VARIAE CONSIDERATIONES
CIRCA
SERIES HYPERGEOMETRICAS.

Auctore
L. EVLERO.

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1.

Proposito hoc producto in infinitum excurrente:

$$\frac{a(a+2b)}{(a+b)(a+b)} \cdot \frac{(a+2b)(a+4b)}{(a+3b)(a+3b)} \cdot \frac{(a+4b)(a+6b)}{(a+5b)(a+5b)} \cdot \frac{(a+6b)(a+8b)}{(a+7b)(a+7b)} \text{ \&c.} = \frac{P}{Q},$$

constat esse

$$P = \int \frac{x^{a+b-1} \partial x}{\sqrt{(1-x^{2b})}} \text{ et } Q = \int \frac{x^{a-1} \partial x}{\sqrt{(1-x^{2b})}},$$

his integralibus ab $x = 0$ ad $x = 1$ extensis; vbi notetur illius producti membrum indici i respondens esse

$$\frac{[a+(2i-2)b](a+2ib)}{[a+(2i-1)b][a+(2i-1)b]}.$$

§. 2. Iam occasione istius producti consideremus sequens productum indefinitum, in quo factorum numerus sit $= n$, ac ponatur

$$a(a+2b)(a+4b)(a+6b) \ldots\ldots [a+(2n-2)b] = \Delta : n,$$

siquidem hoc productum, ob a & b numeros datos, spectari potest tanquam certa functio ipsius n; ex eius igitur natura perspicuum est fore

$$\Delta : (n+1) = \Delta : n \,.\, (a+2nb),$$

similique modo

$$\Delta : (n+2) = \Delta : (n+1) \,.\, [a+(2n+2)b],$$

et ita porro.

 Hinc

Hinc si i denotet numerum infinite magnum, erit

$$\Delta : i = a(a+2b)(a+4b)(a+6b)\ldots\ldots[a+(2i-2)b]$$

vnde pariter colligitur fore

$$\Delta : (i+1) = \Delta : i \,.\, (a+2ib),$$

$$\Delta : (i+2) = \Delta : i\,(a+2ib)\,[a+(2i+2)b],$$

$$\Delta : (i+3) = \Delta : i\,(a+2ib)\,[a+(2i+2)b]\,[a+(2i+4)b]$$

ubi factores insuper accedentes tanquam inter se aequales spe- ctari poterunt; quamobrem in genere statui poterit $\Delta : (i+$ $= \Delta : i\,(a+2ib)^n$, vbi cum $(a+2ib)$ sit factor proxi sequens, eodem iure quilibet sequentium sumi potuisset, quo adhuc generalius statuere poterimus

$$\Delta : (i+n) = (\alpha+2ib)^n\, \Delta : i,$$

denotante α numerum quemcunque finitum, quippe qui pr $2ib$ euanescit.

§. 3. In computum nunc ducamus casum producti definiti, quo $n = \frac{1}{2}$, ac vocemus $\Delta : \frac{1}{2} = k$, quem valorem o methodi interpolationum semper vero proxime assignare lic Hinc igitur per superiora erit

$$\Delta : (1+\tfrac{1}{2}) = k\,(a+b);$$

$$\Delta : (2+\tfrac{1}{2}) = k\,(a+b)(a+3b),$$

$$\Delta : (3+\tfrac{1}{2}) = k\,(a+b)(a+3b)(a+5b),$$

vnde in infinitum progrediendo erit

$$\Delta : (i+\tfrac{1}{2}) = k(a+b)(a+3b)(a+5b)\ldots\ldots[a+(2i-1)b]$$

§. 4. Cum igitur supra iam dederimus formulam p $\Delta : (i+n)$, posito nunc $n = \frac{1}{2}$ habebimus quoque

$$\Delta : (i+\tfrac{1}{2}) = \Delta : i\,\sqrt{(\alpha+2ib)},$$

sicque pro eadem formula $\Delta : (i+\frac{1}{2})$ nacti sumus duas diue sas expressiones, ex iisque conficitur ista aequatio:

$$\Delta : i \sqrt{(\alpha + 2ib)} = k(a+b)(a+3b)(a+5b) \ldots [a+(2i-1)b],$$

atque hinc concludere poterimus, valorem ipsius producti infiniti

$$(a+b)(a+3b)(a+5b) \ldots [a+(2i-1)b] = \frac{\Delta : i\sqrt{(\alpha+2ib)}}{k},$$

sicque innotescit relatio inter hoc productum et id quod supra per $\Delta : i$ expressimus. Hic autem probe notandum est factores huius producti eos ipsos esse, qui denominatorem producti initio propositi constituunt, quamobrem tam numeratorem illius producti quam denominatorem per valores modo inuentos $\Delta : i$ et $\frac{\Delta : i\sqrt{(\alpha+2ib)}}{k}$ exprimere poterimus.

§. 5. Numerator autem producti propositi in infinitum expansus ita repraesentari potest:

$$a(a+2b)^2(a+4b)^2 \ldots [a+(2i-2)b]^2(a+2ib),$$

vbi factores primus et vltimus sunt solitarii, reliqui vero omnes quadrati. Cum igitur sit

$$(\Delta : i)^2 = (a)^2(a+2b)^2(a+4b)^2(a+6b)^2 \ldots [a+(2i-2)b]^2,$$

euidens est illum numeratorem esse $\frac{(\Delta:i)^2}{a}(a+2ib)$. Pro denominatore autem per se manifestum est, eum esse aequalem quadrato producti alterius $(a+b)(a+3b)$ &c., cuius valor cum repertus sit $\frac{\Delta : i\sqrt{(\alpha+2ib)}}{k}$, denominator erit $\frac{(\Delta:i)^2(\alpha+2ib)}{kk}$, quamobrem his valoribus substitutis pro fractione supra exposita $\frac{P}{Q}$ affecuti sumus hanc aequationem:

$$\frac{P}{Q} = \frac{\frac{(\Delta:i)^2(a+2ib)}{a}}{\frac{(\Delta:i)^2(\alpha+2ib)}{kk}} = \frac{kk(a+2ib)}{a(\alpha+2ib)} = \frac{kk}{a},$$

Ex hac aequatione igitur statim innotescit verus valor formulae interpolatae $k = \Delta : \frac{1}{2}$, quandoquidem erit $\Delta : \frac{1}{2} = \sqrt{\frac{aP}{Q}}$, atque hinc porro sequentes:

$$\Delta:(1+\tfrac{1}{2}) = (a+b)\sqrt{\tfrac{aP}{Q}},$$
$$\Delta:(2+\tfrac{1}{2}) = (a+b)(a+3b)\sqrt{\tfrac{aP}{Q}},$$
$$\Delta:(3+\tfrac{1}{2}) = (a+b)(a+3b)(a+5b)\sqrt{\tfrac{aP}{Q}}, \text{ etc.}$$

haecque interpolatio eo magis eſt notatu digna, quod ſine approximatione ſtatim verum valorem horum terminorum interpolatorum ſuppeditat.

§. 6. Quod ſi inſuper iſtud productum infinitum, in quo vtrique factores coniunguntur, contemplemur, ac ſtatuamus

$$a(a+b)(a+2b)(a+3b)\ldots\ldots[a+(i-1)b] = \Gamma:i, \text{ erit}$$
$$\Gamma:2i = a(a+b)(a+2b)(a+3b)\ldots\ldots[a+(2i-1)b],$$

quod manifeſto eſt productum ex binis ſuperioribus, ita vt ſit $\Gamma:2i = \frac{(\Delta:i)^2\sqrt{(a+2ib)}}{k}$; vnde ſi forma $\Gamma:2i$ vti voluerimus, valores amborum praecedentium ex eo aſſignare poterimus, cum ſit $\Delta:i = \sqrt{\frac{k.\Gamma:2i}{\sqrt{(a+2ib)}}}$, qui eſt ipſe valor prioris producti

$$a(a+2b)(a+4b)(a+6b), \text{ etc.}$$

alterius vero producti

$$(a+b)(a+3b)(a+5b) \text{ etc. valor erit } \sqrt{\frac{\Gamma:2i(\sqrt{a+2ib})}{k}}.$$

§. 7. Hactenus igitur tria producta in infinitum excurrentia atque inter ſe affinia ſumus contemplati, quae, quoniam ea accuratius ſumus perſcrutaturi, hic denuo ob oculos exponamus

I. $a(a+b)(a+2b)(a+3b)\ldots\ldots[a+(i-1)b] = \Gamma:i,$

II. $a(a+2b)(a+4b)(a+6b)\ldots\ldots[a+(2i-2)b] = \Delta:i,$

III. $(a+b)(a+3b)(a+5b)\ldots\ldots[a+(2i-1)b] = \Theta:i,$

atque iam inuenimus eſſe $\Theta:i = \frac{\Delta:i\sqrt{(a+2ib)}}{k}$; tum vero tam $\Delta:i$ quam $\Theta:i$ ſequenti modo per functionem $\Gamma:2i$ expreſſimus:

$\Delta:i$

$$\Delta : i = \sqrt{\frac{k \,.\, \Gamma : 2i}{\sqrt{(a+2ib)}}} \text{ et } \Theta : i = \sqrt{\frac{\Gamma : 2i\sqrt{(a+2ib)}}{k}},$$

quandoquidem manifeſtum eſt eſſe $\Gamma : 2i = \Delta : i \,.\, \Theta : i$; vbi meminiſſe oportet eſſe $k = \Delta : \frac{1}{2}$, quod ſcilicet ex forma ſecunda definiri debet, conſiderando ſeriem

$$a,\ a(a+2b),\ a(a+2b)(a+4b),\ a(a+2b)(a+4b)(a+6b), \text{ etc.}$$

cuius terminum indici $\frac{1}{2}$ reſpondentem deſignauimus hac litterà k.

§. 8. Iam ad iſtas formas accommodemus methodum generalem ſummandi omnis generis progreſſiones per terminum earum generalem, quae ita ſe habet, vt propoſita ſerie quacunque A, B, C, D, E, etc. cuius terminus indici inde finito x reſpondens ſit $= X$, eius ſumma

$$A + B + C + D + \ldots\ldots\ldots + X,$$

quam vocemus $= S$, ſit

$$S = \int X \partial x + \tfrac{1}{2}X + \frac{1}{1.2.3} \cdot \frac{1}{2}\frac{\partial X}{\partial x} - \frac{1}{1\ldots5} \cdot \frac{1}{6}\frac{\partial^3 X}{\partial x^3} + \frac{1}{1\ldots7} \cdot \frac{1}{6}\frac{\partial^5 X}{\partial x^5} - \text{etc.}$$

vbi fractiones $\frac{1}{2}, \frac{1}{6}, \frac{1}{6}, \frac{3}{10}, \frac{5}{6}$, etc. ſunt numeri Bernoulliani.

Euolutio formae primae.

$$a(a+b)(a+2b)(a+3b)\ldots\ldots[a+(i-1)b] = \Gamma : i.$$

§. 9. Cum numerus factorum hic conſideretur vt infinitus, quo methodum ſummandi ad eam applicare valeamus, conſideremus eandem formam numero terminorum finito $= x$ conſtantem, ac ſtatuamus ſimili modo

$$a(a+b)(a+2b)(a+3b)\ldots\ldots\ldots[a+(x-1)b] = \Gamma : x.$$

Nunc vero vt loco huius producti ſeriem ſummandam nanciſcamur, ſumamus logarithmos, eritque

$$l\Gamma : x = la + l(a+b) + l(a+2b) + l(a+3b) \ldots\ldots l[a+(x-1)b],$$

cuius ergo ſumma cum fuerit explorata, dabit logarithmum formulae

mulae $\Gamma : x$, ideoque ipſam formulam $\Gamma : x$, in qua ſi deinceps ſtatuatur $x = i$, obtinebitur formula $\Gamma : i$, quem valorem in ſu- perioribus potiſſimum ſpectauimus. Hinc igitur, comparatione cum ſerie generaliſſima inſtituta, erit $X = l\,[a + (x - 1)\,b]$, ſiue erit $X = l\,(a - b + b\,x)$, atque ipſa ſumma $S = l\,\Gamma : x$, vnde colligitur $\int X\,\partial x = \int \partial x\, l\,(a - b + b\,x)$.

§. 10. Cum igitur ſit $\int \partial z\, l\, z = z\, l\, z - z$, atque

$$\int \partial y\, l\,(a + y) = (a + y)\, l\,(a + y) - (a + y),$$

nunc loco y ſcribendo $b\,x$ erit

$$\int b\,\partial x\, l\,(a + b\,x) = (a + b\,x)\, l\,(a + b\,x) - a - b\,x,$$

ideoque

$$\int \partial x\, l\,(a + b\,x) = \frac{(a + b\,x}{b}\, l\,(a + b\,x) - \frac{a}{b} - x,$$

vnde colligitur pro noſtro caſu fore:

$$\int X\,\partial x = \frac{(a - b + b\,x)}{b}\, l\,(a - b + b\,x) - \frac{a}{b} + 1 - x,$$

vbi in vltima parte membrum conſtans $\frac{a}{b} - 1$ omittere lice[t], quia expreſſio conſtantem quantitatem indefinitam per ſe poſſ[it] lat, quam deinceps ex indole ſeriei definiri oportet. Dein[de] vero erit $\frac{\partial X}{\partial x} = \frac{b}{a - b + b\,x}$; tum vero porro

$$\frac{\partial^3 X}{\partial x^3} = \frac{2\,b^3}{(a - b + b\,x)^3};\quad \frac{\partial^5 X}{\partial x^5} = \frac{2.\,3.\,4\,b^5}{(a - b + b\,x)^5};\ \text{etc.}$$

quibus valoribus in vſum vocatis erit

$$l\,\Gamma : x = A + \left(\frac{a}{b} - \frac{1}{2} + x\right) l\,(a - b + b\,x) - x + \frac{1}{1.\,2.\,3}\cdot\frac{1}{2}\cdot\frac{b}{a - b + b\,x}$$
$$- \frac{1}{3.\,4.\,5}\cdot\frac{1}{6}\cdot\frac{b^3}{(a - b + b\,x)^3} + \frac{1}{5.\,6.\,7}\cdot\frac{1}{6}\cdot\frac{b^5}{(a - b + b\,x)^5}$$
$$- \frac{1}{7.\,8.\,9}\cdot\frac{3}{10}\cdot\frac{b^7}{(a - b + b\,x)^7} + \frac{1}{9.\,10.\,11}\cdot\frac{5}{6}\cdot\frac{b^9}{(a - b + b\,x)^9} - \text{etc.}$$

vbi littera A conſtantem denotat ex indole ſeriei definiendam.

§. 11. Conſtans autem iſta A ex caſu quo ſumma ſeriei eſt cognita determinari debet, quod ergo fieri poſſet e[x] ca[ſu] casu $x = 0$, quippe summa etiam nihilo aequalis prodire debet; foret igitur hinc

$$A = \left(\frac{a}{b} - \frac{1}{2}\right) l\,(a-b) + \frac{1}{1.2.3} \cdot \frac{1}{2} \cdot \frac{b}{a-b} - \frac{1}{3.4.5} \cdot \frac{1}{6}\,\frac{b^3}{(a-b)^3} + \frac{1}{5.6.7} \cdot \frac{1}{6}\,\frac{b^5}{(a-b)^5} - \text{etc.}$$

Quoniam autem haec series parum conuergit, atque adeo casu $b = a$ omnes termini fierent infiniti, hinc nihil plane lucrari licet. Sin autem vellemus sumere $x = 1$, ista summa prodire deberet $= l\,a$; vnde pariter vix quicquam pro instituto nostro concludere liceret, quia semper ad seriem infinitam perveniretur, cuius summam demum explorare oporteret, in quo quidem negotio forsitan ea, quae olim de seriebus numeros Bernoullianos inuoluentibus sum commentatus, aliquem vsum praestare possent, cui autem labori nunc immorari non vacat.

§. 12. Quia enim in praesenti instituto potissimum ad valorem $\Gamma : i$ respicimus, sufficiet statim loco x numerum infinitum statui. Sit igitur $x = i$, denotante i numerum infinite magnum, et aequatio nostra hanc induet formam:

$$l\,\Gamma : i = A + \left(\frac{a}{b} - \frac{1}{2} + i\right) l\,(a - b + b\,i) - i,$$

vnde constans ista A sponte determinatur, quam idcirco quasi iam cognitam spectabimus. Hinc ergo ad numeros regrediendo, vbi quidem loco A scriptum intelligamus l A, perueniemus ad hanc expressionem:

$$\Gamma : i = A\,(a - b + b\,i)^{\frac{a}{b} - \frac{1}{2} + i} \cdot e^{-i}.$$

Hic quidem conueniet potestatem exponentis i seorsim repraesentare hoc modo:

$$\Gamma : i = A\,(a - b + b\,i)^{\frac{a}{b} - \frac{1}{2}}\,(a - b + b\,i)^{i} \cdot e^{-i}.$$

Euolutio binarum reliquarum formularum.

§. 13. Secunda forma a prima in eo tantum discre pat, quod loco b hic scribi oportet $2b$, vnde noua euolutio ne carere possumus; at vero loco constantis A hic scribam B, quandoquidem nondum constat, quemadmodum littera b constantem A ingrediatur. Hoc modo igitur statim habebim

$$\Delta : i = B\,(a - 2b + 2bi)^{\frac{a}{2b} - \frac{1}{2}}\,(a - 2b + 2bi)^i\, e^{-i}$$

Simili modo euidens est, ex hac secunda forma oriri tertia si modo loco a scribatur $a + b$, vnde loco B constantem introducendo, habebimus

$$\Theta : i = C\,(a - b + 2bi)^{\frac{a}{2b}}\,(a - b + 2bi)^i\, e^{-i}.$$

Vbi notetur litteram e hic positam esse pro numero cuius garithmus hyperbolicus $= 1$.

Conclusiones hinc oriundae.

§. 14. Videamus nunc quomodo istae nouae deter nationes se respectu relationum supra inuentarum sint hab rae; quare cum ex his nouis valoribus sit

$$\Gamma : 2i = A\,(a - b + 2bi)^{\frac{a}{b} - \frac{1}{2}}\,(a - b + 2bi)^{2i}\, e^{-}$$

quia inuenimus:

$$\Gamma : 2i = \Delta : i \times \Theta : i,$$

si vbique hic valores modo inuentos substituamus, habebi pro ista aequatione primo productum:

$$\Delta : i\,.\,\Theta : i = B\,C\,(a - 2b + 2bi)^{\frac{a}{2b} - \frac{1}{2}}\,(a - b + 2bi$$
$$(a - 2b + 2bi)^i\,(a - b + 2bi)^i\, e^{-2i},$$

quod cum esse debeat illi valori $\Gamma : 2i$ aequale, si per fa res quos habeat communes vtrinque diuidamus, prodibit aequatio:

$$A(a-b+2bi)^{\frac{a}{2b}-\frac{1}{2}}(a-b+2bi)^{i}$$
$$=BC(a-2b+2bi)^{\frac{a}{2b}-\frac{1}{2}}(a-2b+2bi)^{i}.$$

§. 15. Diuidamus hanc aequationem vtrinque per $(a-2b+2bi)^{i}$, et cum sit

$$\frac{a-b+2bi}{a-2b+2bi}=1+\frac{b}{a-2b+2bi}=1+\frac{1}{2i},$$

ob i numerum infinitum, per resolutionem ordinariam erit $(1+\frac{1}{2i})^{i}=e^{\frac{1}{2}}$, vnde aequatio nostra reducitur ad hanc formam:

$$A(a-b+2bi)^{\frac{a}{2b}-\frac{1}{2}}\times e^{\frac{1}{2}}=BC(a-2b+2bi)^{\frac{a}{2b}-\frac{1}{2}},$$

vbi vltimi factores quoque se tollunt, cum sit

$$\left(\frac{a-b+2bi}{a-2b+2bi}\right)^{\frac{a}{2b}-\frac{1}{2}}=\left(1+\frac{1}{2i}\right)^{\frac{a}{2b}-\frac{1}{2}}=1,$$

ita vt peruentum sit ad hanc simplicem aequalitatem:

$$Ae^{\frac{1}{2}}=BC.$$

§. 16. Deinde cum supra inuenerimus esse

$$\Theta:i=\frac{\Delta i\sqrt{(\alpha+2ib)}}{k},\ \text{siue}\ \frac{\Theta:i}{\Delta:i}=\frac{\sqrt{(\alpha+2ib)}}{k},$$

diuidamus valorem pro $\Theta:i$ inuentum per $\Delta:i$, ac reperiemus:

$$\frac{\Theta:i}{\Delta:i}=\frac{C}{B}\sqrt{(a-2b+2bi)}\left(\frac{a-b+2bi}{a-2b+2bi}\right)^{i}=\frac{C}{B}\sqrt{e(a-2b+2bi)}.$$

Erit ergo

$$\frac{\sqrt{(\alpha+2ib)}}{k}=\frac{C}{B}\sqrt{e(a-2b+2bi)},\ \text{siue}$$
$$\frac{1}{k}=\frac{C}{B}\sqrt{\frac{e(a-2b+2bi)}{\alpha+2ib}}=\frac{C}{B}\sqrt{e},$$

siue erit $B=Ck\sqrt{e}$.

§. 17. Nacti ergo sumus eiusmodi binas relationes in-
ter ternas illas constantes A, B, C, vt si earum vnica esset
cognita, ex ea binae reliquae definiri possent. Cum enim sit
$A = \frac{BC}{\sqrt{e}}$ et $B = Ck\sqrt{e}$, si constantem A spectemus vt iam
cognitam, binae reliquae sequenti modo determinabuntur: Cum
sit $B = Ck\sqrt{e}$, hic valor in priore aequatione substitutus da
$A = CCk$, vnde elicitur $C = \sqrt{\frac{A}{k}}$, hincque porro $B = \sqrt{kAe}$
Interim tamen hinc non patet, quomodo constans A absolute
determinari queat; ideoque recurrendum erit ad ipsam illam
summationem seriei logarithmicae, quam littera A supra ind
cauimus; vbi autem loco A scribendum erit lA. Atque hi
tantum sumus lucrati, vt si binae reliquae formae simili mod
euoluantur per series logarithmicas, constantes ibi adhibenda
scilicet lB et lC simul innotescant.

§. 18. Superest vt adhuc pauca addamus de valo
litterae k, quam per interpolationem inueniri debere iam sup
monuimus. Interim tamen haec littera etiam ex ipsa comp
ratione formularum $\Delta : i$ et $\Theta : i$ absolute per certas quadr
turas determinari potest. Cum enim sit

$$k = \frac{\Delta : i}{\Theta : i} \sqrt{(\alpha + 2ib)}, \text{ ideoque}$$

$$kk = \frac{(\Delta : i)^2 (\alpha + 2ib)}{(\Theta : i)^2},$$

si loco $\Delta : i$ et $\Theta : i$ ipsa producta infinita substituamus, e
quoniam vtrumque ex i factoribus constat, hic autem in nu
meratore vnus factor $\alpha + 2ib$ insuper accedit, primum nu
meratoris factorem seorsim exprimamus, hoc pacto perueniem
ad sequens productum determinatum:

$$kk = a \cdot \frac{a(a+2b)(a+2b)(a+4b)(a+4b)(a+6b)}{(a+b)(a+b)(a+3b)(a+3b)(a+5b)(a+5b)} \text{ etc.}$$

§. 19

§. 19. Vt autem huius producti infiniti verum valorem eruamus, recordandum est, si litterae P et Q denotent sequentes formulas integrales:

$$P = \int \frac{x^{p-1}\,\partial x}{(1-x^n)^{1-\frac{m}{n}}} \text{ et } Q = \int \frac{x^{q-1}\,\partial x}{(1-x^n)^{1-\frac{m}{n}}},$$

quae integralia ab $x=0$ ad $x=1$, extendi sunt intelligenda, tum per productum infinitum fore:

$$\frac{P}{Q} = \frac{q(m+p)}{p(m+q)} \cdot \frac{(q+n)(m+p+n)}{(p+n)(m+q+n)} \cdot \frac{(q+2n)(m+p+2n)}{(p+2n)(m+q+2n)} \text{ etc.}$$

quod productum facile ad nostram formam reducitur, sumendo $q=a$, $p=a+b$, $m=b$, $n=2b$, ita vt pro nostro casu fiat

$$P = \int \frac{x^{a+b-1}\,\partial x}{\sqrt{(1-x^{2b})}} \text{ et } Q = \int \frac{x^{a-1}\,\partial x}{\sqrt{(1-x^{2b})}},$$

tum vero erit $kk = \frac{aP}{Q}$, ideoque $k = \sqrt{\frac{aP}{Q}}$, sicque eundem valorem k alio modo elicuimus, quem iam supra attulimus.

§. 20. Quemadmodum autem est $k = \Delta : \frac{1}{2}$, simili modo pro binis reliquis formis poterimus assignare valores $\Gamma : \frac{1}{2}$ et $\Theta : \frac{1}{2}$. Cum enim forma Γ oriatur ex forma Δ, si in hac loco b scribatur $\frac{1}{2}b$, forma autem Θ oriatur ex Δ, si loco a scribatur $a+b$, his obseruatis erit

$$\Gamma : \tfrac{1}{2} = \sqrt{a\,\frac{\int x^{a+\frac{1}{2}b-1}\,\partial x : \sqrt{(1-x^b)}}{\int x^{a-1}\,\partial x : \sqrt{(1-x^b)}}} \text{ et}$$

$$\Theta : \tfrac{1}{2} = \sqrt{(a+b)\,\frac{\int x^{a+2b-1}\,\partial x : \sqrt{(1-x^{2b})}}{\int x^{a+b-1}\,\partial x : \sqrt{(1-x^{2b})}}}.$$

Facile autem intelligitur, valorem $\Theta : \frac{1}{2}$ aeque in nostros calculos introduci potuisse ac $\Delta : \frac{1}{2} = k$, cum sit $\Delta : \frac{1}{2} \cdot \Theta : \frac{1}{2} = a$.

 Ductis

Ductis enim in ſe inuicem illis valoribus integralibus prodit

$$\Delta : \tfrac{1}{2} . \Theta : \tfrac{1}{2} = \sqrt{\frac{a(a+b)\int x^{a+2b-1}\partial x : \sqrt{(1-x^{2b})}}{\int x^{a-1}\partial x : (1-x^{2b})}}$$

ex notiſſima autem reductione talium integralium conſtat eſſe

$$\int \frac{x^{a+2b-1}\partial x}{\sqrt{(1-x^{2b})}} = \frac{a}{a+b}\int \frac{x^{a-1}\partial x}{\sqrt{(1-x^{2b})}},$$

pro terminis ſcilicet integrationis $x = 0$ et $x = 1$, ſicque perſpicuum eſt fore $\Delta : \tfrac{1}{2} . \Theta : \tfrac{1}{2} = a$. Quemadmodum autem valor $\Gamma : \tfrac{1}{2}$ ad binos reliquos referatur, nullo modo defini poteſt.