

EVOLV TIO FORMVLAE INTEGRALIS

$$\int \frac{\partial z (3 + zz)}{(1 + zz) \sqrt[4]{(1 + 6zz + z^4)}}.$$

PER LOGARITHMOS ET ARCVS CIRCVLARES.

Audore

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§. 1.

Videtur hoc nullo alio modo fieri posse, nisi statuatur

$$z = \frac{1+x}{1-x}. \quad \text{Hinc autem siet}$$

$$\partial z = \frac{2\partial x}{(1-x)^2} \quad \text{et} \quad \frac{3+zz}{1+zz} = \frac{2(1-x+xx)}{1+xx} = \frac{2(1+x^3)}{(1+x)(1+xx)},$$

tum vero

$$1 + 6zz + z^4 = \frac{8(1+x^4)}{(1-x^4)}, \quad \text{ideoque}$$

$$\sqrt[4]{(1 + 6zz + z^4)} = 2^{\frac{3}{4}} \sqrt[4]{\frac{1+x^4}{(1-x)}},$$

quibus substitutis formula proposita induet hanc formam:

$$\frac{1}{2^4} \int \frac{\partial x (1+x^3)}{(1-x^4) \sqrt[4]{(1+x^4)}}.$$

§. 2.

§. 2. Disceramus istam formam in has duas partes:
 $\frac{1}{x^4}(P+Q)$, ita vt fit:

$$P = \int \frac{\partial x}{(1-x^4)\sqrt[4]{(1+x^4)}} \text{ et } Q = \int \frac{x^3 \partial x}{(1-x^4)\sqrt[4]{(1+x^4)}},$$

quas seorsim euoluamus. Pro priore quidem parte statuamus
 $\frac{x}{\sqrt[4]{(1+x^4)}} = t$, vt fit $P = \int \frac{t \partial x}{x(1-x^4)}$, tum autem fiet
 $\frac{x^4}{1+x^4} = t^4$, hincque $x^4 = \frac{t^4}{1-t^4}$, ergo $1-x^4 = \frac{1-2t^4}{1-t^4}$; deinde ob $4l x = 4l t - l(1-t^4)$, erit $\frac{\partial x}{x} = \frac{\partial t}{l(1-t^4)}$, hocque ergo modo probabit

$$P = \int \frac{\partial t}{1-2t^4}.$$

At pro altera parte Q ponatur $1+x^4 = u^4$, vt fiat
 $\sqrt[4]{(1+x^4)} = u$ et $x^3 \partial x = u^3 \partial u$; vnde deducitur:

$$Q = \int \frac{u^3 \partial u}{2-u^4},$$

sicque totum negotium ad formulas rationales est reductum.

§. 3. Quo nunc has formulas commodius tractare queamus, pro priore ponamus $t = \frac{p}{\sqrt[4]{2}}$, hocque modo erit

$$P = \frac{1}{\sqrt[4]{2}} \int \frac{\partial p}{1-p^4}. \text{ Nunc vero est}$$

$$\frac{1}{1-p^4} = \frac{1}{2} \cdot \frac{1}{1-p^2} + \frac{1}{2} \cdot \frac{1}{1+p^2},$$

vnde fit

$$P = \frac{1}{2\sqrt[4]{2}} \int \frac{\partial p}{1-p^2} + \frac{1}{2\sqrt[4]{2}} \int \frac{\partial p}{1+p^2},$$

ideoque integrando;

$$P =$$

$$P = \frac{1}{4\sqrt{2}} \sqrt{\frac{1+p}{1-p}} + \frac{x}{2\sqrt{2}} A \tan p,$$

ficque erit pars prior:

$$P = \frac{1}{2} \sqrt{\frac{1+p}{1-p}} + A \tan p.$$

Vbi notetur esse $p = t^{\frac{1}{2}}$, porro vero $t = \frac{x}{\sqrt{(1+x^4)}}$; et quoniam possumus $z = \frac{1+x}{1-x}$, erit $x = \frac{z-1}{z+1}$, sicque tota haec pars prior integralis quaesiti per z exprimi poterit.

etiam ~~quod~~ Pro altera parte Q ponatur $u = q^{\frac{1}{2}}$, fiet

$$\text{que } Q = \frac{1}{4\sqrt{2}} \int \frac{qq \partial q}{1-q^4}, \text{ Nunc vero est}$$

$$(1 - \frac{qq}{1-q^4}) = \frac{1}{2} \cdot \frac{1}{1-qq} - \frac{1}{2} \cdot \frac{1}{1+qq},$$

vnde fiet

$$\int \frac{qq \partial q}{1-q^4} = \frac{1}{2} \int \frac{\partial q}{1-qq} - \frac{1}{2} \int \frac{\partial q}{1+qq} = \frac{1}{2} l \frac{1+q}{1-q} - \frac{1}{2} A \tan q.$$

Hoc ergo modo prodit

$$Q = \frac{1}{4\sqrt{2}} \sqrt{\frac{1+q}{1-q}} - \frac{1}{2\sqrt{2}} A \tan q,$$

consequenter ipsa altera pars integralis erit

$$\frac{5}{24} Q = \frac{1}{2} l \frac{1+q}{1-q} - A \tan q.$$

Vbi est $q = \frac{u}{\sqrt{2}}$, porro vero $u = \sqrt[4]{(1+x^4)}$, denique vero, vt vidimus, est $x = \frac{z-1}{z+1}$.

§. 5. Quoniam igitur omnes isti valores sunt cogniti, formulae propositae integrale quae situm erit

$$\int \frac{\partial z (z + z^2)}{(z + z^2) \sqrt[4]{(z + 6zz + z^4)}} + \frac{1}{2} \sqrt{\frac{1+p}{1-p}} + \frac{1}{2} \sqrt{\frac{1+q}{1-q}} + A \operatorname{tang.} p - A \operatorname{tang.} q,$$

ybi notetur esse

$$p = \frac{z - 1}{\sqrt[4]{(z + 6zz + z^4)}} \text{ et } q = \sqrt[4]{(z + 6zz + z^4)}.$$

§. 6. His ergo valoribus substitutis nostrum integrale erit

$$\begin{aligned} & \frac{1}{2} \sqrt{\frac{\sqrt[4]{(z + 6zz + z^4)} + z - 1}{\sqrt[4]{(z + 6zz + z^4)} - z + 1}} + \frac{1}{2} \sqrt{\frac{1 + \sqrt[4]{(z + 6zz + z^4)}}{1 - \sqrt[4]{(z + 6zz + z^4)}}} \\ & + A \operatorname{tang.} \frac{z - 1}{\sqrt[4]{(z + 6zz + z^4)}} - A \operatorname{tang.} \sqrt[4]{(z + 6zz + z^4)}. \end{aligned}$$

Vbi notetur ambos arcus circulares ita in unum colligi posse, vt prodeat

$$A \operatorname{tang.} \frac{z - 1 - \sqrt[4]{(z + 6zz + z^4)}}{z \sqrt[4]{(z + 6zz + z^4)}}.$$

Ambo autem logarithmi ita in unum colligi poterunt:

$$\frac{\frac{1}{2} \sqrt{\frac{z \sqrt[4]{(z + 6zz + z^4)} + z - 1 + \sqrt[4]{(z + 6zz + z^4)}}{z \sqrt[4]{(z + 6zz + z^4)} - z + 1 - \sqrt[4]{(z + 6zz + z^4)}}}}{z \sqrt[4]{(z + 6zz + z^4)}}.$$

§. 7. Haec adhuc commodius exprimi poterunt. Si enim breuitatis gratia ponamus $\sqrt[4]{(1+6zz+z^4)} = v$, pars logarithmica nostri integralis erit

$$\frac{1}{2} \operatorname{I} \frac{vz+z-r+v}{vz-z+r-v} = \frac{1}{2} \operatorname{I} \frac{(1+v)(z-r+v)}{(v-r)(z-r-v)},$$

altera vero pars circularis est

$$\text{A tang. } \frac{z-r-v}{vz}.$$