quae forma redigitur ad hanc

$$z = \frac{\alpha \sqrt{[-x + \sqrt{(1 + xx)}]}}{\sqrt{(1 + xx)}}.$$

2.) Methodus nova investigandi omnes casus, quibus hanc aequationem differentio-differentialem

$$\partial \partial y (1 - axx) - bx \partial x \partial y - cy \partial x^2 = 0$$

resolvere licet. *M. S. Academiae exhib. die* 13 *Januarii*, 1780.

§. 16. Hic quidem in usum vocari posset methodus a me et ab aliis jam passim exposita, qua valor ipsius y per seriem infinitam exprimitur. Tunc enim omnibus casibus, quibus haec series alicubi abrumpitur, habebitur integrale particulare aequationis propositae; unde quidem haud difficulter integrale completum erui poterit. Verum etsi hoc modo infiniti casus integrabiles reperiuntur, tamen non omnes innotescunt, sed dantur praeterea infiniti alii casus, qui resolutionem admittunt. Quamobrem hic methodum prorsus singularem proponam, cujus ope omnes plane casus integrabiles elici poterunt. Haec autem methodus ita est comparata, ut cognito casu quocunque resolutionem admittente, ex eo innumerabiles alii deduci queant.

§. 17. Statim autem se offerunt duo casus simplicissimi, quibus resolutio succedit, quorum alter est, si $c = 0$, alter vero si $b = a$, quos ergo binos casus principales ante omnia evolvi oportet.

Casus prior principalis
quo $c = 0$.

§. 18. Hoc igitur casu aequatio nostra erit

$$\partial\partial y(1 - axx) = bx\partial x\partial y,$$

quae posito $\partial y = p\partial x$, abit in hanc

$$\partial p(1 - axx) = bpx\partial x, \text{ sive}$$

$$\frac{\partial p}{p} = \frac{bx\partial x}{1 - axx},$$

cujus integrale est

$$lp = -\frac{b}{2a}l(1 - axx) + lC,$$

sicque erit

$$p = C(1 - axx)^{-\frac{b}{2a}} = \frac{\partial y}{\partial x},$$

unde obtinetur

$$y = C\int\partial x(1 - axx)^{-\frac{b}{2a}}:$$

ubi notasse juvabit istum valorem fieri algebraicum quoties fuerit $-\frac{b}{2a}$ numerus integer positivus, sive $b = -2ia$ denotante i numerum integrum quemcunque. Tum vero valor integralis etiam algebraicus evadit, quando fuerit $-\frac{b}{2a}$, sive $-\frac{3}{2}$, sive $-\frac{5}{2}$, sive $-\frac{7}{2}$, etc. ideoque in genere $\frac{b}{a} = 2i + 1$, ubi esse nequit $i = 0$.

Casus principalis alter
quo $b = a$.

§. 19. Hoc ergo casu aequatio nostra per $2\partial y$ multiplicata erit

$$2\partial y\partial\partial y(1 - axx) - 2ax\partial x\partial y^2 - 2cy\partial y\partial x^2 = 0,$$

quae sponte est integrabilis, ejus enim integrale erit

$$\partial y^2(1-axx)-cyy\partial x^2=C\partial x^2.$$

Ex hac igitur aequatione erit

$$\partial y\sqrt{(1-axx)}=\partial x\sqrt{(C+cyy)},$$

separatione ergo facta erit

$$\frac{\partial x}{\sqrt{(1-axx)}}=\frac{\partial y}{\sqrt{(C+cyy)}}.$$

In hac ergo forma iterum continentur casus algebraici, ad quos eruendos faciamus $a=-\alpha.\alpha$, $c=\gamma\gamma$ et $C=\beta.\beta$; ut habeamus

$$\frac{\partial x}{\sqrt{(1+\alpha\alpha xx)}}=\frac{\partial y}{\sqrt{(\beta\beta+\gamma\gamma yy)}},$$

cujus integrale est

$$\frac{1}{\alpha}l[\alpha x+\sqrt{(1+\alpha\alpha xx)}]=\frac{1}{\gamma}l[\gamma y+\sqrt{(\beta\beta+\gamma\gamma yy)}]-\frac{\gamma}{1}l\Delta,$$

unde ad numeros ascendendo erit

$$\gamma y+\sqrt{(\beta\beta+\gamma\gamma yy)}=\Delta[\alpha x+\sqrt{(1+\alpha\alpha xx)}]^{\frac{\gamma}{\alpha}}.$$

Posito ergo V pro hac expressione posteriore erit

$$V-\gamma y=\sqrt{(\beta\beta+\gamma\gamma yy)},$$

et sumtis quadratis $y=\frac{VV-\beta\beta}{2\gamma V}$. Cum igitur sit

$$V=\Delta[\alpha x+\sqrt{(1+\alpha\alpha xx)}]^{\frac{\gamma}{\alpha}}, \text{ erit}$$

$$2\gamma y=\Delta[\alpha x+\sqrt{(1+\alpha\alpha xx)}]^{\frac{\gamma}{\alpha}}-$$

$$\frac{\beta\beta}{\Delta}[\alpha x+\sqrt{(1+\alpha\alpha xx)}]^{-\frac{\gamma}{\alpha}},$$

ubi est $\beta\beta=C$, exponens vero $\frac{\gamma}{\alpha}=\sqrt{\frac{c}{a}}$, sicque, quoties $\sqrt{\frac{c}{a}}$ fuerit numerus rationalis, integrale semper erit algebraicum.

§. 20. His duobus casibus principalibus expeditis duplicem tradam viam aequationem propositam in infinitas alias ejusdem generis transformandi, ita ut semper aequatio hujus formae

$$\partial\partial Y(1 - axx) - Bx\partial x\partial Y - CY\partial x^2 = 0$$

prodeat, quae cum resolutionem admittat casibus vel $C = 0$ vel $B = a$, iisdem casibus etiam ipsa aequatio proposita erit resolubilis. Duplices igitur hasce transformationes jam sum expositurus.

Transformationes prioris ordinis.

§. 21. Statuo $y = \frac{\partial v}{\partial x}$, unde ob

$$\partial y = \frac{\partial\partial v}{\partial x} \text{ et } \partial\partial y = \frac{\partial^3 v}{\partial x},$$

aequatio nostra induet hanc formam

$$\partial^3 v(1 - axx) - bx\partial x\partial\partial v - c\partial x^2\partial v = 0,$$

cujus singuli termini integrationem admittunt: erit enim

$$\int \partial x^2 \partial v = v\partial x^2,$$
$$\int x\partial x\partial\partial v = x\partial x\partial v - v\partial x^2,$$
$$\int \partial^3 v(1 - axx) = \partial\partial v(1 - axx) + 2ax\partial x\partial v - 2av\partial x^2.$$

His partibus colligendis, aequatio nostra erit

$$\partial\partial v(1 - axx) - (b - 2a)x\partial x\partial v - (c - b + 2a)v\partial x^2,$$

quae cum propositae prorsus sit similis, integrabilis erit his duobus casibus $c - b + 2a = 0$ et $b = 3a$, sive quoties fuerit $c = b - 2a$ vel $b = 3a$, atque integratione pro utroque casu instituta, ita ut v exprimatur per x, tum pro ipsa aequatione proposita erit $y = \frac{\partial v}{\partial x}$; unde patet, si integralia pro v inventa fuerint algebraica, fore quoque valorem ipsius y algebraicum.

§. 22. Quod si ulterius simili modo statuamus $v = \frac{\partial v'}{\partial x}$, quoniam per operationem praecedentem litterae b et c transibunt in $b - 2a$ et $c - b + 2a$, nunc ista aequatio proveniet

$$\partial \partial v' (1 - a x x) - (b - 4 a) x \partial x \partial v' - (c - 2 b + 6 a) v' \partial x^2 = 0,$$

quae ergo integrabilis erit, si fuerit vel $b = 5a$ vel $c = 2b - 6a$. Atque inventis valoribus pro v' fiet $y = \frac{\partial \partial v'}{\partial x^2}$, scilicet differentialia secunda ipsius v' dabunt y: sicque, si pro v' valor algebraicus prodierit, etiam y adipiscetur valorem algebraicum.

§. 23. Quod si eandem substitutionem denuo repetamus ponendo $v' = \frac{\partial v''}{\partial x}$, pro litteris initialibus b et c jam habebimus $b - 6a$ et $c - 3b + 12a$, et aequatio resultans erit

$$\partial \partial v'' (1 - a x x) - (b - 6 a) x \partial x \partial v'' - (c - 3 b + 12 a) v'' \partial x^2 = 0,$$

quae ergo resolutionem admittet, quoties fuerit vel $b = 7a$ vel $c = 3b - 12a$, quibus ergo casibus etiam ipsa aequatio proposita resolutionem admittat necesse est, cum sit $y = \frac{\partial^3 v''}{\partial x^3}$.

§. 24. Quod si ergo easdem has operationes continuo repetamus, perpetuo ad aequationes ejusdem formae perveniemus; ubi notasse sufficiet ambos valores, quos pro litteris b et c in qualibet operatione obtinuerimus, quos una cum valoribus ipsius y in sequenti tabula ob oculos ponamus

	b	c	y
Operatio I.	$b-2a$	$c-b+2a$	$\frac{\partial v}{\partial x}$
II.	$b-4a$	$c-2b+6a$	$\frac{\partial\partial v'}{\partial x^2}$
III.	$b-6a$	$c-3b\,12a$	$\frac{\partial^3 v''}{\partial x^3}$
IV.	$b-8a$	$c-4b+20a$	$\frac{\partial^4 v'''}{\partial x^4}$
—	- - -	- - - - - - -	- - -
—	- - -	- - - - - - -	- - -
—	- - -	- - - - - - -	- - -
i	$b-2ia$	$c-ib+i(i+1)a$	$\frac{\partial^i v^{[i-1]}}{\partial x^i}$

§. 25. Hinc igitur in genere patet, aequationem propositam semper resolutionem admittere, quoties fuerit vel $b=2ia+a$, vel $c=ib-i(i+1)a$, ubi pro i omnes numeros integros positivos accipere licet, ita ut hinc duos ordines innumerabilium casuum integrabilium nanciscamur, quorum posteriores tantum per methodum serierum initio indicatam reperiuntur, priores vero huic methodo prorsus sint inaccessi.

Transformationes posterioris ordinis.

§. 26. Quemadmodum hic per differentialia sumus progressi, nunc per integralia regrediamur, ac primo quidem ponamus $y=\int z\,\partial x$, et aequatio proposita evadet

$$\partial z(1-axx)-bxz\,\partial x-c\,\partial x\int z\,\partial x=0,$$

quae differentiata ad formam propositam reducitur

$$\partial\partial z(1-axx)-(b+2a)x\,\partial x\,\partial z-(c+b)z\,\partial x^2=0,$$

quae ergo secundum casus principales integrationem admittet, casibus $c+b=0$ et $b+2a=a$, sive $c=-b$ et $b=-a$.

Integralibus igitur inventis erit $y = \int z \partial x$; unde patet etiamsi haec integralia fuerint algebraica, tamen valores ipsius y fieri transcendentes.

§. 27. Simili modo statuamus porro $z = \int z' \partial x$, et quia per praecedentem operationem loco b et c adepti sumus $b + 2a$ et $c + b$, nunc perveniemus ad hanc aequationem

$$\partial\partial z'(1 - axx) - (b + 4a)x\partial x\partial z' - (c + 2b + 2a)z'\partial x^2 = 0,$$

quae ergo integrationem admittet, si fuerit vel $c + 2b + 2a = 0$, vel $b + 4a = a$, sive $c = -2b - 2a$ et $b = -3a$. Integralibus autem hinc inventis pro y habebimus $y = \int \partial x \int z' \partial x$, quae ita ad signum integrale simplex reducitur, ut sit

$$y = x \int z' \partial x - \int z' x \partial x.$$

§. 28. Simili modo statuamus porro $z' = \int z'' \partial x$, atque nunc deducemur ad hanc aequationem

$$\partial\partial z''(1 - axx) - (b + 6a)x\partial x\partial z'' - (c + 3b + 6a)z''\partial x^2 = 0,$$

quae igitur integrabilis erit, si fuerit vel $c + 3b + 6a = 0$, vel $b + 6a = a$, hoc est si $c = -3b - 6a$ et $b = -5a$; atque ex his integralibus fiet $y = \int \partial x \int \partial x \int z'' \partial x$, qui valor ex praecedente reduci potest, si is per ∂x multiplicatus denuo integretur et loco z' scribatur z'', obtinetur enim

$$y = \tfrac{1}{2} xx \int z'' \partial x - x \int x z'' \partial x + \tfrac{1}{2} \int xx z'' \partial x.$$

§. 29. Quod si jam has operationes ulterius continuemus, totum negotium huc redibit, ut formulae, quae loco b et c sunt proditurae, rite formentur, simulque valores ipsius y assignentur, quemadmodum sequens tabula indicabit

	b	c	y
Operat. I.	$b+2a$	$c+b$	$\int z\,\partial x$
II.	$b+4a$	$c+2b+2a$	$\int\partial x\int z'\,\partial x$
III.	$b+6a$	$c+3b+6a$	$\int\partial x\int\partial x\int z''\,\partial x$
IV.	$b+8a$	$c+4b+12a$	$\int\partial x\int\partial x\int\partial x\int z'''\,\partial x$
—	- - -	- - - - -	- - - - - - - -
—	- - -	- - - - -	- - - - - - - -
—	- - -	- - - - -	- - - - - - - -
i	$b+2ia$	$c+ib+i(i-1)a$	$\int\partial x\int\partial x\ldots\int z^{[i-1]}\,\partial x$

§. 30. Ex antecedentibus satis manifestum est, quomodo integralia ista complicata ad simplicia reduci queant, unde tantum sequentem tabulam subjungemus

$$\int\partial x\int z'\partial x = x\int z'\partial x - \int z' x\,\partial x$$

$$\int\partial x\int\partial x\int z''\partial x = \tfrac{1}{2}\left(xx\int z''\partial x - 2x\int z'' x\,\partial x + \int z'' xx\,\partial x\right)$$

$$\int\partial x\int\partial x\int\partial x\int z'''\partial x = \tfrac{1}{6}\left\{\begin{array}{l} x^3\int z'''\partial x - 3xx\int z''' x\,\partial x \\ +3x\int z''' xx\,\partial x - \int z''' x^3\,\partial x\end{array}\right\}$$

$$\int\partial x\int\partial x\int\partial x\int\partial x\int z^{\mathrm{IV}}\partial x = \tfrac{1}{24}\left\{\begin{array}{l} x^4\int z^{\mathrm{IV}}\partial x - 4x^3\int z^{\mathrm{IV}} x\,\partial x \\ +6xx\int z^{\mathrm{IV}} xx\,\partial x - 4x\int z^{\mathrm{IV}} x^3\,\partial x \\ +\int z^{\mathrm{IV}} x^4\,\partial x\end{array}\right\}$$

etc. etc.

§. 31. Quod si jam has operationes secundum numerum indefinitum i continuemus, et loco b, c, z, scribamus B, C, Z, aequatio resultans erit

$$\partial\partial Z(1-axx) - Bx\,\partial x\,\partial Z - CZ\,\partial x^2 = 0,$$

ubi erit, uti jam indicavimus

$$B = b + 2ia \quad\text{et}\quad C = c + ib + 2ia;$$

quamobrem haec aequatio integrationem admittet, quoties fuerit vel $C = 0$ hoc est $c = -ib - i(i-i)a$, vel $B = a$ hoc est $b = -(2i-1)a$: quae formulae ab illis quas supra pro priori transformationum ordine invenimus, tantum in hoc discrepant, quod hic littera i valorem negativum accepit; unde adjungatur sequens

Conclusio generalis.

§. 32. Si littera i hic denotet omnes numeros integros sive positivos sive negativos, aequatio proposita differentio-differentialis

$$\partial\partial y(1 - axx) - bx\partial x\partial y - cy\partial x^2 = 0$$

semper integrationem sive resolutionem admittet, quoties fuerit

1°.) $0 = ib - i(i+1)a$, vel

2°.) $b = (2i+1)a$:

ubi asseverare licet, omnes plane casus resolubiles in hac duplici forma contineri, ita ut nullus plane casus integrationem admittens exhiberi queat, qui non in alterutra harum duarum formularum comprehendatur, dum contra methodus per series procedens, cujus initio mentionem fecimus, tantum casus integrabiles priores ostendit, ita ut inde infinitus numerus casuum pariter resolubilium inde excludatur.

Corollarium 1.

§. 33. Transformetur aequatio proposita in aequationem differentialem primi gradus ponendo $y = e^{\int u\partial x}$, ac perveniemus ad hanc aequationem

$$\partial u + uu\partial x - \frac{bux\partial x - c\partial x}{1 - axx} = 0,$$

quae ergo etiam integrationem admittet casibus quibus vel $b =$

$(2i+1)a$ vel $c = ib - i(i+1)a$, denotante i numerum quemcunque integrum sive positivum sive negativum.

Corollarium 2.

§. 34. Quod si porro ponatur $u = (1 - axx)^n v$, posito brevitatis gratia $n = -\frac{b}{2a}$, pervenietur ad hanc aequationem ad genus *Riccatianum* referendam

$$(1 - axx)^n \partial v + (1 - axx)^{2n} vv \partial x = \frac{c \partial x}{1 - axx},$$

quae per $(1 - axx)^n$ divisa abit in hanc

$$\partial v + (1 - axx)^n vv \partial x = \frac{c \partial x}{(1 - axx)^{n+1}},$$

quae ergo iisdem casibus integrationem admittet.

Corollarium 3.

§. 35. Quod si sumamus $a = 0$, orietur ista aequatio

$$\partial u + uu \partial x = bux \partial x + c \partial x,$$

quae ergo integrabilis erit, si fuerit vel $b = 0$ vel $c = ib$, quorum quidem prior casus per se est manifestus, quia tum erit $\partial x = \frac{\partial u}{c - uu}$. Haec forma autem commodius exprimi poterit, ponendo

$$u = \tfrac{1}{2}bx + v, \text{ unde } \partial v + vv \partial x = (c - \tfrac{1}{2}b)\partial x + \tfrac{1}{4}bbxx \partial x,$$

sive ponendo $b = 2f$, ut fiat

$$\partial v + vv \partial x = (c - f)\partial x + ffxx \partial x,$$

eritque haec aequatio integrabilis, quoties fuerit $c = 2if$, ita ut sequens aequatio semper integrationem admittat

$$\partial v + vv \partial x = (2i - 1)f \partial x + ffxx \partial x,$$

quicunque numerus integer sive positivus sive negativus pro i accipiatur; hoc est, si in penultimo termino f multiplicetur per numerum imparem quemcunque sive positivum sive negativum, qui casus eo erunt abstrusiores, quo major accipiatur numerus i; atque adeo vix alia via patere videtur integralia eruendi, nisi ut ad aequationem differentialem secundi gradus propositam regrediamur atque easdem operationes instituamus quas supra docuimus. Interim tamen observavi, omnes istos casus etiam immediate ex ipsa aequatione per fractiones continuas derivari posse. Si enim proposita fuerit haec aequatio

$$\partial v + vv\partial x = g\partial x + ffxx\partial x,$$

valor ipsius v duplici modo per fractionem continuam exprimi potest. Est enim priori modo

$$v = fx + \cfrac{g - f}{2fx + \cfrac{g - 3f}{2fx + \cfrac{g - 5f}{2fx + \text{etc.}}}}$$

Altero vero modo est

$$v = -fx - \cfrac{(g - f)}{2fx + \cfrac{g + 3f}{2fx + \cfrac{g + 5f}{2fx + \cfrac{g + 7f}{2fx + \text{etc.}}}}}$$

quarum prior abrumpitur, quoties fuerit $g = (2i + 1)f$, posterior vero, quoties fuerit $g = -(2i + 1)f$, qui sunt ipsi casus integrabiles ante inventi.