SUPPLEMENTUM II.

AD TOM. I. CAP. III.

DE

INTEGRATIONE FORMULARUM DIFFERENTIALIUM PER SERIES INFINITAS.

De resolutione formulae integralis, $\int x^{m-1} \partial x (\Delta + x^n)^\lambda$ in seriem semper convergentem. Ubi simul plura insignia artificia circa serierum summationem explicantur. *M. S. Academiae exhib. die* 12 *Aug.* 1779.

§. 1. Obtulit se mihi nuper haec formula integralis $\int \partial x \sqrt{(\Delta + x^4)}$, cujus valor, cum casu quo $\Delta = 0$ sit $\frac{1}{3} x^3$, in mentem mihi venit, eos ejus valores investigare, quos induit, quando Δ est quantitas valde parva. Mox autem vidi, hoc vulgari evolutione praestari neutiquam posse. Cum enim sit

$$\sqrt{(\Delta + x^4)} = \sqrt{\Delta} \times \left(1 + \frac{x^4}{\Delta}\right)^{\frac{1}{2}},$$

ideoque per seriem

$$\sqrt{(\Delta + x^4)} = \sqrt{\Delta} \times \left(1 + \frac{1}{2} \cdot \frac{x^4}{\Delta} - \frac{1 \cdot 1}{2 \cdot 4} \cdot \frac{x^8}{\Delta^2} + \frac{1 \cdot 1 \cdot 3}{4 \cdot 5 \cdot 6} \cdot \frac{x^{12}}{\Delta^3} - \text{etc.}\right)$$

erit valor formulae hujus integralis

$$\int \partial x \sqrt{(\Delta + x^4)} = x \sqrt{\Delta} \times \left(1 + \frac{1}{2} \cdot \frac{x^4}{5 \Delta} - \frac{1 \cdot 1}{2 \cdot 4} \cdot \frac{x^8}{9 \Delta^2} + \frac{1 \cdot 1 \cdot 3}{2 \cdot 4 \cdot 6} \cdot \frac{x^{12}}{13 \Delta^3} - \text{etc.}\right)$$

quae series ergo manifesto maxime divergit, quoties Δ fuerit quantitas valde parva, atque adeo, quoties fractio $\frac{x^n}{\Delta}$ unitatem superaverit.

§. 2. Ut igitur ad scopum propositum pertingerem, ipsam hanc quaestionem sub hac forma sum contemplatus: *Valorem formulae integralis* $\int \partial x \sqrt{(\Delta + x^4)}$ *a termino* $x = 0$ *usque ad terminum* $x = a$ *extensum per seriem semper convergentem exprimere, quicunque valor litterae* Δ *tribuatur.* Hunc in finem formulam $\Delta + x^4$ sub hac specie repraesento

$$\Delta + a^4 - (a^4 - x^4),$$

sive hac

$$(\Delta + a^4)\left(1 - \frac{a^4 - x^4}{\Delta + a^4}\right).$$

Hinc igitur erit

$$\sqrt{(\Delta + x^4)} = \sqrt{(\Delta + a^4)} \times \left[1 - \tfrac{1}{2} \cdot \frac{a^4 - x^4}{\Delta + a^4} - \frac{1.1}{2.4}\left(\frac{a^4 - x^4}{\Delta + a^4}\right)^2 - \text{etc.}\right].$$

Sicque totum negotium huc redit ut harum formularum integralium

$$\int \partial x\, (a^4 - x^4),\ \int \partial x\, (a^4 - x^4)^2,\ \int \partial x\, (a^4 - x^4)^3,\ \text{etc.}$$

valores ab $x = 0$ usque ad $x = a$ extensi investigentur, unde primus terminus $\int \partial x$ dabit a.

§. 3. Pro secundo termino habebitur integrando

$$\int \partial x\, (a^4 - x^4) = a^4 x - \tfrac{1}{5} x^5,$$

cujus valor sumto $x = a$ erit $\frac{4}{5} a^5$. Pro tertio termino erit

$$\int \partial x\, (a^4 - x^4)^2 = a^8 x - \tfrac{2}{5} a^4 x^5 + \tfrac{1}{9} x^9,$$

quae expressio posito $x = a$ abit in $\frac{4.8}{5.9} a^9$. Simili modo pro quarto termino habebimus

$$\int \partial x\, (a^4 - x^4)^3 = a^{13}\left(1 - \tfrac{3}{5} + \tfrac{3}{9} - \tfrac{1}{13}\right)^3 = \frac{4.8.12}{5.9.13} a^{13}.$$

Eodemque modo reperitur fore

$$\int \partial x\, (a^4 - x^4)^4 = \tfrac{4\,.\,8\,.\,12\,.\,16}{5\,.\,9\,.\,13\,.\,17}\, a^{17},$$

et ita porro. Hanc autem elegantem progressionis legem infra sum demonstraturus.

§. 4. His igitur valoribus substitutis, totus valor integralis quaesitus reperietur fore

$$a\sqrt{(\Delta + a^4)} \times [1 - \tfrac{1}{2}\cdot\tfrac{4}{5}\cdot \tfrac{a^4}{\Delta + a^4} - \tfrac{1\,.\,1}{2\,.\,4}\cdot\tfrac{4\,.\,8}{5\,.\,9}\left(\tfrac{a^4}{\Delta + a^4}\right)^2 - \tfrac{1\,.\,1\,.\,3}{2\,.\,4\,.\,6}\cdot\tfrac{4\,.\,8\,.\,12}{5\,.\,9\,.\,13}\left(\tfrac{a^4}{\Delta + a^4}\right)^3 - \text{etc.}].$$

Quoniam hic duplices coëfficientes occurrunt, si singulos factores priorum tam supra quam infra duplicemus, ista series contrahetur in sequentem

$$a\sqrt{(\Delta + a^4)} \times [1 - \tfrac{2}{5}\cdot \tfrac{a^4}{\Delta + a^4} - \tfrac{2\,.\,2}{5\,.\,9}\left(\tfrac{a^4}{\Delta + a^4}\right)^2 - \tfrac{2\,.\,2\,.\,6}{5\,.\,9\,.\,13}\left(\tfrac{a^4}{\Delta + a^4}\right)^3 - \text{etc.}],$$

quae series manifesto semper convergit, propterea quod non solum coëfficientes haud mediocriter decrescunt, sed etiam formula $\frac{a^4}{\Delta + a^4}$ unitate est minor.

§. 5. Jam nihil obstat quo minus loco a restituamus ipsam quantitatem variabilem x, sicque valor hujus formulae integralis $\int \partial x \sqrt{(\Delta + x^4)}$ exprimetur per sequentem seriem semper convergentem

$$x\sqrt{(\Delta + x^4)} \times [1 - \tfrac{2}{5}\cdot \tfrac{x^4}{\Delta + x^4} - \tfrac{2\,.\,2}{5\,.\,9}\left(\tfrac{x^4}{\Delta + x^4}\right)^2 - \tfrac{2\,.\,2\,.\,6}{5\,.\,9\,.\,13}\left(\tfrac{x^4}{\Delta + x^4}\right)^3 - \text{etc.}].$$

Hic casus quo ista series minime convergit, est ille ipse, quem initio commemoravimus, quo $\Delta = 0$, ipsumque integrale $= \frac{1}{3}x^3$. Posito igitur $\Delta = 0$ pervenimus ad sequentem seriem maxime notatu dignam

$$x^3\,(1 - \tfrac{2}{5} - \tfrac{2\,.\,2}{5\,.\,9} - \tfrac{2\,.\,2\,.\,6}{5\,.\,9\,.\,13} - \tfrac{2\,.\,2\,.\,6\,.\,10}{5\,.\,9\,.\,13\,.\,17} - \text{etc.}),$$

cujus adeo summam novimus esse $= \frac{1}{3}x^3$, ita ut jam habeamus hanc summationem

$$\tfrac{1}{3} = 1 - \tfrac{2}{5} - \tfrac{2}{5}\cdot\tfrac{2}{9} - \tfrac{2}{5}\cdot\tfrac{2}{9}\cdot\tfrac{6}{13} - \tfrac{2}{5}\cdot\tfrac{2}{9}\cdot\tfrac{6}{13}\cdot\tfrac{10}{17} - \text{etc.}$$

cujus demonstratio altioris indaginis videtur. Interim tamen quoniam ejus summa est cognita, veritas sequenti modo ostendi potest. Hinc enim erit

$$\tfrac{2}{5} + \frac{2 \cdot 2}{3 \cdot 9} + \frac{2 \cdot 2 \cdot 6}{5 \cdot 9 \cdot 13} + \text{etc.} = \tfrac{2}{3}.$$

quae aequatio in $\frac{5}{2}$ ducta dat

$$1 + \tfrac{2}{9} + \frac{2 \cdot 6}{9 \cdot 13} + \frac{2 \cdot 6 \cdot 10}{9 \cdot 13 \cdot 17} + \text{etc.} = \tfrac{5}{3}.$$

Transponatur hic primus terminus in alteram partem, et multiplicando per $\frac{9}{2}$ prodibit

$$1 + \tfrac{6}{13} + \frac{6 \cdot 10}{13 \cdot 17} + \frac{6 \cdot 10 \cdot 14}{13 \cdot 17 \cdot 21} + \text{etc.} = \tfrac{9}{3}.$$

Translato iterum primo termino ad alteram partem factaque multiplicatione per $\frac{13}{6}$, colligitur

$$1 + \tfrac{10}{17} + \frac{10 \cdot 14}{17 \cdot 21} + \frac{10 \cdot 14 \cdot 18}{17 \cdot 21 \cdot 25} + \text{etc.} = \tfrac{13}{3}.$$

Simili modo progrediendo prodibit

$$1 + \tfrac{14}{21} + \frac{14 \cdot 18}{21 \cdot 25} + \frac{14 \cdot 18 \cdot 22}{21 \cdot 25 \cdot 29} + \text{etc.} = \tfrac{17}{3}.$$

$$1 + \tfrac{18}{25} + \frac{18 \cdot 22}{25 \cdot 29} + \frac{18 \cdot 22 \cdot 26}{25 \cdot 29 \cdot 33} + \text{etc.} = \tfrac{21}{3}.$$

Sicque innumerabiles nacti sumus series, quarum summa est cognita, et quoniam lege aequabili ulterius progrediuntur, signum hoc certum est summam primo datam esse justam. Hanc autem insignem veritatem infra, ubi rem in genere persequemur, accuratius demonstrabimus.

Problema generale.

Formulae integralis $\int x^{m-1} \partial x (\Delta + x^n)^\lambda$ *valorem a termino* $x = 0$ *usque ad* $x = a$ *extensum per seriem semper convergentem exprimere.*

Solutio.

§. 6. Formulam $\Delta + x^n$ sub hac forma repraesentemus $\Delta + a^n - (a^n - x^n)$, quae reducitur ad hanc

$$(\Delta + a^n)\left(1 - \frac{a^n - x^n}{\Delta + a^n}\right),$$

sicque formula integralis proposita erit

$$(\Delta + a^n)^\lambda \int x^{m-1} \partial x \left(1 - \frac{a^n - x^n}{\Delta + a^n}\right)^\lambda.$$

At facta evolutione est

$$\left(1 - \frac{a^n - x^n}{\Delta + a^n}\right)^\lambda = 1 - \frac{\lambda}{1}\left(\frac{a^n - x^n}{\Delta + a^n}\right) + \frac{\lambda(\lambda - 1}{2}\left(\frac{a^n - x^n}{\Delta + a^n}\right)^2 - \text{etc.}$$

quae ergo series ducta in $x^{m-1}\partial x$ ita integrari debet, ut integrale ab $x = 0$ ad $x = a$ extendatur. Hinc patet totum negotium reduci ad hanc integrationem $\int x^{m-1}\partial x (a^n - x^n)^\theta$, cujus valor casu quo $\theta = 0$ manifesto est $\frac{x^m}{m} = \frac{a^m}{m}$. Casu vero quo $\theta = 1$ erit

$$\int x^{m-1}\partial x (a^n - x^n) = \frac{a^n x^m}{m} - \frac{x^{m+n}}{m+n},$$

qui valor, posito $x = a$, evadit $\frac{n}{m(m+n)} a^{m+n}$. Ac casu quo $\theta = 2$ erit

$$\int x^{m-1}\partial x (a^n - x^n)^2 = a^{2n}\frac{x^m}{m} - 2a^n \frac{x^{m+n}}{m+n} + \frac{x^{m+2n}}{m+2n},$$

quae expressio posito $x = a$ abit in hanc $\frac{n \cdot 2n}{m(m+n)(m+2n)} a^{m+2n}$. Simili modo calculo subducto reperietur

$$\int x^{m-1}\partial x (a^n - x^n)^3 = \frac{n \cdot 2n \cdot 3n}{m(m+n)(m+2n)(m+3n)} a^{m+3n}.$$

Ne autem hic inductioni nimium tribuamus, hanc progressionem sequenti modo accuratius demonstrabimus.

§. 7. Ponamus formulae $\int x^{m-1}\partial x (a^n - x^n)^\theta$ valorem jam esse inventum $= V$, hincque quaeramus valorem formulae sequentis $\int x^{m-1}\partial x(a^n - x^n)^{\theta+1}$. Hunc in finem ponamus

$$\int x^{m-1}\partial x(a^n - x^n)^{\theta+1} = A\int x^{m-1}\partial x(a^n - x^n)^{\theta} + Bx^m(a^n - x^n)^{\theta+1},$$

quae formula differentiata et per $x^{m-1}\partial x(a^n - x^n)^{\theta}$ divisa praebet

$$a^n - x^n = A + mB(a^n - x^n) - (\theta + 1)nBx^n;$$

unde nascuntur hae duae determinationes

$$A + mBa^n = a^n \quad \text{et} \quad mB + (\theta + 1)nB = 1,$$

qui praebent

$$A = \frac{(\theta + 1)na^n}{m + (\theta + 1)n} \quad \text{et} \quad B = \frac{1}{m + (\theta + 1)n}.$$

§. 8. Quoniam igitur post integrationem fieri debet $x = a$, membrum littera B affectum evanescit, eritque

$$\int x^{m-1}\partial x(a^n - x^n)^{\theta+1} = \frac{(\theta + 1)na^n}{m + (\theta + 1)n}. V.$$

Cum igitur casu $\theta = 0$ sit $V = \frac{a^m}{m}$, erit

$$\int x^{m-1}\partial x(a^n - x^n) = \frac{n}{m(m+n)}a^{m+n},$$

$$\int x^{m-1}\partial x(a^n - x^n)^2 = \frac{n.2n}{m(m+n)(m+2n)}a^{m+2n},$$

$$\int x^{m-1}\partial x(a^n - x^n)^3 = \frac{n.2n.3n}{m(m+n)(m+2n)(m+3n)}a^{m+3n}.$$

Unde patet ordinem supra observatum in ipsa rei natura esse fundatum.

§. 9. Quia hic integralia ita capi debent, ut evanescant posito $x = 0$, in reductione generali, qua sumus usi ubi postremum membrum erat $Bx^m(a^n - x^n)^{\theta+1}$, evidens est, hoc membrum non evanescere, nisi fuerit $m > 0$; quamobrem, si forte ejusmodi formulae occurrant, ubi exponens m fuerit vel 0 vel adeo negativus, reductiones hic inventae locum habere nequeunt.

§. 10. Singuli hi termini factorem involvunt comunem $\frac{a^m}{m}$, qui si cum multiplicatore generali conjungatur, series per integrationem orta erit

$$\frac{a^m}{m}(\Delta + a^n)^\lambda \left\{ \begin{array}{l} 1 - \frac{\lambda}{1} \cdot \frac{n}{m+n} \left(\frac{a^n}{\Delta + a^n}\right) \\ + \frac{\lambda(\lambda-1)}{1.2} \cdot \frac{m.2n}{(m+n)(m+2n)} \left(\frac{a^n}{\Delta + a^n}\right)^2 - \text{etc.} \end{array} \right\}$$

ubi coëfficientes sequenti modo contrahi poterunt

$$\frac{a^m}{m}(\Delta + a^n)^\lambda \left\{ \begin{array}{l} 1 - \frac{\lambda n}{m+n} \left(\frac{a^n}{\Delta + a^n}\right) + \frac{\lambda n}{m+n} \cdot \frac{(\lambda-1)n}{m+2n} \left(\frac{a^n}{\Delta + a^n}\right)^2 \\ - \frac{\lambda n}{m+n} \cdot \frac{(\lambda-1)n}{m+2n} \cdot \frac{(\lambda-2)n}{m+3n} \left(\frac{a^n}{\Delta + a^n}\right)^3 + \text{etc.} \end{array} \right\}.$$

Quod si jam hic loco a substituamus ipsam quantitatem variabilem x, haec series

$$\frac{x^m}{m}(\Delta + x^n)^\lambda \left\{ \begin{array}{l} 1 - \frac{\lambda n}{m+n} \left(\frac{x^n}{\Delta + x^n}\right) + \frac{\lambda n}{m+n} \cdot \frac{(\lambda-1)n}{m+2n} \left(\frac{x^n}{\Delta + x^n}\right)^2 \\ - \frac{\lambda n}{m+n} \cdot \frac{(\lambda-1)n}{m+2n} \cdot \frac{(\lambda-2)n}{m+3n} \left(\frac{x^n}{\Delta + x^n}\right)^3 + \text{etc.} \end{array} \right\}$$

exprimet valorem formulae integralis $\int x^{m-1} \partial x (\Delta + x^n)^\lambda$ a termino $x = 0$ sumtum.

§. 11. Casibus quibus exponens λ est numerus integer positivus, veritas seriei inventae sponte elucescit; uti his casibus

1°) Si $\lambda = 1$, erit

$$\int x^{m-1} \partial x (\Delta + x^n) = \frac{x^m}{m}(\Delta + x^n)\left(1 - \frac{n}{m+n} \cdot \frac{x^n}{\Delta + x^n}\right),$$

quae expressio reducitur ad hanc $\frac{x^m}{m}(\Delta + x^n - \frac{n}{m+n} x^n)$: integrale vero ordinario modo sumtum erit $\frac{\Delta x^m}{m} + \frac{x^{m+n}}{m+n}$, quod cum praecedente convenit.

2°) Si fuerit $\lambda = 2$, erit

$$\int x^{m-1}\partial x(\Delta+x^n)^2 = \frac{x^m}{m}(\Delta+x^n)^2\left[1 - \frac{2n}{m+n}\left(\frac{x^n}{\Delta+x^n}\right) + \frac{2n}{m+n}\cdot\frac{2}{m+2n}\left(\frac{x^n}{\Delta+x^n}\right)^2\right]$$

quae expressio reducitur ad hanc

$$\frac{x^m}{m}\left\{\begin{matrix} \Delta\Delta + 2\,\Delta x^n & + x^{2n} & \\ -\frac{2n}{m+n}\Delta x^n & -\frac{2n}{m+n}x^{2n} & \\ & +\frac{n\,.\,2n}{(m+n)(m+2n)}x^{2n} & \end{matrix}\right\},$$

sive ad hanc concinniorem

$$\frac{x^m}{m}\left(\Delta\Delta + \frac{2n}{m+n}\Delta x^n + \frac{m}{m+2n}x^{2n}\right)$$

quod egregie convenit cum integrali more solito sumto. Caeterum hic meminisse juvabit, haec integralia locum habere non posse, nisi m fuerit nihilo major, quia alioquin integrale non ita sumi posset, ut evanesceret casu $x = 0$.

§. 12. Sin autem exponens λ non fuerit numerus integer, series inventa in infinitum progreditur, ejusque veritas non amplius in oculos incurrit. His autem casibus forma nostri integralis simplicior et concinnior reddetur, si statuamus $\lambda = -\frac{\mu}{n}$; tum enim hujus formulae $\int x^{m-1}\partial x(\Delta + x^n)^{-\frac{\mu}{n}}$ integrale erit

$$\frac{x^m}{m(\Delta+x^n)^{\frac{\mu}{n}}}\left\{\begin{matrix} 1 + \frac{\mu}{m+n}\left(\frac{x^n}{\Delta+x^n}\right) + \frac{\mu}{m+n}\cdot\frac{\mu+n}{m+2n}\left(\frac{x^n}{\Delta+x^n}\right)^2 \\ + \frac{\mu}{m+n}\cdot\frac{\mu+n}{m+2n}\cdot\frac{\mu+2n}{m+3n}\left(\frac{x^n}{\Delta+x^n}\right)^3 + \text{etc.} \end{matrix}\right\}.$$

§. 13. Hinc jam summam hujus seriei generalis assignare licebit

$$1 + \frac{a}{b}\chi + \frac{a}{b}\cdot\frac{a+n}{b+n}\chi^2 + \frac{a}{b}\cdot\frac{a+n}{b+n}\cdot\frac{a+2n}{b+2n}\chi^3 + \text{etc.}$$

Si enim hanc seriem cum inventa comparemus, erit $\mu = a$, et $m + n = b$, ideoque $m = b - n$; tum vero erit $\chi = \frac{x^n}{\Delta + x^n}$, unde relatio inter χ et x innotescit. Tum igitur hujus seriei summa aequabitur huic formulae integrali $\int \frac{x^{b-n-1}\partial x}{(\Delta + x^n)^{\frac{a}{n}}}$ divisae per hanc quantitatem $\frac{x^{b-n}}{(b-n)(\Delta + x^n)^{\frac{a}{n}}}$; ideoque ista summa erit

$$\frac{(b-n)(\Delta + x^n)^{\frac{a}{n}}}{x^{b-n}} \cdot \int \frac{x^{b-n-1}\partial x}{(\Delta + x^n)^{\frac{a}{n}}},$$

quae autem summa subsistere nequit, nisi fuerit $b > n$. Caeterum evidens est, istam seriem semper esse convergentem, cum non solum fractio $\frac{x^n}{\Delta + x^n}$ sit unitate minor, sed etiam coëfficientes omnes sint uninate minores.

§. 14. Casus autem maxime memorabilis, qui hic occurrit, est quando $\Delta = 0$; tum enim nostra formula integralis erit

$$\int x^{m-\mu-1}\partial x = \frac{x^{m-\mu}}{m-\mu},$$

huic ergo quantitati semper aequabitur sequens series

$$\frac{x^{m-\mu}}{m}\left(1 + \frac{\mu}{m+n} + \frac{\mu}{m+n}\cdot\frac{\mu+n}{m+2n} + \frac{\mu}{m+n}\cdot\frac{\mu+n}{m+2n}\cdot\frac{\mu+2n}{m+3n} + \text{etc.}\right),$$

si modo fuerit m numerus positivus, uti jam aliquoties est animadversum. Consequenter hujus seriei

$$1 + \frac{\mu}{m+n} + \frac{\mu(\mu+n)}{(m+n)(m+2n)} + \frac{\mu(\mu+n)(\mu+2n)}{(m+n)(m+2n)(m+3n)} + \text{etc.}$$

summa est $\frac{m}{m-\mu}$, quae summatio est eo magis notatu digna, quod vix ulla via patet, ejus veritatem investigandi.

§. 15. Statim autem apparet, hanc summam subsistere non posse, nisi tam n quam $m - \mu$ fuerit numerus positivus. Cum enim formula nostra integralis casu $\Delta = 0$ sit $\int x^{m-1-\mu}\partial x$, quam ab $x = 0$ inchoari oportet, evidens est hoc fieri non posse, nisi $m - \mu$ fuerit numerus positivus; praeterea etiam notandum est, exponentem n necessario positivum esse debere. Cum enim in Analysi supra exposita hoc integrale occurrat $\int x^{m-1}\partial x\,(a^n - x^n)^\theta$, manifestum est, si n esset numerus negativus, integrationem non ita institui posse, ut casu $x = 0$ evanescat. His notatis istam seriem accuratius sum contemplaturus et quoniam ejus indoles non parum abscondita videtur, ejus veritatem duplici modo sum ostensurus. Primo scilicet ostendam, summam assignatam revera aequari summae totius progressionis; deinde analysin prorsus singularem apperiam, quae non solum directe ad ipsam hanc seriem perducet, sed etiam ejus summam indicabit.

Demonstratio hujus summationis:

$$\frac{m}{m-\mu} = 1 + \frac{\mu}{m+n} + \frac{\mu}{m+n}\cdot\frac{\mu+n}{m+2n} + \frac{\mu}{m+n}\cdot\frac{\mu+n}{m+2n}\cdot\frac{\mu+2n}{m+3n} + \text{etc.}$$

§. 16. Hic scilicet ostendam, si omnes hujus seriei termini a summa inventa $\frac{m}{m-\mu}$ successive subtrahantur, tandem revera nihil relictum iri. Subtracto enim primo termino 1 remanet $\frac{\mu}{m-\mu}$. Hinc terminus secundus ablatus relinquet $\frac{\mu(\mu+n)}{(m-\mu)(m+n)}$. Hinc porro subtrahatur tertius terminus ac remanebit

$$\frac{\mu(\mu+n)(\mu+2n)}{(m-\mu)(m+n)(m+2n)}.$$

Hinc jam quartus terminus ablatus residuum praebet sequens

$$\frac{\mu(\mu+n)(\mu+2n)(\mu+3n)}{(m-\mu)(m+n)(m+2n)(m+3n)}.$$

Unde jam satis liquet, omnibus terminis ablatis tandem remansurum esse hoc productum in infinitum excurrens

$$\frac{\mu(\mu+n)(\mu+2n)(\mu+3n)(\mu+4n)(etc.}{(m-\mu)(m+n)(m+2n)(m+3n)(m+4n)(etc.}.$$

§. 17. Facile autem intelligitur valorem hujus producti revera nihilo esse aequalem. Omisso enim primo factore $\frac{\mu}{m-\mu}$, omnes reliqui factores sunt fractiones unitate minores, quia $\mu < m$, et quoniam tam numeratores quam denominatores in arithmetica progressione increscunt, jam satis constat, valorem talis producti revera evanescere. Hic autem probe tenendum est, ut productum infinitarum talium fractionum in nihilum abeat, non sufficere, ut singulae fractiones sint unitate minores, veluti evenit in hac forma

$$\frac{3}{4}\cdot\frac{8}{9}\cdot\frac{15}{16}\cdot\frac{24}{23}\cdot\frac{35}{36}\cdot\frac{48}{49}\cdot \text{etc.}$$

cujus producti in infinitum protensi valor facile ostenditur esse $=\frac{1}{2}$.

§. 18. Quoniam in nostro producto singuli denominatores superant suos numeratores eadem quantitate $m=\mu$, istam formam generaliorem considerabo

$$\frac{a}{a+\Delta}\cdot\frac{b}{b+\Delta}\cdot\frac{c}{c+\Delta}\cdot\frac{d}{d+\Delta}\cdot\frac{e}{e+\Delta}\cdot \text{etc.}$$

et perscrutabor, sub quibusnam conditionibus ejus valor in infinitum extensus, qui sit Π, revera in nihilum sit abiturus. Evidens autem est, hoc evenire, si eadem forma inversa

$$\frac{1}{\Pi} = \frac{a+\Delta}{a} \cdot \frac{b+\Delta}{b} \cdot \frac{c+\Delta}{c} \cdot \frac{d+\Delta}{d} \cdot \text{etc.}$$

in infinitum excrescit. Sin autem ejus valor fuerit infinitus, etiam ejus logarithmus infinitus evadat necesse est. Cum igitur sit

$$l\frac{1}{\Pi} = l\frac{a+\Delta}{a} + l\frac{b+\Delta}{b} + l\frac{c+\Delta}{c} + l\frac{d+\Delta}{d} + \text{etc.}$$

facta evolutione reperietur

$$\begin{aligned} l\frac{1}{\Pi} = & \Delta \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{1}{e} + \text{etc.}\right) \\ & - \tfrac{1}{2}\Delta^2 \left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} + \frac{1}{d^2} + \frac{1}{e^2} + \text{etc.}\right) \\ & + \tfrac{1}{3}\Delta^3 \left(\frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{c^3} + \frac{1}{d^3} + \frac{1}{e^3} + \text{etc.}\right) \\ & - \text{etc.} \end{aligned}$$

quae expressio semper erit infinita, quoties summa primae seriei $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} +$ etc. fuerit infinita. Hanc autem summam semper esse infinitam, quoties numeri a, b, c, d, etc. in progressione arithmetica crescunt, jam notum est et per se perspicuum, quod cum in nostra serie contingat, certum est, illius producti infiniti valorem esse evanescentem.

§. 19. Circa nostram autem seriem id imprimis notatu dignum occurrit, quod ejus summa $\frac{m}{m-\mu}$ litteram n non involvat, ita ut ejus summa semper maneat eadem, quicunque valores litterae n tribuantur, quod quidem pro casu $n = 0$ per se statim fit manifestum, quandoquidem tum series nostra evadit

$$1 + \frac{\mu}{m} + \frac{\mu^2}{m^2} + \frac{\mu^3}{m^3} + \text{etc.}$$

quae cum sit progressio geometrica, ejus summa erit $\frac{m}{m-\mu}$. Quod vero summa perpetuo maneat eadem, quicunque valores ipsi n tribuantur, non tam facile perspicitur. etsi veritas a nobis jam sit demonstrata.

§. 20. Quin etiam demonstratio hic tradita multo latius patet, cum adeo in eadem forma valores ipsius n variare liceat. Ita si posito α loco n, pro ejus multiplis $2n$, $3n$, $4n$, $5n$, etc. scribamus novas litteras β, γ, δ, ε, etc. ut habeatur ista series

$$1 + \frac{\mu}{m+\alpha} + \frac{\mu}{m+\alpha} \cdot \frac{\mu+\alpha}{m+\beta} + \frac{\mu}{m+\alpha} \cdot \frac{\mu+\alpha}{m+\beta} \cdot \frac{\mu+\beta}{m+\gamma} + \text{etc.}$$

ejus summa etiam nunc erit $\frac{m}{m-\mu}$. Subtracto enim termino primo remanet $\frac{\mu}{m-\mu}$. Hinc terminus secundus subtractus relinquit

$$\frac{\mu(\mu+\alpha)}{(m-\mu)(m+\alpha)}.$$

Hinc tertius terminus subtractus

$$\frac{\mu(\mu+\alpha)(\mu+\beta)}{(m-\mu)(m+\alpha)(m+\beta)}.$$

Unde jam patet, in infinitum tandem prodiri productum

$$\frac{\mu(\mu+\alpha)(\mu+\beta)(m+\gamma)(m+\delta)(etc.}{(m-\mu)(m+\alpha)(m+\beta)(m+\gamma)(m+\delta)(etc.},$$

cujus valor semper erit evanescens, si modo haec series

$$\frac{1}{\mu+\alpha} + \frac{1}{\mu+\beta} + \frac{1}{\mu+\gamma} + \frac{1}{\mu+\delta} + \text{etc.}$$

habuerit summam infinite magnam, uti modo ante ostendimus.

Analysis singularis directe ad seriem supra inventam perducens.

§. 21. Ponamus

$$x^m(1-x^n)^\theta = A\int x^{m-1}\partial x(1-x^n)^\theta + B\int x^{m-1}\partial x(1-x^n)^{\theta-1},$$

et reperietur $A = m + \theta n$ et $B = -\theta n$; hinc ergo si ponamus

$$\int x^{m-1}\partial x(1-x^n)^\theta = P \text{ et } \int x^{m-1}\partial x(1-x^n)^{\theta-1} = Q,$$

erit

$$x^m (1 - x^n)^\theta = (m + \theta n) P - \theta n Q, \text{ ideoque}$$

$$Q = \frac{m + \theta n}{\theta n} \cdot P - \frac{1}{\theta n} x^m (1 - x^n)^\theta.$$

Quodsi jam ambo integralia P et Q a termino $x = 0$ usque ad $x = 1$ extendamus, erit $Q = \frac{m + \theta n}{\theta n} \times P$; si modo fuerit tam $m > 0$ quam $\theta > 0$.

§. 22. Cum jam sit $\partial Q = \frac{\partial P}{1 - x^n}$, denominatore in seriem evoluto erit

$$\partial Q = \partial P (1 + x^n + x^{2n} + x^{3n} + x^{4n} + \text{etc.}):$$

consequenter habebitur

$$Q = P + \int x^n \partial P + \int x^{2n} \partial P + \int x^{3n} \partial P + \text{etc.}$$

quae singula integralia ita sunt comparata, ut quodlibet ad praecedens reduci possit, ope hujus reductionis

$$x^\alpha (1 - x^n)^{\theta + 1} = A \int x^{\alpha + n - 1} \partial x (1 - x^n)^\theta + B \int x^{\alpha - 1} \partial x (1 - x^n)^\theta,$$

pro qua reperitur $A = -\alpha - n(\theta + 1)$ et $B = \alpha$.

§. 23. Si etiam haec duo integralia a termino $x = 0$ usque ad $x = 1$ extendantur, fiet

$$0 = -[\alpha + n(\theta + 1)] \int x^{\alpha + n - 1} \partial x (1 - x^n)^\theta + \alpha \int x^{\alpha - 1} \partial x (1 - x^n)^\theta,$$

si modo fuerit $\alpha > 0$ et $\theta + 1 > 0$. Faciamus nunc $\alpha = m + \lambda n$, et quia ante posueramus $x^{m-1} \partial x (1 - x^n)^\theta = \partial P$, haec aequatio abibit in hanc formam

$$-[\alpha + n(\theta + 1)] \int x^{(\lambda + 1) n} \partial P + \alpha \int x^{\lambda n} \partial P,$$

quocirca habebimus hanc reductionem

$$\int x^{\lambda n + n} \partial P = \frac{\alpha}{\alpha + n(\theta + 1)} \int x^{\lambda n} \partial P = \frac{m + \alpha n}{m + n(\lambda + \theta + 1)} \int x^{\lambda n} \partial P.$$

§. 24. Haec formula generalis nobis jam suppeditat sequentes integrationes speciales

1°) Si $\lambda = 0$ | $\int x^n \partial P = \frac{m}{m+n(\theta+1)} P$,

2°) Si $\lambda = 1$ | $\int x^{2n} \partial P = \frac{m+n}{m+n(\theta+2)} \int x^n \partial P$, ideoque

$\int x^{2n} \partial P = \frac{m}{m+n(\theta+1)} \cdot \frac{m+n}{m+n(\theta+2)} P$,

3°) Si $\lambda = 3$ | $\int x^{3n} \partial P = \frac{m+2n}{m+n(\theta+3)} \int x^{2n} \partial P$, ideoque

$\int x^{3n} \partial P = \frac{m}{m+n(\theta+1)} \cdot \frac{m+n}{m+n(\theta+2)} \cdot \frac{m+2n}{m+n(\theta+3)} P$,

4°) Si $\lambda = 4$ | $\int x^{4n} \partial P = \frac{m}{m+n(\theta+1)} \cdot \frac{m+n}{m+n(\theta+2)} \cdot \frac{m+2n}{m+n(\theta+3)} \cdot \frac{m+3n}{m+n(\theta+4)} P$.

etc. | etc.

§. 25. Cum igitur ex superioribus fuisset

$$Q = \int \partial P (1 + x^n + x^{2n} + x^{3n} + x^{4n} + \text{etc.}),$$

si pro singulis terminis valores modo inventos substituamus, atque utrinque per P dividamus, nanciscemur hanc aequationem

$$\frac{Q}{P} = 1 + \frac{m}{m+n(\theta+1)} + \frac{m}{m+n(\theta+1)} \cdot \frac{m+n}{m+n(\theta+2)} + \frac{m}{m+n(\theta+1)} \cdot \frac{m+n}{m+n(\theta+2)} \cdot \frac{m+2n}{m+n(\theta+3)} + \text{etc.}$$

supra autem ostendimus esse $\frac{Q}{P} = \frac{m+\theta n}{\theta n}$, quae ergo fractio est summa istius seriei infinitae.

§. 26. Ut jam consensum hujus seriei cum supra inventa ostendamus, primo loco θ scribamus $\frac{\mu}{n}$, atque series nostra inventa hanc induet formam

$$\frac{m+\mu}{\mu} = \frac{m}{m+n+\mu} + \frac{m}{m+n+\mu} \cdot \frac{m+n}{m+2n+\mu} + \frac{m}{m+n+\mu} \cdot \frac{m+n}{m+2n+\mu} \cdot \frac{m+2n}{m+3n+\mu} + \text{etc.}$$

cujus veritas eodem modo quo ante fueram usus, demonstrari potest. Si enim a summa subtrahatur terminus primus relinquitur $\frac{m}{\mu}$. Subtracto hinc termino secundo remanet $\frac{m(m+n)}{\mu(m+n+\mu)}$. Hinc porro tertius terminus subtractus relinquit $\frac{m(m+n)(m+2n)}{\mu(m+n+\mu)(m+2n+\mu)}$ et ita porro, quae operatio si in infinitum continuetur, producti hujus resultantis valor est $= 0$. Tum vero evidens est, seriem hanc inventam in eam ipsam quam supra dedimus transmutari, si hic loco m scribatur μ, at vero $m - \mu$ loco μ.

§. 27. Coronidis loco hic subjungam seriem multo generaliorem ejusdem generis, cujus summam pariter assignare licet, quam sequenti problemate sum complexurus.

Problema 1.

§. 28. *Proposita hac serie* $A + B\frac{\alpha}{a} + C\frac{\alpha\beta}{ab} + D\frac{\alpha\beta\gamma}{abc} + etc.$ *investigare conditiones, sub quibus ejus summam assignare liceat.*

Solutio.

Haec ergo series involvit ternas series: primam litterarum α, β, γ, δ, etc. quae numeratores seriei propositae constituunt; secundam litterarum a, b, c, d, etc. ex quibus denominatores formantur; tertiam litterarum A, B, C, D, etc. quae coëfficientes terminorum exhibent. Quemadmodum igitur ternae istae series comparatae esse debeant, ut seriei propositae summam per expressionem finitam atque adeo rationalem assignare liceat, hic investigabo.

§. 29. Statuamus hujus seriei summam esse $\frac{S}{t}$, atque eadem methodo utamur quam jam supra adhibuimus, scilicet ab hac summa primo subtrahamus primum terminum A et cum remaneat $\frac{S - At}{t}$, statuamus $S - At = \alpha$, ut habeamus $\frac{\alpha}{t}$; hinc subtrahamus secundum terminum $B\frac{\alpha}{a}$, et residuum erit $\frac{\alpha(a - Bt)}{t.a}$. Hic jam faciamus $a - Bt = \beta$, ut habeamus $\frac{\alpha\beta}{t.a}$; unde si subtrahatur tertius terminus $C\frac{\alpha\beta}{ab}$, residuum erit $\frac{\alpha\beta(b - Ct)}{t.ab}$. Fiat hic $b - Ct = \gamma$, ut habeamus $\frac{\alpha\beta\gamma}{tab}$, unde terminus quartus ablatus relinquit $\frac{\alpha\beta\gamma(c - Dt)}{t.abc}$. Fiat hic iterum $c - Dt = \delta$, ut habeamus $\frac{\alpha\beta\gamma\delta}{t.abc}$, unde quintum terminum subtrahendo colligitur $\frac{\alpha\beta\gamma\delta(d-Et)}{t\,.\,abcd}$. Haecque operationes in infinitum continuari intelligantur,

§. 30. Ex his igitur determinationibus tam littera S quam litterae a, b, c, d, etc. sequenti modo definientur

$$S = \alpha + At;\quad a = \beta + Bt;\quad b = \gamma + Ct;\quad c = \delta + Dt;\ \text{etc.}$$

Atque his valoribus introductis residuum, postquam omnes seriei termini fuerint a formula $\frac{S}{t}$ ablati, remanebit hoc productum in infinitum excurrens $\frac{\alpha\beta\gamma\delta\varepsilon\zeta\ \text{etc.}}{t.\,abcdef\ \text{etc.}}$, quod ergo productum si in nihilum abeat, tum summa seriei propositae revera erit $= \frac{S}{t}$. Videamus igitur sub quibusnam conditionibus hoc productum evanescat.

§. 31. Designemus hoc productum littera Π, ut substitutis pro a, b, c, etc. valoribus inventis erit

$$\Pi = \frac{S}{t}\left(\frac{\alpha}{\alpha + At} \cdot \frac{\beta}{\beta + Bt} \cdot \frac{\gamma}{\gamma + Ct} \cdot \frac{\delta}{\delta + Dt} \cdot \text{etc}\right)$$

ubi scilicet factorem $\frac{S}{t}$ praefiximus. Nunc igitur quaeritur sub quibusnam conditionibus istud productum in infinitum continuatum in nihilum sit abiturum. Evidens autem est hoc evenire, si productum istud invertatur, ejusque logarithmus eveniat infinite magnus. Hoc ergo locum inveniet, quando summa horum logarithmorum

$$l\left(1 + \frac{At}{\alpha}\right) + l\left(1 + \frac{Bt}{\beta}\right) + l\left(1 + \frac{Ct}{\gamma}\right) + l\left(1 + \frac{Dt}{\delta}\right) + \text{etc.} = \infty;$$

id quod semper continget, si sumtis tantum primis terminis, qui omnes factorem comunem habent t, series haec

$$\frac{A}{\alpha} + \frac{B}{\beta} + \frac{C}{\gamma} + \frac{D}{\delta} + \text{etc.}$$

habuerit summam infinite magnam, tum igitur nostrae seriei propositae summa semper erit $\frac{\alpha + At}{t}$.

§. 32. Neque vero absolute necesse est, ut productum Π penitus evanescat, sed quemcunque habuerit valorem scilicet Π, quoniam is oritur postquam tota summa seriei propositae, quam ponamus $= S$, ablata fuerit a formula $\frac{S}{t}$, ita ut sit $\Pi = \frac{S}{t} - S$, unde manifesto fit $S = \frac{S}{t} - \Pi$.

§. 33. Ut hoc exemplo illustramus, litteris $\alpha, \beta, \gamma, \delta$, etc. hos tribuemus valores $\alpha = 3$, $\beta = 15$, $\gamma = 35$, $\delta = 63$, etc. praeterea vero sit $t = 1$, atque insuper $A = B = C = D = \text{etc.} = 1$: hinc ergo determinationes inventae praebebunt

$$S = 4, \; a = 16, \; b = 36, \; c = 64, \; d = 100, \text{ etc.}$$

Sicque series nostra jam erit

$$1 + \frac{3}{16} + \frac{3 \cdot 15}{16 \cdot 36} + \frac{3 \cdot 15 \cdot 35}{16 \cdot 36 \cdot 64} + \frac{3 \cdot 15 \cdot 35 \cdot 63}{16 \cdot 36 \cdot 64 \cdot 100} + \text{etc.}$$

pro cujus summa notetur esse $\Pi = 4 \cdot \frac{3 \cdot 15 \cdot 35 \cdot 63 \cdot 99}{4 \cdot 16 \cdot 36 \cdot 64 \cdot 100}$ etc. Constat autem ex quadratura circuli *Wallisiana* esse $\frac{3 \cdot 15 \cdot 35 \cdot 63 \cdot 99 \text{ etc}}{4 \cdot 16 \cdot 36 \cdot 64 \cdot 100 \text{ etc.}} = \frac{2}{\pi}$, existente π peripheria circuli cujus diameter est unitas. Hinc ergo erit $\Pi = \frac{8}{\pi}$, ideoque summa nostrae seriei $S = 4 - \frac{8}{\pi}$, ideoque proxime $\frac{16}{11}$.

§. 34. At vero series generalis, quam hoc modo sumus adepti, maxime est faecunda in formatione innumerabilium serierum specialium, cum non modo tam seriem litterarum $\alpha, \beta, \gamma, \delta$, etc. sed etiam litterarum A, B, C, D, etc. prorsus pro lubitu assumere liceat, quandoquidem inde litterae a, b, c, d, etc. sponte determinantur; tum autem talium serierum summam semper assignare licebit, si modo valor producti in infinitum excurrentis, quod littera Π indicavimus definiri poterit, ubi perinde est, utrum iste valor fuerit rationalis sive adeo transcendens quadraturam quamcunque involvens.