

EXERCITATIO ANALYTICA;

VBI IMPRIMIS SERIEI MAXIME GENERALIS

SVMMATIO TRADITVR.

Auctore

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§. 1.

Cum nuper ostendissem, naturam linearum curuarum maxime adaequate per relationem inter ipsum curuae arcum eiusque amplitudinem exprimi posse, in mentem mihi venit, hanc rationem ad hyperbolam aequilateram, hac aequatione expressam: $y y = x x - 1$, accommodare. Sit igitur CB axis huius hyperbolae, punctum C eius centrum et A vertex, voceturque semiaxis $CA = 1$. Iam pro puncto curuae quocunque M, ducta applicata MP, vocetur abscissa $CP = x$ et applicata $PM = y$, eritque $y = \sqrt{(x x - 1)}$. Sit porro recta CV huius hyperbolae affymptota, cum axe angulum semirectum constituens, vnde producta applicata PM vsque ad affymptotam in S, erit $PS = CP = x$ et $CS = x \sqrt{2}$.

Tab. I.

Fig. 1.

§. 2. Vocetur nunc insuper arcus hyperbolae $AM = s$, et ducta ad curuam normali MN, vocetur amplitudo arcus AM, seu angulus $ANM = \phi$. Cum nunc fit $y \partial y = x \partial x$,

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erit

erit subnormalis $PN = x = CP$, hincque $\text{tang. } \Phi = \frac{x}{\sqrt{x^2 - 1}}$; unde per amplitudinem quantitas x ita exprimetur, ut fit $xx = \frac{\text{cof. } \Phi^2}{\text{cof. } \Phi^2 - \text{fin. } \Phi^2}$, ideoque $x = \frac{\text{cof. } \Phi}{\sqrt{\text{cof. } 2\Phi}}$, ex quo colligitur fore $\partial x = \frac{\partial \Phi (\text{cof. } \Phi \text{ fin. } 2\Phi - \text{fin. } \Phi \text{ cof. } 2\Phi)}{\text{cof. } 2\Phi \sqrt{\text{cof. } 2\Phi}} = \frac{\partial \Phi \text{ fin. } \Phi}{\text{cof. } 2\Phi \sqrt{\text{cof. } 2\Phi}}$.

Quoniam igitur est $\frac{\partial x}{\partial s} = \text{fin. } \Phi$, erit $\partial s = \frac{\partial x}{\text{fin. } \Phi}$, ideoque $\partial s = \frac{\partial \Phi}{\text{cof. } 2\Phi \sqrt{\text{cof. } 2\Phi}}$, quae est aequatio differentialis inter arcum curvae s eiusque amplitudinem Φ .

§. 3. Quo nunc facilius longitudinem arcus s per eius amplitudinem Φ exprimere valeamus, statuamus

$$s = \frac{z}{\sqrt{\text{cof. } 2\Phi}}, \text{ eritque } \partial s = \frac{\partial z \text{ cof. } 2\Phi + z \partial \Phi \text{ fin. } 2\Phi}{\text{cof. } 2\Phi \sqrt{\text{cof. } 2\Phi}},$$

unde quantitatem z ex hac aequatione elici oportebit:

$$\partial \Phi = \partial z \text{ cof. } 2\Phi + z \partial \Phi \text{ fin. } 2\Phi,$$

pro cuius integration fingamus hanc seriem:

$$z = A \text{ fin. } 2\Phi + B \text{ fin. } 6\Phi + C \text{ fin. } 10\Phi + \text{etc.}$$

Calculus enim tentanti mox patebit angulos per 4Φ continuo augeri debere. Quoniam igitur esse debet

$$1 = \frac{\partial z}{\partial \Phi} \text{ cof. } 2\Phi + z \text{ fin. } 2\Phi,$$

habebimus primo

$$\frac{\partial z}{\partial \Phi} = 2A \text{ cof. } 2\Phi + 6B \text{ cof. } 6\Phi + 10C \text{ cof. } 10\Phi + \text{etc.}$$

unde cum in genere fit

$$\begin{aligned} \text{cof. } 2\Phi \text{ cof. } n\Phi &= \frac{1}{2} \text{cof. } (n-2)\Phi + \frac{1}{2} \text{cof. } (n+2)\Phi, \text{ erit} \\ \frac{\partial z}{\partial \Phi} \text{ cof. } 2\Phi &= A + A \text{ cof. } 4\Phi + 3B \text{ cof. } 8\Phi + 5C \text{ cof. } 12\Phi \text{ etc.} \\ &\quad + 3B \quad + 5C \quad + 7D \end{aligned}$$

Simili modo cum fit

$$\text{fin. } 2\Phi \text{ fin. } n\Phi = \frac{1}{2} \text{cof. } (n-2)\Phi - \frac{1}{2} \text{cof. } (n+2)\Phi,$$

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$$z \sin. 2\Phi = \frac{1}{2}A - \frac{1}{2}A \cos. 4\Phi - \frac{1}{2}B \cos. 8\Phi - \frac{1}{2}C \cos. 12\Phi \text{ etc.} \\ + \frac{1}{2}B \quad + \frac{1}{2}C \quad + \frac{1}{2}D$$

his igitur seriebus coniunctis prodibit sequens aequatio :

$$1 = \frac{1}{2}A + (\frac{1}{2}A + \frac{1}{2}B) \cos. 4\Phi + (\frac{1}{2}B + \frac{1}{2}C) \cos. 8\Phi \\ + (\frac{1}{2}C + \frac{1}{2}D) \cos. 12\Phi + (\frac{1}{2}D + \frac{1}{2}E) \cos. 16\Phi + \text{etc.}$$

§. 4. Aequalitate igitur rite constituta sequentes colligentur coefficientium determinationes:

$$A = \frac{2}{3}, B = -\frac{1}{7}A, C = -\frac{5}{11}B, D = -\frac{9}{15}C, E = -\frac{13}{19}D, \text{ etc.}$$

quibus valoribus substitutis nanciscemur hinc valorem:

$$z = A (\sin. 2\Phi - \frac{1}{7} \sin. 6\Phi + \frac{1}{7} \cdot \frac{5}{11} \sin. 10\Phi - \frac{1}{7} \cdot \frac{5}{11} \cdot \frac{9}{15} \sin. 14\Phi + \text{etc.})$$

existente $A = \frac{2}{3}$, quae series pro evanescente amplitudine Φ praebet $z = 0$, ideoque etiam $s = 0$, vti natura rei postulat. Quod si autem curva in infinitum producat, quoniam tum curva cum affymptota confunditur, fiet amplitudo $\Phi = 45^\circ$, vnde ob $\sin. 2\Phi = 1$, $\sin. 6\Phi = -1$, $\sin. 10\Phi = +1$, $\sin. 14\Phi = -1$, et ita porro, pro hoc casu valor ipfius z erit:

$$z = A (1 + \frac{1}{7} + \frac{1}{7} \cdot \frac{5}{11} + \frac{1}{7} \cdot \frac{5}{11} \cdot \frac{9}{15} + \frac{1}{7} \cdot \frac{5}{11} \cdot \frac{9}{15} \cdot \frac{13}{19} + \text{etc.})$$

cuius seriei summa manifesto est finita, nihilo tamen minus ipse arcus s fiet utique infinite magnus, quemadmodum evidens est ex aequatione $s = \frac{z}{\sqrt{\cos. 2\Phi}}$, ob $\cos. 2\Phi = \cos. 90^\circ = 0$.

§. 5. Referat nunc in figura punctum E terminum infinite remotum in hyperbola, cui in affymptota respondeat punctum V, ita vt totus arcus $AE = \frac{z}{\sqrt{\cos. 2\Phi}}$, existente $2\Phi = 90^\circ$; tum igitur, cum sit in affymptota spatium indefinitum $CS = x \sqrt{2} = \frac{\cos. \Phi \sqrt{2}}{\sqrt{\cos. 2\Phi}}$, facto pariter $\Phi = 45^\circ$, erit longitudo

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infinita $CV = \frac{1}{\sqrt{\cos. 2\Phi}}$. Constat autem differentiam inter curvam AME et rectam CV esse finitam, quandoquidem curua AE manifesto est minor quam recta CV , at [si ex A in CV ducatur perpendicularis AD] maior quam recta VD . Sit igitur $CV - AE = \Delta$, vbi sufficit nosse, Δ esse quantitatem finitam. Hinc igitur erit $\frac{1-z}{\sqrt{\cos. 2\Phi}} = \Delta$, ideoque $z = 1 - \Delta \sqrt{\cos. 2\Phi}$, consequenter, ob $\sqrt{\cos. 2\Phi} = 0$, erit $z = 1$; ex quo concludimus, summam seriei pro z inuentae, casu $\Phi = 45^\circ$, praecise vnitati esse aequalem; vnde per $\frac{3}{2}$ multiplicando habebimus

$$1 + \frac{1}{7} + \frac{1}{7} \cdot \frac{5}{11} + \frac{1}{7} \cdot \frac{5}{11} \cdot \frac{9}{15} + \text{etc.} = \frac{3}{2},$$

ablata igitur vtrinque vnitatem erit

$$\frac{1}{7} + \frac{1}{7} \cdot \frac{5}{11} + \frac{1}{7} \cdot \frac{5}{11} \cdot \frac{9}{15} + \text{etc.} = \frac{1}{2},$$

quae summatio mihi eo magis memorabilis est visa, quod non memini, eam vsquam consignatam inuenisse.

§. 6. Quo igitur certior fierem de summationis huius veritate, eam sequenti modo mihi familiari sum perscrutatus. Pono

$$s = \frac{1}{7} x^7 + \frac{1}{7} \cdot \frac{5}{11} x^{11} + \frac{1}{7} \cdot \frac{5}{11} \cdot \frac{9}{15} x^{15} + \text{etc.}$$

ita vt posito $x = 1$ summa quaesita resultet. Hinc ergo erit differentiando:

$$\frac{\partial s}{\partial x} = x^6 + \frac{1}{7} \cdot 5 x^{10} + \frac{1}{7} \cdot \frac{5}{11} \cdot 9 x^{14} + \text{etc.}$$

Deinde vero cum fit

$$\frac{s}{x} = \frac{1}{7} x^5 + \frac{1}{7} \cdot \frac{5}{11} x^9 + \frac{1}{7} \cdot \frac{5}{11} \cdot \frac{9}{15} x^{13} + \text{etc.}$$

erit itidem differentiando

$$\frac{1}{x} \cdot \partial \cdot \frac{s}{x} = \frac{5}{7} x^4 + \frac{1}{7} \cdot \frac{5}{11} \cdot 9 x^8 + \frac{1}{7} \cdot \frac{5}{11} \cdot \frac{9}{15} \cdot 13 x^{12} + \text{etc.}$$

quae series, ducta in x^6 et a priore serie differentiali ablata, relinquit

quod $\frac{\partial s}{\partial x} - \frac{x^6}{\partial x} \cdot \partial \cdot \frac{s}{xx} = x^6$, sicque habetur aequatio finita,
ex qua valorem ipsius s erui oportet.

§. 7. Facta igitur evolutione deducti fumus ad hanc
aequationem differentialem:

$$\partial s - x^4 \partial s + 2 x^3 s \partial x = x^6 \partial x.$$

Diuidatur haec aequatio per $s(1 - x^4)$, vt prodeat

$$\frac{\partial s}{s} + \frac{2 x^3 \partial x}{1 - x^4} = \frac{x^6 \partial x}{s(1 - x^4)},$$

cuius aequationis membrum prius sponte est integrabile.
Integrale enim est

$$l s - \frac{1}{2} l(1 - x^4) = l \frac{s}{\sqrt{1 - x^4}},$$

vnde aequatio hoc modo referatur:

$$\partial \cdot l \frac{s}{\sqrt{1 - x^4}} = \frac{x^6 \partial x}{s(1 - x^4)}.$$

Manebit igitur haec aequatio integrabilis, si multiplicetur
per $\frac{s}{\sqrt{1 - x^4}}$, quo pacto simul membrum dextrum integratio-
nem admittet. Sic enim erit

$$\frac{s}{\sqrt{1 - x^4}} \partial \cdot l \frac{s}{\sqrt{1 - x^4}} = \frac{x^6 \partial x}{(1 - x^4)^{\frac{3}{2}}},$$

quae aequatio, posito breuitatis gratia $\frac{s}{\sqrt{1 - x^4}} = v$, induet hanc
formam:

$$v \partial \cdot l v = \partial v = \frac{x^6 \partial x}{(1 - x^4)^{\frac{3}{2}}},$$

vnde ergo colligitur

$$v = \frac{s}{\sqrt{1 - x^4}} = \int \frac{x^6 \partial x}{(1 - x^4)^{\frac{3}{2}}}.$$

§. 8. Facile autem patet hanc postremam formam integrationem algebraice non admittere. Interim tamen haec reductio ad formam simpliciore cum successu adhiberi poterit. Ponatur scilicet

$$\int \frac{x^6 \partial x}{(1-x^4)^{\frac{3}{2}}} = \frac{\alpha x^3}{\sqrt{1-x^4}} + \beta \int \frac{x x \partial x}{\sqrt{1-x^4}},$$

et sumtis differentialibus erit

$$\frac{x^6 \partial x}{(1-x^4)^{\frac{3}{2}}} = \frac{3\alpha x x \partial x}{\sqrt{1-x^4}} + \frac{2\alpha x^6 \partial x}{(1-x^4)^{\frac{3}{2}}} + \frac{\beta x x \partial x}{\sqrt{1-x^4}},$$

quae aequatio per $(1-x^4)^{\frac{3}{2}}$ multiplicata praebet

$$x^6 = (3\alpha + \beta) x x - (\alpha + \beta) x^6,$$

vnde patet capi debere $\alpha + \beta = -1$ et $3\alpha + \beta = 0$, sicque erit $\alpha = \frac{1}{2}$ et $\beta = -\frac{3}{2}$, hocque pacto fiet

$$\int \frac{x^6 \partial x}{(1-x^4)^{\frac{3}{2}}} = \frac{x^3}{2\sqrt{1-x^4}} - \frac{3}{2} \int \frac{x x \partial x}{\sqrt{1-x^4}},$$

ideoque

$$s = \frac{1}{2} x^3 - \frac{3}{2} \sqrt{1-x^4} \int \frac{x x \partial x}{\sqrt{1-x^4}}.$$

§. 9. Quanquam autem hic postremum integrale $\int \frac{x x \partial x}{\sqrt{1-x^4}}$ expediri nequit: tamen facile perspicitur, istud integrale, casu $x=1$, finitum valorem esse habiturum, id quod ad praesens nostrum institutum sufficit. Sit igitur iste valor finitus formulae $\int \frac{x x \partial x}{\sqrt{1-x^4}} = \Delta$, et aequatio inuenta, posito $x=1$, dabit $s = \frac{1}{2}$, quae est ea ipsa summa, quam nobis praecedens summatio suppeditavit.

§. 10. Similis igitur operatio nos ad plures alias summationes huiusmodi serierum magis generalium perducere valebit, vnde sequens Problema suscipiamus:

Problema.

Invenire summam huius seriei in infinitum excurrentis:

$$s = \frac{a}{b} + \frac{a}{b} \cdot \frac{a+\theta}{b+\theta} + \frac{a}{b} \cdot \frac{a+\theta}{b+\theta} \cdot \frac{a+2\theta}{b+2\theta} + \text{etc.}$$

Solutio.

§. 11. Statuamus igitur vt ante

$$s = \frac{a}{b} x^b + \frac{a}{b} \cdot \frac{a+\theta}{b+\theta} x^{b+\theta} + \frac{a}{b} \cdot \frac{a+\theta}{b+\theta} \cdot \frac{a+2\theta}{b+2\theta} x^{b+2\theta} + \text{etc.}$$

cuius ergo valor quaeritur casu $x=1$. Nunc vero haec series differentiatia dabit

$$\frac{\partial s}{\partial x} = a x^{b-1} + \frac{a}{b} \cdot (a+\theta) x^{b+\theta-1} + \frac{a}{b} \cdot \frac{a+\theta}{b+\theta} \cdot (a+2\theta) x^{b+2\theta-1} + \text{etc.}$$

Iam ipsa series proposita ducatur in $x^{a-b+\theta}$, eritque

$$x^{a-b+\theta} \cdot s = \frac{a}{b} x^{a+\theta} + \frac{a}{b} \cdot \frac{a+\theta}{b+\theta} x^{a+2\theta} + \frac{a}{b} \cdot \frac{a+\theta}{b+\theta} \cdot \frac{a+2\theta}{b+2\theta} x^{a+3\theta} + \text{etc.}$$

quae etiam differentiatia praebet:

$$\frac{\partial}{\partial x} \cdot x^{a-b+\theta} \cdot s = \frac{a}{b} (a+\theta) x^{a+\theta-1} + \frac{a}{b} \cdot \frac{a+\theta}{b+\theta} (a+2\theta) x^{a+2\theta-1} + \text{etc.}$$

quae series ducta in x^{b-a} et a superiore subtrahita destruet omnes terminos post primum; eritque

$$\frac{\partial s}{\partial x} - \frac{x^{b-a}}{\partial x} \cdot \frac{\partial}{\partial x} \cdot x^{a-b+\theta} \cdot s = a x^{b-1},$$

quae ergo est aequatio finita, vnde incognitam s erui oportet.

§. 12. Facta igitur evolutione deducimur ad hanc aequationem differentialem:

$$\partial s (1-x) - (a-b+\theta) s x^{\theta-1} \partial x = a x^{b-1} \partial x,$$

quam per $s(1-x^{\theta})$ diuidamus, vt obtineamus

∂s

$$\frac{\partial s}{s} = \frac{(a-b+\theta) x^{\theta-1} \partial x}{1-x^{\theta}} = \frac{a x^{b-1} \partial x}{s(1-x^{\theta})},$$

vbi prioris membri integrale est

$$l s + \frac{a-b+\theta}{\theta} l(1-x^{\theta}), \text{ siue}$$

$$l s = \frac{b-a-\theta}{\theta} l(1-x^{\theta}).$$

Mox enim videbimus summam seriei finitam non esse, nisi sit $b > a + \theta$. Hanc ob rem statuamus

$$\frac{s}{(1-x^{\theta})^{\frac{b-a-\theta}{\theta}}} = v,$$

vt membrum finiftrum sit $\partial . l v$, et aequatio erit

$$\partial . l v = \frac{a x^{b-1} \partial x}{s(1-x^{\theta})},$$

quae per v multiplicata euadet integrabilis; prodit enim

$$v . \partial . l v = \partial v = \frac{a x^{b-1} \partial x}{(1-x^{\theta})^{\frac{b-a}{\theta}}},$$

cuius integrale est

$$v = \frac{s}{(1-x^{\theta})^{\frac{b-a-\theta}{\theta}}} = a \int \frac{x^{b-1} \partial x}{(1-x^{\theta})^{\frac{b-a}{\theta}}}.$$

§. 13. Ponamus breuitatis gratia $\frac{b-a-\theta}{\theta} = n$, vt sit $b = n\theta + a + \theta$, siue $a = b - \theta - n\theta$, ficque aequatio nostra erit

$$\frac{s}{(1-x^{\theta})^n} = a \int \frac{x^{b-1} \partial x}{(1-x^{\theta})^{n+1}}.$$

Iam adhibeamus reductionem supra vfurpatam, ponendo

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$$\int \frac{x^{b-\theta-1} \partial x}{(1-x^\theta)^{n+1}} = \frac{a x^{b-\theta}}{(1-x^\theta)^n} + \beta \int \frac{x^{b-\theta-1} \partial x}{(1-x^\theta)^n},$$

vnde facta evolutione prodit

$$\frac{x^{b-1}}{(1-x^\theta)^{n+1}} = \frac{a(b-\theta) x^{b-\theta-1}}{(1-x^\theta)^{n+1}} + \frac{n a \theta x^{b-1}}{(1-x^\theta)^{n+1}} + \frac{\beta x^{b-\theta-1}}{(1-x^\theta)^n},$$

quae aequatio, ducta in $(1-x^\theta)^{n+1}$, praebet:

$$x^{b-1} = [a(b-\theta) + \beta] x^{b-\theta-1} - [a(b-\theta) - n a \theta + \beta] x^{b-1},$$

vnde sequitur sumi debere $a = \frac{1}{n\theta}$ et $\beta = -\frac{b-\theta}{n\theta}$. Hinc

igitur aequatio nostra x , per $(1-x^\theta)^n$ multiplicata, dabit

$$\frac{a x^{b-1}}{(1-x^\theta)^n} = \frac{a(b-\theta)}{n\theta} \frac{x^{b-\theta-1}}{(1-x^\theta)^n} + \frac{a(b-\theta)}{n\theta} (1-x^\theta)^n \int \frac{x^{b-\theta-1} \partial x}{(1-x^\theta)^n}.$$

§. 14. Quod autem hic ad postremum membrum attinet, etsi summatio institui nequit, tamen, quia in differentiali habetur denominator $(1-x^\theta)^n$, certum est, in integrali, si exhiberi posset, denominatorem tantum fore $(1-x^\theta)^{n-1}$, quippe cuius potestas unitate minor est quam praecedentis. Hinc tuto assumere licebit, hoc integrale tantum habere

formam $\frac{Q}{(1-x^\theta)^{n-1}}$, vbi nosse sufficiet in Q non amplius contineri denominatorem $1-x^\theta$; quo valore substituto habebimus:

$$s = \frac{a}{n\theta} x^{b-\theta} - \frac{a(b-\theta)}{n\theta} (1-x^\theta) Q.$$

§. 15. Inuenta iam summa seriei generalis tractatae, inde summam seriei propositae eliciemus, si faciamus $x = 1$; tum autem prodibit $s = \frac{a}{n\theta}$, ita vt formula incognita Q prorsus e calculo excefferit. Quoniam igitur breuitatis gratia

posuimus $n = \frac{b-a-\theta}{\theta}$, erit summa nostrae seriei $= \frac{a}{b-a-\theta}$,
vnde deducimus sequens

Theorema.

Quodsi proposita fuerit ista series infinita :

$$\frac{a}{b} + \frac{a}{b} \cdot \frac{a+\theta}{b+\theta} + \frac{a}{b} \cdot \frac{a+\theta}{b+\theta} \cdot \frac{a+2\theta}{b+2\theta} + \text{etc.}$$

eius summa semper erit $= \frac{a}{b-a-\theta}$; vnde sequitur, nisi fuerit $b > a + \theta$, hanc seriem summam finitam non habere, sed fore infinitam.

§. 16. Cum igitur ista summatio latissime pateat, notasse inuabit, in ea contineri series maxime cognitae, quae scilicet ex evolutione binomii nascuntur. Si enim potestatem binomiale $(1-x)^{\frac{a}{b}}$ euoluamus, ac fuerit $b > a$, sequens formabitur series:

$$(1-x)^{\frac{a}{b}} = 1 - \frac{a}{b}x + \frac{a(b-a)}{b \cdot 2b}x^2 - \frac{a(b-a)(2b-a)}{b \cdot 2b \cdot 3b}x^3 - \text{etc.}$$

Hinc igitur si ponamus $x = 1$, orietur ista summatio

$$1 = \frac{a}{b} + \frac{a}{b} \cdot \frac{b-a}{2b} + \frac{a}{b} \cdot \frac{b-a}{2b} \cdot \frac{2b-a}{3b} + \text{etc.}$$

quae egregie conuenit cum nostro theoremate. Si enim multiplicemus per $\frac{b}{a}$, fiet

$$\frac{b}{a} = 1 + \frac{b-a}{2b} + \frac{b-a}{2b} \cdot \frac{2b-a}{3b} + \text{etc.},$$

ideoque

$$\frac{b}{a} - 1 = \frac{b-a}{2b} + \frac{b-a}{2b} \cdot \frac{2b-a}{3b} + \text{etc.}$$

Si haec series cum nostra generali comparetur, quod nobis erat a , hic est $b-a$, quod nobis erat b , hic est ab , at quod nobis erat θ , hic est b . Vnde cum summa fuisset $\frac{a}{b-a-\theta}$, praesenti casu summa erit $\frac{b-a}{a}$, id quod pulcherrime congruit.

§. 17. Quin etiam summa assignari poterit serierum multo magis generalium, in quibus adeo innumerabiles litterae arbitrariae occurrunt, quemadmodum in sequente Theoremate plenius sum ostensurus.

Theorema generale.

Si litterae a, b, c, d , etc. cum θ pro lubitu numeros quoscunque denotent, huius progressionis, siue in infinitum excurrentis, siue alicubi terminatae:

$$\frac{a}{b+\theta} + \frac{a}{b+\theta} \cdot \frac{b}{c+\theta} + \frac{a}{b+\theta} \cdot \frac{b}{c+\theta} \cdot \frac{c}{d+\theta} + \text{etc.}$$

summa semper est $\frac{a}{\theta}$.

Hanc veritatem neutiquam methodo supra adhibita per Analysin Infinitorum ostendere licet; at vero ex principis Algebrae communis geminam demonstrationem sum traditurus: priore scilicet, ex ipsa summae expressione $\frac{a}{\theta}$ seriem propositam deriuare docebo, altera vero ex consideratione ipsius seriei eius summam.

Demonstratio prior.

§. 18. Ista demonstratio per sequentes considerationes planissimas absoluetur. Ponatur scilicet:

I°. $\frac{a}{\theta} = \frac{a}{b+\theta} + \frac{p}{\theta}$, eritque $p = \frac{a b}{b+\theta}$,

II°. $\frac{p}{\theta} = \frac{p}{c+\theta} + \frac{q}{\theta}$, eritque $q = \frac{c p}{c+\theta}$,

III°. $\frac{q}{\theta} = \frac{q}{d+\theta} + \frac{r}{\theta}$, eritque $r = \frac{d q}{d+\theta}$,

etc.

hocque modo quovsque libuerit procedere licet.

§. 19. Quod si iam statim in ipsa littera p substituamus, habebimus hanc aequationem: $\frac{a}{\theta} = \frac{a}{b+\theta} + \frac{a}{b+\theta} \cdot \frac{b}{\theta}$.

Sin autem in littera q substituamus, quia $\frac{p}{\theta} = \frac{p}{c+\theta} + \frac{cp}{(c+\theta)\theta}$,
erit

$$\frac{a}{\theta} = \frac{a}{b+\theta} + \frac{a}{b+\theta} \cdot \frac{b}{c+\theta} + \frac{a}{b+\theta} \cdot \frac{b}{c+\theta} \cdot \frac{c}{\theta}$$

Sin autem demum in r substituamus erit:

$$\frac{a}{\theta} = \frac{a}{b+\theta} + \frac{a}{b+\theta} \cdot \frac{b}{c+\theta} + \frac{a}{b+\theta} \cdot \frac{b}{c+\theta} \cdot \frac{c}{d+\theta} + \frac{a}{b+\theta} \cdot \frac{b}{c+\theta} \cdot \frac{c}{d+\theta} \cdot \frac{d}{\theta}$$

et ita porro. Vnde patet istam summationem semper locum habere, ad quotcunque terminos progressio continuetur, id quod adeo in infinitum valebit. Id tantum hic est monendum, quoniam postremus terminus a forma reliquorum aliquantillum discrepat, terminum infinitesimum hic plane non in censum venire. Nam, quia omnes factores numeratoris minores sunt factoribus denominatoris, evidens est, valorem termini infinitesimi prorsus evanescere.

Altera demonstratio.

§. 20. Hic igitur ipsam seriem tanquam datam spectemus atque in eius summam, quae fit $= s$, inquiremus. Ad hoc singulorum terminorum factores postremos in duas partes discerpamus, sequenti modo:

$$\begin{aligned} \text{I}^{\circ}. \quad \frac{a}{b+\theta} &= \frac{a}{\theta} - \frac{ab}{(b+\theta)\theta}, \\ \text{II}^{\circ}. \quad \frac{b}{c+\theta} &= \frac{b}{\theta} - \frac{bc}{(c+\theta)\theta}, \\ \text{III}^{\circ}. \quad \frac{c}{d+\theta} &= \frac{c}{\theta} - \frac{cd}{(d+\theta)\theta}, \\ &\text{etc.} \end{aligned}$$

§. 20. Quod si iam hos valores introducamus, loco cuiusque factoris ultimi singuli termini nostrae seriei in binas partes diidentur, quas sequenti modo sibi inuicem subscribamus:

$$s = \frac{a}{\theta} + \frac{a}{b+\theta} \cdot \frac{b}{\theta} + \frac{a}{b+\theta} \cdot \frac{b}{c+\theta} \cdot \frac{c}{\theta} \text{ etc.}$$

$$- \frac{a b}{(b+\theta)\theta} - \frac{a}{b+\theta} \cdot \frac{b c}{(c+\theta)\theta} - \frac{a}{b+\theta} \cdot \frac{b}{c+\theta} \cdot \frac{c d}{(d+\theta)\theta}$$

vbi evidens est quemlibet terminum negativum a sequenti positivo destrui, ita vt tandem primus positivus et vltimus negativus relinquuntur. Quodsi autem haec series in infinitum continetur, modo observauimus, terminum vltimum negativum in nihilum abire, propterea quod eius numerator infinites minor erit quam denominator; quo notato tota summa manifesto redigitur ad $s = \frac{a}{\theta}$.

§. 22. Hanc vltimam summationem iam olim in adversariis meis consignatam reperiō: non autem memini, eius mentionem vnquam publice fecisse. In commercio quidem epistolico quod iam ante quadraginta annos cum Ill. Goldbachio beatae mem. colui, plurima huc pertinentia occurrunt, quamobrem non dubito, quin Geometrae hanc evolutionem benigne sint accepturi.