$$\int \frac{x^{n+1}\partial x}{\sqrt{(aa-2bx+cxx)}} = \left[\frac{(2n+1)bN}{(n+1)c} - \frac{naaM}{(n+1)c}\right]\Delta$$
$$- \frac{(2n+1)b\mathfrak{N}}{(n+1)c} + \frac{naa\mathfrak{M}}{(n+1)c}.$$

Hoc igitur modo has integrationes, quousque libuerit, continuare licet, dum ex binis quibusque sequens ope hujus regulae formatur, ita ut omnia haec integralia vel a logarithmis vel ab arcubus circularibus pendeant, prouti coëfficiens c fuerit vel positivus vel negativus. Manifestum autem est istos valores assignari non posse, nisi exponens n fuerit numerus integer positivus.

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3) De integratione formulae $\int \frac{\partial x \sqrt{(1+x^4)}}{1-x^4}$, aliarumque ejusdem generis, per logarithmos et arcus circulares. *M. S. Academiae exhib. die* 16 *Sept.* 1776.

§. 56. Cum mihi non ita pridem contigisset, integrale hujus formulae $\int \frac{\partial x \sqrt{(1+x^4)}}{1-x^4}$ per arcum circularem et logarithmum exprimere, haec integratio eo magis mihi visa est notatu digna, quod nullo modo perspiciebam, eam ad rationalitatem perduci posse, quandoquidem certum est, istam formulam, quae simplicior videatur, $\int \partial x \sqrt{(1+x^4)}$, neutiquam ad rationalitatem revocari posse, neque enim videbam, accessionem denominatoris $1-x^4$ hanc reductionem promovere posse, hincque concludebam dari ejusmodi formulas differentiales irrationales, quarum integralia per logarithmos et arcus circulares exhibere liceat, etiamsi nulla substitutione ab irrationalitate liberari queant: quaequidem conclusio utique valet pro formulis compositis, quanquam enim istae formulae

$$\int \frac{\partial x}{\sqrt[3]{(1+x^3)}} \text{ et } \int \frac{\partial x}{\sqrt[4]{(1+x^4)}}$$

ad rationalitatem reduci possunt, tamen formula ex iis composita

$$\int \partial x \left[\frac{A}{\sqrt[3]{(1+x^3)}} + \frac{B}{\sqrt[4]{(1+x^4)}}\right]$$

per nullam plane substitutionem ad aliam formulam rationalem reduci potest; propterea quod utraque pars peculiarem substitutionem postulat.

§. 57. Interim tamen cum formulam propositam

$$\int \frac{\partial x \sqrt{(1+x^4)}}{1-x^4} = S$$

attentius essem contemplatus, inveni, eam ab irrationalitate liberari posse, ope hujus substitutionis prorsus singularis

$$x = \frac{\sqrt{(1+tt)}+\sqrt{(1-tt)}}{t\sqrt{2}}.$$

Hinc enim fit

$$\partial x = -\frac{\partial t}{tt\sqrt{2(1+tt)}} - \frac{\partial t}{tt\sqrt{2(1-tt)}},$$

quae duae partes ad eundem denominatorem reductae dant

$$\partial x = -\frac{\partial t}{tt\sqrt{2(1-t^4)}}[\sqrt{(1-tt)}+\sqrt{(1+tt)}].$$

Cum igitur sit

$$\sqrt{(1+tt)}+\sqrt{(1-tt)} = tx\sqrt{2},$$

hoc valore substituto fiet

$$\partial x = -\frac{x\,\partial t}{t\sqrt{(1-t^4)}},$$

ita ut sit

$$\partial S = -\frac{x\,\partial t\sqrt{(1+x^4)}}{t(1-x^4)\sqrt{(1-t^4)}}.$$

§. 58. Porro autem sumtis quadratis erit

$$xx = \frac{1 + \sqrt{(1 - t^4)}}{tt},$$

unde colligimus

$$1 + xx = \frac{1 + tt + \sqrt{(1 - t^4)}}{tt} = \frac{\sqrt{(1 + tt)}}{tt} [\sqrt{(1 + tt)} + \sqrt{(1 - tt)}],$$

sicque ob

$$\sqrt{(1 + tt)} + \sqrt{(1 - tt)} = tx\sqrt{2}, \text{ erit}$$

$$1 + xx = \frac{x\sqrt{2}(1 + tt)}{t}.$$

Simili modo erit

$$1 - xx = -\left(\frac{1 - tt + \sqrt{(1 - t^4)}}{tt}\right)$$

$$= -\frac{\sqrt{(1 - tt)}}{tt} [\sqrt{(1 - tt)} + \sqrt{(1 + tt)}] = -\frac{x\sqrt{2}(1 - tt)}{t}.$$

Hinc igitur sequitur fore

$$1 - x^4 = -\frac{2xx\sqrt{(1 - t^4)}}{tt},$$

qui valor in nostra formula substitutus praebet

$$\partial S = + \frac{t\partial i \sqrt{(1 + x^4)}}{2x(1 - t^4)}.$$

§. 59. Deinde sumtis quadratis habebimus

$$(1 + xx)^2 = \frac{2xx(1 + tt)}{tt} \text{ et}$$

$$(1 - xx)^2 = \frac{2xx(1 - tt)}{tt},$$

quibus additis prodibit

$$(1 + xx)^2 + (1 - xx)^2 = 2(1 + x^4) = \frac{4xx}{tt},$$

unde fit

$$\sqrt{(1 + x^4)} = \frac{x\sqrt{2}}{t};$$

quo valore substituto nostra formula abit in hanc

$$\partial S = \frac{1}{\sqrt{2}} \cdot \frac{\partial t}{1 - t^4};$$

quae ergo formula est rationalis et solam variabilem t complectitur.

§. 60. Cum igitur porro sit

$$\frac{1}{1-t^4} = \frac{1}{2} \cdot \frac{1}{1+tt} + \frac{1}{2} \cdot \frac{1}{1-tt},$$

tum vero integrando reperiatur

$$\int \frac{\partial t}{1+tt} = \text{Arc. tang. } t, \quad \text{et}$$

$$\int \frac{\partial t}{1-tt} = \frac{1}{2} l \frac{1+t}{1-t} = l \frac{1+t}{\sqrt{1-tt}},$$

quibus valoribus substitutis reperietur

$$S = \frac{1}{2\sqrt{2}} \text{Arc. tang. } t + \frac{1}{2\sqrt{2}} l \frac{1+t}{\sqrt{(1-tt)}}.$$

Quare cum regrediendo sit $t = \frac{x\sqrt{2}}{\sqrt{(1+x^4)}}$, supra autem invenerimus

$$1 + x^4 = \frac{2xx}{tt}, \quad \text{erit} \quad tt = \frac{2xx}{1+x^4},$$

hincque

$$1 - tt = \frac{(1-xx)^2}{1+x^4}, \quad \text{ideoque} \quad \sqrt{(1-tt)} = \frac{1-xx}{\sqrt{(1+x^4)}},$$

his valoribus substitutis, integrale quaesitum per ipsam variabilem x sequenti modo exprimetur

$$\int \frac{\partial x \sqrt{(1+x^4)}}{1-x^4} = \frac{1}{2\sqrt{2}} \text{Arc. tang.} \frac{x\sqrt{2}}{\sqrt{(1+x^4)}} + \frac{1}{2\sqrt{2}} l \frac{x\sqrt{2}+\sqrt{(1+x^4)}}{1-xx}.$$

§. 61. Hic autem merito quaeretur, quonam artificio ad substitutionem illam, quae primo intuitu a scopo prorsus aliena videtur pertigerim? quandoquidem nemo certe in eam incidisset, neque etiam ipse memini, quanam ratione ad eam sim perductus. Verum postquam omnia momenta accuratius perpendissem, methodum multo planiorem detexi, qua istud negotium sine tot ambagibus absolvi potest, quam igitur hic perspicue proponi conveniet.

Methodus planior et magis naturalis, formulam integralem propositam tractandi.

§. 62. Quo ex formula $\partial S = \frac{\partial x \sqrt{(1+x^4)}}{1-x^4}$ irrationalitatem saltem apparenter tollamur, ponamus $\sqrt{(1+x^4)} = px$, ut fiat $\partial S = \frac{px\,\partial x}{1-x^4}$. Cum igitur sit $1 + x^4 = ppxx$, erit radicem extrahendo

$$xx = \tfrac{1}{2}pp + \sqrt{(\tfrac{1}{4}p^4 - 1)}.$$

Ponatur hic $\frac{1}{2}pp = q$, ut habeamus

$$xx = q + \sqrt{(qq - 1)}, \text{ et}$$
$$2lx = l\,[q + \sqrt{(qq - 1)}],$$

hincque differentiando $\frac{2\,\partial x}{x} = \frac{\partial q}{\sqrt{(qq-1)}}$: ergo loco q restituto valore $\frac{1}{2}pp$, erit $\frac{2\,\partial x}{x} = \frac{2p\,\partial p}{\sqrt{(p^4-4)}}$, sicque fiet $\partial x = \frac{xp\,\partial p}{\sqrt{(p^4-4)}}$, quo valore substituto fit $\partial S = \frac{p^2 x^2\,\partial p}{(1-x^4)\sqrt{(p^4-4)}}$.

§. 63. Ut nunc hinc quantitatem x penitus ejiciamus, quoniam invenimus

$$xx = \frac{pp + \sqrt{(p^4-4)}}{2}, \text{ erit}$$
$$x^4 = \frac{p^4 - 2 + pp\sqrt{(p^4-4)}}{2}, \text{ hincque}$$
$$1 - x^4 = \frac{4 - p^4 - pp\sqrt{(p^4-4)}}{2} = -\frac{\sqrt{(p^4-4)}\,[pp + \sqrt{(p^4-4)}]}{2}.$$

Unde colligitur fore $\frac{xx}{1-x^4} = -\frac{1}{\sqrt{(p^4-4)}}$, quo valore substituto impetramus formulam differentialem rationalem per novam variabilem p expressam, quae est

$$\partial S = -\frac{pp\,\partial p}{p^4-4}, \text{ existente } p = \frac{\sqrt{(1+x^4)}}{x};$$

unde idem integrale, quod ante nacti sumus, deducitur. Similis autem substitutio cum successu adhiberi potest in formulis integralibus multo magis generalibus; veluti in sequente problemate ostendemus.

Problema 16.

§. 64. *Propositam formulam integralem* $S = \int \frac{\partial x \sqrt{(a + bxx + cx^4)}}{a - cx^4}$ *ope idoneae substitutionis ab omni irrationalitate liberare.*

Solutio.

Ad speciem saltem irrationalitatis tollendam, ponamus

$$\sqrt{(a + bxx + cx^4)} = px,$$

ut habeamus $S = \int \frac{px \partial x}{a - cx^4}$. Cum igitur sit

$$p = \frac{\sqrt{(a + bxx + cx^4)}}{x}, \text{ erit}$$

$$\partial p = - \frac{a\partial x + cx^4 \partial x}{xx \sqrt{(a + bxx + cx^4)}} = - \frac{a\partial x + cx^4 \partial x}{px^3},$$

unde erit

$$\partial x = - \frac{px^3 \partial p}{a - cx^4},$$

quo valore substituto fiet

$$\partial S = - \frac{ppx^4 \partial p}{(a - cx^4)^2}.$$

§. 65. Deinde cum sit

$$a + cx^4 = (pp - b)\, xx,$$

hincque porro

$$(a + cx^4)^2 = (pp - b)^2\, x^4,$$

aufferatur $4acx^4$, ac remanebit

$$(a - cx^4)^2 = [(pp - b)^2 - 4ac]\, x^4,$$

quo substituto formula nostra fiet

$$\partial S = - \frac{pp \partial p}{(pp - b)^2 - 4ac}.$$

Sicque quantitas variabilis x penitus e calculo est extrusa, ac deducti sumus ad formulam differentialem prorsus rationalem, cujus ergo integratio per logarithmos et arcus circulares nulla amplius laborat difficultate. Quin etiam formulae adhuc generaliores eodem modo feliciter tractari poterunt.

Problema 17.

§. 66. *Propositam hanc formulam integralem*

$$S = \int \frac{x^{n-2}\partial x \sqrt[n]{(a + bx^n + cx^{2n})}}{a - cx^{2n}}$$

ope idoneae substitutionis ab omni irrationalitate liberare.

Solutio.

Utamur igitur hac substitutione

$$\sqrt[n]{(a + bx^n + cx^{2n})} = px,$$

ut formula proposita hanc induat formam

$$\partial S = \frac{px^{n-1}\partial x}{a - cx^{2n}};$$

tum vero cum sit

$$p^n = \frac{a + bx^n + cx^{2n}}{x^n},$$

erit differentiando

$$p^{n-1}\partial p = -\frac{\partial x(a - cx^{2n})}{x^{n+1}},$$

unde fit

$$\partial x = -\frac{p^{n-1}x^{n+1}\partial p}{a - cx^{2n}},$$

quo valore substituto formula nostra induet hanc formam

$$\partial S = -\frac{p^n x^{2n}\partial p}{(a - cx^{2n})^2}.$$

§. 67. Deinde cum sit

$$a + cx^{2n} = (p^n - b)\, x^n, \text{ erit}$$

$$(a + cx^{2n})^2 = (p^n - b)^2\, x^{2n};$$

hinc subtrahatur $4acx^{2n}$, et remanebit

$$(a - cx^{2n})^2 = [(p^n - b)^2 - 4ac]\, x^{2n},$$

substituto igitur hoc valore fiet

$$\partial S = -\frac{p^n \partial p}{(p^n - b)^2 - 4ac},$$

quae ergo omnino est rationalis, atque adeo integratio per logarithmos et arcus circulares facile expeditur.

Problema 18.

§. 68. *Invenire formulas integrales adhuc generaliores, quae ope substitutionis*

$$\sqrt[n]{(a + bx^n + cx^{2n})} = px$$

ad rationalitatem perduci queant.

Solutio.

Quoniam in praecedente problemate invenimus, hanc formulam differentialem

$$\frac{x^{n-2}\partial x \sqrt[n]{(a + bx^n + cx^{2n})}}{a - cx^{2n}}$$

ope hujus substitutionis reduci ad istam formulam rationalem

$$-\frac{p^n \partial p}{(p^n - b)^2 - 4ac}, \text{ erit}$$

$$\frac{\mathrm{P}x^{n-2}\partial x \sqrt[n]{(a + bx^n + cx^{2n})}}{a - cx^{2n}} = -\frac{\mathrm{P}p^n \partial p}{(p^n - b)^2 - 4ac},$$

ubi loco P functiones quaecunque ipsius x accipi possunt ejusmodi, ut facta substitutione praebeant functiones rationales ipsius p, id quod infinitis modis fieri poterit, quorum praecipuos hic percurramus.

§. 69. Cum vi substitutionis sit

$$\frac{\sqrt[n]{(a + bx^n + cx^{2n})}}{x} = p,$$

loco P potestas quaecunque ipsius p assumi poterit, quae sit p^λ. Sumatur igitur $P = p^\lambda Q$, eritque etiam

$$P = \frac{Q\sqrt[n]{(a + bx^n + cx^{2n})^\lambda}}{x^\lambda};$$

quibus valoribus substitutis prodibit ista aequatio

$$\frac{Qx^{n-\lambda-2}\,\partial x\,\sqrt[n]{(a + bx^n + cx^{2n})^{\lambda+1}}}{a - cx^{2n}} = -\frac{Qp^{n+\lambda}\,\partial p}{(p^n - b)^2 - 4ac}$$

quae posterior formula denuo est rationalis.

§. 70. Deinde in praecedente problemate quoque invenimus esse

$$\frac{(a - cx^{2n})^2}{x^{2n}} = (p^n - b)^2 - 4ac,$$

quam ob rem pro Q sumamus potestatem exponentis i harum quantitatum, vel potius harum quantitatum reciprocam, scilicet capiatur

$$Q = \frac{x^{2in}}{(a - cx^{2n})^{2i}} = \frac{1}{[(p^n - b)^2 - 4ac]^i}.$$

Quibus valoribus substitutis obtinebimus formulam latissime patentem hanc

$$\frac{x^{(2i+1)n-\lambda-2}\partial x\sqrt[n]{(a+bx^n+cx^{2n})^{\lambda+1}}}{(a-cx^{2n})^{2i+1}} = -\frac{p^{n+\lambda}\partial p}{[(p^n-b)^2-4ac]^{i+1}};$$

ubi pro litteris λ et i numeros quoscunque integros sive positivos sive negativos accipere licet, perpetuo enim formula differentialis per p expressa manebit rationalis.

§. 71. Quin etiam haec reductio multo generalior reddi potest, propterea quod necessum non est ut λ sit numerus integer: Quaecunque enim fractio pro λ assumatur, formula per p expressa semper facile ad rationalitatem reduci poterit. Si enim ponamus $\lambda = \frac{\mu}{\nu}$, membrum dextrum fiet

$$-\frac{p^{\frac{\nu n+\mu}{\nu}}\partial p}{[(p^n-b)^2-4ac]^{i+1}},$$

quae rationalis redditur ponendo $p = q^\nu$, erit enim $\partial p = \nu q^{\nu-1}\partial q$, ideoque hoc membrum

$$-\frac{\nu q^{\mu+\nu n+\nu-1}\partial q}{[(q^{\nu n}-b)^2-4ac]^{i+1}}.$$

Nunc autem uti oportebit hac substitutione

$$\sqrt[n]{(a+bx^n+cx^{2n})} = q^\nu x,$$

atque habebitur ista reductio

$$\frac{x^{(2i+1)n-\frac{\mu}{\nu}-2}\partial x\sqrt[n]{(a+bx^n+cx^{2n})^{\frac{\mu+\nu}{\nu}}}}{(a-cx^{2n})^{2i+1}}$$
$$= -\frac{\nu q^{\mu+\nu n+\nu-1}\partial q}{[(q^{n\nu}-b)^2-4ac]^{i+1}},$$

quae postrema formula utique est rationalis.

§. 72. Ut etiam in membro sinistro exponentes fractos ipsius x tollamus, ponamus $x = y^\nu$, eritque

$$\frac{y^{(2i+1)n\nu-\mu-\nu-1}\partial y \sqrt[n\nu]{(a+by^{n\nu}+cy^{2n\nu})^{\mu+\nu}}}{(a-cy^{2n\nu})^{2i+1}}$$

$$= -\frac{q^{\mu+n\nu+\nu-1}\partial q}{[(q^{n\nu}-b)^2-4ac]^{i+1}},$$

quae expressio autem multo generalior videtur, quam revera est. Si enim loco $n\nu$ ubique scribamus n resultat ista aequatio

$$\frac{y^{(2i+1)n-\mu-\nu-1}\partial y \sqrt[n]{(a+by^{n}+cy^{2n})^{\mu+\nu}}}{(a-cy^{2n})^{2i+1}}$$

$$= -\frac{q^{\mu+\nu+n-1}\partial q}{[(q^{n}-b)^2-4ac]^{i+1}};$$

haec autem aequatio manifesto non discrepat ab illa §. 70. allata; si enim hic loco $\mu+\nu-1$ scribamus λ et loco y et q ut ante x et p, ipsa praecedens aequatio reperitur, sicque sufficiet loco λ numeros integros assumere.

Corollarium.

§. 73. Quo clarius indoles harum formularum perspiciatur, sumamus $n = 2$, et formula differentialis variabilem x involvens erit

$$\frac{x^{4i-\lambda}\partial x \sqrt{(a+bxx+cx^4)^{\lambda+1}}}{(a-cx^4)^{2i+1}},$$

quae facta substitutione $\sqrt{(a+bxx+cx^4)} = px$, transmutatur in hanc rationalem

$$-\frac{p^{\lambda+2}\partial p}{[(pp-b)^2-4ac]^{i-1}},$$

unde sumendo $\lambda = 4i$ resultat ista aequatio

$$\frac{\partial x \sqrt{(a + bxx + cx^4)^{4i+1}}}{(a - cx^4)^{ci+1}} = - \frac{p^{4i+2}\,\partial p}{[(pp - b)^2 - 4ac]^{i+1}},$$

in qua si porro ponatur $i = 0$, fiet

$$\frac{\partial x \sqrt{(a + bxx + cx^4)}}{a - cx^4} = - \frac{pp\,\partial p}{(pp - b)^2 - 4ac};$$

quae si insuper ponatur $a = 1$, $b = 0$ et $c = 1$, praebet

$$\frac{\partial x \sqrt{(1 + x^4)}}{1 - x^4} = - \frac{pp\,\partial p}{p^4 - 4},$$

quae est ipsa reductio, quae supra §. 63. fuerat inventa.

Corollarium 2.

§. 74. Si sumamus $n = 3$, prodibit ista reductio generalis

$$\frac{x^{6i-\lambda+1}\,\partial x \sqrt[3]{(a + bx^3 + cx^6)^{\lambda+1}}}{(a - cx^6)^{2i+1}} = - \frac{p^{\lambda+3}\,\partial p}{[(p^3 - b)^2 - 4ac]^{i+1}},$$

quae ponendo $i = 0$ migrat in hanc

$$\frac{x^{-\lambda+1}\,\partial x \sqrt[3]{(a + bx^3 + cx^6)^{\lambda+1}}}{a - cx^6} = - \frac{p^{\lambda+3}\,\partial p}{(p^3 - b)^2 - 4ac};$$

posito vero $b = 0$, haec prodit formula concinnior

$$\frac{x^{-\lambda+1}\,\partial x \sqrt[3]{(a+cx^6)^{\lambda+1}}}{a - cx^6} = - \frac{p^{\lambda+3}\,\partial p}{p^6 - 4ac},$$

cujus duos casus evolvisse juvabit.

I. Sit $\lambda = 0$, eritque

$$\frac{x\partial x \sqrt[3]{(a + cx^6)}}{a - cx^6} = - \frac{p^3\,\partial p}{p^6 - 4ac};$$

quae concinnior redditur ponendo $xx = y$, reperietur enim

$$\frac{\partial y \sqrt[3]{(a + cy^3)}}{a - cy^3} = - \frac{2p^3 \partial p}{p^6 - 4ac}.$$

II. Sumto autem $\lambda = 1$, ista prodit expressio

$$\frac{\partial x \sqrt[3]{(a + cx^6)^2}}{a - cx^6} = - \frac{p^4 \partial p}{p^6 - 4ac}.$$

Scholion.

§. 75. Ex his exemplis satis intelligitur, quam egregie reductiones ex nostris formulis generalibus deduci queant, quarum resolutio, nisi methodus nostra adhibeatur, omnes vires analyseos superare videatur.

4.) Memorabile genus formularum differentialium maxime irrationalium, quas tamen ad rationalitatem perducere licet. *M. S. Academiae exhib. d. 15. Maii 1777.*

§. 76. Cum nuper hanc formulam differentialem

$$\frac{\partial x}{(1 - xx) \sqrt[4]{(2xx - 1)}}$$

tractassem eamque singulari modo ad rationalitatem perduxissem, mox vidi eandem methodum succedere in hac generaliori $\frac{\partial x}{(a + bxx) \sqrt[4]{(a + 2bxx)}}$, atque adeo in hac multo generaliori