etiam utrinque tang. $n\,\phi$ imaginariis inquinatur, unde etiam in valores X et Y imaginaria ingredientur; quamobrem reductio ad realitatem plerumque maximum laborem exigere poterit, proque autem negotio praecepta necessaria jam satis sunt cognita.

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2) Theorema maxime memorabile circa formulam integralem $\int \frac{\partial \phi \cos.\lambda\,\phi}{(1 + aa - 2a\cos.\phi)^{n+1}}$. *M. S. Academiae exhib. die* 13. *Augusti* 1778.

§. 21. Haec formula aliam restrictionem non postulat nisi quod littera λ numeros tantum integros designat sive positivos sive negativos. Evidens autem est valores negativos non discrepare a positivis, cum semper sit $\cos. -\phi = \cos. +\phi$. Hoc notato si istius formulae integrale a termino $\phi = 0$ usque ad terminum $\phi = 180^\circ$ sive $\phi = \pi$ porrigatur, ejus valor semper sequenti formula exprimetur $\frac{\pi a}{(1 - aa)^{2n+1}}$. V, existente

$$V = \left(\frac{n-\lambda}{0}\right)\left(\frac{n+\lambda}{\lambda}\right) + \left(\frac{n-\lambda}{1}\right)\left(\frac{n+\lambda}{\lambda+1}\right) aa$$
$$+ \left(\frac{n-\lambda}{2}\right)\left(\frac{n+\lambda}{\lambda+2}\right) a^4 + \left(\frac{n-\lambda}{3}\right)\left(\frac{n+\lambda}{\lambda+3}\right) a^6$$
$$+ \left(\frac{n-\lambda}{4}\right)\left(\frac{n+\lambda}{\lambda+4}\right) a^8 + \left(\frac{n-\lambda}{5}\right)\left(\frac{n+\lambda}{\lambda+5}\right) a^{10} \text{ etc.}$$

Ubi formulae uncinulis inclusae non fractiones, sed eos characteres designant, quibus unciae potestatum Binomii designari solent, ita ut sit

$$\left(\frac{\alpha}{\beta}\right) = \frac{\alpha}{1} \cdot \frac{\alpha-1}{2} \cdot \frac{\alpha-2}{3} \quad . \quad . \quad . \quad . \quad . \quad . \quad \frac{\alpha-\beta+1}{\beta},$$

quae expressio quoniam nostro casu β ubique est numerus integer, determinatum valorem facile quovis casu exhibendam declarat, ubi notasse sufficiet, quoties fuerit $\beta = 0$ semper fore $(\frac{\alpha}{0}) = 1$; sin autem fuerit β numerus negativus, valorem hujus characteris in nihilum abire; tum vero etiam observari convenit, si fuerit $\beta = \alpha$ fore $(\frac{\alpha}{\alpha}) = 1$, et si $\beta > \alpha$ pariter valores evanescere. Cum semper sit $(\frac{\alpha}{\beta}) = (\frac{\alpha}{\alpha - \beta})$.

§. 22. His explicatis evolvamus praecipuos casus quibus exponenti n valores simpliciores 0, 1, 2, 3, 4 etc. tribuuntur.

Casus I.

quo $n = 0$, et formula integralis haec proponitur

$$\int \frac{\partial \phi \cos. \lambda \phi}{1 + \alpha\alpha - 2\alpha \cos. \phi} \begin{bmatrix} b\,x = 0 \\ a\,d\,x = \pi \end{bmatrix}.$$

Quia hic $n = 0$, pro prioribus factoribus quantitatis V habebimus

$$(\frac{0 - \lambda}{0}) = 1;\ (\frac{0 - \lambda}{1}) = -\lambda;\ (\frac{0 - \lambda}{2}) = \frac{\lambda}{1} \cdot \frac{\lambda + 1}{2};$$
$$(\frac{0 - \lambda}{3}) = -\frac{\lambda}{1} \cdot \frac{\lambda + 1}{2} \cdot \frac{\lambda + 2}{3};\ (\frac{0 - \lambda}{4}) = \frac{\lambda}{1} \cdot \frac{\lambda + 1}{2} \cdot \frac{\lambda + 2}{3} \cdot \frac{\lambda + 3}{4};\ \text{etc.}$$

Pro posterioribus vero factoribus habebimus

$$(\frac{0 + \lambda}{\lambda}) = 1;\ (\frac{0 + \lambda}{\lambda + 1}) = 0;\ (\frac{0 + \lambda}{\lambda + 2}) = 0\ \text{etc.}$$

hic scilicet omnes isti factores praeter primum evanescunt; unde colligitur valor quantitatis $V = 1$, ideoque integrale quaesitum hujus casus erit $= \frac{\pi a^{\lambda}}{1 - a a}$.

Hinc ergo si fuerit $n = 0$, erit $\int \frac{\partial \phi}{1 + a a - 2 a \cos. \phi} = \frac{\pi}{1 - a a}$ quod egregie consentit cum integratione satis cognita

$$\int \frac{\partial \phi}{\alpha + \beta \cos. \phi} = \frac{1}{\sqrt{(\alpha\alpha - \beta\beta)}} \text{Arc. cos.} \frac{\alpha \cos. \phi + \beta}{\alpha + \beta \cos. \phi},$$

quod integrale jam sponte evanescit sumto $\Phi = 0$. Statuatur igitur, ut hic perpetuo assumimus, $\Phi = 180^\circ = \pi$, atque ob cos. $\Phi = -1$, erit istud integrale

$$\frac{1}{\sqrt{(\alpha\alpha - \beta\beta)}} \text{Arc. cos.} -1 = \frac{\pi}{\sqrt{(\alpha\alpha - \beta\beta)}}.$$

Jam nostro casu est $\alpha = 1 + aa$ et $\beta = -2a$, unde fit $\sqrt{(\alpha\alpha - \beta\beta)} = 1 - aa$.

Casus II.

quo $n = 1$, et formula integralis haec proponitur

$$\int \frac{\partial\Phi \cos.\lambda\Phi}{(1 + aa - 2a\cos.\Phi)^2} \begin{bmatrix} a\ \Phi = 0 \\ ad\ \Phi = \pi \end{bmatrix}.$$

Quia hic est $n = 1$, erit pro prioribus factoribus quantitatis V

$$\left(\frac{1-\lambda}{0}\right) = 1;\ \left(\frac{1-\lambda}{1}\right) = -(\lambda - 1);$$
$$\left(\frac{1-\lambda}{2}\right) = \frac{\lambda(\lambda-1)}{1 \cdot 2}.$$

Pro posterioribus vero factoribus habebimus

$$\left(\frac{1+\lambda}{\lambda}\right) = \lambda + 1;\ \left(\frac{1+\lambda}{\lambda+1}\right) = 1;$$

sequentes vero formulae evanescunt, sicque erit

$$V = \lambda + 1 - (\lambda - 1)aa;$$

quocirca valor integralis propositi erit

$$\frac{\pi a^\lambda}{(1-aa)^3}\left[(\lambda+1) - (\lambda-1)aa\right]:$$

hinc ergo sequentes casus speciales apposuisse juvabit, ubi brevitatis gratia loco formulae $1 + aa - 2a\cos.\Phi$ characterem Δ scribamus

$$\int \frac{\partial\Phi}{\Delta^2} = \frac{\pi(1+aa)}{(1-aa)^3},$$
$$\int \frac{\partial\Phi \cos.\Phi}{\Delta^2} = \frac{2\pi a}{(1-aa)^3},$$

$$\int \frac{\partial \Phi \cos. 2\Phi}{\Delta^2} = \frac{\pi a^2 (3 - aa)}{(1 - aa)^3},$$
$$\int \frac{\partial \Phi \cos. 3\Phi}{\Delta^2} = \frac{\pi a^3 (4 - 2aa)}{(1 - aa)^3},$$
$$\int \frac{\partial \Phi \cos. 4\Phi}{\Delta^2} = \frac{\pi a^4 (5 - 3aa)}{(1 - aa)^3},$$
$$\int \frac{\partial \Phi \cos. 5\Phi}{\Delta^2} = \frac{\pi a^5 (6 - 4aa)}{(1 - aa)^3},$$
$$\int \frac{\partial \Phi \cos. 6\Phi}{\Delta^2} = \frac{\pi a^6 (7 - 5aa)}{(1 - aa)^3},$$
etc. etc.

Casus III.

quo $n = 2$, et formula integralis haec proponitur
$$\int \frac{\partial \Phi \cos. \lambda \Phi}{(1 + aa - 2a \cos. \Phi)^3} \begin{bmatrix} a\ \Phi = 0 \\ ad\ \Phi = \pi \end{bmatrix}.$$

Hic factores priores, qui in valore quantitatis V occurrunt, erunt
$$\left(\frac{2-\lambda}{0}\right) = 1;\ \left(\frac{2-\lambda}{1}\right) = -(\lambda - 2);\ \left(\frac{2-\lambda}{2}\right) = \frac{(\lambda-2)(\lambda-1)}{1.\quad 2};$$
$$\left(\frac{2-\lambda}{3}\right) = \frac{\lambda-2.\lambda-1.\lambda}{1.\quad 2.\quad 3} \text{ etc.}$$
factores autem posteriores erunt
$$\left(\frac{2+\lambda}{\lambda}\right) = \frac{\lambda+2}{1}\cdot\frac{\lambda+1}{2};\ \left(\frac{2+\lambda}{\lambda+1}\right) = \lambda + 2;\ \left(\frac{2+\lambda}{\lambda+2}\right) = 1;$$
et sequentes omnes evanescunt; hinc ergo colligimus
$$V = \frac{(\lambda+2)(\lambda+1)}{1.\quad 2} - (\lambda\lambda - 4)\,aa + \frac{(\lambda-2)(\lambda-1)}{1.\quad 2}\,a^4,$$
hocque valore invento erit integrale quaesitum $\frac{\pi a^\lambda}{(1-aa)^5}\cdot V$, unde sequentes casus speciales, statuendo ut ante $1 + aa - 2a\cos.\Phi = \Delta$, evolvamus
$$\int \frac{\partial \Phi}{\Delta^3} = \frac{\pi}{(1-aa)^5}(1 + 4aa + a^4),$$
$$\int \frac{\partial \Phi \cos. \Phi}{\Delta^3} = \frac{3\pi a}{(1-aa)^5}(1 + aa),$$
$$\int \frac{\partial \Phi \cos. 2\Phi}{\Delta^3} = \frac{6\pi a^2}{(1-aa)^5},$$

$$\int \frac{\partial \phi \cos. 3\phi}{\Delta^3} = \frac{\pi a^3}{(1-aa)^5}(10 - 5aa + a^4),$$
$$\int \frac{\partial \phi \cos. 4\phi}{\Delta^3} = \frac{3\pi a^4}{(1-aa)^5}(5 - 4aa + a^4),$$
$$\int \frac{\partial \phi \cos. 5\phi}{\Delta^3} = \frac{3\pi a^5}{(1-aa)^5}(7 - 7aa + 2a^4),$$
$$\int \frac{\partial \phi \cos. 6\phi}{\Delta^3} = \frac{2\pi a^6}{(1-aa)^5}(14 - 16aa + 5a^4),$$
etc. etc.

Casus IV.

quo $n = 3$, et formula integralis haec proponitur

$$\int \frac{\partial \phi \cos. \lambda \phi}{(1 + aa - 2a\cos.\phi)^4} \left[\begin{matrix} a\,\phi = 0 \\ ad\,\phi = \pi \end{matrix}\right].$$

Hic pro prioribus factoribus quantitatis V habebimus

$$\left(\frac{3-\lambda}{0}\right) = 1;\ \left(\frac{3-\lambda}{1}\right) = -(\lambda - 3);\ \left(\frac{3-\lambda}{2}\right) = \frac{3-\lambda}{1}\cdot\frac{2-\lambda}{2};$$
$$\left(\frac{3-\lambda}{3}\right) = \frac{3-\lambda}{1}\cdot\frac{2-\lambda}{2}\cdot\frac{1-\lambda}{3};\ \ \left(\frac{3-\lambda}{4}\right) = \frac{3-\lambda}{1}\cdot\frac{2-\lambda}{2}\cdot\frac{1-\lambda}{3}\cdot\frac{-\lambda}{4};$$

factores autem posteriores erunt

$$\left(\frac{3+\lambda}{\lambda}\right) = \frac{3+\lambda}{1}\cdot\frac{2+\lambda}{2}\cdot\frac{1+\lambda}{3};\ \ \left(\frac{3+\lambda}{\lambda+1}\right) = \frac{3+\lambda}{1}\cdot\frac{2+\lambda}{2};$$
$$\left(\frac{3+\lambda}{\lambda+2}\right) = 3+\lambda;\ \left(\frac{3+\lambda}{\lambda+3}\right) = 1;$$

et sequentes omnes evanescunt, hinc ergo colligimus

$$V = \frac{(\lambda+1)}{1.}\frac{(\lambda+2)}{2.}\frac{(\lambda+3)}{3} - \frac{(\lambda+2)}{1.}\frac{(\lambda\lambda-9)}{2}aa + \frac{(\lambda-2)}{1.}\frac{(\lambda\lambda-9)}{2}a^4$$
$$- \frac{(\lambda-1)}{1.}\frac{(\lambda-2)}{2.}\frac{(\lambda-3)}{3}a^6.$$

Quo valore invento colligimus integrale quaesitum $= \frac{\pi a^\lambda}{(1-aa)^7}\cdot V$, hincque sequentes casus speciales, ponendo ut hactenus $1 + aa - 2a\cos.\phi = \Delta$, evolvamus

$$\int \frac{\partial \phi}{\Delta^4} = \frac{\pi}{(1-aa)^7}(1 + 9aa + 9a^4 + a^6),$$
$$\int \frac{\partial \phi \cos.\phi}{\Delta^4} = \frac{4\pi a}{(1-aa)^7}(1 + 3aa + a^4),$$
$$\int \frac{\partial \phi \cos. 2\phi}{\Delta^4} = \frac{10\pi a^2}{(1-aa)^7}(1 + aa),$$

$$\int \frac{\partial \phi \cos. 3\phi}{\Delta^4} = \frac{20 \pi a^3}{(1-aa)^7},$$
$$\int \frac{\partial \phi \cos. 4\phi}{\Delta^4} = \frac{\pi a^4}{(1-aa)^7} (35 - 21\, aa + 7\, a^4 - a^6),$$
etc. etc.

§. 23. Hic longius progredi superfluum foret, cum forma generalis pro V inventa totum negotium facillime conficiat; verum haud inutile erit, litterae n etiam valores negativos tribuere, quibus casibus tota integratio per methodos consuetas haud difficulter expeditur, unde jucundum erit pulcherrimum consensum nostrae formae generalis perspicere.

Casus I.

quo $n = -1$, et formula integralis haec proponitur

$$\int \partial \phi \cos. \lambda \phi \begin{bmatrix} a\,\phi = 0 \\ ad\,\phi = \pi \end{bmatrix}.$$

Haec formula absolute est integrabilis, cum sit

$$\int \partial \phi \cos. \lambda \phi = \frac{1}{\lambda} \sin. \lambda \phi,$$

quae formula cum jam evanescat posito $\phi = 0$; sumendo $\phi = \pi$, ob λ numerum integrum iste valor semper erit $= 0$, solo casu excepto $\lambda = 0$. Spectato enim λ tanquam infinite parvo, erit sin. $\lambda \pi = \lambda \pi$, ideoque hoc casu valor erit $= \pi$. Nunc autem forma generalis pro quantitate V data erit

$$V = \left(\frac{-1-\lambda}{0}\right)\left(\frac{-1+\lambda}{\lambda}\right) + \left(\frac{-1-\lambda}{1}\right)\left(\frac{-1+\lambda}{\lambda+1}\right) a^2$$
$$+ \left(\frac{-1-\lambda}{2}\right)\left(\frac{-1+\lambda}{\lambda+2}\right) a^4 + \left(\frac{-1-\lambda}{3}\right)\left(\frac{-1+\lambda}{\lambda+3}\right) a^6$$
$$+ \left(\frac{-1-\lambda}{4}\right)\left(\frac{-1+\lambda}{\lambda+4}\right) a^8 + \left(\frac{-1-\lambda}{5}\right)\left(\frac{-1+\lambda}{\lambda+5}\right) a^{10}$$
etc. etc.

Cujus expressionis factores posteriores omnes evanescunt, quoties fuerit vel $\lambda = 1$ vel $\lambda > 1$, propterea quod numeri inferiores majores, quam superiores, utrique vero positivi; quae conclusio autem non valet, quando superior numerus evadit negativus, uti evenit casu $\lambda = 0$, quem ergo solum perpendisse necesse est; hoc autem casu factores priores evadent

$$\left(\frac{-1}{0}\right) = 1;\ \left(\frac{-1}{1}\right) = -1;\ \left(\frac{-1}{2}\right) = +1;$$
$$\left(\frac{-1}{3}\right) = -1;\ \left(\frac{-1}{4}\right) = +1; \text{ etc.}$$

at vero valores posteriores eosdem determinationes recipiunt; sicque habebimus

$$V = 1 + aa + a^4 + a^6 + a^8 + a^{10} + \text{etc.}$$

quae series cum sit geometrica, erit $V = \frac{1}{1-aa}$ quare cum, ob $n = -1$ et $\lambda = 0$, valor quaesitus per nostram formam generalem sit $\pi\ (1-aa)\ V$, iste valor nunc ob $V = \frac{1}{1-aa}$, abit in π, uti supra.

Casus II.

quo $n = -2$, et formula integralis haec proponitur

$$\int \partial\Phi \cos.\lambda\Phi\ (1 + aa - 2a\cos.\Phi) \left[\begin{smallmatrix} a\ \Phi = 0 \\ ad\ \Phi = \pi \end{smallmatrix}\right].$$

Per formam nostram generalem integrale quaesitum erit $\pi\ a^\lambda\ (1-aa)^3\ V$, existente

$$V = \left(\frac{-2-\lambda}{0}\right)\left(\frac{-2+\lambda}{\lambda}\right) + \left(\frac{-2-\lambda}{1}\right)\left(\frac{-2+\lambda}{\lambda+1}\right) aa + \left(\frac{-2-\lambda}{2}\right)\left(\frac{-2+\lambda}{\lambda+2}\right) a^4$$
$$+ \left(\frac{-2-\lambda}{3}\right)\left(\frac{-2+\lambda}{\lambda+3}\right) a^6 + \left(\frac{-2-\lambda}{4}\right)\left(\frac{-2+\lambda}{\lambda+4}\right) a^8 + \left(\frac{-2-\lambda}{5}\right)\left(\frac{-2+\lambda}{\lambda+5}\right) a^{10}$$
$$\text{etc.} \qquad\qquad \text{etc.}$$

Ubi iterum evidens est, si fuerit vel $\lambda = 2$ vel $\lambda > 2$, omnes factores posteriores evanescere, ideoque fieri $V = 0$, ita ut etiam valor integralis quaesitus semper evanescat, id quod ex ipsa natura formulae sponte sequitur, quippe cujus integrale, ob

$$\cos.\Phi\cos.\lambda\Phi = \tfrac{1}{2}\cos.(\lambda-1)\Phi + \tfrac{1}{2}\cos.(\lambda+1)\Phi,$$

in genere erit

$$\frac{1+aa}{\lambda}\sin.\lambda\Phi - \frac{a}{\lambda-1}\sin.(\lambda-1)\Phi - \frac{a}{\lambda+1}\sin.(\lambda+1)\Phi,$$

quod quia $\lambda > 1$ casu $\phi = \pi$ manifesto evanescit; unde duos casus perpendere superest, alterum quo $\lambda = 0$, et alterum quo $\lambda = 1$.

I°. Sit $\lambda = 0$, et integrale $\pi\,(1 - aa)^3\,V$, ubi pro V factores posteriores evadunt

$$\left(\frac{-2}{0}\right) = 1;\ \left(\frac{-2}{1}\right) = -2;\ \left(\frac{-2}{2}\right) = 3;\ \left(\frac{-2}{3}\right) = -4;$$
$$\left(\frac{-2}{4}\right) = +5;\ \left(\frac{-2}{5}\right) = -6;\ \text{etc.}$$

simili modo priores factores erunt

$$\left(\frac{-2}{0}\right) = 1;\ \left(\frac{-2}{1}\right) = -2;\ \left(\frac{-2}{2}\right) = 3;\ \text{etc.}$$

unde colligitur fore

$$V = 1 + 4aa + 9a^4 + 16a^6 + 25a^8 + 36a^{10} + \text{etc.}$$

Pro qua serie summanda, inde subtrahatur series $V\,a\,a$, et remanebit

$$V(1 - aa) = 1 + 3aa + 5a^4 + 7a^6 + 9a^8 + \text{etc.}$$

Multiplicitur denuo utrinque per $1 - aa$, ac prodibit

$$V(1 - aa)^2 = 1 + 2aa + 2a^4 + 2a^6 + 2a^8 + \text{etc.}$$

quae denuo ducta in $1 - aa$ praebet

$$V(1 - aa)^3 = 1 + aa,\ \text{ideoque}\ V = \frac{1 + aa}{(1 - aa)^3}.$$

Consequenter integrale quaesitum erit $= \pi(1 + aa)$, id quod utique oritur ex integratione actuali, cum sit

$$\int \partial\phi\,(1 + aa - 2a\cos.\phi) = (1 + aa)\,\phi - 2a\sin.\phi,$$

quod facto $\phi = \pi$ abit in $(1 + aa)\,\pi$.

II°. Sit $\lambda = 1$, et integrale quaesitum $\pi a(1 - aa)^3 V$; ubi pro factoribus posterioribus est

$$\left(\frac{-1}{1}\right) = -1;\ \left(\frac{-1}{2}\right) = +1;\ \left(\frac{-1}{3}\right) = -1;$$
$$\left(\frac{-1}{4}\right) = +1;\ \left(\frac{-1}{5}\right) = -1;\ \text{etc.}$$

Factores vero priores evadunt

$$(\tfrac{-3}{0}) = 1;\ (\tfrac{-3}{1}) = -3;\ (\tfrac{-3}{2}) = 6;\ (\tfrac{-3}{3}) = -10;$$
$$(\tfrac{-3}{4}) = 15;\ (\tfrac{-3}{5}) = -21;\ (\tfrac{-3}{6}) = 28;$$
$$(\tfrac{-3}{7}) = -36;\ \text{etc.}$$

hinc igitur habebimus

$$V = -1 - 3aa - 6a^4 - 10a^6 - 15a^8 - 21a^{10} - 28a^{12} - 36a^{14} - \text{etc.}$$

Pro cujus summatione multiplicetur utrinque per $1 - aa$, et prodibit

$$V(1-aa) = -1 - 2aa - 3a^4 - 4a^6 - 5a^8 - 6a^{10} - 7a^{12} - 8a^{14} - \text{etc.}$$

multiplicando denuo per $1 - aa$, prodit

$$V(1-aa)^2 = -1 - aa - a^4 - a^6 - a^8 - a^{10} - a^{12} - a^{14} - \text{etc.}$$

et multiplicando rursus per $1 - aa$, erit

$$V(1-aa)^3 = -1, \text{ ita ut sit } V = -\frac{1}{(1-aa)^3},$$

consequenter integrale quaesitum $= -\pi a$. Ipsa autem integratio ob $\cos.\Phi^2 = \frac{1}{2} + \frac{1}{2}\cos.2\Phi$ praebet

$$\int \partial\Phi \cos.\Phi (1 + aa - 2a\cos.\Phi) = (1 + aa)\sin.\Phi$$
$$- a\Phi - \tfrac{1}{2}a \sin.2\Phi,$$

unde statuendo $\Phi = \pi$, oritur integrale $= -a\pi$.

Casus III.

quo $n = -3$, et formula integralis haec proponitur

$$\int \partial\Phi \cos.\lambda\Phi (1 + aa - 2a\cos.\Phi)^2 \left[\begin{smallmatrix} a\,\Phi = 0 \\ ad\,\Phi = \pi \end{smallmatrix}\right].$$

Hoc ergo casu ex forma generali erit integrale

$\pi a^\lambda (1 - aa)^5 V$, existente

$$V = (\tfrac{-3-\lambda}{0})(\tfrac{-3+\lambda}{\lambda}) + (\tfrac{-3-\lambda}{1})(\tfrac{-3+\lambda}{\lambda+1})a^2$$
$$+ (\tfrac{-3-\lambda}{2})(\tfrac{-3+\lambda}{\lambda+2})a^4 + (\tfrac{-3-\lambda}{3})(\tfrac{-3+\lambda}{\lambda+3})a^6$$

etc. etc.

ubi factores posteriores manifesto omnes evanescunt, quando fuerit vel $\lambda = 3$ vel $\lambda > 3$, quibus ergo casibus totum integrale evanescit, ut cuilibet calculum instituenti facile patebit: tres autem casus considerandi restant, quibus $\lambda < 3$.

I. Sit $\lambda = 0$, atque tam priores quam posteriores factores convenient, eruntque

$$(\tfrac{-3}{0}) = 1;\ (\tfrac{-3}{1}) = -3;\ (\tfrac{-3}{2}) = 6;\ (\tfrac{-3}{3}) = -10;$$
$$(\tfrac{-3}{4}) = 15;\ (\tfrac{-3}{5}) = -21;\ (\tfrac{-3}{6}) = 28;\ \text{etc.}$$

unde colligitur

$$V = 1 + 9\,aa + 36\,a^4 + 100\,a^6 + 225\,a^8 + 441\,a^{10} + \text{etc.}$$

quae series cum tandem perducat ad differentias constantes, simili modo ut hactenus summari poterit, prima enim multiplicatio per $1 - aa$ praebet

$$V(1-aa) = 1 + 8aa + 27a^4 + 64\,a^6 + 125\,a^8 + 216\,a^{10} + 343\,a^{12} + \text{etc.}$$

Secunda multiplicatio per $1 - aa$ praebet

$$V(1-aa)^2 = 1 + 7\,aa + 19\,a^4 + 37\,a^6 + 61\,a^8 + 91\,a^{10} + 127\,a^{12} + \text{etc.}$$

Tertia multiplicatio dat

$$V(1-aa)^3 = 1 + 6aa + 12\,a^4 + 18\,a^6 + 24\,a^8 + 30\,a^{10} + \text{etc.}$$

Quarta multiplicatio dat

$$V(1-aa)^4 = 1 + 5\,aa + 6\,a^4 + 6\,a^6 + 6\,a^8 + 6\,a^{10} + \text{etc.}$$

ac denique

$$V(1-aa)^5 = 1 + 4\,aa + a^4,\ \text{ita ut sit}\ V = \frac{1+4aa+a^4}{(1-aa)^6};$$

consequenter valor integralis quaesitus hoc casu erit $\pi\,(1 + 4aa + a^4)$, quod egregie cum integrali more solito invento congruit.

II. Sit $\lambda = 1$, quo casu priores factores ipsius V erunt

$$(\frac{-4}{0}) = 1;\ (\frac{-4}{1}) = -4;\ (\frac{-4}{2}) = 10;\ (\frac{-4}{3}) = -20;$$
$$(\frac{-4}{4}) = 35;\ (\frac{-4}{5}) = -56;\ (\frac{-4}{6}) = 84;\ (\frac{-4}{7}) = -120 \text{ etc.}$$

posteriores vero ita se habent

$$(\frac{-2}{1}) = -2;\ (\frac{-2}{2}) = +3;\ (\frac{-2}{3}) = -4;\ (\frac{-2}{4}) = +5;$$
$$(\frac{-2}{5}) = -6;\ (\frac{-2}{6}) = +7;\ (\frac{-2}{7}) = -8:\ (\frac{-2}{8}) = +9; \text{ etc.}$$

ideoque

$$V = -2 - 12\,a^2 - 40\,a^4 - 100\,a^6 - 210\,a^8 - 392\,a^{10}$$
$$- 672\,a^{12} - 1080\,a^{14} - \text{etc.}$$

quae series cum tandem perducat ad differentias constantes, simili modo ut ante summari poterit; prima enim multiplicatio per $1 - aa$ dat

$$V(1 - aa) = -2 - 10\,a^2 - 28\,a^4 - 60\,a^6 - 110\,a^8$$
$$- 182\,a^{10} - 280\,a^{12} - \text{etc.}$$

Secunda multiplicatio per $1 - aa$ praebet

$$V(1 - aa)^2 = -2 - 8\,a^2 - 18\,a^4 - 32\,a^6 - 50\,a^8$$
$$- 72\,a^{10} - 98\,a^{12} - \text{etc.}$$

Tertia multiplicatio dat

$$V(1 - aa)^3 = -2 - 6\,a^2 - 10\,a^4 - 14\,a^6 - 18\,a^8$$
$$- 22\,a^{10} - 26\,a^{12} - \text{etc.}$$

Quarta multiplicatio dat

$$V(1 - aa)^4 = -2 - 4a^2 - 4\,a^4 - 4\,a^6 - 4\,a^8$$
$$- 4\,a^{10} - 4\,a^{12} - \text{etc.}$$

ac denique quinta multiplicatio per $1 - aa$ praebet

$$V(1 - aa)^5 = -2 - 2\,aa = -2\,(1 + aa);$$

unde colligitur $V = -\frac{2(1+aa)}{(1-aa)^5}$, ideoque valor integralis quaesitus erit $= -2\pi a(1 + aa)$, qui egregie cum integrali more solito invento congruit.

III. Sit $\lambda = 2$, atque factores priores ipsius V erunt

$$(\tfrac{-5}{0}) = 1;\ (\tfrac{-5}{1}) = -5;\ (\tfrac{-5}{2}) = 15;\ (\tfrac{-5}{3}) = -35;$$
$$(\tfrac{-5}{4}) = 70;\ (\tfrac{-5}{5}) = -126;\ (\tfrac{-5}{6}) = 210;$$
$$(\tfrac{-5}{7}) = -330 \text{ etc.}$$

posteriores vero factores ita se habebunt

$$(\tfrac{-1}{2}) = 1;\ (\tfrac{-1}{3}) = -1;\ (\tfrac{-1}{4}) = 1;\ (\tfrac{-1}{5}) = -1;$$
$$(\tfrac{-1}{6}) = 1;\ (\tfrac{-1}{7}) = -1;\ (\tfrac{-1}{8}) = 1;\ (\tfrac{-1}{9}) = -1 \text{ etc.}$$

unde colligitur

$$V = 1 + 5a^2 + 15a^4 + 35a^6 + 70a^8 + 126a^{10} + 210a^{12}$$
$$+ 330a^{14} + \text{etc.}$$

quae series eodem modo ut ante summata praebet $V = + \frac{1}{(1-aa)^5}$, unde colligitur valor integralis quaesitus $= \pi aa$, qui cum integrali more solito invento utique egregie congruit.

§. 24. Quodsi haec integralia quibus n est numerus negativus cum iis comparemus, quibus n est numerus positivus, insignis analogia deprehenditur inter valores harum formularum

$$\int \Delta^n \partial \phi \cos. \lambda \phi \text{ et } \int \frac{\partial \phi \cos. \lambda \phi}{\Delta^{n+1}},$$

quae affinitas, si per plures casus exploretur, sequens nobis suppeditat theorema maxime notabile.

Theorema.

§. 25. Posito brevitatis gratia $\Delta = 1 + aa - 2a\cos. \phi$, atque integralia a termino $\phi = 0$ usque ad terminum $\phi = 180^\circ$ extendantur, semper locum habebit sequens proportio

$$\int \Delta^n \partial \phi \cos. \lambda \phi : \int \frac{\partial \phi \cos. \lambda \phi}{\Delta^{n+1}} = (\tfrac{n}{\lambda})(1-aa)^n : (\tfrac{-n-1}{\lambda})(1-aa)^{-n-1},$$

vel si statuamus

$$\frac{\Delta}{1-aa} = \frac{1+aa-2a\cos.\Phi}{1-aa} = \Gamma,$$

simplicius erit

$$\int \Gamma^n \partial\Phi \cos.\lambda\Phi : \int \frac{\partial\Phi \cos.\lambda\Phi}{\Gamma^{n+1}} = (\tfrac{n}{\lambda}) : (\tfrac{-n-1}{\lambda}).$$

§. 26. Ita exempla gratia si ponamus $n=2$, erit ex priore proportione

$$\int \Delta^2 \partial\Phi \cos.\lambda\Phi : \int \frac{\partial\Phi \cos.\lambda\Phi}{\Delta^3} = (\tfrac{2}{\lambda})(1-aa)^2 : (\tfrac{-3}{\lambda})(1-aa)^{-3}$$

unde si $\lambda = 0$, ob $(\frac{n}{0}) = 1$ et $(\frac{-3}{0}) = 1$, erit

$$\int \Delta^2 \partial\Phi : \int \frac{\partial\Phi}{\Delta^3} = (1-aa)^2 : \frac{1}{(1-aa)^3} = 1 : \frac{1}{(1-aa)^5},$$

ideoque erit

$$\int \frac{\partial\Phi}{\Delta^3} = \frac{1}{(1-aa)^5} \int \Delta^2 \partial\Phi.$$

Cum igitur sit

$$\int \Delta^2 \partial\Phi = \pi(1+4aa+a^4), \text{ erit}$$

$$\int \frac{\partial\Phi}{\Delta^3} = \frac{\pi}{(1-aa)^5}(1+4aa+a^4).$$

§. 27. Manente $n=2$, sit $\lambda=1$, ob $(\frac{2}{1}) = 1$ et $(\frac{-3}{1}) = -3$, erit

$$\int \Delta^2 \partial\Phi \cos.\Phi : \int \frac{\partial\Phi \cos.\Phi}{\Delta^3} = 2(1-aa)^2 : -3(1-aa)^{-3} = 1 : \frac{-3}{2(1-aa)^5}$$

unde fit

$$\int \frac{\partial\Phi \cos.\Phi}{\Delta^3} = \frac{-3}{2(1-aa)^5} \int \Delta^2 \partial\Phi \cos.\Phi;$$

cum igitur sit

$$\int \Delta^2 \partial\Phi \cos.\Phi = -2\pi a(1+aa), \text{ erit}$$

$$\int \frac{\partial\Phi \cos.\Phi}{\Delta^3} = \frac{+3\pi a(1+aa)}{(1-aa)^5}.$$

§. 28. Simili modo sumatur $\lambda = 2$, et ob $(\frac{2}{2}) = 1$ et $(\frac{-3}{2}) = 6$, erit

$$\int \Delta^2 \partial \Phi \cos. 2\Phi : \int \frac{\partial \Phi \cos. 2\Phi}{\Delta^3} = (1-aa)^2 : 6(1-aa)^{-3} = 1 : \frac{6}{(1-aa)^5},$$

unde fit

$$\int \frac{\partial \Phi \cos. 2\Phi}{\Delta^3} = \frac{6}{(1-aa)^5} \int \Delta^2 \partial \Phi \cos. 2\Phi.$$

Erat autem

$$\int \Delta^2 \partial \Phi \cos. 2\Phi = \pi a a,$$

consequenter

$$\int \frac{\partial \Phi \cos. 2\Phi}{\Delta^3} = \frac{6\pi a a}{(1-aa)^5}.$$

§. 29. Cum character $(\frac{n}{\lambda})$ fiat $= 1$ casu $\lambda = n$, casibus vero quibus $\lambda > n$ semper sit $(\frac{n}{\lambda}) = 0$, siquidem λ fuerit numerus integer, uti hic perpetuo assumimus, evidens est istis casibus, quibus $\lambda > n$, semper valorem formulae $\int \Delta^n \partial \Phi \cos. \lambda \Phi$ in nihilum abire.

§. 30. Theorema, quod hic proposuimus, non solum ob simplicitatem rationis omni attentione est dignum, sed etiam quod id tantum per plures casus sola inductione conclusimus, neque adhuc ulla via patere videtur, qua ejus veritas directe demonstrari queat; hujusmodi autem theoremata summam Geometrarum attentionem merentur. Evolvamus autem adhuc alios quosdam casus memorabiles nostri theorematis initio propositi.

Evolutio casus quo $\lambda = n$, et formula integralis proposita

$$\int \frac{\partial \Phi \cos. n\Phi}{\Delta^{n+1}}.$$

Ex forma generali hoc casu integrale erit $\frac{\pi a^n}{(1-aa)^{2n+1}} V$, existente

$$V = (\tfrac{0}{0})(\tfrac{2n}{n}) + (\tfrac{0}{1})(\tfrac{2n}{n+1})\,aa + (\tfrac{0}{2})(\tfrac{2n}{n+2})\,a^4 + \text{etc.}$$

ubi manifesto omnes termini praeter primum evanescunt, ita ut sit $V = (\frac{2n}{n})$, ideoque nostrum integrale

$$\int \frac{\partial \Phi \cos. n \Phi}{\Delta^{n+1}} = \frac{\pi a^n}{(1 - aa)^{2n+1}} \cdot (\tfrac{2n}{n});$$

ubi notetur, valores characteris $(\frac{2n}{n})$ pro variis valoribus numeri n sequenti modo se habere

n	0,	1,	2,	3,	4,	5,	6,	7	
$(\frac{2n}{n})$	1,	2,	6,	20,	70,	252,	924,	3432	etc.

quae series facillime per hos factores continuatur

$$\tfrac{2}{1} \cdot \tfrac{6}{2} \cdot \tfrac{10}{3} \cdot \tfrac{14}{4} \cdot \tfrac{18}{5} \cdot \tfrac{22}{6} \cdot \tfrac{26}{7} \text{ etc.}$$

Postremum vero theorema inventum ad hunc casum applicatum praebebit hanc proportionem

$$\int \Delta^n \partial \Phi \cos. n \Phi : \int \frac{\partial \Phi \cos. n \Phi}{\Delta^{n+1}} = (1 - aa)^n : (\tfrac{-1-n}{n})(1 - aa)^{-n-1},$$

unde fit

$$\int \Delta^n \partial \Phi \cos. n \Phi = \frac{\pi a^n}{(\frac{-n-1}{n})} \cdot (\tfrac{2n}{n}) = (\tfrac{2n}{n})\,\pi a^n : (\tfrac{-n-1}{n};$$

ubi notetur valores characteris $(\frac{-n-1}{n})$ pro variis valoribus ipsius n esse

n	0,	1,	2,	3,	4,	5,	6	
$(\frac{-n-1}{n})$	−1,	−2,	6,	−20,	70,	−252,	924	etc.

unde patet esse $(\frac{-n-n}{n}) = \pm (\frac{2n}{n})$, dum signum superius valet, quando n est numerus par, contra vero signum inferius, quando n est numerus impar; hinc ergo erit

$$\int \Delta^n \partial \Phi \cos. n \Phi = \pm \pi a^n.$$

His notatis evolvamus casus simpliciores pro utraque formula integrali

$n = 0$	$\int \frac{\partial \Phi}{\varDelta} = \frac{\pi}{1 - aa}$	$\int \partial \Phi = + \pi$
$n = 1$	$\int \frac{\partial \Phi \cos. \Phi}{\varDelta^2} = \frac{2 \pi a}{(1 - aa)^3}$	$\int \varDelta \partial \Phi \cos. \Phi = - \pi a$
$n = 2$	$\int \frac{\partial \Phi \cos. 2 \Phi}{\varDelta^3} = \frac{2 \pi a^2}{(1 - aa)^5}$	$\int \varDelta^2 \partial \Phi \cos. 2 \Phi = + \pi a^2$
$u = 3$	$\int \frac{\partial \Phi \cos. 3 \Phi}{\varDelta^4} = \frac{20 \pi a^3}{(1 - aa)^7}$	$\int \varDelta^3 \partial \Phi \cos. 3 \Phi = - \pi a^3$
$n = 4$	$\int \frac{\partial \Phi \cos. 4 \Phi}{\varDelta^5} = \frac{70 \pi a^4}{(1 - aa)^9}$	$\int \varDelta^4 \partial \Phi \cos. 4 \Phi = + \pi a^4$
$n = 5$	$\int \frac{\partial \Phi \cos. 5 \Phi}{\varDelta^6} = \frac{252 \pi a^5}{(1 - aa)^{11}}$	$\int \varDelta^5 \partial \Phi \cos. 5 \Phi = - \pi a^5$
$n = 6$	$\int \frac{\partial \Phi \cos. 6 \Phi}{\varDelta^7} = \frac{924 \pi a^6}{(1 - aa)^{13}}$	$\int \varDelta^6 \partial \Phi \cos. 6 \Phi = + \pi a^6$
	etc.	etc.

Hic imprimis notatu dignum occurrit, quod his casibus $\lambda = n$ integralia tam succincte exprimuntur; nunc autem alios perpendamus casus, quibus litterae λ successive valores 0, 1, 2, 3 etc. tribuantur.

Evolutio casus

quo $\lambda = 0$, et formula integralis proposita

$$\int \frac{\partial \Phi}{\Delta^{n+1}}.$$

§. 31. Cum hic sit $\lambda = 0$, integrale quaesitum ex nostra formula erit $\frac{\pi}{(1 - aa)^{2n+1}}$ V, existente

$$V = \left(\tfrac{n}{0}\right)^2 + \left(\tfrac{n}{1}\right)^2 aa + \left(\tfrac{n}{2}\right)^2 a^4 + \left(\tfrac{n}{3}\right)^2 a^6 + \left(\tfrac{n}{4}\right)^2 a^8 + \text{etc.}$$

simul vero hinc etiam assignari poterit valor hujus formulae $\int \varDelta^n \partial \Phi$, cum sit

$$\int \Delta^n \partial \Phi : \int \frac{\partial \Phi}{\Delta^{n+1}} = (1-aa)^n : (1-aa)^{-n-1} = (1-aa)^{2n+1} : 1,$$

ex qua proportione colligitur

$$\int \Delta^n \partial \Phi = \pi \, . \, V.$$

Percurramus igitur simpliciores casus pro exponente n, quos sequenti tabula subjungamus

$$n = 0 \begin{cases} \int \frac{\partial \Phi}{\Delta} = \frac{\pi}{1-aa} \\ \int \partial \Phi = \pi \end{cases}$$

$$n = 1 \begin{cases} \int \frac{\partial \Phi}{\Delta^2} = \frac{\pi}{(1-aa)^3} (1+aa) \\ \int \Delta \partial \Phi = \pi \, (1+aa) \end{cases}$$

$$n = 2 \begin{cases} \int \frac{\partial \Phi}{\Delta^3} = \frac{\pi}{(1-aa)^5} (1+2^2 aa + a^4) \\ \int \Delta^2 \partial \Phi = \pi (1+2^2 aa + a^4) \end{cases}$$

$$n = 3 \begin{cases} \int \frac{\partial \Phi}{\Delta^4} = \frac{\pi}{(1-aa)^7} (1+3^2 aa + 3^2 a^4 + a^6) \\ \int \Delta^3 \partial \Phi = \pi (1+3^2 aa + 3^2 a^4 + a^6) \end{cases}$$

$$n = 4 \begin{cases} \int \frac{\partial \Phi}{\Delta^5} = \frac{\pi}{(1-aa)^9} (1+4^2 aa + 6^2 a^4 + 4^2 a^6 + a^8) \\ \int \Delta^4 \partial \Phi = \pi (1+4^2 aa + 6^2 a^4 + 4^2 a^6 + a^8) \end{cases}$$

etc. etc.

Evolutio casuum

quibus $\lambda = 1$, et formula integralis proposita

$$\int \frac{\partial \Phi \cos. \Phi}{\Delta^{n+1}}$$

§. 32. Hoc igitur casu integrale quaesitum erit

$$\frac{\pi a}{(1-aa)^{2n+1}} \, . \, V$$

existente

$$V = \left(\frac{n-1}{0}\right)\left(\frac{n+1}{1}\right) + \left(\frac{n-1}{1}\right)\left(\frac{n+1}{2}\right) a a$$
$$+ \left(\frac{n-1}{2}\right)\left(\frac{n+1}{3}\right) a^4 + \left(\frac{n-1}{3}\right)\left(\frac{n+1}{4}\right) a^6$$
$$+ \left(\frac{n-1}{4}\right)\left(\frac{n+1}{5}\right) a^6 + \left(\frac{n-1}{5}\right)\left(\frac{n+1}{6}\right) a^8 + \text{etc.}$$

Tum vero cum ob $\lambda = 1$ fit

$$\int \Delta^n \partial \Phi \cos. \Phi : \int \frac{\partial \Phi \cos. \Phi}{\Delta^{n+1}} = n(1-aa)^n : -(n+1)(1-aa)^{-n-1}$$

unde fit

$$\int \Delta^n \partial \Phi \cos. \Phi = -\frac{n}{n+1} . \pi a V.$$

Pro casibus ergo simplicioribus ipsius n sequentem tabulam subjungamus

$$n = 0 \begin{cases} \int \frac{\partial \Phi \cos. \Phi}{\Delta} = \frac{\pi a}{1-a a} \\ \int \partial \Phi \cos. \Phi = 0 \end{cases}$$

$$n = 1 \begin{cases} \int \frac{\partial \Phi \cos. \Phi}{\Delta^2} = \frac{2 \pi a}{(1-aa)^3} \\ \int \Delta \partial \Phi \cos. \Phi = -\pi a \end{cases}$$

$$n = 2 \begin{cases} \int \frac{\partial \Phi \cos. \Phi}{\Delta^3} = \frac{\pi a}{(1-aa)^5} (1.3 + 1.3 aa) \\ \int \Delta^2 \partial \Phi \cos. \Phi = -\frac{2}{3} \pi a (1.3 + 1.3 aa) \end{cases}$$

$$n = 3 \begin{cases} \int \frac{\partial \Phi \cos. \Phi}{\Delta^4} = \frac{\pi a}{(1-aa)^7} (1.4 + 2\,6 aa + 1.4 a^4) \\ \int \Delta^3 \partial \Phi \cos. \Phi = -\frac{3}{4} \pi a (1.4 + 2.6 aa + 1.4 a^4) \end{cases}$$

$$n = 4 \begin{cases} \int \frac{\partial \Phi \cos. \Phi}{\Delta^5} = \frac{\pi a}{(1-aa)^9} (1.5 + 3.10 aa + 3.10 a^4 + 1.5.a^6) \\ \int \Delta^4 \partial \Phi \cos. \Phi = -\frac{4}{5} \pi a (1.5 + 3.10 aa + 3.10 a^4 + 1.5.a^6) \end{cases}$$

$$n = 5 \begin{cases} \int \frac{\partial \Phi \cos. \Phi}{\Delta^6} = \frac{\pi a}{(1-aa)^{11}} (1.6 + 4.15 aa + 6.20 a^4 + 4.15 a^6 + 1.6 a^8) \\ \int \Delta^5 \partial \Phi \cos. \Phi = -\frac{5}{6} \pi (1.6 + \text{etc.}) \end{cases}$$

$$n = 6 \begin{cases} \int \frac{\partial \Phi \cos. \Phi}{\Delta^7} = \frac{\pi a}{(1-aa)^{13}} (1.7 + 5.21 aa + 10.35 a^4 + 10.35 a^6 + \text{etc.}) \\ \int \Delta^6 \partial \Phi \cos. \Phi = -\frac{6}{7} \pi a (1.7 + \text{etc.}) \end{cases}$$

Evolutio casuum

quibus $\lambda = 2$, et formula integralis proposita

$$\int \frac{\partial \Phi \cos. 2 \Phi}{\Delta^{n+1}}.$$

§. 33. Hoc ergo casu integrale quaesitum erit

$$\frac{\pi a^2}{(1 - aa)^{2n+1}} . V$$

existente

$$V = \left(\frac{n-2}{0}\right)\left(\frac{n+2}{2}\right) + \left(\frac{n-2}{1}\right)\left(\frac{n+2}{3}\right) aa + \left(\frac{n-2}{2}\right)\left(\frac{n+2}{4}\right) a^4$$
$$+ \left(\frac{n-2}{3}\right)\left(\frac{n+2}{5}\right) a^6 + \left(\frac{n-2}{4}\right)\left(\frac{n+2}{6}\right) a^8 + \text{etc.}$$

tum vero erit altera forma

$$\int \Delta^n \partial \Phi \cos. 2 \Phi = \frac{n(n-1)}{(n+1)(n+2)} \pi a a V.$$

Percurramus ergo ut hactenus casus simpliciores, et quia integratio formulae $\int \Delta^n \partial \Phi \cos. 2 \Phi$ sponte patet ex ultima formula, superfluum foret haec integralia allegare

$$n = 0 : \int \frac{\partial \Phi \cos. 2 \Phi}{\Delta} = \frac{\pi a a}{1 - aa}$$

$$n = 1 : \int \frac{\partial \Phi \cos. 2 \Phi}{\Delta^2} = \frac{\pi a a}{(1 - aa)^3} (1 . 3 - 1 . 1 . aa)$$

$$n = 2 : \int \frac{\partial \Phi \cos. 2 \Phi}{\Delta^3} = \frac{\pi a a}{(1 - aa)^5} (1 . 6)$$

$$n = 3 : \int \frac{\partial \Phi \cos. 2 \Phi}{\Delta^4} = \frac{\pi a a}{(1 - aa)^7} (1 . 10 + 1 . 10\, aa)$$

$$n = 4 : \int \frac{\partial \Phi \cos. 2 \Phi}{\Delta^5} = \frac{\pi a a}{(1 - aa)^9} (1 . 15 + 2 . 20\, aa + 1 . 15\, a^4)$$

$$n = 5 : \int \frac{\partial \Phi \cos. 2 \Phi}{\Delta^6} = \frac{\pi a a}{(1 - aa)^{11}} (1 . 21 + 3 . 35\, a^2 + 3 . 35\, a^4 + 1 . 21\, a^6)$$

$$n = 6 : \int \frac{\partial \Phi \cos. 2 \Phi}{\Delta^7} = \frac{\pi a a}{(1 - aa)^{13}} (1 . 28 + 4 . 56\, aa + 6 . 70\, a^4 + 4 . 56\, a^6 - 1 . 28\, a^8)$$

etc. etc.

Evolutio casuum quibus $\lambda = 3$ et formula integralis proposita

$$\int \frac{\partial \phi \cos. 3 \phi}{\Delta^{n+1}}.$$

§. 34. Hoc ergo casu integrale erit

$$\frac{\pi a^3}{(1-aa)^{2n+1}} . V,$$

existente

$$V = \left(\frac{n-3}{0}\right)\left(\frac{n+3}{3}\right) + \left(\frac{n-3}{1}\right)\left(\frac{n+3}{4}\right) aa + \left(\frac{n-3}{2}\right)\left(\frac{n+3}{5}\right) a^4$$
$$+ \left(\frac{n-3}{3}\right)\left(\frac{n+3}{6}\right) a^6 + \text{etc.}$$

pro altera autem formula habebimus

$$\int \Delta^n \partial \phi \cos. 3 \phi = - \frac{n(n-1)(n-2)}{(n+1)(n+2)(n+3)} \pi a^3 V.$$

Pro praecipuis igitur casibus habebimus sequentem tabellam

$n = 0 : \int \frac{\partial \phi \cos. 3 \phi}{\Delta} = \frac{\pi a^3}{1-aa}$

$n = 1 : \int \frac{\partial \phi \cos. 3 \phi}{\Delta^2} = \frac{\pi a^3}{(1-aa)^3} (1.4 - 2.1 aa)$

$n = 2 : \int \frac{\partial \phi \cos. 3 \phi}{\Delta^3} = \frac{\pi a^3}{(1-aa)^5} (1.10 - 1.5 aa)$

$n = 3 : \int \frac{\partial \phi \cos. 3 \phi}{\Delta^4} = \frac{\pi a^3}{(1-aa)^7} (1.20)$

$n = 4 : \int \frac{\partial \phi \cos. 3 \phi}{\Delta^5} = \frac{\pi a^3}{(1-aa)^9} (1.35 + 1.35 aa)$

$n = 5 : \int \frac{\partial \phi \cos. 3 \phi}{\Delta^6} = \frac{\pi a^3}{(1-aa)^{11}} (1.56 + 2.70 aa + 1.56 a^4)$

$n = 6 : \int \frac{\partial \phi \cos. 3 \phi}{\Delta^7} = \frac{\pi a^3}{(1-aa)^{13}} (1.84 + 3.126 aa + 3.126 a^4 + 1.84 a^6)$

etc. etc.

Observatio circa valores negativos ipsius λ.

§. 35. Jam initio monuimus, pro littera λ tantum numeros integros positivos sumi oportere, qua conditione generalitas nostrae quaestionis non restringitur cum semper sit $\cos. -\lambda\,\phi =$ $\cos.\lambda\,\phi$. Interim tamen hic ingens paradoxon se offert, quod solutiones supra inventae evadant falsae, quando ipsi λ valores negativi tribuuntur; quod quo clarius pateat consideremus casum $n = 0$; pro quo supra invenimus

$$\int \frac{\partial\phi \cos.\lambda\phi}{\Delta} = \frac{\pi a^\lambda}{1-aa},$$

unde videtur sequi debere, casu $\lambda = -i$ fore

$$\int \frac{\partial\phi \cos. i\,\phi}{\Delta} = \frac{\pi}{a^i(1-aa)},$$

quod autem manifesto est falsum, cum verum integrale utique sit $\frac{\pi a^i}{1-aa}$, perinde ac si esset $\lambda = +i$. At vero ista restrictio tantum est apparens, atque solutio nostra generalis nihilo minus veritati est consentanea, etiamsi litterae λ valores negativi tribuantur, dummodo fuerint integri; quandoquidem perpetuo assumimus, casu $\phi = \pi$ semper esse $\sin.\lambda\phi = 0$; hoc igitur maxime operae erit pretium clarius ostendisse.

§. 36. Sufficiet autem, casum quo $n = 0$ perpendisse, pro quo nostra solutio generalis praebet

$$\int \frac{\partial\phi \cos.\lambda\phi}{\Delta} = \frac{\pi a^\lambda}{1-aa}V,$$

existente

$$V = (\tfrac{-\lambda}{0})(\tfrac{\lambda}{\lambda}) + (\tfrac{-\lambda}{1})(\tfrac{\lambda}{\lambda+1})aa + (\tfrac{-\lambda}{2})(\tfrac{\lambda}{\lambda+2})a^4$$
$$+ (\tfrac{-\lambda}{3})(\tfrac{\lambda}{\lambda+3})a^6 + \text{etc.}$$

Cujus expressionis tantum prima pars remanet, quando λ est numerus positivus integer, propterea quod tum formulae $(\frac{\lambda}{\lambda+1})$, $(\frac{\lambda}{\lambda+2})$, $(\frac{\lambda}{\lambda+3})$, etc. evanescunt; longe secus autem se res habet, quando pro λ assumitur numerus negativus, veluti si ponamus $\lambda = -i$ tum erit

$$V = \left(\frac{i}{0}\right)\left(\frac{-i}{-i}\right) + \left(\frac{i}{1}\right)\left(\frac{-i}{1-i}\right) a a + \left(\frac{i}{2}\right)\left(\frac{-i}{2-i}\right) a^4$$
$$+ \left(\frac{i}{3}\right)\left(\frac{-i}{3-i}\right) a^6 + \text{etc.}$$

ubi notetur, omnium horum characterum, quamdiu denominator est negativus, valores evanescere; quoniam vero denominatores continuo crescunt, tandem evadent positivi, atque adeo valores determinatos exhibebunt. Ad hoc ostendendum ponamus primo $\lambda = -1$ sive $i = +1$, eritque $V = -a a$ ubi primum membrum sine dubio est $= 0$, secundum vero

$$\left(\frac{1}{1}\right)\left(\frac{+1}{0}\right) a a = a a,$$

Cum igitur sit $V = a a$ casu $\lambda = -1$, nostra formula praebet hoc integrale

$$\int \frac{\partial \Phi \cos. - \Phi}{\Delta} = \frac{\pi a^{-1}}{1 - a a} \cdot a a = \frac{\pi a}{1 - a a},$$

id quod prorsus convenit.

§. 37. Sumamus nunc $\lambda = -2$ sive $i = 2$, manente $n = 0$, eritque

$$V = \left(\frac{2}{0}\right)\left(\frac{-2}{-2}\right) + \left(\frac{2}{1}\right)\left(\frac{-2}{-1}\right) a a + \left(\frac{2}{2}\right)\left(\frac{-2}{0}\right) a^4,$$

ubi sequentes termini manifesto evanescunt; ob factores priores autem bini termini initiales etiam evanescunt ob denominatores negativos; tertius autem terminus ob $\left(\frac{-2}{0}\right) = 1$ praebet $V = a^4$, consequenter casu $\lambda = -2$ habebimus

$$\int \frac{\partial \Phi \cos. - 2 \Phi}{\Delta} = \frac{\pi a^{-2}}{1 - a a} \cdot a^4 = \frac{\pi a a}{1 - a a},$$

prorsus atque invenimus pro $\int \frac{\partial \Phi \cos. 2 \Phi}{\Delta}$.

§. 38. Simili modo facile intelligitur, casu $\lambda = -3$ proditurum esse $V = a^6$, eodemque modo casu $\lambda = -4$ reperietur $V = a^8$, atque adeo in genere casu $\lambda = -i$ obtinebitur $V = a^{2i}$, sicque hujus formulae $\int \partial \Phi \frac{\cos. -i\Phi}{\Delta}$ integrale erit

$$\frac{\pi a^{-i}}{1 - aa} \cdot a^{2i} = \frac{\pi a^i}{1 - aa},$$

quod ipsum est integrale formulae $\int \frac{\partial \Phi \cos. i \Phi}{\Delta}$, uti natura rei postulat.

§. 39. Talis autem egregius consensus locum habebit pro omnibus valoribus ipsius n. Sit enim verbi gratia $n = 2$, et integratio nostra

$$\int \frac{\partial \Phi \cos. \lambda \Phi}{\Delta^3} = \frac{\pi a^\lambda}{(1 - aa)^5} \cdot V$$

existente

$$V = (\tfrac{2-\lambda}{0})(\tfrac{2+\lambda}{\lambda}) + (\tfrac{2-\lambda}{1})(\tfrac{2+\lambda}{\lambda+1})\, aa + (\tfrac{2-\lambda}{2})(\tfrac{2+\lambda}{\lambda+2})\, a^4 + \text{etc.}$$

quare sumto $\lambda = -3$, ut forma nostra sit

$$\int \frac{\partial \Phi \cos. -3\Phi}{\Delta^3} = \frac{\pi a^{-3}}{(1 - aa)^5} \cdot V,$$

existente

$$V = (\tfrac{5}{0})(\tfrac{-1}{-3}) + (\tfrac{5}{1})(\tfrac{-1}{-2})\, aa + (\tfrac{5}{2})(\tfrac{-1}{-1})\, a^4 + (\tfrac{5}{3})(\tfrac{-1}{0})\, a^6$$
$$+ (\tfrac{5}{4})(\tfrac{-1}{1})\, a^8 + (\tfrac{5}{5})(\tfrac{-1}{2})\, a^{10},$$

ubi tria priora membra evanescunt, sequentia autem ob

$$(\tfrac{-1}{0}) = 1, \quad (\tfrac{-1}{1}) = -1, \quad (\tfrac{-1}{2}) = 1, \text{ erit}$$
$$V = 10 a^6 - 5 a^8 + a^{10} = a^6 (10 - 5 aa + a^4),$$

consequenter nostrum integrale fit

$$\int \frac{\partial \Phi \cos. -3\Phi}{\Delta^3} = \frac{\pi a^3}{(1 - aa)^5} (10 - 5 aa + a^4),$$

prorsus uti supra invenimus pro casu $\int\frac{\partial\Phi\cos.3\Phi}{\Delta^3}$; talis autem consensus perpetuo deprehendi debebit.

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3) Disquisitio conjecturalis super formula integrali

$$\int\frac{\partial\Phi\cos.i\Phi}{(a+\beta\cos.\Phi)^n}.$$

M. S. Academiae exhib. die 31. *Augusti* 1778.

§. 40. Incipiamus a casu simplicissimo quo $i=0$ et $n=1$, et formula integranda proponitur haec $\int\frac{\partial\Phi}{\alpha+\beta\cos.\Phi}$, ad quod praestandum commodissime in subsidium vocatur haec substitutio tang. $\frac{1}{2}\Phi=t$, unde statim fit $\partial\Phi=\frac{2\partial t}{1+tt}$: porro vero cum hinc sit

$$\sin.\tfrac{1}{2}\Phi=\frac{t}{\sqrt{(1+tt)}} \text{ et } \cos.\tfrac{1}{2}\Phi=\frac{1}{\sqrt{(1+tt)}},$$

erit $\cos.\Phi=\frac{1-tt}{1+tt}$, ideoque denominator nostrae formulae

$$\alpha+\beta\cos.\Phi=\frac{\alpha+\beta+(\alpha-\beta)tt}{1+tt},$$

sicque nostra formula integranda erit

$$\int\frac{2\partial t}{\alpha+\beta+(\alpha-\beta)tt}.$$

§. 41. Constat autem ex elementis esse

$$\int\frac{\partial t}{f+gtt}=\frac{1}{\sqrt{fg}}\text{ Arc. tang. } t\sqrt{\frac{g}{f}}.$$

Quare cum pro nostro casu sit $f=\alpha+\beta$ et $g=\alpha-\beta$, habebimus hanc integrationem

$$\int\frac{\partial\Phi}{\alpha+\beta\cos.\Phi}=\frac{2}{\sqrt{(\alpha\alpha-\beta\beta)}}\text{ Arc. tang. } t\sqrt{\frac{\alpha-\beta}{\alpha+\beta}},$$

existente $t=$ tang. $\frac{1}{2}\Phi$; quod ergo integrale evanescit casu $t=0$, ideoque casu $\Phi=0$. Quodsi ergo hoc integrale extendere velimus