\quad - g(\lambda-1) \cos^2 \Phi \sin \Phi^2 \cos \Phi^{\lambda-2} - \lambda f \sin^2 \Phi \cos \Phi \cos \Phi^{\lambda-1} \\ &\quad - 2 g \sin^2 \Phi \cos \Phi \cos \Phi^{\lambda-1} + h \cos^2 \Phi \cos \Phi^{\lambda-2}. \end{aligned}$$

Hic

ac iam primo termini, qui continent fin. $\circ \Phi$, tolli debent,
de fit $g = -\frac{\lambda f}{2}$; tum vero remanebit haec aequatio, post-
am loco fin. Φ^2 scriptum fuerit $i - \cos. \Phi^2$, divisione per
 $\circ \Phi$ facta,

$$\cos. \Phi^\lambda = -\frac{f(\lambda \lambda - 4)}{2} \cos. \Phi^\lambda + \frac{\lambda f}{2} (\lambda - i) \cos. \Phi^{\lambda-2} + h \cos. \Phi^{\lambda-2},$$

de manifesto fit $f = \frac{-2}{\lambda \lambda - 4}$, hincque $h = \frac{\lambda(\lambda - i)}{\lambda \lambda - 4}$, sicque
audio generalis ita habebit:

$$\begin{aligned} \partial \Phi \cos. \circ \Phi \cos. \Phi^\lambda &= \frac{-2}{\lambda \lambda - 4} \text{fin. } \circ \Phi \cos. \Phi^\lambda \\ &+ \frac{\lambda}{\lambda \lambda - 4} \cos. \circ \Phi \text{fin. } \Phi \cos. \Phi^{\lambda-2} + \frac{\lambda(\lambda - i)}{\lambda \lambda - 4} \partial \Phi \cos. \circ \Phi \cos. \Phi^{\lambda-2}. \end{aligned}$$

ac iam, posito $\Phi = \pi$, erit secundum Lemma

$$\int \partial \Phi \cos. \circ \Phi \cos. \Phi^\lambda = \frac{\lambda(\lambda - i)}{\lambda \lambda - 4} \int \partial \Phi \cos. \circ \Phi \cos. \Phi^{\lambda-2}.$$

§. 15. Tribuamus nunc exponenti λ successive ordines
valores $0, 1, 2, 3, 4$, etc. ac pro $\lambda = 0$ erit

$$\int \partial \Phi \cos. \circ \Phi = \frac{1}{2} \text{fin. } \circ \Phi = 0,$$

casu $\lambda = 1$ ipsum Lemma praebet $= 0$; at vero pro
casu $\lambda = 2$ usus Lemmatis cessat: tradenda ergo erit ipsa
formula $\int \partial \Phi \cos. \circ \Phi \cos. \Phi^2$, quae ob $\cos. \Phi^2 = \frac{1}{2} + \frac{1}{2} \cos. \circ \Phi$
sit in hanc: $+ \frac{1}{2} \int \partial \Phi \cos. \circ \Phi^2$, quae ob $\cos. \circ \Phi^2 = \frac{1}{2} + \frac{1}{2} \cos. 4\Phi$,
sit in $\frac{1}{4} \int \partial \Phi (1 + \cos. 4\Phi) = \frac{1}{4} \pi$, sicque pro casu $\lambda = 2$ erit
 $\Phi \cos. \circ \Phi \cos. \Phi^2 = \frac{\pi}{4}$.

§. 16. His igitur casibus simplicioribus expeditis
quentes ope Lemmatis facile conficiuntur; reperiemus enim:

$$1. \int \partial \Phi \cos. \circ \Phi \cos. \Phi^3 = c.$$

$$2. \int \partial \Phi \cos. \circ \Phi \cos. \Phi^4 = \frac{4 \cdot 3}{2 \cdot 6} \cdot \frac{\pi}{4}.$$

$$3. \int \partial \Phi \cos. \circ \Phi \cos. \Phi^5 = c.$$

$$4. \int \partial \Phi \operatorname{cof.} 2 \Phi \operatorname{cof.} \Phi^6 = \frac{4 \cdot 3}{2 \cdot 6} \cdot \frac{6 \cdot 5}{4 \cdot 8} \cdot \frac{\pi}{4}.$$

$$5. \int \partial \Phi \operatorname{cof.} 2 \Phi \operatorname{cof.} \Phi^8 = \frac{4 \cdot 3}{2 \cdot 6} \cdot \frac{6 \cdot 5}{4 \cdot 8} \cdot \frac{8 \cdot 7}{6 \cdot 10} \cdot \frac{\pi}{4}.$$

etc.

etc.

Cum igitur invenerimus $\frac{1}{2}\pi C = \int \Phi \partial \Phi \operatorname{cof.} 2 \Phi$, si integralia modo inventa introducantur, ac per π dividantur, repetiuntur:

$$\frac{1}{2}C = \frac{1}{4}(2) + \frac{1}{4} \cdot \frac{4 \cdot 3}{2 \cdot 6}(4) + \frac{1}{4} \cdot \frac{4 \cdot 3}{2 \cdot 6} \cdot \frac{6 \cdot 5}{4 \cdot 8}(6)$$

$$+ \frac{1}{4} \cdot \frac{4 \cdot 3}{2 \cdot 6} \cdot \frac{6 \cdot 5}{4 \cdot 8} \cdot \frac{8 \cdot 7}{6 \cdot 10}(8) + \text{etc.}$$

quae concinnius hoc modo exprimi potest:

$$\frac{1}{2}C = \frac{1}{2}(2) + \frac{3}{2 \cdot 6}(4) + \frac{3 \cdot 5}{2 \cdot 4 \cdot 8}(6) + \frac{3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 10}(8)$$

$$+ \frac{3 \cdot 5 \cdot 7 \cdot 9}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 12}(10) + \text{etc.}$$

quae adhuc elegantius ita referri potest:

$$\frac{1}{2}C = \frac{1}{4} \cdot (2) + \frac{1}{6} \cdot \frac{3}{2}(4) + \frac{1}{8} \cdot \frac{3 \cdot 5}{2 \cdot 4}(6) + \frac{1}{10} \cdot \frac{3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6}(8)$$

$$+ \frac{1}{12} \cdot \frac{3 \cdot 5 \cdot 7 \cdot 9}{2 \cdot 4 \cdot 6 \cdot 8}(10) + \text{etc.}$$

five adhuc elegantius ita:

$$2C = (2) + \frac{4}{4}(4) + \frac{6 \cdot 5}{4 \cdot 8}(6) + \frac{7 \cdot 6 \cdot 5}{4 \cdot 8 \cdot 12}(8)$$

$$+ \frac{9 \cdot 8 \cdot 7 \cdot 6}{4 \cdot 8 \cdot 12 \cdot 16}(10) + \text{etc.}$$

§. 17. Pro sequentibus terminis stabiliamus istud Lemma generale:

$$\int \partial \Phi \operatorname{cof.} i \Phi \operatorname{cof.} \Phi^\lambda \left(\frac{a \Phi^{\infty}}{\operatorname{ad} \Phi - \pi} \right) = \frac{\pi(\lambda-1)}{\lambda \lambda - i^2} \int \partial \Phi \operatorname{cof.} i \Phi \operatorname{cof.} \Phi^{\lambda-2},$$

pro quo demonstrando statuamus in genere

$$\begin{aligned} \int \partial \Phi \operatorname{cof.} i \Phi \operatorname{cof.} \Phi^\lambda &= f \operatorname{fin.} i \Phi \operatorname{cof.} \Phi^\lambda + g \operatorname{cof.} i \Phi \operatorname{fin.} \Phi \operatorname{cof.} \Phi^{\lambda-1} \\ &+ h \int \partial \Phi \operatorname{cof.} i \Phi \operatorname{cof.} \Phi^{\lambda-2}, \end{aligned}$$

unde

unde differentiatio perducit ad hanc aequationem:

$$\begin{aligned} \text{cof. } i\phi \text{ cof. } \Phi^\lambda &= (fi + g) \text{cof. } i\phi \text{ cof. } \Phi^\lambda - (\lambda f + gi) \text{sin. } i\phi \text{ cof. } \Phi^{\lambda-1} \\ &\quad - g(\lambda-1) \text{cof. } i\phi \text{sin. } \Phi^2 \text{cof. } \Phi^{\lambda-2} + h \text{cof. } i\phi \text{cof. } \Phi^{\lambda-2} \end{aligned}$$

Hic iam primo termini, qui continent sin. $i\phi$, tolli debent, unde fit $g = -\frac{\lambda f}{i}$, quo valore substituto, per cof. $i\phi$ dividendo, postquam loco sin. Φ^2 scriptum fuerit $1 - \text{cof. } \Phi^2$, prodit ista aequatio:

$$\text{cof. } \Phi^\lambda = -\frac{f(\lambda\lambda - i)}{i} \text{cof. } \Phi^\lambda + \frac{\lambda f}{i}(\lambda-1) \text{cof. } \Phi^{\lambda-2} + h \text{cof. } \Phi^{\lambda-2},$$

unde manifesto fit $f = \frac{-i}{\lambda\lambda - ii}$, hincque $h = \frac{\lambda(\lambda-1)}{\lambda\lambda - ii}$, sicque reducio generalis ita se habebit:

$$\begin{aligned} \int \partial \phi \text{cof. } i\phi \text{cof. } \Phi^\lambda &= \frac{-i}{\lambda\lambda - ii} \text{sin. } i\phi \text{cof. } \Phi^\lambda \\ &\quad + \frac{\lambda}{\lambda\lambda - ii} \text{cof. } \phi \text{sin. } i\phi \text{cof. } \Phi^{\lambda-2} \\ &\quad + \frac{\lambda(\lambda-1)}{\lambda\lambda - ii} \int \partial \phi \text{cof. } i\phi \text{cof. } \Phi^{\lambda-2}, \end{aligned}$$

quae saepenumero maximum utilitatem habere potest; posito autem $\phi = \pi$ manifesto prodat effatum Lemmatis.

§. 18. Hoc Lemmate constituto pro littera D definita sumi debet $i = 3$, eritque

$$\int \partial \phi \text{cof. } 3\phi \text{cof. } \Phi^\lambda = \frac{\lambda(\lambda-1)}{(\lambda-3)(\lambda+3)} \int \partial \phi \text{cof. } 3\phi \text{cof. } \Phi^{\lambda-2},$$

unde statim patet, casibus $\lambda = 0$ et $\lambda = 1$ formulam istam evanescere, posito scilicet $\phi = \pi$, ita ut sit

$$\int \partial \phi \text{cof. } 3\phi = 0 \text{ et } \int \partial \phi \text{cof. } 3\phi \text{cof. } \Phi = 0.$$

Hinc autem porro patet, casu quoque $\lambda = 2$ fore

$$\int \partial \phi \text{cof. } 3\phi \text{cof. } \Phi^2 = 0.$$

At vero casu $\lambda = 3$ Lemma dabit

$$\int \partial \phi$$

$$\int \partial \Phi \cos. 3 \Phi \cos. \Phi^3 = \frac{0}{0},$$

cuius ergo valor peculiari modo investigari debet; neque vero artifia cognita hic ullum usum praestari poterunt.

§. 19. Ad ipsam ergo indolem formulae propositae $\int \partial \Phi \cos. 3 \Phi \cos. \Phi^3$ respicere debemus, resolvendo potestatem $\cos. \Phi^3$ in hanc formam: $\frac{\cos. 3 \Phi + 3 \cos. \Phi}{4}$; tum igitur exit

$$\cos. 3 \Phi \cos. \Phi^3 = \frac{1}{8} + \frac{3}{8} \cos. 2\Phi + \frac{3}{8} \cos. 4\Phi + \frac{1}{8} \cos. 6\Phi,$$

quae forma duila in $\partial \Phi$ et integrata dat

$$\begin{aligned} \int \partial \Phi \cos. 3 \Phi \cos. \Phi^3 &= \frac{1}{8}\Phi + \frac{3}{16} \sin. 2\Phi + \frac{3}{32} \sin. 4\Phi \\ &\quad + \frac{1}{48} \sin. 6\Phi, \end{aligned}$$

unde sumto $\Phi = \pi$ valor exsurgit $= \frac{1}{8}\pi$, ita ut fit

$$\int \partial \Phi \cos. 3 \Phi \cos. \Phi^3 = \frac{1}{8}\pi.$$

§. 20. Ab hoc autem valore $\lambda = 3$ pendent sequentes: $\lambda = 5$; $\lambda = 7$; $\lambda = 9$; etc., qui ergo ita se habebunt:

$$\int \partial \Phi \cos. 3 \Phi \cos. \Phi^5 = \frac{\pi}{8},$$

$$\int \partial \Phi \cos. 3 \Phi \cos. \Phi^7 = \frac{5 \cdot 4}{2 \cdot 8} \cdot \frac{\pi}{8},$$

$$\int \partial \Phi \cos. 3 \Phi \cos. \Phi^9 = \frac{5 \cdot 4}{2 \cdot 8} \cdot \frac{7 \cdot 6}{4 \cdot 10} \cdot \frac{\pi}{8},$$

$$\int \partial \Phi \cos. 3 \Phi \cos. \Phi^{11} = \frac{5 \cdot 4}{2 \cdot 8} \cdot \frac{7 \cdot 6}{4 \cdot 10} \cdot \frac{97 \cdot 8}{6 \cdot 12} \cdot \frac{\pi}{8},$$

$$\int \partial \Phi \cos. 3 \Phi \cos. \Phi^{13} = \frac{5 \cdot 4}{2 \cdot 8} \cdot \frac{7 \cdot 6}{4 \cdot 10} \cdot \frac{9 \cdot 8}{6 \cdot 12} \cdot \frac{11 \cdot 10}{8 \cdot 14} \cdot \frac{\pi}{8},$$

etc.

etc.

reliqui vero casus omnes evanescunt.

§. 21. Cum nunc sit $D = \frac{2}{\pi} \int \Phi \partial \Phi \cos 3\Phi$, existente
 $\Phi = (0) + (1) \cos \Phi + (2) \cos \Phi^2 + (3) \cos \Phi^3 + (4) \cos \Phi^4 + \text{etc.}$
 singulos valores integrales in unam summam colligendo
 reperietur:

$$D = \frac{1}{4} [(3) + \frac{5 \cdot 4}{2 \cdot 8} (5) + \frac{5 \cdot 4 \cdot 7 \cdot 6}{2 \cdot 8 \cdot 4 \cdot 10} (7) + \frac{5 \cdot 4 \cdot 7 \cdot 6}{2 \cdot 8 \cdot 4 \cdot 10} \cdot \frac{9 \cdot 8}{6 \cdot 12} (9) + \text{etc.}]$$

quae quidem expressio in plures alias formas transfundi pos-
 set, quarum elegantissima est haec:

$$\begin{aligned} 4D &= (3) + \frac{5}{4}(5) + \frac{7 \cdot 6}{4 \cdot 8}(7) + \frac{9 \cdot 3 \cdot 7}{4 \cdot 8 \cdot 12}(9) + \frac{11 \cdot 10 \cdot 9 \cdot 8}{4 \cdot 8 \cdot 12 \cdot 16}(11) \\ &\quad + \frac{13 \cdot 12 \cdot 11 \cdot 9}{4 \cdot 8 \cdot 12 \cdot 16 \cdot 20}(13) + \text{etc.} \end{aligned}$$

§. 22. Pro littera porro E invenienda poni debet
 $i = 4$, et Lemma praemissum dabit:

$$\int \partial \Phi \cos 4\Phi \cos \Phi^\lambda = \frac{-\lambda(\lambda-1)}{(\lambda-4)(\lambda+4)} \int \partial \Phi \cos 4\Phi \cos \Phi^{\lambda-2},$$

undé iterum patet casibus $\lambda = 0$ et $\lambda = 1$ valorem evane-
 scere, quod propterea etiam contingit casibus $\lambda = 2$ et $\lambda = 3$;
 at vero casus $\lambda = 4$ peculiarem evolutionem postulat. Quo-
 niam vero ante vidimus esse $\cos \Phi^3 = \frac{1}{4} \cos 3\Phi + \frac{3}{4} \cos \Phi$,
 si denuo per $\cos \Phi$ multiplicemus, prodibit $\cos \Phi^4 = \frac{3}{8} +$
 $\frac{1}{16} \cos 2\Phi + \frac{1}{8} \cos 4\Phi$, quae forma porro in $\cos 4\Phi$ ducta dabit:

$$\begin{aligned} \cos 4\Phi \cos \Phi^4 &= \frac{1}{16} + \frac{1}{16} \cos 2\Phi + \frac{6}{16} \cos 4\Phi \\ &\quad + \frac{1}{16} \cos 6\Phi + \frac{1}{16} \cos 8\Phi. \end{aligned}$$

Haec iam formula ducatur in $\partial \Phi$ et integratur, tunc vero
 falso $\Phi = \pi$ manifesto resultabit valor quae fitus $= \frac{1}{16}\pi$, qui
 eatenus tantum prodiit, quotentus potestas $\cos \Phi^4$ per resolu-
 tionem dederat $\cos 4\Phi$.

§. 23. Cum igitur casu $\lambda = 4$ prodierat valor $\frac{\pi}{16}$, re-
 liqui independentes vi Lématis sequentes accipient valores:

$$\int \circ \Phi$$

$$\int \partial \Phi \cos. 4 \Phi \cos. \Phi^4 = \frac{\pi}{16},$$

$$\int \partial \Phi \cos. 4 \Phi \cos. \Phi^6 = \frac{6 \cdot 5}{2 \cdot 10} \cdot \frac{\pi}{16},$$

$$\int \partial \Phi \cos. 4 \Phi \cos. \Phi^8 = \frac{6 \cdot 5}{2 \cdot 10} \cdot \frac{8 \cdot 7}{4 \cdot 12} \cdot \frac{\pi}{16},$$

$$\int \partial \Phi \cos. 4 \Phi \cos. \Phi^{10} = \frac{6 \cdot 5}{2 \cdot 10} \cdot \frac{8 \cdot 7}{4 \cdot 12} \cdot \frac{10 \cdot 9}{6 \cdot 14} \cdot \frac{\pi}{16}.$$

etc.

etc.

Cum igitur sit $E = \frac{2}{\pi} \int \Phi \partial \Phi \cos. 4 \Phi$, existente

$$\Phi = (0) + (1) \cos. \Phi + (2) \cos. \Phi^2 + (3) \cos. \Phi^3 \\ + (4) \cos. \Phi^4 + \text{etc.}$$

praemissae reductiones nobis suppeditabunt sequentem valorem:

$$E = \frac{1}{8} [(4) + \frac{5 \cdot 6}{2 \cdot 10} (6) + \frac{5 \cdot 6 \cdot 7 \cdot 8}{2 \cdot 10 \cdot 4 \cdot 12} (8) + \frac{5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10}{2 \cdot 10 \cdot 4 \cdot 12 \cdot 6 \cdot 14} (10) + \text{etc.}]$$

quae forma haud difficulter in sequentem transfunditur:

$$8E = (4) + \frac{6}{4}(6) + \frac{8 \cdot 7}{4 \cdot 8}(8) + \frac{10 \cdot 9 \cdot 8}{4 \cdot 8 \cdot 12}(10) + \frac{13 \cdot 11 \cdot 10 \cdot 9}{4 \cdot 8 \cdot 12 \cdot 16}(12) + \text{etc.}$$

His casibus evolutis iam rem in genere exsequi possumus pro quounque numero i , ubi totum negotium ad casum $\lambda = i$ reducitur, quem ergo in peculiari problemate resolvamus.

Problema.

Denotante i numerum integrum quemcunque investigare valorem huius formulae integralis: $\int \partial \Phi \cos. i \Phi \cos. \Phi^i$, si quidem post integrationem statuatur $\Phi = \pi$.

Solutio.

§. 24. Ut solutionem ex primis principiis repeta-mus, ponamus

$$\cos. \Phi + \sqrt{-1} \sin. \Phi = p \text{ et } \cos. \Phi - \sqrt{-1} \sin. \Phi = q,$$

erit

eritque primo $p q = 1$, deinde vero erit $\text{col. } \Phi = \frac{p+q}{2}$, et quia porro est

$$p^n = \text{col. } n \Phi + \sqrt{-1} \sin. n \Phi \text{ et}$$

$$q^n = \text{col. } n \Phi - \sqrt{-1} \sin. n \Phi,$$

erit $\text{col. } i \Phi = \frac{p^i + q^i}{2}$, praeterea vero erit $\Phi^i = \frac{(p+q)^i}{2^i}$.

§. 25. Evolvatur iam potestas $(p+q)^i$ more solito; verum termini postremi cum primis iuncti repraesententur hoc modo:

$$(p+q)^i = +p^i + \frac{i}{1} p^{i-1} q + \frac{i(i-1)}{1 \cdot 2} p^{i-2} q q + \frac{i(i-1)(i-2)}{1 \cdot 2 \cdot 3} p^{i-3} q^3 + \text{etc.}$$

$$+ q^i + \frac{i}{1} p q^{i-1} + \frac{i(i-1)}{1 \cdot 2} p p q^{i-2} + \frac{i(i-1)(i-2)}{1 \cdot 2 \cdot 3} p^3 q^{i-3} + \text{etc.}$$

quae series, ob $p q = 1$, in hanc formam commodiorem redigitur:

$$(p+q)^i = p^i + q^i + \frac{i}{1} \cdot (p^{i-2} + q^{i-2}) + \frac{i(i-1)}{1 \cdot 2} (p^{i-4} + q^{i-4})$$

$$+ \frac{i(i-1)(i-2)}{1 \cdot 2 \cdot 3} (p^{i-6} + q^{i-6}) \text{ etc.}$$

Hic tantum notari oportet casibus, quibus i est numerus par, terminum dari medium solitarium, qui continebit quantitatem constantem, quam ergo duplicare non decet.

§. 26. Cum igitur ad angulos regrediendo sit in genere $p^n + q^n = 2 \text{ col. } n \Phi$, erit nunc:

$$(p+q)^i = 2 \text{ col. } i \Phi + \frac{2i}{1} \text{ col. } (i-2) \Phi + \frac{2i(i-1)}{1 \cdot 2} \text{ col. } (i-4) \Phi$$

$$+ \frac{2i(i-1)(i-2)}{1 \cdot 2 \cdot 3} \text{ col. } (i-6) \Phi + \text{etc.}$$

Quoniam vero est $p + q = 2 \text{ col. } \Phi$, erit

$$2^{i-1} \text{ col. } \Phi^i = \text{col. } i \Phi + \frac{i}{1} \text{ col. } (i-2) \Phi + \frac{i(i-1)}{1 \cdot 2} \text{ col. } (i-4) \Phi$$

$$+ \frac{i(i-1)(i-2)}{1 \cdot 2 \cdot 3} \text{ col. } (i-6) \Phi + \text{etc.}$$

Multiplicetur nunc utrinque per $2 \cos i\Phi$ et per notissimas reductiones reperiatur:

$$2\cos i\Phi \cos \Phi^i = 1 + \frac{i}{1} \cos 2\Phi + \frac{i(i-1)}{1 \cdot 2} \cos 4\Phi + \frac{i(i-1)(i-2)}{1 \cdot 2 \cdot 3} \cos 6\Phi + \dots$$

$$+ \cos 2i\Phi + \frac{i}{2} \cos (2i-2)\Phi + \frac{i(i-1)}{2 \cdot 2} \cos (2i-4)\Phi$$

$$+ \frac{i(i-1)(i-2)}{2 \cdot 2 \cdot 3} \cos (2i-6)\Phi + \dots$$

§. 27. Multiplicetur nunc utrinque per $\partial \Phi$ et integretur, prodibiique

$$2^i \int \partial \Phi \cos i\Phi \cos \Phi^i = \Phi + \frac{i}{2} \sin 2\Phi + \frac{i(i-2)}{4 \cdot 2} \sin 4\Phi + \dots$$

$$+ \frac{1}{2i} \sin 2i\Phi + \frac{i}{2i-2} \sin (2i-2)\Phi + \frac{i(i-2)}{4(2i-4)2} \sin (2i-4)\Phi + \dots$$

quae formula iam evanescit posito $\Phi = c$. Statuatur ergo $\Phi = \pi$, atque proveniet $\int \partial \Phi \cos i\Phi \cos \Phi^i = \pi$, quocirca valor in problemate quaesitus erit:

$$= \int \partial \Phi \cos i\Phi \cos \Phi^i = \frac{\pi}{2^i}.$$

§. 28. Quodsi iam ponamus in serie quamquaerimus
 $A + B \cos \Phi + C \cos 2\Phi + D \cos 3\Phi + E \cos 4\Phi + \dots$
 coëfficientem, ipsius $\cos i\Phi$ effe I, ita ut sit

$$I = \frac{2}{\pi} \int \Phi \partial \Phi \cos i\Phi, \text{ existente}$$

$\Phi = (c) + (1) \cos \Phi + (2) \cos 2\Phi + (3) \cos 3\Phi + (4) \cos 4\Phi + \dots$
 evidens est ex singulis terminis initialibus nihil prodire donec perveniat ad $\lambda = i$, quippe quo casu modo vidimus effe $\int \partial \Phi \cos i\Phi \cos \Phi^i = \frac{\pi}{2^i}$, a quo valore pendent casus sequentes per binarium ascéndentes, $\lambda = i+2; \lambda = i+4; \lambda = i+6$; etc. Scilicet vi Lemmatis erit:

$\int \partial \Phi$

$$\int \partial \Phi \text{cof. } i \Phi \text{cof. } \Phi^{i+2} = \frac{\pi}{2^i} \cdot \frac{\lambda(\lambda-1)}{(\lambda-i)(\lambda+i)} = \frac{(i+2)}{4} \cdot \frac{\pi}{2^i};$$

$$\int \partial \Phi \text{cof. } i \Phi \text{cof. } \Phi^{i+4} = \frac{(i+4)(i+3)}{4 \cdot 8} \cdot \frac{\pi}{2^i};$$

$$\int \partial \Phi \text{cof. } i \Phi \text{cof. } \Phi^{i+6} = \frac{(i+6)(i+5)(i+4)}{4 \cdot 8 \cdot 12} \cdot \frac{\pi}{2^i};$$

$$\int \partial \Phi \text{cof. } i \Phi \text{cof. } \Phi^{i+8} = \frac{(i+8)(i+7)(i+6)(i+5)}{4 \cdot 8 \cdot 12 \cdot 16} \cdot \frac{\pi}{2^i};$$

$$\int \partial \Phi \text{cof. } i \Phi \text{cof. } \Phi^{i+10} = \frac{(i+10)(i+9)(i+8)(i+7)(i+6)}{4 \cdot 8 \cdot 12 \cdot 16 \cdot 20} \cdot \frac{\pi}{2^i};$$

etc.

etc.

§. 29. His iam terminis colligendis et multiplicando per 2^{i-1} orietur sequens expressio:

$$2^{i-1} I = (i) + \frac{i+2}{4}(i+2) + \frac{(i+4)(i+3)}{4 \cdot 8}(i+4) + \frac{(i+6)(i+5)(i+4)}{4 \cdot 8 \cdot 12}(i+6) \\ + \frac{(i+8)(i+7)(i+6)(i+5)}{4 \cdot 8 \cdot 12 \cdot 16}(i+8) + \text{etc.}$$

quae forma iam continet determinationem omnium terminorum seriei, in quam formulam:

$\Phi = (0) + (1) \text{cof. } \Phi + (2) \text{cof. } \Phi^2 + (3) \text{cof. } \Phi^3 + (4) \text{cof. } \Phi^4 + \text{etc.}$
evolvere erat propositum, quam hoc modo hactenus representavimus:

$$\Phi = A + B \text{cof. } \Phi + C \text{cof. } 2 \Phi + D \text{cof. } 3 \Phi + E \text{cof. } 4 \Phi + \text{etc.}$$

cuius singuli termini per sequentes series exprimentur:

$$A = (0) + \frac{2}{4}(2) + \frac{4 \cdot 3}{4 \cdot 8}(4) + \frac{6 \cdot 5 \cdot 4}{4 \cdot 8 \cdot 12}(6) + \frac{8 \cdot 7 \cdot 6 \cdot 5}{4 \cdot 8 \cdot 12 \cdot 16}(8) + \text{etc.}$$

$$1. B = (1) + \frac{3}{4}(3) + \frac{5 \cdot 4}{4 \cdot 8}(5) + \frac{7 \cdot 6 \cdot 5}{4 \cdot 8 \cdot 12}(7) + \frac{9 \cdot 8 \cdot 7 \cdot 6}{4 \cdot 8 \cdot 12 \cdot 16}(9) + \text{etc.}$$

$$2. C = (2) + \frac{4}{4}(4) + \frac{6 \cdot 5}{4 \cdot 8}(6) + \frac{8 \cdot 7 \cdot 6}{4 \cdot 8 \cdot 12}(8) + \frac{10 \cdot 9 \cdot 8 \cdot 7}{4 \cdot 8 \cdot 12 \cdot 16}(10) + \text{etc.}$$

132.

$$4. D = (3) + \frac{5}{4}(5) + \frac{7 \cdot 6}{4 \cdot 8}(7) + \frac{9 \cdot 8 \cdot 7}{4 \cdot 8 \cdot 12}(9) + \frac{11 \cdot 10 \cdot 9 \cdot 8}{4 \cdot 8 \cdot 12 \cdot 16}(11) + \text{etc.}$$

$$8. E = (4) + \frac{6}{4}(6) + \frac{8 \cdot 7}{4 \cdot 8}(8) + \frac{10 \cdot 9 \cdot 8}{4 \cdot 8 \cdot 12}(10) + \frac{12 \cdot 11 \cdot 10 \cdot 9}{4 \cdot 8 \cdot 12 \cdot 16}(12) + \text{etc.}$$

$$16. F = (5) + \frac{7}{4}(7) + \frac{9 \cdot 8}{4 \cdot 8}(9) + \frac{11 \cdot 10 \cdot 9}{4 \cdot 8 \cdot 12}(11) + \frac{13 \cdot 12 \cdot 11 \cdot 10}{4 \cdot 8 \cdot 12 \cdot 16}(13) + \text{etc.}$$

etc.

atque in genere, si in serie quae sita terminus indici i re-
spondens fuerit I cos. $i\Phi$ erit:

$$2^{i-1} I = (i) + \frac{i+2}{4}(i+2) + \frac{i+4}{4} \cdot \frac{i+3}{8}(i+4) + \frac{i+6}{4} \cdot \frac{i+5}{8} \cdot \frac{i+4}{12}(i+6) \\ + \frac{i+8}{4} \cdot \frac{i+7}{8} \cdot \frac{i+6}{12} \cdot \frac{i+5}{16}(i+8) + \text{etc.}$$

INVE