

S U P P L E M E N T U M I X.

AD SECT. I. TOM. II.

DE

RESOLUTIONE AEQUATIONUM DIFFERENTIALIUM SECUNDI GRADUS, DUAS TANTUM VARIABILES INVOLVENTIUM.

1). Methodus singularis resolvendi aequationes differentiales secundi gradus. *M. S. Academiae exhib. die 19 Jan. 1779.*

§. 1. Si p et q fuerint functiones quaecunque ipsius x , atque proposita fuerit haec aequatio inter binas variables x et z

$$2 p \partial z + z \partial p = \frac{\partial x}{q} \int \frac{z \partial x}{q},$$

evidens est ejus integrale facile inveniri posse, si ea multiplicetur per z , ut habeatur

$$2 p z \partial z + z z \partial p = \frac{z \partial x}{q} \int \frac{z \partial x}{q}.$$

Prioris enim membri integrale est $p z z$, posterius verò membrum posito $\int \frac{z \partial x}{q} = v$, abit in $v \partial v$, cujus integrale est $\frac{1}{2} v v + C$, ita ut hinc nanciscamur istam aequationem integram $p z z = \frac{1}{2} v v + C$, unde fit $v v = 2 p z z - 2 C$, hincque

$$v = \int \frac{z \partial x}{q} = \sqrt{(2 p p z z - C)},$$

quae differentiata dat

$$\frac{z \partial x}{q} = \frac{2 p z \partial z + z z \partial p}{\sqrt{(2 p z z - C)}},$$

facto ergo divisione per z , erit

$$\frac{\partial x}{q} = \frac{2 p \partial z + z \partial p}{\sqrt{(2 p z z - C)}},$$

quemadmodum autem hinc valor ipsius z per x , ejusque functiones p et q exprimi queat, non liquet. Ut autem istum scopum obtineamus, posito ut fecimus $\int \frac{z \partial x}{q} = v$ ut sit $v v = 2 p z z - C$, retineamus quantitatem v in calculo, et cum sit

$$\frac{z \partial x}{q} = \partial v, \text{ erit } Z = \frac{q \partial v}{\partial x},$$

quo valore substituto habebimus

$$v v = \frac{2 p q q \partial v^2}{\partial x^2} - C,$$

unde colligitur

$$\partial v = \frac{\partial x \sqrt{(v v + C)}}{q \sqrt{2 p}},$$

quae sponte separationem admittit, cum sit

$$\frac{\partial v}{\sqrt{(v v + C)}} = \frac{\partial x}{q \sqrt{2 p}}, \text{ ideoque}$$

$$\int \frac{\partial v}{\sqrt{(v v + C)}} = \int \frac{\partial x}{q \sqrt{2 p}},$$

cujus valor, quoniam p et q sunt functiones ipsius x , tanquam cognitus spectari potest.

§. 2. Statuamus ergo hoc integrale

$$\int \frac{\partial x}{q \sqrt{2 p}} = I X,$$

ut habeamus

$$\int \frac{\partial v}{\sqrt{(v v + C)}} = I X,$$

quare cum constet esse

$$\int \frac{\partial v}{\sqrt{(v v + C)}} = I [v + \sqrt{(v v + C)}], \text{ erit}$$

$$v + \sqrt{(v v + C)} = X,$$

unde colligitur $v = \frac{x^2 - C}{2X}$, ideoque per quantitatem X definitur.

§. 3. Cum igitur supra invenerimus $2pz = vv + C$,
erit

$$2pz = \frac{(x^2 - C)^2}{4XX} + C = \frac{(xx + C)^2}{4XX},$$

consequenter erit

$$z\sqrt{2p} = \frac{x^2 + C}{2X},$$

sicque quantitas z ita per X exprimitur, ut sit

$$z = \frac{x^2 + C}{2X\sqrt{2p}},$$

ubi meminisse oportet esse

$$lX = \int \frac{\partial x}{q\sqrt{2p}}, \text{ sive } X = e^{\int \frac{\partial x}{q\sqrt{2p}}}.$$

§. 4. Manifestum autem est, aequationem nostram propositam, si a signo integrali liberetur, abire in aequationem differentialem secundi gradus, cujus ergo integrale completum modo elicuimus. Facta enim multiplicatione per q fiet

$$2pq\partial z + qz\partial p = \partial x \int \frac{z\partial x}{q},$$

et differentiatio sumto elemento ∂x constante praebebit sequentem aequationem differentialem secundi gradus

$$\left. \begin{aligned} 2pq\partial\partial z + 2p\partial q\partial z + z\partial q\partial p \\ + 3q\partial p\partial z + qz\partial\partial p \end{aligned} \right\} = \frac{z\partial x^2}{q},$$

cujus ergo aequationis non parum absurdum novimus esse integrale completum

$$z = \frac{x^2 + C}{2X\sqrt{2p}}, \text{ existente } X = e^{\int \frac{\partial x}{q\sqrt{2p}}},$$

ita ut ista quantitas X etiam constantem arbitrariam involvat.

§. 5. Cum autem haec aequatio non parum sit complicata, sequenti modo concinnius repraesentari potest; cum enim sit

$$\frac{q}{z} \partial . p z z = \partial x \int \frac{z \partial x}{q},$$

erit differentiationem tantum indicando

$$\partial . \frac{q \partial . p z z}{z} = \frac{z \partial x^2}{q},$$

quae manifesto integrabilis evadit, si multiplicetur per $\frac{2 q \partial . p z z}{z}$, quodsi enim brevitatis gratia statuatur $\frac{q \partial . p z z}{z} = s \partial x$, membrum sinistrum fit

$$2 s \partial x . \partial s \partial x = 2 \int \partial s \partial x^2,$$

ejusque ergo integrale $s s \partial x^2$: at vero ex parte dextra habebimus $2 \partial x^2 \partial . p z z$, cujus igitur integrale est

$$2 p z z \partial x^2 + C \partial x^2,$$

ita ut integratio nobis praebeat $s s = 2 p z z + C$.

§. 6. Quo nunc hanc aequationem penitus evolvamus, statuamus ut ante $p z z = v$, ita ut sit $\frac{q \partial v}{z} = s \partial x$, eritque nostrum integrale inventum

$$s s = \frac{q q \partial v^2}{z z \partial x^2} = 2 v + C,$$

quae ob $z z = \frac{v}{p}$ abit in hanc

$$\frac{p q q \partial v^2}{v \partial x^2} = 2 v + C,$$

unde eruitur propemodum ut ante

$$\frac{\partial v}{\sqrt{v(2v+C)}} = \frac{\partial x}{q \sqrt{p}},$$

quae a forma ante inventa non discrepat.

§. 7. Simili modo etiam aliae aequationes differentiales magis complicatae resolvi poterunt, veluti si proponatur

ista aequatio

$$3 p \partial z + z \partial p = \frac{\partial x}{q} \int \frac{z z \partial x}{q},$$

ubi iterum p et q denotant functiones quascunque ipsius x . Cum enim sit

$$3 p \partial z + z \partial p = \frac{\partial \cdot p z^3}{z z},$$

erit per $z z$ multiplicando

$$\partial \cdot p z^3 = \frac{z z \partial x}{q} \int \frac{z z \partial x}{q},$$

quae posito $\int \frac{z z \partial x}{q} = v$ abit in $\partial \cdot p z^3 = v \partial v$, ideoque integrando $2 p z^3 = v v + C$.

§. 8. Quoniam autem posuimus $\int \frac{z z \partial x}{q} = v$, erit

$$z z = \frac{q \partial v}{\partial x}, \text{ hincque } z^3 = \frac{q \partial v}{\partial x} \sqrt{\frac{q \partial v}{\partial x}},$$

unde fit

$$\frac{2 p q \partial v}{\partial x} \sqrt{\frac{q \partial v}{\partial x}} = v v + C.$$

Sumtis ergo quadratis erit

$$\frac{4 p p q^3 \partial v^3}{\partial x^3} = (v v + C)^2, \text{ ideoque}$$

$$\frac{\partial v^3}{(v v + C)^2} = \frac{\partial x^3}{4 p p q^3},$$

cujus radix cubica praebet

$$\frac{\partial v}{\sqrt[3]{(v v + C)^2}} = \frac{\partial x}{q \sqrt[3]{4 p p}}.$$

Hinc igitur quantitas v per x definitur, ita ut jam v spectare queamus tanquam veram functionem ipsius x , qua inventa erit

$$z^3 = \frac{v v + C}{2 p}, \text{ hincque } z = \sqrt[3]{\frac{v v + C}{2 p}}.$$

§. 9. Eadem ista aequatio adhuc alio modo resolvi poterit, quandoquidem per q multiplicata ita repraesentatur

$$\frac{q \partial \cdot p z^3}{z z} = \partial x \int \frac{z z \partial x}{q}, \text{ sive}$$

$$\partial \cdot \frac{q \partial \cdot p z^3}{z z} = \frac{z z \partial x^2}{q},$$

quae manifesto integrabilis redditur, multiplicando per $\frac{2 q \partial \cdot p z^3}{z z}$, prodit enim

$$\left(\frac{q \partial \cdot p z^3}{z z} \right) = 2 p z^3 \partial x^2 + C \partial x^2.$$

§. 10. Jam ponatur $p z^3 = v$, ita ut sit

$$z^3 = \frac{v}{p}, \text{ et } z^4 = \frac{v}{p} \sqrt[3]{\frac{v}{p}},$$

quo valore substituto habebimus

$$\frac{p q q \partial v^2 \sqrt[3]{p}}{v \sqrt[3]{v}} = 2 v \partial x^2 + C \partial x^2,$$

unde concluditur

$$\frac{\partial v^2}{v (2 v + C) \sqrt[3]{v}} = \frac{\partial x^2}{p q q \sqrt[3]{p}}, \text{ sive}$$

$$\frac{\partial v}{\sqrt[3]{v} (2 v + C) \sqrt[3]{v}} = \frac{\partial v}{v^{\frac{2}{3}} \sqrt[3]{(2 v + C)}} = \frac{\partial x}{q \sqrt[3]{p p}},$$

haec aequatio simplicior evadit, ponendo $v = u^3$, scilicet

$$\frac{3 \partial u}{\sqrt[3]{(2 u^3 + C)}} = \frac{\partial x}{q \sqrt[3]{p p}}.$$

Hinc intelligitur, innumerabilia exempla per has formulas expediri posse.

§. 11. Quin etiam hujusmodi aequationes multo generaliores tractari poterunt; namque aequatio generalior ita potest repraesentari

$$\frac{\partial \cdot p z^m}{z^n} = \frac{\partial x}{q} \int \frac{z^n \partial x}{q},$$

quae evoluta dat

$$m p z^{m-n-1} \partial z + z^{m-n} \partial p = \frac{\partial x}{q} \int \frac{z^n \partial x}{q}.$$

Facta autem multiplicatione per z^n , prodit aequatio sponte integrabilis

$$\partial \cdot p z^m = \frac{z^n \partial x}{q} \int \frac{z^n \partial x}{q},$$

si quidem prodit

$$2 p z^m = \left(\int \frac{z^n \partial x}{q} \right)^2 + C.$$

§. 12. Ad hanc aequationem ulterius evolvendam statuamus

$$\int \frac{z^n \partial x}{q} = v, \text{ eritque } z^n = \frac{q \partial v}{\partial x},$$

unde primo $2 p z^m = v v + C$, et hinc porro

$$(2 p)^{\frac{n}{m}} \cdot z^n = (2 p)^{\frac{n}{m}} \cdot \frac{q \partial v}{\partial x} = (v v + C)^{\frac{n}{m}},$$

quae cum sponte sit separabilis, dabit

$$\frac{\partial v}{(v v + C)^{\frac{n}{m}}} = \frac{\partial x}{q (2 p)^{\frac{n}{m}}},$$

unde ergo quantitas v per x determinabitur, qua inventa ipsa quantitas quaesita z ita exprimetur, ut sit $z^m = \frac{v v + C}{2 p}$.

§. 13. Illustremus haec unico exemplo a primo casu petito, sumendo scilicet $p = 1 + xx$ et $q = \sqrt{2}$, ita ut aequatio proposita sit

$$2 \partial z (1 + xx) + 2 zx \partial x = \frac{\partial x}{2} \int z \partial x,$$

quae in hanc aequationem secundi gradus evolvitur

$$4 \partial \partial z (1 + xx) + 12 x \partial x \partial z + 3 z \partial x^2 = 0,$$

cujus ergo integrale quaeritur.

§. 14. Faciamus ergo applicationem solutionis supra §. 3. inventae, ubi cum hic sit $p = 1 + xx$ et $q = \sqrt{2}$, erit

$$IX = \frac{1}{2} \int \frac{\partial x}{\sqrt{(1+xx)}} = \frac{1}{2} l [x + \sqrt{(1+xx)}] - \frac{1}{2} l a,$$

unde fit

$$X = \frac{\sqrt{[x + \sqrt{(1+xx)}]}}{\sqrt{a}},$$

hoc igitur valore substituto habebimus

$$z = \frac{aC + x + \sqrt{(1+xx)}}{2\sqrt{2a(1+xx)}[x + \sqrt{(1+xx)}]},$$

quae hoc modo simplicius exprimitur

$$z = \frac{[aC + x + \sqrt{(1+xx)}] \sqrt{[-x + \sqrt{(1+xx)}]}}{2\sqrt{2a(1+xx)}}.$$

Ubi ergo duae quantitates constantes arbitrariae sunt involutae, atque adeo hoc integrale completum algebraice determinetur. Posito ergo $C = 0$, integrale particulare erit ex prima forma petitem

$$z = \frac{\sqrt{[x + \sqrt{(1+xx)}]}}{2\sqrt{2a(1+xx)}}.$$

§. 15. Aliud integrale particulare hinc exhiberi potest, constantes ita sumendo ut sit aC infinitum, at vero $C\sqrt{a}$ finitum $= b$, tum enim erit

$$z = \frac{aC}{2\sqrt{2a(1+xx)}[x + \sqrt{(1+xx)}]} = \frac{b}{2\sqrt{2(1+xx)}[x + \sqrt{(1+xx)}]},$$

quae forma redigitur ad hanc.

$$z = \frac{\alpha \sqrt{[-x + \sqrt{(1+xx)}]}}{\sqrt{(1+xx)}}$$

- 2.) Methodus nova investigandi omnes casus, quibus hanc aequationem differentio-differentialem

$$\partial \partial y (1 - \bar{a} x x) - b x \partial x \partial y - c y \partial x^2 = 0$$

resolvere licet. *M. S. Academiae exhib. die 13 Januarii, 1780.*

§. 16. Hic quidem in usum vocari posset methodus a me et ab aliis jam passim exposita, qua valor ipsius y per seriem infinitam exprimitur. Tunc enim omnibus casibus, quibus haec series alicubi abruptitur, habebitur integrale particulare aequationis propositae; unde quidem haud difficulter integrale completum erui poterit. Verum etsi hoc modo infiniti casus integrabiles reperiuntur, tamen non omnes innotescunt, sed dantur praeterea infiniti alii casus, qui resolutionem admittunt. Quamobrem hic methodum prorsus singularem proponam, cujus ope omnes plane casus integrabiles elici poterunt. Haec autem methodus ita est comparata, ut cognito casu quocunque resolutionem admittente, ex eo innumerabiles alii deduci queant.

§. 17. Statim autem se offerunt duo casus simplicissimi, quibus resolutio succedit, quorum alter est, si $c = 0$, alter vero si $b = a$, quos ergo binos casus principales ante omnia evolvi oportet.

Casus prior principalis
quo $c = 0$.

§. 18. Hoc igitur casu aequatio nostra erit.

$$\partial \partial y (1 - a x x) = b x \partial x \partial y,$$

quae posito $\partial y = p \partial x$, abit in hanc

$$\partial p (1 - a x x) = b p x \partial x, \text{ sive}$$

$$\frac{\partial p}{p} = \frac{b x \partial x}{1 - a x x},$$

cujus integrale est

$$l p = -\frac{b}{2a} l(1 - a x x) + l C,$$

sicque erit

$$p = C (1 - a x x)^{-\frac{b}{2a}} = \frac{\partial y}{\partial x},$$

unde obtinetur

$$y = C \int \partial x (1 - a x x)^{-\frac{b}{2a}};$$

ubi notasse juvabit istum valorem fieri algebraicum quoties fuerit $-\frac{b}{2a}$ numerus integer positivus, sive $b = -2i a$ denotante i numerum integrum quemcunque. Tum vero valor integralis etiam algebraicus evadit, quando fuerit $-\frac{b}{2a}$, sive $-\frac{3}{2}$, sive $-\frac{5}{2}$, sive $-\frac{7}{2}$, etc. ideoque in genere $\frac{b}{a} = 2i + 1$, ubi esse nequit $i = 0$.

Casus principalis alter
quo $b = a$.

§. 19. Hoc ergo casu aequatio nostra per $2 \partial y$ multiplicata erit.

$$2 \partial y \partial \partial y (1 - a x x) - 2 a x \partial x \partial y^2 - 2 c y \partial y \partial x^2 = 0,$$

quae sponte est integrabilis, ejus enim integrale erit

$$\partial y^2 (1 - a x x) - c y y \partial x^2 = C \partial x^2.$$

Ex hac igitur aequatione erit

$$\partial y \sqrt{(1 - a x x)} = \partial x \sqrt{(C + c y y)},$$

separatione ergo facta erit

$$\frac{\partial x}{\sqrt{(1 - a x x)}} = \frac{\partial y}{\sqrt{(C + c y y)}}.$$

In hac ergo forma iterum continentur casus algebraici, ad quos eruendos faciamus $a = -\alpha \alpha$, $c = \gamma \gamma$ et $C = \beta \beta$; ut habeamus

$$\frac{\partial x}{\sqrt{(1 + \alpha \alpha x x)}} = \frac{\partial y}{\sqrt{(\beta \beta + \gamma \gamma y y)}},$$

cujus integrale est

$$\frac{1}{\alpha} l [a x + \sqrt{(1 + \alpha \alpha x x)}] = \frac{1}{\gamma} l [\gamma y + \sqrt{(\beta \beta + \gamma \gamma y y)}] - \frac{\gamma}{\alpha} l \Delta,$$

unde ad numeros ascendendo erit

$$\gamma y + \sqrt{(\beta \beta + \gamma \gamma y y)} = \Delta [a x + \sqrt{(1 + \alpha \alpha x x)}]^{\frac{\gamma}{\alpha}}.$$

Posito ergo V pro hac expressione posteriore erit

$$V - \gamma y = \sqrt{(\beta \beta + \gamma \gamma y y)},$$

et sumtis quadratis $y = \frac{V V - \beta \beta}{2 \gamma V}$. Cum igitur sit

$$V = \Delta [a x + \sqrt{(1 + \alpha \alpha x x)}]^{\frac{\gamma}{\alpha}}, \text{ erit}$$

$$2 \gamma y = \Delta [a x + \sqrt{(1 + \alpha \alpha x x)}]^{\frac{\gamma}{\alpha}} -$$

$$\frac{\beta \beta}{\Delta} [a x + \sqrt{(1 + \alpha \alpha x x)}]^{-\frac{\gamma}{\alpha}},$$

ubi est $\beta \beta = C$, exponens vero $\frac{\gamma}{\alpha} = \sqrt{\frac{c}{a}}$, sicque, quoties $\sqrt{\frac{c}{a}}$ fuerit numerus rationalis, integrale semper erit algebraicum.

§. 20. His duobus casibus principalibus expeditis duplicem tradam viam aequationem propositam in infinitas alias ejusdem generis transformandi, ita ut semper aequatio hujus formae

$$\partial \partial Y (1 - a x x) - B x \partial x \partial Y - C Y \partial x^2 = 0$$

prodeat, quae cum resolutionem admittat casibus vel $C = 0$ vel $B = a$, iisdem casibus etiam ipsa aequatio proposita erit resolubilis. Duplices igitur hasce transformationes jam sum expositurus.

Transformationes prioris ordinis.

§. 21. Statuo $y = \frac{\partial v}{\partial x}$, unde ob

$$\partial y = \frac{\partial \partial v}{\partial x} \text{ et } \partial \partial y = \frac{\partial^3 v}{\partial x^2},$$

aequatio nostra induet hanc formam

$$\partial^3 v (1 - a x x) - b x \partial x \partial \partial v - c \partial x^2 \partial v = 0,$$

cujus singuli termini integrationem admittunt: erit enim

$$\int \partial x^2 \partial v = v \partial x^2,$$

$$\int x \partial x \partial \partial v = x \partial x \partial v - v \partial x^2,$$

$$\int \partial^3 v (1 - a x x) = \partial \partial v (1 - a x x) + 2 a x \partial x \partial v - 2 a v \partial x^2.$$

His partibus colligendis, aequatio nostra erit

$$\partial \partial v (1 - a x x) - (b - 2 a) x \partial x \partial v - (c - b + 2 a) v \partial x^2,$$

quae cum propositae prorsus sit similis, integrabilis erit his duobus casibus $c - b + 2 a = 0$ et $b = 3 a$, sive quoties fuerit $c = b - 2 a$ vel $b = 3 a$, atque integration pro utroque casu instituta, ita ut v exprimatur per x , tum pro ipsa aequatione proposita erit $y = \frac{\partial v}{\partial x}$; unde patet, si integralia pro v inventa fuerint algebraica, fore quoque valorem ipsius y algebraicum.

§. 22. Quod si ulterius simili modo statuamus $v = \frac{\partial v'}{\partial x}$, quoniam per operationem praecedentem litterae b et c transibunt in $b - 2a$ et $c - b + 2a$, nunc ista aequatio proveniet

$$\begin{aligned} \partial \partial v' (1 - axx) - (b - 4a)x \partial x \partial v' \\ - (c - 2b + 2a)v' \partial x^2 = 0, \end{aligned}$$

quae ergo integrabilis erit, si fuerit vel $b = 5a$ vel $c = 2b - 6a$. Atque inventis valoribus pro v' fiet $y = \frac{\partial \partial v'}{\partial x^2}$, scilicet differentialia secunda ipsius v' dabunt y : sicque, si pro v' valor algebraicus prodierit, etiam y adipiscetur valorem algebraicum.

§. 23. Quod si eandem substitutionem denuo repetamus ponendo $v' = \frac{\partial v''}{\partial x}$, pro litteris initialibus b et c jam habebimus $b - 6a$ et $c - 3b + 12a$, et aequatio resultans erit

$$\begin{aligned} \partial \partial v'' (1 - axx) - (b - 6a)x \partial x \partial v'' \\ - (c - 3b + 12a)v'' \partial x^2 = 0, \end{aligned}$$

quae ergo resolutionem admittet, quoties fuerit vel $b = 7a$ vel $c = 3b - 12a$, quibus ergo casibus etiam ipsa aequatio proposita resolutionem admittat necesse est, cum sit $y = \frac{\partial^3 v''}{\partial x^3}$.

§. 24. Quod si ergo easdem has operationes continuo repetamus, perpetuo ad aequationes ejusdem formae perveniemus; ubi notasse sufficiet ambos valores, quos pro litteris b et c in qualibet operatione obtinuerimus, quos una cum valoribus ipsius y in sequenti tabula ob oculos ponamus

	b	c	y
Operatio I.	$b - 2a$	$c - b + 2a$	$\frac{\partial v}{\partial x}$
II.	$b - 4a$	$c - 2b + 6a$	$\frac{\partial^2 v}{\partial x^2}$
III.	$b - 6a$	$c - 3b + 12a$	$\frac{\partial^3 v}{\partial x^3}$
IV.	$b - 8a$	$c - 4b + 20a$	$\frac{\partial^4 v}{\partial x^4}$
—	—	—	—
—	—	—	—
—	—	—	—
i	$b - 2ia$	$c - ib + i(i+1)a$	$\frac{\partial^i v^{[i-1]}}{\partial x^i}$

§. 25. Hinc igitur in genere patet, aequationem propositam semper resolutionem admittere, quoties fuerit vel $b = 2ia + a$, vel $c = ib - i(i+1)a$, ubi pro i omnes numeros integros positivos accipere licet, ita ut hinc duos ordines innumerabilium casuum integrabilium nanciscamur, quorum posteriores tantum per methodum serierum initio indicatam reperiuntur, priores vero huic methodo prorsus sint inaccessi.

Transformationes posterioris ordinis.

§. 26. Quemadmodum hic per differentialia sumus progressi, nunc per integralia regrediamur, ac primo quidem ponamus $y = \int z \partial x$, et aequatio proposita evadet

$$\partial z (1 - axx) - bxz \partial x - c \partial x \int z \partial x = 0,$$

quae differentiata ad formam propositam reducitur

$$\partial \partial z (1 - axx) - (b + 2a)x \partial x \partial z - (c + b)z \partial x^2 = 0,$$

quae ergo secundum casus principales integrationem admittet, casibus $c + b = 0$ et $b + 2a = a$, sive $c = -b$ et $b = -a$.

Integralibus igitur inventis erit $y = \int z \partial x$; unde patet etiamsi haec integralia fuerint algebraica, tamen valores ipsius y fieri transcendentes.

§. 27. Simili modo statuamus porro $z = \int z' \partial x$, et quia per praecedentem operationem loco b et c adepti sumus $b + 2a$ et $c + b$, nunc perveniemus ad hanc aequationem

$$\partial \partial z' (1 - axx) - (b + 4a)x \partial x \partial z' - (c + 2b + 2a)z' \partial x^2 = 0,$$
 quae ergo integrationem admittet, si fuerit vel $c + 2b + 2a = 0$, vel $b + 4a = a$, sive $c = -2b - 2a$ et $b = -3a$. Integralibus autem hinc inventis pro y habebimus $y = \int \partial x \int z' \partial x$, quae ita ad signum integrale simplex reducitur, ut sit

$$y = x \int z' \partial x - \int z' x \partial x.$$

§. 28. Simili modo statuamus porro $z' = \int z'' \partial x$, atque nunc deducemur ad hanc aequationem

$$\partial \partial z'' (1 - axx) - (b + 6a)x \partial x \partial z'' - (c + 3b + 6a)z'' \partial x^2 = 0,$$
 quae igitur integrabilis erit, si fuerit vel $c + 3b + 6a = 0$, vel $b + 6a = a$, hoc est si $c = -3b - 6a$ et $b = -5a$; atque ex his integralibus fiet $y = \int \partial x \int \partial x \int z'' \partial x$, qui valor ex praecedente reduci potest, si is per ∂x multiplicatus denuo integretur et loco z' scribatur z'' , obtinetur enim

$$y = \frac{1}{2} x x \int z'' \partial x - x \int x z'' \partial x + \frac{1}{2} \int x x z'' \partial x.$$

§. 29. Quod si jam has operationes ulterius continuemus, totum negotium huc redibit, ut formulae, quae loco b et c sunt proditurae, rite formentur, simulque valores ipsius y assignentur, quemadmodum sequens tabula indicabit

	b	c	y
Operat. I.	$b+2a$	$c+b$	$\int z \partial x$
II.	$b+4a$	$c+2b+2a$	$\int \partial x \int z' \partial x$
III.	$b+6a$	$c+3b+6a$	$\int \partial x \int \partial x \int z'' \partial x$
IV.	$b+8a$	$c+4b+12a$	$\int \partial x \int \partial x \int \partial x \int z''' \partial x$
—	—	—	—
—	—	—	—
—	—	—	—
i	$b+2ia$	$c+ib+i(i-1)a$	$\int \partial x \int \partial x \dots \int z^{[i-1]} \partial x$

§. 30. Ex antecedentibus satis manifestum est, quomodo integralia ista complicata ad simplicia reduci queant, unde tantum sequentem tabulam subjungemus

$$\begin{aligned}
 \int \partial x \int z' \partial x &= x \int z' \partial x - \int z' x \partial x \\
 \int \partial x \int \partial x \int z'' \partial x &= \frac{1}{2} (xx \int z'' \partial x - 2x \int z'' x \partial x + \int z'' xx \partial x) \\
 \int \partial x \int \partial x \int \partial x \int z''' \partial x &= \frac{1}{6} \left\{ x^3 \int z''' \partial x - 3xx \int z''' x \partial x \right. \\
 &\quad \left. + 3x \int z''' xx \partial x - \int z''' x^3 \partial x \right\} \\
 \int \partial x \int \partial x \int \partial x \int \partial x \int z^{IV} \partial x &= \frac{1}{24} \left\{ x^4 \int z^{IV} \partial x - 4x^3 \int z^{IV} x \partial x \right. \\
 &\quad \left. + 6xx \int z^{IV} xx \partial x - 4x \int z^{IV} x^3 \partial x \right. \\
 &\quad \left. + \int z^{IV} x^4 \partial x \right\} \\
 &\text{etc.} \qquad \text{etc.}
 \end{aligned}$$

§. 31. Quod si jam has operationes secundum numerum indefinitum i continuemus, et loco b, c, z , scribamus B, C, Z , aequatio resultans erit

$$\partial \partial Z (1 - axx) - Bx \partial x \partial Z - CZ \partial x^2 = 0,$$

ubi erit, uti jam indicavimus

$$B = b + 2ia \text{ et } C = c + ib + 2ia;$$

quamobrem haec aequatio integrationem admittet, quoties fuerit vel $C \equiv 0$ hoc est $c \equiv -ib - i(i - i)a$, vel $B \equiv a$ hoc est $b \equiv -(2i - 1)a$: quae formulae ab illis quas supra prior transformationum ordine invenimus, tantum in hoc discrepant, quod hic littera i valorem negativum accepit; unde adjungatur sequens

Conclusio generalis.

§. 32. Si littera i hic denotet omnes numeros integros sive positivos sive negativos, aequatio proposita differentio-differentialis

$$\partial \partial y (1 - axx) - bx \partial x \partial y - cy \partial x^2 \equiv 0$$

semper integrationem sive resolutionem admittet, quoties fuerit

$$1^{\circ}) 0 \equiv ib - i(i + 1)a, \text{ vel}$$

$$2^{\circ}) b \equiv (2i + 1)a:$$

ubi asseverare licet, omnes plane casus resolubiles in hac duplici forma contineri, ita ut nullus plane casus integrationem admittens exhiberi queat, qui non in alterutra harum duarum formularum comprehendatur, dum contra methodus per series procedens, cujus initio mentionem fecimus, tantum casus integrabiles priores ostendit, ita ut inde infinitus numerus casuum pariter resolubilium inde excludatur.

Corollarium 1.

§. 33. Transformetur aequatio proposita in aequationem differentialem primi gradus ponendo $y = e^{\int u \partial x}$, ac perveniemus ad hanc aequationem

$$\partial u + uu \partial x - \frac{bx \partial x - c \partial x}{1 - axx} \equiv 0,$$

quae ergo etiam integrationem admittet casibus quibus vel $b \equiv$

$(2i + 1)a$ vel $c = ib - i(i + 1)a$, denotante i numerum quemcunque integrum sive positivum sive negativum.

COROLLARIUM 2.

§. 34. Quod si porro ponatur $u = (1 - axx)^n v$, posito brevitate gratia $n = -\frac{b}{2a}$, pervenietur ad hanc aequationem ad genus *Riccatianum* referendam

$$(1 - axx)^n \partial v + (1 - axx)^{2n} v v \partial x = \frac{c \partial x}{1 - axx},$$

quae per $(1 - axx)^n$ divisa abit in hanc

$$\partial v + (1 - axx)^n v v \partial x = \frac{c \partial x}{(1 - axx)^{n+1}},$$

quae ergo iisdem casibus integrationem admittet.

COROLLARIUM 3.

§. 35. Quod si sumamus $a = 0$, orietur ista aequatio

$$\partial u + uu \partial x = bux \partial x + c \partial x,$$

quae ergo integrabilis erit, si fuerit vel $b = 0$ vel $c = ib$, quorum quidem prior casus per se est manifestus, quia tum erit $\partial x = \frac{\partial u}{c - uu}$. Haec forma autem commodius exprimi poterit, ponendo

$$u = \frac{1}{2}bx + v, \text{ unde } \partial v + vv \partial x = (c - \frac{1}{2}b) \partial x + \frac{1}{4}bbxx \partial x,$$

sive ponendo $b = 2f$, ut fiat

$$\partial v + vv \partial x = (c - f) \partial x + ffx \partial x,$$

eritque haec aequatio integrabilis, quoties fuerit $c = 2if$, ita ut sequens aequatio semper integrationem admittat

$$\partial v + vv \partial x = (2i - 1)f \partial x + ffx \partial x,$$

quicunque numerus integer sive positivus sive negativus pro i accipiatur; hoc est, si in penultimo termino f multiplicetur per numerum imparem quemcunque sive positivum sive negativum, qui

casus eo erunt abstrusiores, quo major accipiat numerus i ; atque adeo vix alia via patere videtur integralia eruendi, nisi ut ad aequationem differentialem secundi gradus propositam regrediamur atque easdem operationes instituamus quas supra docuimus. Interim tamen observavi, omnes istos casus etiam immediate ex ipsa aequatione per fractiones continuas derivari posse. Si enim proposita fuerit haec aequatio

3.) De formulis integralibus implicatis, earumque evolutione et transformatione. *M. S. Academiae exhib. die 20 Aprilis 1778.*

§. 36. Talium formularum implicatarum forma generalis ita exhiberi potest

$$\int p \partial x \int q \partial x \int r \partial x \int s \partial x \text{ etc.}$$

ubi quodvis signum integrale omnia sequentia in se complectitur. Ita ad valorem hujus expressionis inveniendum a fine est incipiendum, positoque integrali $\int s \partial x = S$ erit

$$\int r \partial x \int s \partial x = \int S r \partial x,$$

cujus valor si ponatur $= R$, erit

$$\int q \partial x \int r \partial x \int s \partial x = \int R q \partial x,$$

quod integrale si ponatur $= Q$, valor ipsius formulae propositae erit $= \int Q p \partial x$, ubi per se intelligitur, in qualibet integratione more solito constantem arbitrariam in calculum introduci posse.

* §. 37. Hic scilicet probe tenendum est, istam expressionem $\int p \partial x \int q \partial x$ non significare productum ex formula $\int p \partial x$ in formulam $\int q \partial x$, sed integrale quod oritur, si tota formula differentialis $p \partial x \int q \partial x$ integretur: at vero si velimus productum talium duarum formularum integralium designare, id interpositione puncti fieri solet hoc modo $\int p \partial x . \int q \partial x$, ubi scilicet punctum declarat praecedentia signa integralia non ultra hunc terminum extendi debere, ita haec forma

$$\int p \partial x \int q \partial x . \int r \partial x \int s \partial x$$

exprimit productum, quod oritur si formula $\int p \partial x \int q \partial x$ multiplicetur per $\int r \partial x \int s \partial x$.

§. 38. Hic igitur signandi nos prorsus contrarius usu est receptus, atque in formulis differentialibus observari solet, ubi talis expressio $\partial x \partial y \partial z$ denotat productum trium differentialium ∂x , ∂y et ∂z , ita ut singula signa differentiationis tantum litteras immediate sequentes afficiant: at si velimus verbi gratia differentiale hujus expressionis $x \partial y \partial z$ exprimere, hoc interpositione puncti fieri solet $\partial . x \partial y \partial z$, ubi punctum significat, praefixum ∂ complecti totam expressionem sequentem.

§. 39. Tales autem formulae integrales implicatae potissimum nascuntur ex continua integratione aequationum integralium linearium, quarum forma in genere est

$$p z + \frac{q \partial z}{\partial x} + \frac{r \partial \partial z}{\partial x^2} + \frac{s \partial^3 z}{\partial x^3} + \text{etc.} = X,$$

ubi litterae p , q , r , s , etc. sunt functiones datae variabilis x , cujus etiam functio quaecunque sit littera X , altera vero variabilis z ubique una tantum tenet dimensionem, prouti haec forma generalis hic exhibetur, ad ordinem tertium differentialium refertur, ideoque ternas integrationes postulat, totidemque constantes arbitrarias involvere est censenda, hic scilicet ad methodum integrandi maxime naturalem respicio, quae per ternas integrationes successivas integrale desideratum producat.

§. 40. Tali scilicet aequatione proposita ante omnia nosse oportet multiplicatorem, quo ea reddatur integrabilis, quem ergo supponamus esse $= \partial P$, atque integratione peracta prodeat ista aequatio

$$p' z + \frac{q' \partial z}{\partial x} + \frac{r' \partial \partial z}{\partial x^2} = \int X \partial P,$$

quae aequatio jam est ordinis secundi; quodsi jam ponamus hujus multiplicatorem idoneum esse $= \partial P'$, facta integration oriatur haec aequatio primi ordinis, quae sit

$$p'' z + \frac{q'' \partial z}{\partial x} = \int \partial P' \int X \partial P,$$

pro qua si $\partial P''$ fuerit multiplicator idoneus, completum integrale induet hanc formam

$$p''' z = \int \partial P'' \int \partial P' \int X \partial P.$$

Sicque quantitas z exprimetur per formulam integram implicatam.

§. 41. Tali autem forma pro integrali inventa praecipuum negotium huc redit, ut ea ita evolvatur, ut formula continens functionem indefinitam X , quae hic terna signa integralia habet praefixa, plus unico ante se non habeat, quamobrem quemadmodum talis reductio commodissime institui queat, hic ostendere constitui, siquidem nisi certa artificia adhibeantur, hujusmodi operatio calculos maxime molestos postularet.

§. 42. In genere autem hujusmodi formulas implicatas ita repraesentemus

$$\int \partial p \int \partial q \int \partial r \int \partial s \int \partial t \text{ etc.}$$

pro cujus evolutione a casu duorum signorum integralium inchoemus, et quia erit $\int \partial p \int \partial q = \int q \partial p$, reductio vulgaris dat $p q - \int p \partial q$. Jam loco p et q iterum scribamus $\int \partial p$ et $\int \partial q$, atque evolutio ita se habebit

$$\int \partial p \int \partial q = \int \partial p \cdot \int \partial q - \int \partial q \int \partial p,$$

ubi in genere hanc aequalitatem notasse juvabit

$$\int \partial p \int \partial q - \int \partial p \cdot \int \partial q + \int \partial q \int \partial p = 0.$$

§. 43. Consideremus nunc formulam tria signa integralia involventem $= \int \partial p \int \partial q \int \partial r$, et quia ut modo vidimus est $\int \partial q \int \partial r = q r - \int q \partial r$, nostra formula in has partes discerpitur $\int q r \partial p - \int \partial p \int q \partial r$, quae posterior pars reducitur ad hanc formam $p \int q \partial r - \int p q \partial r$, sicque formula nostra erit

$\int q r \partial p - p \int q \partial r + \int p q \partial r$. Quoniam nunc requiritur, ut elementum ∂r in singulis partibus unicum tantum signum integrale habeat praefixum; ponamus $q \partial p = \partial v$ ut sit

$$v = \int q \partial p = \int \partial p \int \partial q, \text{ eritque}$$

$$\int q r \partial p = \int r \partial v = r v - \int v \partial r,$$

hincque colligitur

$$\int p q \partial r - \int v \partial r = \int \partial r (p q - v) = \int \partial r \int p \partial q.$$

Jam loco litterarum finitarum differentialia rursus introducentur, atque valor quaesitus formulae $\int \partial p \int \partial q \int \partial r$ sequenti modo exprimetur

$$\int \partial p \int \partial q \int \partial r - \int \partial p \cdot \int \partial r \int \partial q + \int \partial r \int \partial q \int \partial p,$$

ubi in singulis membris elemento ∂r unicum signum integrale est praefixum.

§. 44. Inter terna igitur elementa ∂p , ∂q et ∂r sequentem relationem notari operae erit pretium

$$\int \partial p \int \partial q \int \partial r - \int \partial p \int \partial q \cdot \int \partial r + \int \partial p \cdot \int \partial r \int \partial q - \int \partial r \int \partial q \int \partial p = 0,$$

quodsi autem similem reductionem pro casibus plurium signorum integralium exsequi vellemus, in calculos molestissimos ac taediosissimos delaberemur; interim tamen totum hoc negotium per sequentia theoremata facillime et planissime expedietur, et quoniam singula membra ope puncti in duos factores resolvi convenit, ubi talis factor deest, ejus locum unitate supplebimus.

Theorema 1.

§. 45. Pro unico elemento ∂p haec relatio habetur $\int \partial p \cdot 1 - 1 \cdot \int \partial p = 0$, maxime obvia.

Theorema 2.

§. 46. Inter bina elementa ∂p et ∂q semper locum habebit haec relatio

$$\int \partial p \int \partial q . 1 - \int \partial p . \int \partial q + 1 . \int \partial q \int \partial p = 0.$$

Demonstratio.

Ad hoc demonstrandum sufficiet ostendisse, differentiale hujus aequationis esse $= 0$, quoniam vero singula membra binis constant factoribus, seorsim considerentur differentialia ex factoribus prioribus et posterioribus oriunda, hic igitur ex factoribus prioribus oritur differentiale $\partial p (\int \partial q . 1 - 1 . \int \partial q) = 0$ per theorema 1. At ex factoribus posterioribus oritur differentiale $-\partial q (\int \partial p . 1 - 1 . \int \partial p) = 0$.

Theorema 3.

§. 47. Inter terna elementa ∂p , ∂q et ∂r semper haec relatio locum habet

$$\int \partial p \int \partial q \int \partial r . 1 - \int \partial p \int \partial q . \int \partial r + \int \partial p . \int \partial r \int \partial q - 1 . \int \partial r \int \partial q \int \partial p = 0.$$

Demonstratio.

Hic iterum seorsim perpendantur differentialia tam ex prioribus quam ex posterioribus factoribus oriunda; ex prioribus autem oritur

$\partial p (\int \partial q \int \partial r . 1 - \int \partial q . \int \partial r + 1 . \int \partial r \int \partial q)$,
cujus valor manifesto ad nihilum redigitur per theorema 2. si scilicet litterae p et q uno gradu promoveantur; tum vero differentiale ex factoribus posterioribus ortum est

$-\partial r (\int \partial p \int \partial q . 1 - \int \partial p . \int \partial q + 1 . \int \partial q \int \partial p)$,
cujus valor pariter per theorema praecedens evanescit, quoniam

igitur ambo differentialia sunt $\equiv 0$, etiam ipsa forma nihilo vel etiam constanti aequalis esse debet, evidens autem est constantem sponte involvi in signis integralibus.

Theorema 4.

§. 48. Inter quaterna elementa ∂p , ∂q , ∂r et ∂s semper ista relatio locum habet

$$\left. \begin{aligned} & \int \partial p \int \partial q \int \partial r \int \partial s . 1 - \int \partial p \int \partial q \int \partial r . \int \partial s \\ & + \int \partial p \int \partial q . \int \partial s \int \partial r - \int \partial p . \int \partial s \int \partial r \int \partial q \\ & + 1 . \int \partial s \int \partial r \int \partial q \int \partial p \end{aligned} \right\} = 0.$$

Demonstratio.

Differentiatio factorum priorum suppeditat sequentem expressionem

$\partial p (\int \partial q \int \partial r \int \partial s . 1 - \int \partial q \int \partial r . \int \partial s + \int \partial q . \int \partial r \int \partial s - 1 \int \partial q \int \partial r \int \partial s),$
 quae ob theorema praecedens ad nihilum reducitur. Simili modo differentiatio factorum posteriorum praebet hanc expressionem
 $-\partial s (\int \partial p \int \partial q \int \partial r . 1 - \int \partial p \int \partial q . \int \partial r + \int \partial p . \int \partial r \int \partial q - 1 . \int \partial r \int \partial q \int \partial p),$
 quae ob theorema 3. iterum est $\equiv 0$.

Theorema 5.

§. 49. Inter quina elementa ∂p , ∂q , ∂r , ∂s et ∂t semper haec relatio locum habet

$$\left. \begin{aligned} & \int \partial p \int \partial q \int \partial r \int \partial s \int \partial t . 1 - \int \partial p \int \partial q \int \partial r \int \partial s . \int \partial t \\ & + \int \partial p \int \partial q \int \partial r . \int \partial t \int \partial s - \int \partial p \int \partial q . \int \partial t \int \partial s \int \partial r \\ & + \int \partial p . \int \partial t \int \partial s \int \partial r \int \partial q - 1 . \int \partial t \int \partial s \int \partial r \int \partial q \int \partial p \end{aligned} \right\} = 0.$$

D e m o n s t r a t i o.

Hujus theorematis demonstratio prorsus eodem modo se habet ac theorematum praecedentium; sicque clarissime jam est evictum tales relationes perpetuo veritati esse consentaneas, quotcunque etiam elementis fuerint composita.

§. 50. Quo vis horum theorematum clarius perspicatur, operae pretium erit, ea per exempla determinata illustrasse; ponamus igitur esse

$$\partial p = x^{\alpha-1} \partial x, \quad \partial q = x^{\beta-1} \partial x, \quad \partial r = x^{\gamma-1} \partial x, \\ \partial s = x^{\theta-1} \partial x, \quad \partial t = x^{\epsilon-1} \partial x,$$

atque ex theoremate primo statim aequatio identica nascitur

$$\frac{x^{\alpha}}{\alpha} - \frac{x^{\alpha}}{\alpha} = 0. \quad \text{Verum theorema secundum nobis praebet hanc}$$

aequationem

$$\frac{x^{\alpha+\beta}}{\beta(\alpha+\beta)} - \frac{x^{\alpha+\beta}}{\alpha\beta} + \frac{x^{\alpha+\beta}}{\alpha(\alpha+\beta)} = 0,$$

unde per $x^{\alpha+\beta}$ dividendo prodit haec aequalitas

$$\frac{1}{\beta(\alpha+\beta)} - \frac{1}{\alpha\beta} + \frac{1}{\alpha(\alpha+\beta)} = 0,$$

cujus veritas satis facile in oculos incurrit.

§. 51. Hac porro positiones in theoremate tertio introductae producent hanc aequationem

$$\frac{x^{\alpha+\beta+\gamma}}{\gamma(\alpha+\beta+\gamma)(\beta+\gamma)} - \frac{x^{\alpha+\beta+\gamma}}{\beta\gamma(\alpha+\beta)} \\ + \frac{x^{\alpha+\beta+\gamma}}{\alpha\beta(\beta+\gamma)} - \frac{x^{\alpha+\beta+\gamma}}{\alpha(\alpha+\beta)(\alpha+\beta+\gamma)},$$

unde per $x^{\alpha+\beta+\gamma}$ dividendo prodit haec egregia aequalitas

$$\frac{1}{\gamma(\beta+\gamma)(\alpha+\beta+\gamma)} - \frac{1}{\beta\gamma(\alpha+\beta)} + \frac{1}{\alpha\beta(\beta+\gamma)} - \frac{1}{\alpha(\alpha+\beta)(\alpha+\beta+\gamma)} = 0,$$

§. 52. Hae positiones iterum in theoremate quarto substitutae dant hanc aequationem

$$\left. \begin{aligned} & \frac{x^{\alpha+\beta+\gamma+\delta}}{\delta(\delta+\gamma)(\delta+\gamma+\beta)(\delta+\gamma+\beta+\alpha)} - \frac{x^{\alpha+\beta+\gamma+\delta}}{\gamma\delta(\gamma+\beta)(\gamma+\beta+\alpha)} \\ & + \frac{x^{\alpha+\beta+\gamma+\delta}}{\beta\gamma(\beta+\alpha)(\gamma+\delta)} - \frac{x^{\alpha+\beta+\gamma+\delta}}{\alpha\beta(\beta+\gamma)(\beta+\gamma+\delta)} \\ & + \frac{x^{\alpha+\beta+\gamma+\delta}}{\alpha(\alpha+\beta)(\alpha+\beta+\gamma)(\alpha+\beta+\gamma+\delta)} \end{aligned} \right\} = 0,$$

quae per $x^{\alpha+\beta+\gamma+\delta}$ divisa producit hanc aequationem

$$\left. \begin{aligned} & \frac{1}{\delta(\delta+\gamma)(\delta+\gamma+\beta)(\delta+\gamma+\beta+\alpha)} - \frac{1}{\delta\gamma(\gamma+\beta)(\gamma+\beta+\alpha)} \\ & + \frac{1}{\gamma\beta(\beta+\alpha)(\gamma+\delta)} - \frac{1}{\alpha\beta(\beta+\gamma)(\beta+\gamma+\delta)} \\ & + \frac{1}{\alpha(\alpha+\beta)(\alpha+\beta+\gamma)(\alpha+\beta+\gamma+\delta)} \end{aligned} \right\} = 0.$$

§. 53. Denique eadem positiones in theoremate quinto substitutae producant hanc aequationem

$$\left. \begin{aligned} & \frac{x^{\alpha+\beta+\gamma+\delta+\epsilon}}{\epsilon(\epsilon+\delta)(\epsilon+\delta+\gamma)(\epsilon+\delta+\gamma+\beta)(\epsilon+\delta+\gamma+\beta+\alpha)} \\ & - \frac{x^{\alpha+\beta+\gamma+\delta+\epsilon}}{\epsilon\delta(\delta+\gamma)(\delta+\gamma+\beta)(\delta+\gamma+\beta+\alpha)} \\ & + \frac{x^{\alpha+\beta+\gamma+\delta+\epsilon}}{\delta\gamma(\gamma+\beta)(\gamma+\beta+\alpha)(\delta+\epsilon)} \\ & - \frac{x^{\alpha+\beta+\gamma+\delta+\epsilon}}{\beta\gamma(\beta+\alpha)(\gamma+\delta)(\gamma+\delta+\epsilon)} \\ & + \frac{x^{\alpha+\beta+\gamma+\delta+\epsilon}}{\alpha\beta(\beta+\gamma)(\beta+\gamma+\delta)(\beta+\gamma+\delta+\epsilon)} \\ & - \frac{x^{\alpha+\beta+\gamma+\delta+\epsilon}}{\alpha(\alpha+\beta)(\alpha+\beta+\gamma)(\alpha+\beta+\gamma+\delta)(\alpha+\beta+\gamma+\delta+\epsilon)} \end{aligned} \right\} = 0,$$

quae per $x^{\alpha} + \beta + \gamma + \delta + \varepsilon$ divisa dat hanc aequationem maxime notatu dignam

$$\left. \begin{aligned} & \frac{1}{\varepsilon(\varepsilon + \delta)(\varepsilon + \delta + \gamma)(\varepsilon + \delta + \gamma + \beta)(\varepsilon + \delta + \gamma + \beta + \alpha)} \\ & - \frac{1}{\delta\varepsilon(\delta + \gamma)(\delta + \gamma + \beta)(\delta + \gamma + \beta + \alpha)} \\ & + \frac{1}{\gamma\delta(\gamma + \beta)(\gamma + \beta + \alpha)(\delta + \varepsilon)} \\ & - \frac{1}{\beta\gamma(\beta + \alpha)(\gamma + \delta)(\gamma + \delta + \varepsilon)} \\ & + \frac{1}{\alpha\beta(\beta + \gamma)(\beta + \gamma + \delta)(\beta + \gamma + \delta + \varepsilon)} \\ & - \frac{1}{\alpha(\alpha + \beta)(\alpha + \beta + \gamma)(\alpha + \beta + \gamma + \delta)(\alpha + \beta + \gamma + \delta + \varepsilon)} \end{aligned} \right\} = 0.$$

§. 54. Haec theorematibus eo magis sunt memorabilia, quod eorum veritas non nisi per plures ambages in numeris explorari potest, ideoque multo majorem attentionem merentur, quam aliud simile theorema, ad quod nuper sum perductus, quippe cujus demonstratio haud difficulter exhiberi potest, quod ita se habet.

Theorema numericum.

Suntis pro lubitu quocunque numeris veluti quatuor $\alpha, \beta, \gamma, \delta$, si hinc totidem alii sequenti modo formentur

$$a = \alpha, \quad b = \alpha + \beta,$$

$$c = \alpha + \beta + \gamma \quad \text{et} \quad d = \alpha + \beta + \gamma + \delta,$$

similique modo etiam isti

$$D = \delta, \quad C = \delta + \gamma$$

$$B = \delta + \gamma + \beta \quad \text{et} \quad A = \delta + \gamma + \beta + \alpha,$$

tam semper erit

$$\frac{1}{abcd} - \frac{1}{abcD} + \frac{1}{abCD} - \frac{1}{aBCD} + \frac{1}{ABCD} = 0.$$

Demonstratio.

§. 55. Binae fractiones priores inventae, ob $D - d = -c$, dant fractionem $-\frac{1}{abdD}$, quae cum tertia conjuncta producit $\frac{1}{adCD}$, cui quarta fractio juncta dat $-\frac{1}{dBCD}$, quae [ob $d = A$] a termino ultimo penitus destruitur.

§. 56. Ope superiorum theorematum omnes formulae integrales implicatae, ad quas integratio aequationum linearum perducere solet, facile resolvi poterunt. Pervenitur autem plerumque ad tales formas:

$$Z = \int \partial q \int X \partial p, \quad Z = \int \partial r \int \partial q \int X \partial p,$$

$$Z = \int \partial s \int \partial r \int \partial q \int X \partial p, \quad Z = \int \partial t \int \partial s \int \partial r \int \partial q \int X \partial p \text{ etc.}$$

ubi litterae p, q, r, s, t , etc. sunt functiones datae ipsius X , at vero X functio quaecunque ipsius x ; atque hic tota resolutio ita institui debet, ut in singulis membris functio haec indefinita X unicum tantum signum integrale habeat praefixum: hoc igitur, ope superiorum theorematum, facile praestari poterit, si modo ibi loco elementi ∂p scribamus $X \partial p$, quo observato singulae reductiones sequenti modo se habebunt.

I. Resolutio
formulae integralis

$$\int \partial q \int X \partial p.$$

§. 57. Si loco ∂p scribamus $X \partial p$ theorema secundum §. 46. nobis suppeditat hanc aequationem:

$$\int X \partial p \int \partial q - \int X \partial p \cdot \int \partial q + \int \partial q \int X \partial p = 0,$$

cujus postremum membrum est ipsa nostra forma reducenda Z , consequenter resolutio statim dat

$$Z = \int \partial q \cdot \int X \partial p - \int X \partial p \int \partial q.$$

ideoque ob $f \partial q = q$ habebimus

$$Z = q \int X \partial p - \int X q \partial p.$$

Corollarium.

§. 58. Si fuerit $q = p$, erit

$$Z = p \int X \partial p - \int X p \partial p.$$

II. Resolutio

formulae implicatae

§. 59. Pro hoc casu sumamus theorema 3. §. 47. unde, si loco ∂p scribatur $X \partial p$, deducimus hanc aequationem $\int X \partial p f \partial q f \partial r - \int X \partial p f \partial q \cdot \int \partial r + \int X \partial p \cdot \int \partial r f \partial q - \int \partial r f \partial q \int X \partial p = 0$, cujus postremum membrum est ipsa forma reducenda Z , hincque adeoque colligitur

$$Z = \int \partial r f \partial q \cdot \int X \partial p - \int \partial r \cdot \int X \partial p f \partial q + \int X \partial p f \partial q \int \partial r,$$

quae ergo reducta dat

$$Z = \int q \partial r \cdot \int X \partial p - r \int X q \partial p + \int X \partial p \int r \partial q.$$

Corollarium.

§. 60. Si ergo hic fuerit $q = r = p$, prodibit ista resolutio:

$$Z = \int \partial p f \partial p \int X \partial p = \frac{1}{2} p p \int X \partial p - p \int X p \partial p + \frac{1}{2} \int X p p \partial p.$$

III. Resolutio

hujus formulae implicatae

$$Z = \int \partial s \int \partial r \int \partial q \int X \partial p.$$

§. 61. Pro hoc casu sumamus theorema 4. §. 48. unde si loco ∂p scribatur $X \partial p$ deducimus hanc aequationem

$$\left. \begin{aligned} & \int X \partial p \int \partial q \int \partial r \int \partial s - \int X \partial p \int \partial q \int \partial r. \int \partial s + \int X \partial p \int \partial q. \int \partial s \int \partial r \\ & - \int X \partial p. \int \partial s \int \partial r \int \partial q + \int \partial s \int \partial r \int \partial q \int X \partial p \end{aligned} \right\} = 0,$$

cujus postremum membrum est ipsa nostra formula reducenda Z;
hincque adeo colligimus

$$Z = \left\{ \begin{aligned} & \int \partial s \int \partial r \int \partial q. \int X \partial p - \int \partial s \int \partial r. \int X \partial p \int \partial q \\ & + \int \partial s. \int X \partial p \int \partial q \int \partial r - \int X \partial p \int \partial q \int \partial r \int \partial s, \end{aligned} \right.$$

quae ergo reducta praebet

$$Z = \left\{ \begin{aligned} & \int \partial s \int q \partial r. \int X \partial p - \int r \partial s. \int X q \partial p + s \int X \partial q \int r \partial q \\ & - \int X \partial p \int \partial q \int s \partial r. \end{aligned} \right.$$

Corollarium.

§. 62. Si ponatur $s = r = q = p$, tum prodibit ista
resolutio

$$Z = \left\{ \begin{aligned} & \frac{1}{6} p^3 \int X \partial p - \frac{1}{2} p p \int X p \partial p + \frac{1}{2} p \int X p p \partial p \\ & - \frac{1}{6} \int X p^3 \partial p. \end{aligned} \right.$$

IV. Resolutio

hujus formulae implicatae.

$$Z = \int \partial t \int \partial s \int \partial r \int \partial q \int X \partial p.$$

§. 63. Pro hoc casu sumamus theorema 5. §. 49. unde
si loco ∂p scribatur $X \partial p$, prodibit ista aequatio

$$\left. \begin{aligned} & \int X \partial p \int \partial q \int \partial r \int \partial s \int \partial t - \int X \partial p \int \partial q \int \partial r \int \partial s. \int \partial t \\ & + \int X \partial p \int \partial q \int \partial r. \int \partial t \int \partial s - \int X \partial p \int \partial q. \int \partial t \int \partial s \int \partial r \\ & + \int X \partial p. \int \partial t \int \partial s \int \partial r \int \partial q - \int \partial t \int \partial s \int \partial r \int \partial q \int X \partial p \end{aligned} \right\} = 0,$$

cujus postremum membrum est ipsa nostra forma reducenda Z,
unde ergo prodit

$$Z = \begin{cases} \int \partial t \int \partial s \int \partial r \int \partial q . f X \partial p - \int \partial t \int \partial s \int \partial r . f X \partial p \int \partial q \\ + \int \partial t \int \partial s . f X \partial p \int \partial q \int \partial r - \int \partial t . f X \partial p \int \partial q \int \partial r \int \partial s \\ + \int X \partial p \int \partial q \int \partial r \int \partial s \int \partial t, \end{cases}$$

quae ergo reducta praebet

$$Z = \begin{cases} \int \partial t \int \partial s \int q \partial r . f X \partial p - \int \partial t \int r \partial s . f X q \partial p \\ + \int s \partial t . f X \partial p \int r \partial q - \int t \int X \partial p \int \partial q \int s \partial r \\ + \int X \partial p \int \partial q \int \partial r \int t \partial s, \end{cases}$$

Corollarium.

§. 64. Si hic sumatur $t = s = r = q = p$, tum prodibit ista resolutio

$$Z = \begin{cases} \frac{1}{24} p^4 \int X \partial p - \frac{1}{6} p^3 \int X p \partial p + \frac{1}{4} p p \int X p p \partial p \\ \frac{1}{6} p \int X p^3 \partial p + \frac{1}{24} \int X p^4 \partial p. \end{cases}$$

§. 65. Quo indoles harum resolutionum clarius perspicia-
tūr, quoniam litterae p, q, r, s, t , functiones datas ipsius x de-
notant, ideoque omnes expressiones ex iis formatae pariter ut cog-
nitae spectari possunt, statuamus brevitatis gratia

$$\partial p \int \partial q = \partial p'; \quad \partial p \int \partial q \int \partial r = \partial p''; \quad \partial p \int \partial q \int \partial r \int \partial s = \partial p''';$$

$$\partial p \int \partial q \int \partial r \int \partial s \int \partial t = \partial p'''; \text{ etc.}$$

hocque modo postrema resolutio ita referetur

$$Z = \int \partial t \int \partial s \int \partial r \int \partial q . f X \partial p - \int \partial t \int \partial s \int \partial r . f X \partial p' \\ + \int \partial t \int \partial s . f X \partial p'' - \int \partial t . f X \partial p''' + \int X \partial p''''.$$

Quod si hic porro statuamus

$$\int \partial t \int \partial s = \int s \partial t = t'; \quad \int \partial t \int \partial s \int \partial r = t''; \quad \int \partial t \int \partial s \int \partial r \int \partial q = t''';$$

tota resolutio hoc modo concinne repraesentabitur

$$Z = t''' \int X \partial p - t'' \int X \partial p' + t' \int X \partial p'' - t \int X \partial p''' \\ + \int X \partial p''',$$

quam repraesentationem etiam ad praecedentes resolutiones accommodasse juvabit.

§. 66. Cum igitur integratio formulae implicatae

$$Z = \int \partial t \int \partial s \int \partial r \int \partial q \int X \partial p$$

reducatur ad integrationem sequentium formularum integralium simplicium: $\int x \partial p$; $\int x \partial p'$; $\int x \partial p''$; $\int x \partial p'''$; $\int x \partial p''''$; quaestio hinc oritur non parum curiosa: quemadmodum ex his formulis simplicibus vicissim quantitates q , r , s et t concludi queant? quod sequenti modo facile praestabitur. Cum sit $\partial p' = \partial p \int \partial q$, erit $\int \partial q = q = \frac{\partial p'}{\partial p}$. Ponatur nunc porro $\frac{\partial p''}{\partial p} = q'$; $\frac{\partial p'''}{\partial p} = q''$; $\frac{\partial p''''}{\partial p} = q'''$; etc. quibus valoribus introductis habebimus

$$q' = \int \partial q \int \partial r; \quad q'' = \int \partial q \int \partial r \int \partial s;$$

$$q''' = \int \partial q \int \partial r \int \partial s \int \partial t; \text{ etc.}$$

Quoniam igitur hi valores q , q' , q'' , q''' , q'''' sunt dati, ex prima statim colligimus $\int \partial r = \frac{\partial q'}{\partial q} = r$. Ponamus autem porro $\frac{\partial q''}{\partial q} = r'$; $\frac{\partial q'''}{\partial q} = r''$; etc. eruntque etiam hi valores, r , r' , r'' , etc. dati, quibus substitutis habebitur $r' = \int \partial r \int \partial s$; $r'' = \int \partial r \int \partial s \int \partial t$; ex quarum prima sequitur $\int \partial s = s = \frac{\partial r'}{\partial r}$. Quare si porro fiat $s' = \frac{\partial r''}{\partial r}$, erit quoque $s' = \int \partial s \int \partial t$, hincque $\int \partial t = t = \frac{\partial s'}{\partial s}$. Ex his clare intelligitur, quomodo hae formulae inveniri queant pro casibus adhuc magis complicatis.

§. 67. Superest, ut etiam de transformatione talium formularum integralium implicatarum pauca adjiciamus, quod totum negotium sequenti problemate includi potest.

P r o b l e m a.

§. 68. *Proposita formula implicata terna signa summatoria involvente $\int \partial p \int \partial q \int \partial r$, investigare aliam similem formulam*

$$\int \partial P \int \partial Q \int \partial R,$$

illi aequalem.

S o l u t i o.

Per theorema 2. supra allatum formula proposita ita est resoluta

$$\int \partial q \int \partial r = \int \partial q . \int \partial r - \int \partial r \int \partial q = q \int \partial r - \int q \partial r.$$

Simili modo pro formula quaesita erit

$$\int \partial Q \int \partial R = Q \int \partial R - \int Q \partial R,$$

requiritur igitur ut sit

$$q \partial p \int \partial r - \partial p \int q \partial r = Q \partial P \int \partial R - \partial P \int Q \partial R,$$

quae aequalitas adimpleretur, sumendo $P = p$, $Q = q$ et $R = r$; verum permutandis membris statuamus

$$Q \partial P \int \partial R = - \partial p \int q \partial r \text{ et } \partial P \int Q \partial R = - q \partial p \int \partial r,$$

atque ex priore aequatione deducimus $Q \partial P = - \partial p$, ideoque $\partial P = - \frac{\partial p}{Q}$, tum vero $\partial R = q \partial r$; ex altera vero aequatione habemus $\partial P = - q \partial p$ et $Q \partial R = \partial r$. Cum igitur esset $\partial P = - \frac{\partial p}{Q}$, erit $Q = \frac{1}{q}$, hincque porro $\partial R = q \partial r$, unde ob $Q = \frac{1}{q}$, erit $\partial Q = - \frac{\partial q}{q^2}$. Consequenter formula integralis quaesita proposita $\int \partial p \int \partial q \int \partial r$ aequalis erit

$$\int q \partial p \int \frac{\partial q}{q^2} \int q \partial r,$$

unde patet perpetuo loco formulae $\int \partial p \int \partial q \int \partial r$, scribi posse istam: $\int q \partial p \int \frac{\partial q}{q^2} \int q \partial r$.

Corollarium 1.

§. 69. Quando igitur plura signa integralia sibi invicem fuerint involuta, veluti si habeamus $\int \partial p \int \partial q \int \partial r \int \partial s$, ista transformatio in quibusvis ternis signis se mutuo sequentibus institui poterit, unde in hac formula proposita duplex transformatio adhiberi poterit; prior scilicet in ternis signis prioribus praebebit

$$\int q \partial p \int \frac{\partial q}{q} \int q \partial r \int \partial s,$$

at vero in ternis posterioribus haec transformatio adhibita dabit

$$\int \partial p \int r \partial q \int \frac{\partial r}{rr} \int r \partial s,$$

Corollarium 2.

§. 70. Hinc porro ope ejusdem transformationis aliae insuper fieri possunt, veluti ex postrema forma

$$\int \partial p \int r \partial q \int \frac{\partial r}{rr} \int r \partial s,$$

ut in ternis prioribus signis res expediri queat, loco $r \partial q$ scribamus ∂v , ut habeamus

$$\int \partial p \int \partial v \int \frac{\partial r}{rr} \int r \partial s,$$

quae transformatur in hanc

$$\int v \partial p \int \frac{\partial v}{vv} \int \frac{v \partial r}{rr} \int r \partial s,$$

quae omnes formulae ipsi propositae sunt prorsus aequales.

§. 71. Ut rem exemplo illustremus, sumamus esse $p = x^\alpha$; $q = x^\beta$; $r = x^\gamma$, ita ut formula proposita sit

$$\alpha \beta \gamma \int x^{\alpha-1} \partial x \int x^{\beta-1} \partial x \int x^{\gamma-1} \partial x = \frac{\alpha \beta x^{\gamma+\beta+\alpha}}{(\gamma+\beta)(\gamma+\beta+\alpha)}.$$

Jam pro transformatione erit primo

$$\int q \partial r = \frac{\gamma x^{\beta+\gamma}}{\beta+\gamma}, \text{ ideoque ob } \frac{\partial q}{q \cdot q} = \frac{\beta \partial x}{x^{\beta+1}}, \text{ erit}$$

$$\int \frac{\partial q}{q \cdot q} \int q \partial r = \frac{\beta x^{\gamma}}{\beta+\gamma},$$

quod ductum in $q \partial p$ et integratum producit

$$\frac{\alpha \beta x^{\alpha+\beta+\gamma}}{(\beta+\gamma)(\alpha+\beta+\gamma)}.$$

Patet igitur hanc transformationem latissime patere, atque ad omnes formulas implicatas accommodari posse eo pluribus diversis modis, quo plura signa integralia invicem involvantur.

§. 72. Haud abs re fore judico resolutiones supra traditas ad summationem serierum potestatum reciprocarum applicare, quod fiet si loco X sumamus fractionem $\frac{x}{1-x}$, tum vero pro singulis elementis $\partial p, \partial q, \partial r, \partial s$, scribamus $\frac{\partial x}{x}$, unde corollaria subnexa in usum vocari poterunt, ubi scilicet erit $p = lx$.

§. 73. Cum sit per seriem infinitam

$$X = x + xx + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + \text{etc.}$$

erit

$$\int X \partial p = \int \frac{x \partial x}{x} = x + \frac{1}{2}xx + \frac{1}{3}x^3 + \frac{1}{4}x^4 + \frac{1}{5}x^5 + \frac{1}{6}x^6 + \text{etc.}$$

quam seriem constat exprimere logarithmum fractionis $\frac{1}{1-x}$, quandoquidem est

$$\int \frac{x \partial x}{x} = -l(1-x) = l \frac{1}{1-x}.$$

§. 74. Multiplicetur haec series porro per $\frac{\partial x}{x}$ et integretur, prodibitque

$$\int \frac{\partial x}{x} \int \frac{x \partial x}{x} = x + \frac{1}{4} x x + \frac{1}{9} x^3 + \frac{1}{16} x^4 + \frac{1}{25} x^5 + \text{etc.}$$

at vero hujus formulae integralis resolutio supra §. 57. data praebet

$$\int \frac{\partial x}{x} \int \frac{x \partial x}{x} = l x \int \frac{\partial x}{1-x} - \int \frac{\partial x l x}{1-x},$$

quae quidem integralia ita accipi supponuntur, ut posito $x = 0$ evanescant; hic autem imprimis notetur, casu quo sumitur $x = 1$, ob $l 1 = 0$, hujus seriei

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \text{etc.}$$

summam fore $-\int \frac{\partial x l x}{1-x}$, cujus valorem olim primus inveni esse $= \frac{\pi \pi}{6}$.

§. 75. Ducamus superiorem seriem denuo in $\frac{\partial x}{x}$ et integrando obtinebimus:

$$\int \frac{\partial x}{x} \int \frac{\partial x}{x} \int \frac{\partial x}{1-x} = x + \frac{1}{2^3} x x + \frac{1}{3^3} x^3 + \frac{1}{4^3} x^4 + \frac{1}{5^3} x^5 + \text{etc.}$$

Formula autem haec implicata per §. 59. ita resolvitur

$$\frac{1}{2} (l x)^2 \int \frac{\partial x}{1-x} - l x \int \frac{\partial x l x}{1-x} + \frac{1}{2} \int \frac{\partial x (l x)^2}{1-x}.$$

Casu igitur quo $x = 1$, summa seriei reciprocae cuborum

$$1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \frac{1}{5^3} + \text{etc.}$$

$$\text{erit} = \frac{1}{2} \int \frac{\partial x (l x)^2}{1-x}.$$

§. 76. Simili modo superiorem seriem per $\frac{\partial x}{x}$ multiplicemus et integremus, tum prodibit

$$\int \frac{\partial x}{x} \int \frac{\partial x}{x} \int \frac{\partial x}{x} \int \frac{\partial x}{1-x} = x + \frac{1}{2^4} x x + \frac{1}{3^4} x^3 + \frac{1}{4^4} x^4 + \text{etc.}$$

At vero haec formula implicata per §. 61. reducitur ad hanc formam

$$\begin{aligned} \frac{1}{6} (l x)^3 \int \frac{\partial x}{1-x} - \frac{1}{2} (l x)^2 \int \frac{\partial x l x}{1-x} + \frac{1}{2} l x \int \frac{\partial x (l x)^2}{1-x} \\ - \frac{1}{6} \int \frac{\partial x (l x)^3}{1-x}. \end{aligned}$$

Pro casu ergo quo $x = 1$ hujus seriei reciprocae biquadratorum summa erit $-\frac{1}{6} \int \frac{\partial x (lx)^5}{1-x}$, cujus valorem olim ostendi esse $\frac{\pi^4}{90}$.

§. 77. Multiplicatione denuo per $\frac{\partial x}{x}$ instituta et integratione peracta habebimus:

$$\int \frac{\partial x}{x} \int \frac{\partial x}{x} \int \frac{\partial x}{x} \int \frac{\partial x}{x} \int \frac{\partial x}{1-x} = x + \frac{1}{2} x x + \frac{1}{3} x^3 + \frac{1}{4} x^4 + \frac{1}{5} x^5 + \text{etc.}$$

quae formula implicata per §. 63. reducitur ad hanc formam

$$\begin{aligned} \frac{1}{24} (lx)^4 \int \frac{\partial x}{x} - \frac{1}{6} (lx)^3 \int \frac{\partial x lx}{1-x} + \frac{1}{4} (lx)^2 \int \frac{\partial x (lx)^2}{1-x} \\ - \frac{1}{6} lx \int \frac{\partial x (lx)^3}{1-x} + \frac{1}{24} \int \frac{\partial x (lx)^4}{1-x}. \end{aligned}$$

Hinc ergo casu $x = 1$ hujus seriei reciprocae potestatum quintarum summa erit $\frac{1}{24} \int \frac{\partial x (lx)^4}{1-x}$.

§. 78. Colligamus omnes istas series pro casu $x = 1$, earumque summae sequenti modo per formulam integram simplicem exprimentur:

$$\begin{aligned} 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \text{etc.} &= \int \frac{\partial x}{1-x} = \infty, \\ 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \text{etc.} &= - \int \frac{\partial x lx}{1-x} = \frac{\pi^2}{6}, \\ 1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \frac{1}{5^3} + \text{etc.} &= \frac{1}{2} \int \frac{\partial x (lx)^2}{1-x}, \\ 1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \text{etc.} &= - \frac{1}{6} \int \frac{\partial x (lx)^3}{1-x} = \frac{\pi^4}{90}, \\ 1 + \frac{1}{2^5} + \frac{1}{3^5} + \frac{1}{4^5} + \frac{1}{5^5} + \text{etc.} &= \frac{1}{24} \int \frac{\partial x (lx)^4}{1-x}, \\ 1 + \frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{4^6} + \frac{1}{5^6} + \text{etc.} &= - \frac{1}{120} \int \frac{\partial x (lx)^5}{1-x} = \frac{\pi^6}{945}, \\ 1 + \frac{1}{2^7} + \frac{1}{3^7} + \frac{1}{4^7} + \frac{1}{5^7} + \text{etc.} &= \frac{1}{720} \int \frac{\partial x (lx)^6}{1-x}, \\ 1 + \frac{1}{2^8} + \frac{1}{3^8} + \frac{1}{4^8} + \frac{1}{5^8} + \text{etc.} &= - \frac{1}{5040} \int \frac{\partial x (lx)^7}{1-x} = \frac{\pi^8}{9450}, \\ \text{etc.} & \qquad \qquad \qquad \text{etc.} \end{aligned}$$

§. 79. In genere igitur hujus seriei

$$1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \frac{1}{5^n} + \frac{1}{6^n} + \text{etc.}$$

in infinitum continuatae summa ita exprimitur

$$= \frac{1}{1 \cdot 2 \cdot 3 \dots (n-1)} \int \frac{\partial x (1-x)^{n-1}}{1-x}$$

ubi signum superius $+$ valet, quando exponens n est impar, inferius vero, quando est par. Istas summationes, jam pridem quidem repertas, ideo hic afferre visum est, quod non ita pridem Celeberr. Lorgna easdem has summationes per formulas continuo magis implicatas expressas exhibuit, cum sine dubio istae formulae integrales simplices longe praeferendae videantur.