SUPPLEMENTUM VIII.

AD TOM. I. SECT. II. CAP. VI.

DE

COMPARATIONE QUANTITATUM TRANSCENDENTIUM IN FORMA $\int \frac{P\partial z}{\sqrt{(A+2Bz+Czz+2Dz^3+EZ^4)}}$ CONTENTARUM.

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1). Dilucidationes super methodo elegantissima, qua illustris *de la Grange* usus est, in integranda aequatione differentiali $\frac{\partial x}{\sqrt{X}} = \frac{\partial y}{\sqrt{Y}}$. *Acta Acad. Imp. Sc. Tom. II. P. I. Pag.* 20 — 57.

§. 1. Postquam diu et multum in perscrutanda aequatione differentiali $\frac{\partial x}{\sqrt{X}} = \frac{\partial y}{\sqrt{Y}}$ desudassem, atque imprimis in methodum directam, quae via facili ac plana ad ejus integrale perduceret, nequicquam inquisivissem; penitus obstupui, cum mihi nunciaretur, in volumine quarto *Miscellaneorum Taurinensium* ab illustri *de la Grange* talem methodum esse expositam, cujus ope pro casu, quo

$$X = A + Bx + Cxx + Dx^3 + Ex^4 \text{ et}$$
$$Y = A + By + Cyy + Dy^3 + Ey^4,$$

propositae aequationis differentialis hoc integrale algebraicum atque adeo completum felicissimo successu elicuit

$$\frac{\sqrt{X}+\sqrt{Y}}{x-y} = \sqrt{[\Delta + D(x+y) + E(x+y)^2]}$$

ubi Δ denotat quantitatem constantem arbitrariam per integrationem ingressam.

§. 2. Istud autem egregium inventum eo magis sum admiratus, quod equidem semper putaveram, talem methodum in investigando idoneo factore, quo aequatio proposita integrabilis redderetur, quaeri oportere, cum vulgo omnis methodus integrandi vel in separatione variabilium, vel in idoneo multiplicatore contineri videatur, etiamsi certis casibus quoque ipsa differentiatio ad integrale perducere queat, quemadmodum tam a me ipso quam ab aliis per plurima exempla est ostensum. Ad hanc autem tertiam viam illa ipsa methodus *Grangiana* rite referri posse videtur.

§. 3. Quanquam autem facile est inventis aliquid addere, tamen in re tam ardua plurimum intererit, hanc methodum ab illustri *de la Grange* adhibitam accuratius perpendisse atque ad usum analyticum magis accommodasse; siquidem totum negotium multo facilius ac simplicius expediri posse videtur. Quamobrem, quae de hoc argumento, quod merito maximi momenti est censendum, sum meditatus, hic data opera fusius sum expositurus.

§. 4. Quoniam autem hoc integrale ab illustri *de la Grange* inventum, ab iis formis quas ipse olim dederam, plurimum discrepat, ac simplicitate non mediocriter antecellit, ante omnia visum est scitari, quomodo aequationi differentiali satisfaciat. Hunc in finem pono brevitatis gratia $\sqrt{X} + \sqrt{Y} = V$, ut habeam

$$\frac{V}{x-y} = \sqrt{[\Delta + D(x+y) + E(x+y)^2]},$$

quam aequationem ita differentiare oportet, ut constans arbitraria Δ ex differentiali excedat. Sumtis igitur quadratis erit

$$\frac{V^2}{(x-y)^2} = \Delta + D(x+y) + E(x+y)^2,$$

quae differentiata dat

$$\frac{2V\partial V}{(x-y)^2} - \frac{2VV(\partial x-\partial y)}{(x-y)^3} - D(\partial x+\partial y) - 2E(x+y)(\partial x+\partial y) = 0.$$

§. 5. Quo nunc calculus planior reddatur, seorsim partes vel per ∂x vel per ∂y affectas investigemus. Pro elemento igitur ∂x, si y ut constans spectetur, erit $\partial V = \frac{X'\partial x}{2\sqrt{X}}$, unde singulae partes ita se habebunt

$$\partial x\left[\frac{VX'}{(x-y)^2\sqrt{X}} - \frac{2VV}{(x-y)^3} - D - 2E(x+y)\right]$$

ubi notetur esse $V = \sqrt{X} + \sqrt{Y}$, hincque

$$VV\sqrt{X} = (X+Y)\sqrt{X} + 2X\sqrt{Y}:$$

unde hic duplicis generis termini occurrunt, dum vel per $\sqrt{X}$ vel per $\sqrt{Y}$ sunt affecti. Duo autem termini adsunt $\sqrt{Y}$ affecti, qui sunt

$$-\frac{4X\sqrt{Y}}{(x-y)^3} + \frac{X'\sqrt{Y}}{(x-y)^2},$$

qui ergo junctim sumti dabunt

$$\frac{\sqrt{Y}}{(x-y)^3}[X'(x-y) - 4X],$$

quae forma ob

$$X = A + Bx + Cxx + Dx^3 + Ex^4, \text{ hincque}$$
$$X' = B + 2Cx + 3Dxx + 4Ex^3, \text{ dabit}$$
$$X'(x-y) - 4X = -4A - B(3x+y)$$
$$-2C(xx+xy) - D(x^3+3xxy) - 4Ex^3y.$$

Termini autem per $\sqrt{X}$ affecti sunt

$$\frac{\sqrt{X}}{(x-y)^3}[X'(x-y) - 2(X+Y) - D(x-y)^3 - 2E(x+y)(x-y)^3].$$

Cum igitur sit

$$X + Y = 2A + B(x+y) + C(x^2+y^2)$$
$$+ D(x^3+y^3) + E(x^4+y^4),$$

facta substitutione iste postremus factor erit

$$-4A - B(x+3y) - 2C(xy+yy)$$
$$-D(3xyy+y^3) - 4Exy^3,$$

quae forma a praecedente hoc tantum discrepat, quod litterae x et y sunt permutatae.

§. 6. Quod si ergo brevitatis gratia ponamus

$$M = 4A + B(3x+y) + 2C(xx+xy)$$
$$+ D(x^3+3xxy) + 4Ex^3y,$$
$$N = 4A + B(x+3y) + 2C(yy+xy)$$
$$+ D(y^3+3xyy) + 4Exy^3,$$

hinc pars elemento ∂x affecta ita erit expressa

$$-\frac{\partial x}{(x-y)^3\sqrt{X}}(M\sqrt{Y}+N\sqrt{X}).$$

§. 7. Simili modo ob $\partial V = \frac{Y'\partial Y}{2\sqrt{Y}}$, partes elemento ∂y affectae erunt

$$\frac{\partial y}{\sqrt{Y}}\left[\frac{VY'}{(x-y)^2} + \frac{2VV\sqrt{Y}}{(x-y)^3} - D\sqrt{Y} - 2E(x+y)\sqrt{Y}\right].$$

Haec jam forma ob

$$V = \sqrt{X} + \sqrt{Y} \text{ et } VV\sqrt{Y} = (X+Y)\sqrt{Y} + 2Y\sqrt{X},$$

continebit sequentes terminos per $\sqrt{X}$ affectos

$$\frac{\sqrt{X}}{(x-y)^3}[Y'(x-y)+4Y],$$

quae forma ex priore praecedentis calculi oritur, si litterae x et y permutentur, simulque signa; unde patet hanc expressionem praebere valorem $+N$. Reliqui autem termini per $\sqrt{X}$ affecti erunt

$$\frac{\sqrt{Y}}{(x-y)^3}[Y'(x-y)+2(X+Y)-D(x-y)^3-2E(x+y)(x-y)^3].$$

Haec forma iterum ex permutatione litterarum et signorum ex forma praecedentis calculi oritur, quae ergo cum esset $-N$, haec erit $+M$. Hoc igitur modo partes elementum ∂y continentes erunt

$$\frac{+\partial y}{(x-y)^3 \sqrt{Y}}(N\sqrt{X}+M\sqrt{Y}).$$

§. 8. Conjungendis igitur his membris, aequatio differentialis ex forma *Grangiana* orta erit

$$\left(\frac{\partial y}{\sqrt{Y}}-\frac{\partial x}{\sqrt{Y}}\right)\left[\frac{N\sqrt{X}+M\sqrt{Y}}{(x-y)^3}\right]=0,$$

quae per factorem communem divisa praebet ipsam aequationem differentialem propositam $\frac{\partial x}{\sqrt{X}}=\frac{\partial y}{\sqrt{Y}}$; unde simul patet aequationem integralem exhibitam recte se habere, atque adeo valorem litterae Δ arbitrio nostro penitus relinqui.

§. 9. Antequam autem methodum *Grangianam* ad ipsam aequationem differentialem $\frac{\partial x}{\sqrt{X}}=\frac{\partial y}{\sqrt{Y}}$ in omni extensione acceptam applicemus, a casu simpliciore inchoëmus, quo aequatio adeo rationalis proponitur haec

$$\frac{\partial x}{a+2bx+cxx}=\frac{\partial y}{a+2by+cyy}.$$

Analysis

pro integratione aequationis differentialis

$$\frac{\partial x}{a+2bx+cxx}=\frac{\partial y}{a+2by+cyy}.$$

§. 10. Ponamus brevitatis gratia $a+2bx+cxx=X$ et $a+2by+cyy=Y$, ut fieri debeat $\frac{\partial x}{X}=\frac{\partial y}{Y}$, quae formulae cum inter se debeant esse aequales, utraque per idem elementum ∂t designetur, ita ut nanciscamur has duas formulas $\frac{\partial x}{\partial t}=X$ et $\frac{\partial y}{\partial t}=Y$. Quod si ergo jam statuamus

$$x-y=q, \text{ erit } \frac{\partial q}{\partial t}=X-Y=2bq+cq(x+y),$$

unde per q dividendo erit $\frac{\partial q}{q\partial t}=2b+c(x+y)$.

§. 11. Nunc primas formulas differentiemus, sumto elemento ∂t constante, et facto

$$\partial X = X' \partial x \text{ et } \partial Y = Y' \partial y$$

orientur hae duae aequationes

$$\frac{\partial \partial x}{\partial x \partial t} = X' \text{ et } \frac{\partial \partial y}{\partial y \partial t} = Y',$$

quae invicem additae praebent

$$\frac{\partial \partial x}{\partial x \partial t} + \frac{\partial \partial y}{\partial y \partial t} = X' + Y'.$$

Quare cum sit

$$X' = 2b + 2cx \text{ et } Y' = 2b + 2cy, \text{ erit}$$

$$\frac{1}{\partial t}\left(\frac{\partial \partial x}{\partial x} + \frac{\partial \partial y}{\partial y}\right) = 4b + 2c(x + y).$$

§. 12. Quoniam igitur hic postremus valor duplo major est praecedente $\frac{\partial q}{q \partial t}$, hoc modo deducti sumus ad hanc aequationem

$$\frac{\partial \partial x}{\partial x} + \frac{\partial \partial y}{\partial y} = \frac{2 \partial q}{q},$$

quae integrata dat $l \partial x + l \partial y = 2 l q +$ constans, hincque in numeris erit

$$\partial x \partial y = C q q \partial t^2, \text{ ita ut sit } C = \frac{\partial x \partial y}{q q \partial t^2}.$$

Quare cum sit $\frac{\partial x}{\partial t} = X$ et $\frac{\partial y}{\partial t} = Y$, aequatio integralis erit $\frac{XY}{(x-y)^2} = C$, quae ergo non solum est algebraica, sed etiam completa.

§. 13. Si igitur proposita fuerit haec aequatio differentialis

$$\frac{\partial x}{a + 2bx + cxx} = \frac{\partial y}{a + 2by + cyy},$$

ejus integrale completum ita erit expressum

$$\frac{(a + 2bx + cxx)(a + 2by + cyy)}{(x - y)^2} = C,$$

quae, utrinque addendo $bb - ac$, induet hanc formam

$$\frac{aa+2ab(x+y)+2acxy+bb(x+y)^2+2bcxy(x+y)+ccxxyy}{(x-y)^2} = \Delta\Delta,$$

sicque, extracta radice, integrale hanc formam habebit

$$\frac{a+b(x+y)+cxy}{x-y} = \Delta,$$

quae sine dubio est simplicissima, quandoquidem tam y per x quam x per y facillime exprimi potest, cum sit

$$y = \frac{(\Delta-b)x-a}{\Delta+b+cx} \text{ et } x = \frac{a+(\Delta+b)y}{\Delta-b-cy}.$$

§. 14. Calculum, quo hic usi sumus, perpendenti facile patebit, in his formis X et Y, non ultra quadrata progredi licere. Si enim ipsi X insuper tribuamus terminum $d\,x^3$ et ipsi Y terminum $d\,y^3$, pro priore forma prodit

$$\frac{X-Y}{x-y} = 2b + c(x+y) + d(xx+xy+yy) = \frac{\partial q}{q\,\partial t};$$

pro altera autem forma est

$$X' + Y' = 4b + 2c(x+y) + 3d(xx+yy) = \frac{\partial\partial x}{\partial x\,\partial t} + \frac{\partial\partial y}{\partial y\,\partial t}.$$

Quare si hinc duplum praecedentis aufferamus, colligitur

$$\frac{\partial\partial x}{\partial x\,\partial t} + \frac{\partial\partial y}{\partial y\,\partial t} - \frac{2\,\partial q}{q\,\partial t} = d(x-y)^2,$$

quam aequationem non amplius integrare licet.

§. 15. Facile autem ostendi potest, talem aequationem differentialem, in qua ultra quadratum proceditur, nullo amplius modo algebraice integrari posse. Si enim tantum hic casus proponeretur $\frac{\partial x}{1+x^3} = \frac{\partial y}{1+x^3}$, notum est, utrinque integrale partim logarithmos partim arcus circulares involvere, ideoque quantitates transcendentes diversos, quae nullo modo inter se comparari possunt. Hujusmodi scilicet comparationes iis tantum casibus locum habere possunt, quando utrinque unius generis tantum quantitates transcendentes occurrunt.

Analysis
pro integratione aequationis

$$\frac{\partial x}{a+2bx+cxx}+\frac{\partial y}{a+2by+cyy}=0.$$

§. 16. Quod si hic ut ante ponamus

$$\frac{\partial x}{a+2bx+cxx}=\partial t,$$

statui debebit

$$\frac{\partial y}{a+2by+cyy}=-\partial t,$$

at vero si calculum simili modo quo ante instituere velimus, nihil plane proficimus. Postquam autem omnes difficultates probe perpendissem, tandem in artificium incidi, quo hunc casum expedire licuit, ita ut hinc non contemnendum incrementum methodo *Grangianae* attulisse mihi videar.

§. 17. Quoniam igitur has duas habeo aequationes

$$\frac{\partial x}{\partial t}=\text{X} \text{ et } \frac{\partial y}{\partial t}=-\text{Y},$$

hinc formo istam novam aequationem

$$\frac{y\partial x+x\partial y}{\partial t}=y\text{X}-x\text{Y}.$$

Jam facio $xy=u$, ut habeam

$$\frac{\partial u}{\partial t}=a(y-x)+cxy(x-y),$$

unde posito

$$x-y=q \text{ erit } \frac{\partial u}{\partial t}=q(cu-a),$$

quae aequatio per $cu-a$ divisa ductaque in c praebet

$$\frac{c\partial u}{(cu-a)\partial t}=cq,$$

hocque modo nacti sumus differentiale logarithmicum.

§. 18. Dein vero aequationes principales ut ante differentiemus, et obtinebimus

$$\frac{\partial \partial x}{\partial t \partial x} = X' \text{ et } \frac{\partial \partial y}{\partial t \partial y} = -Y',$$

quae invicem additae dant

$$\frac{1}{\partial t} \cdot \left(\frac{\partial \partial x}{\partial x} + \frac{\partial \partial y}{\partial y}\right) = X' - Y' = 2cq;$$

quare si hinc duplum praecedentis aequationis subtrahamus, remanebit

$$\frac{1}{\partial t} \cdot \left(\frac{\partial \partial x}{\partial x} + \frac{\partial \partial y}{\partial y} - \frac{2c\partial u}{cu-a}\right) = 0,$$

unde per ∂t multiplicando et integrando nanciscimur

$$l\partial x + l\partial y - 2l(cu-a) = lC, \text{ ideoque}$$

$$\frac{\partial x \partial y}{(cu-a)^2} = C\partial t^2.$$

Cum igitur sit

$$\partial x = X\partial t \text{ et } \partial y = -Y\partial t,$$

aequatio integralis nostra erit $-\frac{XY}{(cu-a)^2} = C$.

§. 19. Per hanc ergo analysin deducti sumus ad hanc aequationem integralem aequationis propositae

$$\frac{(a+2bx+cxx)(a+2by+cyy)}{(a-cxy)^2} = C.$$

Quae aequatio, si utrinque unitas subtrahatur, reducitur ad hanc formam

$$\frac{2ab(x+y)+ac(x+y)^2+4bbxy+2bcxy(x+y)}{(a-cxy)^2} = C.$$

§. 20. Illustremus hanc integrationem exemplo, ponendo $a = 1$, $b = 0$ et $c = 1$, ita ut proposita sit haec aequatio differentialis $\frac{\partial x}{1+xx} + \frac{\partial y}{1+yy} = 0$, cujus integrale novimus esse

$$\text{Arc. tang. } x + \text{Arc. tang. } y = \text{Arc. tang. } \frac{x+y}{1-xy} = C,$$

sicque novimus esse $\frac{x+y}{1-xy}=C$. At vero nostra postrema formula dat pro hoc casu

$$\frac{(x+y)^2}{(1-xy)^2}=C, \text{ ideoque } \frac{x+y}{1-xy}=C,$$

quod egregie convenit.

§. 21. Consideremus etiam casum, quo $a=1$, $b=\frac{1}{2}$ et $c=1$, ita ut proponatur haec aequatio

$$\frac{\partial x}{1+x+xx}+\frac{\partial y}{1+y+yy}=0,$$

cujus integrale est

$$\frac{2}{\sqrt{3}}\text{ Arc. tang. }\frac{x\sqrt{3}}{2+x}+\frac{2}{\sqrt{3}}\text{ Arc. tang. }\frac{y\sqrt{3}}{2+y}=C,$$

unde sequitur fore

$$\text{Arc. tang. }\frac{2(x+y+xy)\sqrt{3}}{4+2(x+y)-2xy}=C,$$

ideoque etiam $\frac{x+y+xy}{2+x+y-xy}=C$. At vero forma integralis inventa pro hoc casu dabit

$$\frac{x+y+(x+y)^2+xy+xy(x+y)}{(1-xy)^2}=C,$$

quae in factores resoluta dat

$$\frac{(1+x+y)(x+y+xy)}{(1-xy)^2}=C.$$

Prior vero aequatio

$$\frac{x+y+xy}{2+x+y-xy}=C \text{ inversa praebet } \frac{2+x+y-xy}{x+y+xy}=C,$$

et unitate subtracta $\frac{1-xy}{x+y+xy}=C$, atque haec in praecedentem ducta dat $\frac{1+x+y}{1-xy}=C$.

§. 22. Videamus igitur, utrum haec posteriores aequationes inter se conveniant, et quia constantes utrinque inter se discrepare possunt, ambas aequationes ita referamus

$$\frac{1-xy}{x+y+xy}=\alpha \text{ et } \frac{1+x+y}{1-xy}=\beta;$$

unde cum sit $\frac{1}{\alpha} = \frac{x+y+xy}{1-xy}$, evidens est fore $\beta - \frac{1}{\alpha} = 1$, ex quo pulcherrimus consensus inter ambas formulas elucet. Ex his exemplis intelligitur aequationem generalem supra inventam hoc modo per factores repraesentari posse

$$\frac{[2b+c(x+y)]\,[a(x+y)+2bxy]}{(a-cxy)^2}.$$

Caeterum consideratio harum formularum haud injucundas speculationes suppeditare poterit.

§. 23. Sequenti autem modo forma illa integralis inventa

$$\frac{[2b+c(x+y)]\,[a(x+y)+2bxy]}{(a-cxy)^2} = C,$$

statim ad formam simplicissimam reduci potest; si enim ejus factores statuamus

$$\frac{2b+c(x+y)}{a-cxy} = P \text{ et } \frac{a(x+y)+2bxy}{a-cxy} = Q,$$

ut esse debeat $PQ = C$, erit

$$aP - cQ = \frac{2ab-2bcxy}{a-cxy} = 2b, \text{ unde fit } Q = \frac{aP-2b}{c},$$

sicque quantitati constanti aequari debet haec forma $\frac{aPP-2bP}{c}$; ex quo patet, etiam ipsam quantitatem P constanti aequari debere, ita ut jam aequatio nostra integralis sit

$$\frac{2b+c(x+y)}{a-cxy} = C, \text{ vel etiam } \frac{a(x+y)+2bxy}{a-cxy} = C.$$

Alia solutio facillima ejusdem aequationis

$$\frac{\partial x}{a+2bx+cxx} + \frac{\partial y}{a+2by+cyy} = 0.$$

§. 24. Postrema reductione probe perpensa, comperui, statim ab initio ad formam integralis simplicissimam perveniri posse, atque adeo non necesse esse ad differentialia secunda ascendere. Si enim ut ante ponamus $x+y=p$, $x-y=q$ et $xy=u$, ex formulis

$$\frac{\partial x}{\partial t} = X \text{ et } \frac{\partial y}{\partial t} = - Y$$

statim deducimus

$$\frac{\partial p}{\partial t} = X - Y = 2bq + cpq, \text{ unde fit } \frac{\partial p}{2b + cp} = q\,\partial t.$$

§. 25. Porro vero erit

$$\frac{y\,\partial x + x\,\partial y}{\partial t} = \frac{\partial u}{\partial t} = yX - xY = -aq + cqu,$$

unde fit $\frac{\partial u}{cu - a} = q\,\partial t$, quamobrem hinc statim colligimus hanc aequationem $\frac{\partial u}{2b + cp} = \frac{\partial u}{cu - a}$, cujus integratio praebet

$$l(2b + cp) = l(cu - a) + lC;$$

unde deducitur haec aequatio algebraica $\frac{2b + cp}{cu - a} = C$, quae, restitutis litteris x et y, dat $\frac{2b + c(x + y)}{cxy - a} = C$, quae est forma simplicissima aequationis integralis desideratae. Hic imprimis notatu dignum occurrit, quod casum primum hac ratione resolvere non licet.

§. 26. Ex forma autem integrali inventa facile aliae derivantur veluti, si addamus $\frac{2b}{a}$, orietur haec forma

$$\frac{a(x+y)2 + bxy}{cxy - a} = C,$$

quae per praecedentem divisa denuo novam formam suppeditat, scilicet

$$\frac{2b + c(x + y)}{a(x + y) + 2bxy} = C,$$

quae formae quomodo satisfaciant operae pretium erit ostendisse. Et quidem postrema forma differentiata, erit

$$\frac{-2ab(\partial x + \partial y) - 4bb(y\,\partial x + x\,\partial y) - 2bc(yy\,\partial x + xx\,\partial y)}{[a(x + y) + 2bxy]^2}$$

quae in ordinem redacta praebet

$$\partial x(2ab + 4bby + 2bcyy) + \partial y(2ab + 4bbx + 2bcxx) = 0.$$

Haec per $2b$ divisa et separata dat

$$\frac{\partial x}{a+2bx+cxx}+\frac{\partial y}{a+2by+cyy}=0,$$

quae est ipsa proposita.

Analysis

pro integratione aequationis

$$\frac{\partial x}{\sqrt{(A+Bx+Cxx}}=\frac{\partial y}{\sqrt{(A+By+Cyy}}.$$

§. 27. Introducto novo elemento ∂t, deinceps pro constanti habendo, oriuntur hae duae aequationes

$$\frac{\partial x}{\partial t}=\sqrt{X} \text{ et } \frac{\partial y}{\partial t}=\sqrt{Y},$$

ubi litteris X et Y valores initio assignatos tribuamus. Videbimus autem, pro methodo, qua hic utemur, terminos litteris D et E affectos omitti debere. Sumtis ergo quadratis erit

$$\frac{\partial x^2}{\partial t^2}=X \text{ et } \frac{\partial y^2}{\partial t^2}=Y.$$

§. 28. Nunc istas formulas differentiemus, positoque, ut fieri solet, $\partial X=X'\partial x$ et $\partial Y=Y'\partial y$, nanciscemur has aequationes

$$\frac{2\partial\partial x}{\partial t^2}=X' \text{ et } \frac{2\partial\partial y}{\partial t^2}=Y',$$

ac posito $x+y=p$, fiet $\frac{2\partial\partial p}{\partial t^2}=X'+Y'$. Cum jam sit

$$X'=B+2Cx+3Dxx+4Ex^3 \text{ et}$$
$$Y'=B+2Cy+3Dyy+4Ey^3, \text{ erit}$$
$$X'+Y'=2B+2Cp+3D(xx+yy)$$
$$+4E(x^3+y^3)=\frac{2\partial\partial p}{\partial t^2},$$

quae aequatio manifesto integrationem admittet, si fuerit et $D=0$ et $E=0$, quemadmodum assumsimus. Multiplicando igitur per

∂p et integrando nanciscimur

$$\frac{\partial p^2}{\partial t^2} = \Delta + 2\,B\,p + C\,p\,p,$$

et radicem extrahendo

$$\frac{\partial p}{\partial t} = \sqrt{(\Delta + 2\,B\,p + C\,p\,p)}.$$

Cum igitur sit $\frac{\partial p}{\partial t} = \sqrt{X} + \sqrt{Y}$, aequatio integralis, quam sumus adepti erit

$$\sqrt{X} + \sqrt{Y} = \sqrt{[\Delta + 2\,B(x+y) + C(x+y)^2]},$$

quae adeo est algebraica; ubi notetur esse

$$X = A + Bx + Cxx \text{ et } Y = A + By + Cyy.$$

§. 29. Sumamus igitur quadrata, et nostra aequatio integralis erit

$$2\,A + B(x+y) + C(x^2+y^2) + 2\sqrt{XY}$$
$$= \Delta + 2\,B(x+y) + C(x+y)^2, \text{ sive}$$
$$2\,A - B(x+y) - 2\,Cxy + 2\sqrt{XY} = \Delta,$$

quae penitus ab irrationalitate liberata, posito $\Delta - 2\,A = \Gamma$, praebebit

$$4\,XY = 4\,AA + 4\,AB(x+y) + 4\,AC(xx+yy)$$
$$+ 4\,BBxy + 4\,BCxy(x+y) + 4\,CCxxyy$$
$$= \Gamma^2 + 2\,\Gamma B(x+y) + 4\,\Gamma Cxy + BB(x+y)^2$$
$$+ 4\,BCxy(x+y) + 4\,CCxxyy \text{ sive}$$
$$(4\,AA - \Gamma^2) + 2\,B(2\,A - \Gamma)(x+y) + 4(BB - \Gamma C)xy$$
$$+ 4\,AC(xx+yy) - B^2(x+y)^2 = 0.$$

§. 30. Quod si jam hanc aequationem rationalem cum formula *canonica*, qua olim sum usus ad hujusmodi integrationes expediendas, comparemus, quae erat

$$\alpha + 2\,\beta(x+y) + \gamma(xx+yy) + 2\,\delta xy = 0,$$

dum scilicet loco $(x+y)^2$ scribamus $(xx+yy)+2xy$, reperiemus fore

$$\alpha = 4\,A\,A - \Gamma^2,\ \beta = B(2A - \Gamma),\ \gamma = 4\,A\,C - B^2,$$
$$\delta = B\,B - 2\,\Gamma\,C.$$

§. 31. Alio vero insuper modo eandem aequationem differentialem propositam integrare poterimus, introducendo literam $q = x - y$; tum enim habebimus

$$\frac{2\,\partial\partial q}{\partial t} = X' - Y'.$$

At vero erit

$$X' - Y' = 2\,C\,q + 3\,D\,q\,(x+y),$$

ubi iterum patet, statui debere tam $D = 0$ quam $E = 0$, ut integratio, multiplicando per ∂q, succedat. Hoc autem notato erit integrale $\frac{\partial q^2}{\partial t^2} = \text{Const.} + C\,q\,q$, ideoque

$$\frac{\partial q}{\partial t} = \sqrt{(\Delta + C\,q\,q)}.$$

§. 32. Cum igitur sit $\frac{\partial q}{\partial t} = \sqrt{X} - \sqrt{Y}$, hoc integrale ita erit expressum

$$\sqrt{X} - \sqrt{Y} = \sqrt{(\Delta + C\,q\,q)}$$

quae aequatio sumtis quadratis abit in hanc

$$2\,A + B\,(x+y) + C\,(xx+yy) - 2\sqrt{XY} = \Delta + C\,(x-y)^2, \text{ sive}$$

$$2\,A + B\,(x+y) + 2\,C\,xy - 2\sqrt{XY} = \Delta;$$

unde fit

$$2\sqrt{XY} = 2\,A - \Delta + B\,(x+y) + 2\,C\,xy,$$

ubi si ponatur $2\,A - \Delta = -\Gamma$, aequatio ab ante inventa prorsus non discrepat.

§. 33. Quod si autem proposita fuisset aequatio

$$\frac{\partial x}{\sqrt{(A+Bx+Cxx)}}+\frac{\partial y}{\sqrt{(A+By+Cyy)}}=0,$$

integralia ante inventa ad hunc casum referentur, si modo loco $\sqrt{Y}$ scribatur $-\sqrt{Y}$; unde patet pro hoc casu haberi hanc aequationem

$$\sqrt{X}-\sqrt{Y}=\sqrt{[\Delta+2B(x+y)+C(x+y)^2]},$$

vel etiam

$$\sqrt{X}+\sqrt{Y}=\sqrt{[\Delta+C(x-y)^2]}.$$

§. 34. Hic singularis casus occurrit, quando formulae $A+Bx+Cxx$ sunt quadrata. Sit enim

$$X=(a+bx)^2 \text{ et } Y=(a+by)^2$$

ideoque

$$A=a^2,\ B=2ab,\ C=bb,$$

tum enim prior forma integralis erit

$$b(x-y)=\sqrt{[\Delta+4ab(x+y)+bb(x+y)^2]}$$

sumtisque quadratis

$$-4bbxy=\Delta+4ab(x+y),$$

ideoque

$$\Delta=a(x+y)+bxy,$$

cujus aequationis differentiale est

$$a(\partial x+\partial y)+b(x\partial y+y\partial x)=0$$

ideoque

$$\partial x(a+by)+\partial y(a+bx)=0.$$

Sin autem altera formula utamur, erit

$$2a+b(x+y)=\sqrt{[\Delta+bb(x-y)^2]},$$

unde quadratis sumtis, positoque $\Delta-4aa=\Gamma$, prodit ut ante $\Gamma=a(x+y)+bxy$.

Analysis
pro integranda aequatione

$$\frac{\partial x}{\sqrt{X}} = \frac{\partial y}{\sqrt{Y}}$$

existente

$$X = A + B\,x + C\,x\,x + D\,x^3 + E\,x^4 \text{ et}$$
$$Y = A + B\,y + C\,y\,y + D\,y^3 + E\,y^4.$$

§. 35. Introducto iterum elemento ∂t, ut sit

$$\frac{\partial x}{\partial t} = \sqrt{X} \text{ et } \frac{\partial y}{\partial t} = \sqrt{Y},$$

ideoque sumtis quadratis

$$\frac{\partial x^2}{\partial t^2} = X \text{ et } \frac{\partial y^2}{\partial t^2} = Y,$$

statuamus

$$x + y = p \text{ et } x - y = q,$$

et quia hinc prodit

$$\partial x^2 - \partial y^2 = \partial p\,\partial q, \text{ erit}$$
$$\frac{d\,p\,\partial q}{\partial t^2} = X - Y = B\,(x - y) + C\,(x^2 - y^2)$$
$$+ D\,(x^3 - y^3) + E\,(x^4 - y^4).$$

§. 36. Quoniam igitur est

$$x = \frac{p+q}{2} \text{ et } y = \frac{p-q}{2},$$

his valoribus introductis reperietur

$$X - Y = B\,q + C\,p\,q + \tfrac{1}{4}D\,q\,(3\,p\,p + q\,q)$$
$$+ \tfrac{1}{2}E\,p\,q\,(p\,p + q\,q),$$

unde per q dividendo oritur

$$\frac{\partial p\,\partial q}{q\,\partial t^2} = B + C\,p + \tfrac{1}{4}D\,(3\,p\,p + q\,q)$$
$$+ \tfrac{1}{2}E\,p\,(p\,p + q\,q).$$

§. 37. Nunc etiam formulas quadratas primo exhibitas differentiemus, et statuendo ut ante

$\partial X = X' \partial x$ et $\partial Y = Y' \partial y$, habebimus

$$\frac{2 \partial \partial x}{\partial t^2} = X' \text{ et } \frac{2 \partial \partial y}{\partial t^2} = Y',$$

hincque addendo

$$\frac{2 \partial \partial p}{\partial t^2} = X' + Y'.$$

Cum vero sit

$$X' = B + 2 C x + 3 D x x + 4 E x^3 \text{ et}$$
$$Y' = B + 2 C y + 3 D y y + 4 E y^3, \text{ erit}$$
$$X' + Y' = 2 B + 2 C p + \tfrac{3}{2} D (p p + q q) + E p (p p + 3 q q),$$

ita ut substituto hoc valore fiat

$$\frac{\partial \partial p}{\partial t^2} = B + C p + \tfrac{3}{4} D (p p + q q) + \tfrac{1}{2} E p (p p + 3 q q),$$

a qua aequatione si priorem $\frac{\partial p \partial q}{q \partial t^2}$ subtrahamus, remanebit sequens

$$\frac{\partial \partial p}{\partial t^2} - \frac{\partial p \partial q}{q \partial t^2} = \tfrac{1}{2} D q q + E p q q.$$

§. 38. Haec jam aequatio per $q q$ divisa producit istam

$$\frac{1}{\partial t^2} \cdot \left(\frac{\partial \partial p}{q q} - \frac{\partial p \partial q}{q^3}\right) = \tfrac{1}{2} D + E p,$$

quae ducta in $2 \partial p$ manifesto fit integrabilis: prodit enim

$$\frac{\partial p^2}{q q \partial t^2} = \Delta + D p + E p p,$$

ex qua radice extracta colligitur

$$\frac{\partial p}{q \partial t} = \sqrt{(\Delta + D p + E p p)}.$$

Cum igitur posuerimus

$p = x + y$ et $q = x - y$, erit $\frac{\partial p}{\partial t} = \sqrt{X} + \sqrt{Y}$

unde resultat haec aequatio integralis algebraica

$$\frac{\sqrt{X} + \sqrt{Y}}{x - y} = \sqrt{[\Delta + D (x + y) + E (x + y)^2]}$$

quae est ipsa forma ab illustri *de la Grange* inventa.

§. 39. Evolvamus ulterius hanc formam, ac sumtis quadratis erit

$$\frac{X+Y+2\sqrt{XY}}{(x-y)^2}=\Delta+D(x+y)+E(x+y)^2.$$

Est vero

$$X+Y=2A+B(x+y)+C(xx+yy)$$
$$+D(x^3+y^3)+E(x^4+y^4),$$

unde si auferamus

$$[D(x+y)+E(x+y)^2](x-y)^2$$

remanebit

$$2A+B(x+y)+C(x^2+y^2)+Dxy(x+y)$$
$$+2Exxyy,$$

quo substituto aequatio integralis erit

$$\frac{2A+B(x+y)+C(x^2+y^2)+Dxy(x+y)+2Ex^2y^2+2\sqrt{XY}}{(x-y)^2}=\Delta.$$

§. 40. Haec aequatio aliquanto concinnior reddi potest subtrahendo utrinque C et statuendo $\Delta-C=\Gamma$: habebitur enim hoc facto

$$\frac{2A+B(x+y)+2Cxy+Dxy(x+y)+2Exxyy+2\sqrt{XY}}{(x-y)^2}=\Gamma;$$

unde deducimus

$$2\sqrt{XY}=\Gamma(x-y)^2-2A-B(x+y)-2Cxy$$
$$-Dxy(x+y)-2Exxyy,$$

sive ponendo

$$2A+B(x+y)+2Cxy+Dxy(x+y)+2Exxyy=V,$$

aequatio nostra erit

$$2\sqrt{XY}=\Gamma(x-y)^2-V,$$

quae sumtis quadratis abit in hanc

$$4XY=\Gamma^2(x-y)^4-2\Gamma V(x-y)^2+VV, \text{ sive}$$
$$4XY-VV=\Gamma^2(x-y)^4-2\Gamma V(x-y)^2.$$

§. 41. Facta nunc substitutione erit

$$4XY = 4A^2 + 4AB(x+y) + 4AC(xx+yy)$$
$$+ 4AD(x^3+y^3) + 4AE(x^4+y^4) + 4BBxy$$
$$+ 4BCxy(x+y) + 4BDxy(xx+yy)$$
$$+ 4BExy(x^3+y^3) + 4CCxxyy$$
$$+ 4CDxxyy(x+y) + 4CExxyy(xx+yy)$$
$$+ 4DDx^3y^3 + 4DEx^3y^3(x+y)$$
$$+ 4EEx^4y^4.$$

At vero porro colligitur fore

$$VV = 4AA + 4AB(x+y) + 8ACxy$$
$$+ 4ADxy(x+y) + 8AExxyy + BB(x+y)^2$$
$$+ 4BCxy(x+y) + 2BDxy(x+y)^2$$
$$+ 4BE(x+y)xxyy + 4CCxxyy$$
$$+ 4CD(x+y)xxyy + 8CEx^3y^3$$
$$+ DDxxyy(x+y)^2 + 4DEx^3y^3(x+y)$$
$$+ 4EEx^4y^4$$

§. 42. Quod si jam posteriorem formulam a priore subtrahamus et singulos terminos ordine analogos disponamus, reperiemus

$$4XY - VV = 4AC(x-y)^2 + 4AD(x+y)(x-y)^2$$
$$+ 4AE(x+y)^2(x-y)^2 - B^2(x-y)^2$$
$$+ 2BDxy(x-y)^2 + 4BExy(x+y)(x-y)^2$$
$$+ 4CExxyy(x-y)^2 - DDxxyy(x-y)^2,$$

quae expressio factorem habet communem $(x-y)^2$, per quem ergo si dividamus perveniemus ad hanc aequationem concinniorem

$$
\begin{aligned}
&4AC + 4AD(x+y) + 4AE(x+y)^2 - BB \\
&\quad + 2BDxy + 4BExy(x+y) + (4CE - DD)xxyy \\
&= \Gamma\Gamma(x-y)^2 - 4\Gamma A - 2\Gamma B(x+y) - 4\Gamma Cxy \\
&\quad - 2\Gamma Dxy(x+y) - 4\Gamma Exxyy.
\end{aligned}
$$

§. 43. Transferamus nunc omnes terminos ad partem sinistram, et loco $(x+y)^2$ scribamus $(xx+yy)+2xy$, tum vero $(xx+yy)-2xy$ loco $(x-y)^2$, quo facto talis oritur aequatio meae canonicae respondens

$$
0 = \begin{cases}
4AC + 4AD(x+y) + 4AE(x^2+y^2) + 2BDxy + 4BExy(x+y) + 4CExxyy \\
-BB + 2\Gamma C(x+y) - \Gamma\Gamma(x^2+y^2) + 8AExy + 2\Gamma Cxy(x+y) - DDxxyy \\
+4\Gamma A \qquad\qquad\qquad\qquad + 2\Gamma^2 xy \qquad\qquad\qquad + 4\Gamma Exxyy \\
\qquad\qquad\qquad\qquad\qquad\quad + 4\Gamma Cxy
\end{cases}
$$

§. 44. Hinc ergo pro aequatione canonica litterae graecae α, β, γ, δ, ε, ζ, per latinas A, B, C, D, E, una cum constante Γ sequenti modo determinantur

$$
\begin{aligned}
\alpha &= 4AC + 4\Gamma A - BB \\
\beta &= 2AD + \Gamma B \\
\gamma &= 4AE - \Gamma\Gamma \\
\delta &= BD + 4AE + \Gamma\Gamma + 2\Gamma C \\
\varepsilon &= 2BE + \Gamma C \\
\zeta &= 4CE + 4\Gamma E - DD,
\end{aligned}
$$

ita ut aequatio canonica, qua olim sum usus, sit

$$
\begin{aligned}
&\alpha + 2\beta(x+y) + \gamma(xx+yy) + 2\delta xy \\
&\quad + 2\varepsilon xy(x+y) + \zeta xxyy = 0.
\end{aligned}
$$

§. 45. Haec autem aequatio integralis ad rationalitatem perducta latius patet quam aequatio proposita differentialis $\frac{\partial x}{\sqrt{X}} - \frac{\partial y}{\sqrt{Y}} = 0$:

simul enim complectitur integrale hujus

$$\frac{\partial x}{\sqrt{X}} + \frac{\partial y}{\partial Y} = 0.$$

Scilicet haec aequatio complectitur duos factores, quorum alteruter alterutri satisfacit. Ex genesi autem patet hanc aequationem esse productum ex his factoribus

$$\Delta (x-y)^2 - V + 2\sqrt{XY}, \text{ et}$$

$$\Delta (x-y)^2 - V - 2\sqrt{XY}.$$

§. 46. Supra jam observavimus, ejusdem aequationis differentialis integrale hoc quoque modo exhiberi posse

$$\frac{M\sqrt{Y} + N\sqrt{X}}{(x-y)^3} = C \text{ (vide §. 8. et praec.)}$$

existente

$$M = 4A + B(3x + y) + 2Cx(x + y)$$
$$+ Dxx(x + 3y) + 4Ex^3y,$$
$$N = 4A + B(3y + x) + 2Cy(x + y)$$
$$+ Dyy(y + 3x) + 4Exy^3,$$

ubi notasse juvabit esse

$$M + N = 8A + 4B(x + y) + 2C(x + y)^2$$
$$+ D(x + y)^3 + 4Exy(xx + yy),$$
$$M - N = 2B(x - y) + 2C(x + y)(x - y)$$
$$+ D(x - y)(x^2 + 4xy + y^2)$$
$$+ 4Exy(x + y)(x - y).$$

Interim tamen haud facile intelligitur, quomodo haec forma cum ante inventa consentiat, dum tamen de consensu certi esse possumus.

§. 47. Ex iis, quae hactenus sunt allata, satis liquet, eandem aequationem integralem innumeris modis exhiberi posse, prout constans arbitraria alio atque alio modo repraesentatur; unde plurimum intererit certam legem stabilire, secundum quam quovis casu constantem illam arbitrariam exprimere velimus. Hunc in finem ista regula observetur: ut perpetuo integralia ita capiantur, ut posito $y = 0$ fiat $x = k$, hincque secundum legem compositionis $X = K$, existente

$$K = A + Bk + Ckk + Dk^3 + Ek^4.$$

Hac enim lege observata omnia integralia, utcunque diversa videantur, ad perfectum consensum perduci poterunt. Hoc igitur modo quae hactenus invenimus sequentibus theorematibus complectamur.

Theorema 1.

§. 48. Si haec aequatio differentialis

$$\frac{\partial x}{a + bx + cxx} - \frac{\partial y}{a + by + cyy} = 0,$$

ita integretur, ut posito $y = 0$ fiat $x = k$, integrale ita se habebit

$$\frac{2a + b(x + y) + 2cxy}{x - y} = \frac{2a + bk}{k}.$$

Theorema 2.

§. 49. Si haec aequatio differentialis

$$\frac{\partial x}{a + bx + cxx} + \frac{\partial y}{a + by + cyy} = 0$$

ita integretur, ut posito $y = 0$ fiat $x = k$, integrale supra triplici modo est inventum; erit enim

$$\text{I. } \frac{b + c(x + y)}{cxy - a} = -\frac{b + ck}{a},$$

$$\text{II. } \frac{a(x + y) + bxy}{cxy - a} = -k,$$

$$\text{III. } \frac{b + c(x + y)}{a(x + y) + bxy} = \frac{b + ck}{ak}.$$

Theorema 3.

§. 50. Si haec aequatio differentialis

$$\frac{\partial x}{\sqrt{(A+Bx+Cxx)}}-\frac{\partial y}{\sqrt{(A+By+Cyy)}}=0$$

ita integretur, ut posito $y=0$ fiat $x=k$, integrale erit

$$-B(x+y)-2Cxy+2\sqrt{(A+Bx+Cxx)\times(A+By+Cyy)}=$$
$$-Bk+2\sqrt{A(A+Bk+Ckk)},\ \text{sive}$$
$$B(k-x-y)-2Cxy=2\sqrt{A(A+Bk+Ckk)}$$
$$-2\sqrt{(A+Bx+Cxx)(A+By+Cyy)}.$$

Corollarium.

§. 51. Hinc ergo patet, si aequatio differentialis proposita fuerit ista

$$\frac{\partial x}{\sqrt{(A+Bx+Cxx)}}+\frac{\partial y}{\sqrt{(A+By+Cyy)}}=0,$$

eaque integretur ita, ut posito $y=0$ fiat $x=k$, integrale fore

$$B(k-x-y)-2Cxy=2\sqrt{(A+Bx+Cxx)\times(A+By+Cyy)}$$
$$-2\sqrt{A(A+Bk+Ckk)}.$$

Theorema 4.

§. 52. Si posito brevitatis gratia

$$X=A+Bx+Cxx+Dx^3+Ex^4$$
$$Y=A+By+Cyy+Dy^3+Ey^4$$
$$K=A+Bk+Ckk+Dk^3+Ek^4,$$

haec proponetur aequatio differentialis $\frac{\partial x}{\sqrt{X}}-\frac{\partial y}{\sqrt{Y}}=0$, quae ita integrari debeat, ut posito $y=0$ fiat $x=k$, ejus integrale ita erit expressum

$$\frac{2A+B(x+y)+2Cxy+Dxy(x+y)+2Exxyy+2\sqrt{XY}}{(x+y)^2}=\frac{2A+Bk+2\sqrt{AK}}{kk}$$

Sin autem aequatio proposita fuerit

$$\frac{\partial x}{\sqrt{X}} + \frac{\partial y}{\sqrt{Y}} = 0,$$

ejus integrale erit

$$\frac{2A + B(x+y) + 2Cxy + Dxy(x+y) + 2Exxyy - 2\sqrt{XY}}{(x-y)^2} = \frac{2A + Bk - 2\sqrt{AK}}{kk}.$$

Corollarium 1.

§. 53. Quod si hic ponamus $D = 0$ et $E = 0$, casus oritur theorematis tertii, pro aequatione

$$\frac{\partial x}{\sqrt{(A + Bx + Cxx)}} - \frac{\partial y}{\sqrt{(A + By + Cyy)}} = 0,$$

cujus ergo integrale erit

$$\frac{2A + B(x+y) + 2Cxy + 2\sqrt{(A + Bx + Cxx)(A + By + Cyy)}}{(x-y)^2} = \frac{2A + Bk + 2\sqrt{A(A + Bk + Ckk)}}{kk},$$

quae forma si cum superiori comparetur, formulae irrationales eliminari poterunt. Quoniam enim ex priore est

$$2\sqrt{XY} = 2\sqrt{[A(A + Bk + Ckk)]} - B(k - x - y) + 2Cxy,$$

erit hoc integrale postremum

$$\frac{2A + B(2x + 2y - k) + 4Cxy + 2\sqrt{[A(A + Bk + Ckk)]}}{(x-y)^2} = \frac{2A + Bk + 2\sqrt{A(A + Bk + Ckk)}}{kk},$$

unde statim deduci potest aequatio canonica

$$\alpha + 2\beta(x+y) + \gamma(xx+yy) + 2\delta xy = 0.$$

Corollarium 2.

§. 54. Ponamus nunc esse $A = 0$ et $B = 0$, ut sit

$X = xx(C + Dx + Exx)$ et
$Y = yy(C + Dy + Eyy)$ et
$K = kk(C + Dk + Ekk)$,

aequatio differentialis integranda fiet

$$\frac{\partial x}{x\sqrt{(C+Dx+Exx)}} - \frac{\partial y}{y\sqrt{(C+Dy+Eyy)}} = 0,$$

cujus ergo integrale erit

$$\frac{xy[2C+D(x+y)+2Exy]+2xy\sqrt{(C+Dx+Exx)(C+Dy+Eyy)}}{(x-y)^2} = \Delta,$$

atque hic constantem Δ per k definire non licebit: positio enim $y = 0$ incongruum jam involvit. Interim tamen et haec integratio maxime est memoratu digna.

Corollarium 3.

§. 55. Quod si autem in hac postrema integratione loco x et y scribamus $\frac{1}{x}$ et $\frac{1}{y}$, primo aequatio differentialis erit

$$\frac{\partial y}{\sqrt{(Cyy+Dy+E)}} - \frac{\partial x}{\sqrt{(Cxx+Dx+E)}} = 0;$$

tum vero integrale sequentem induet formam

$$\frac{2Cxy+D(x+y)+2E+2\sqrt{(Cxx+Dx+E)(Cyy+Dy+E)}}{(y-x)^2} = \Delta$$

$$= \frac{Dk+2E+2\sqrt{E(Ckk+Dk+E)}}{kk}.$$

Si igitur hic loco literarum E, D, C, scribamus A, B, C, prodibit aequatio differentialis supra tractata

$$\frac{\partial x}{\sqrt{(A+Bx+Cxx)}} - \frac{\partial y}{\sqrt{(A+By+Cyy)}} = 0$$

cujus ergo integrale erit

$$\frac{2A+B(x+y)+2Cxy+2\sqrt{(A+Bx+Cxx)(A+By+Cyy)}}{(x-y)^2} = \frac{Bk+2A+2\sqrt{A(A+Bk+Ckk)}}{kk},$$

quae egregie convenit cum ea in coroll. 1. data.

Corollarium 4.

§. 56. Contemplemur nunc etiam casum, quo formula $A + Bx + Cxx + Dx^3 + Ex^4$ fit quadratum, quod sit $(a + bx + cxx)^2$, ita ut jam habeamus

$$A = aa, \; B = 2ab, \; C = bb + 2ac, \; D = 2bc, \; E = ec,$$

tum vero

$$\sqrt{X} = a + bx + cxx, \; \sqrt{Y} = a + by + cyy,$$
$$\sqrt{K} = a + bk + ckk,$$

atque aequatio differentialis pro priore casu erit

$$\frac{\partial x}{a + bx + cxx} - \frac{\partial y}{a + by + cyy} = 0,$$

cujus ergo integrale erit

$$\left\{ \begin{array}{l} 2aa + 2ab(x+y) + 2(bb + 2ac)xy + 2bcxy(x+y) \\ \quad + 2ccxxyy + 2(a + bx + cxx)(a + by + cyy) \end{array} \right\} : (x-y)^2 = \Delta,$$

quae reducitur ad

$$\frac{aa + ab(x+y) + (bb+ac)xy + bcxy(x+y) + ccxxyy}{(x-y)^2} = \frac{aa + abk}{kk}.$$

Quod si jam utrinque addamus $\frac{1}{4}bb$, prodibit

$$\frac{[a + \frac{1}{2}b(x+y) + cxy]^2}{(x-y)^2} = \frac{(a + \frac{1}{2}bk)^2}{k^2},$$

unde extracta radice obtinetur forma integralis in theoremate primo assignata.

§. 57. Sin autem hoc modo alterum casum aequationis

$$\frac{\partial x}{a + bx + cxx} + \frac{\partial y}{a + by + cyy} = 0,$$

evolvere velimus, pervenimus ad hanc aequationem

$$\frac{2aa + 2ab(x+y) + 2(bb + 2ac)xy + 2bcxy(x+y) + 2ccxxyy}{(x-y)^2} - \frac{2(a + bx + cxx)(a + by + cyy)}{(x-y)^2} = \Delta,$$

quae evoluta praebet $\Delta = -2ac$, haecque aequatio manifesto est absurda, et nihil circa integrale quaesitum declarat, cujus rationem maximi momenti erit perscrutari.

Insigne Paradoxon.

§. 58. Cum hujus aequationis differentialis

$$\frac{\partial x}{\sqrt{X}} + \frac{\partial y}{\sqrt{Y}} = 0,$$

integrale in genere inventum sit

$$\frac{2A + B(x+y) + y\,C\,xy + D\,xy(x+y) + 2E\,xxyy - 2\sqrt{XY}}{(x-y)^2} = \Delta,$$

casu autem, quo statuitur

$$\sqrt{X} = a + bx + cxx \text{ et}$$
$$\sqrt{Y} = a + by + cyy,$$

aequatio absurda inde oriatur, quaeritur enodatio hujus insignis difficultatis ac praecipue modus, hinc verum integralis valorem investigandi.

Enodatio Paradoxi.

§. 59. Quemadmodum scilicet in analysi ejusmodi formulae occurrere solent, quae certis casibus indeterminatae atque adeo nihil plane significare videntur: ita hic simile quid usu venit, sed longe alio modo, cum neque ad fractionem, cujus numerator et denominator simul evanescunt, neque ad differentiam inter duo infinita perveniatur, quod exemplum eo magis est notatu dignum, quod non memini, similem casum mihi unquam se obtulisse. Istud singulare phaenomenon se nimirum exerit, quando ambae formulae X et Y evadunt quadrata, ad quod ergo resolvendum ad simile artificium recurri oportet, quo formulae X et Y non ipsis quadratis aequales sed ab iis infinite parum discrepare assumuntur.

§. 60. Statuamus igitur

$$X = (a + bx + cxx)^2 + \alpha \text{ et}$$
$$Y = (a + by + cyy)^2 + \alpha,$$

ita ut pro litteris majusculis A, B, C, D, E, fiat $A = aa + \alpha$, $B = 2ab$, $C = 2ac + bb$, $D = 2bc$, $E = cc$, ubi α denotat quantitatem infinire parvam, deinceps nihilo aequalem ponendam. Hinc ergo si brevitatis gratia ponamus

$$a + bx + cxx = R \text{ et } a + by + cyy = S, \text{ erit}$$
$$\sqrt{X} = R + \frac{\alpha}{2R} \text{ et } \sqrt{Y} = S + \frac{\alpha}{2S}.$$

§. 61. Hunc igitur consideremus formam integralis primo inventam, quae erat

$$\frac{\sqrt{X} - \sqrt{Y}}{x - y} = \sqrt{[\Delta + D(x + y) + E(x + y)^2]},$$

pro qua igitur habebimus

$$\sqrt{X} - \sqrt{Y} = R - S - \frac{\alpha(R - S)}{2RS}.$$

Quia vero hic erit

$$R - S = b(x - y) + c(xx - yy), \text{ fiet}$$
$$\frac{R - S}{x - y} = b + c(x + y).$$

At posito brevitatis gratia

$$x + y = p \text{ erit } \frac{R - S}{x - y} = b + cp,$$

unde aequatio nostra erit

$$b + cp - \frac{\alpha(b + cp)}{2RS} = \sqrt{(\Delta + 2bcp + ccpp)}.$$

§. 62. Sumantur nunc utrinque quadrata, et aequatio nostra sequentem induet formam $bb - \frac{\alpha}{RS}(b + cp)^2 = \Delta$. Altiores scilicet potestates ipsius α hic ubique praetermittuntur. Hic ergo ratio nostri paradoxi manifesto in oculos incidit, quia posito $\alpha = 0$ oritur $bb = \Delta$; unde, ut Δ maneat constans arbitraria, evidens est, differentiam inter bb et Δ etiam infinite parvam statui debere; quamobrem ponamus $\Delta = bb - \alpha\Gamma$, ac obtinebitur ista aequatio penitus determinata $\frac{(b + cp)^2}{RS} = \Gamma$, sive

$$[b+c(x+y)]^2 = \Gamma(a+bx+cxx)(a+by+cyy),$$

quae forma non multum discrepat a formula supra inventa.

§. 63. Haec quidem forma magis est complicata quam solutiones §. §. 24. et seqq. inventae: Sequenti autem artificio ad formam simplicissimam redigi poterit. Cum haec fractio $\frac{RS}{(b+cp)^2}$ debeat esse quantitas constans, sit ea $=$ F, ut esse debeat $F(cp+b)^2 = RS$, et quemadmodum hic posuimus $x+y=p$, ponamus porro $xy=u$, fietque

$$RS = aa + abp + ac(pp-2u) + bbu + bcpu + ccuu,$$

atque aequatio jam secundum potestates ipsius p disposita erit

$$\begin{aligned}F(cp+b)^2 = acpp &+ abp + aa \\ &+ bcpu + bbu \\ &\quad - 2acu \\ &\quad + ccuu,\end{aligned}$$

ubi primo utrinque dividamus, quatenus fieri potest, per $cp+b$, ac reperietur

$$F(cp+b) = ap + bu + \frac{(a-cu)^2}{cp+b}.$$

Dividamus nunc porro per $cp+b$, quatenus fieri potest, ac fiet

$$F = \frac{a}{c} - \frac{b}{c} \cdot \frac{(a-cu)}{(cp+b)} + \frac{(a-cu)^2}{(cp+b)^2}.$$

§. 64. Hac forma inventa, si statuamus

$\frac{a-cu}{cp+b} = V$, erit $F = \frac{a}{c} - \frac{b}{c} \cdot V + VV$.

Cum igitur ista expressio aequari debeat quantitati constanti, evidens est, ipsam quantitatem V constantem esse debere, ita ut jam nostrum integrale reductum sit ad hanc formam

$$\frac{a-cu}{cp+b} = \frac{a-cxy}{c(x+y)+b} = \text{Const.}$$

Subtrahamus utrinque $\frac{a}{b}$, fietque

$$\frac{cxy + a(x+y)}{b + c(x+y)} = \text{Const.}$$

quae forma per priorem divisa producit hanc

$$\frac{a(x+y) + cxy}{cxy - a} = \text{Const.}$$

quae formae conveniunt cum supra exhibitis.

Theorema 5.

§. 65. Si in genere haec ratio designandi adhibeatur, ut sit $Z = A + Bz + Czz + Dz^3 + Ez^4$, atque valor hujus formulae integralis $\int \frac{\partial z}{\sqrt{Z}}$, ita sumtus ut evanescat posito $z = 0$, designetur hoc charactere $\Pi : z$; tum, ut fiat $\Pi : k = \Pi : x \pm \Pi : y$, necesse est ut inter quantitates k, x, y, ista relatio subsistat

$$\frac{2A + B(x+y) + 2Cxy + Dxy(x+y) + 2Exxyy \mp 2\sqrt{XY}}{(x-y)^2} = \frac{2A + Bk \mp 2\sqrt{AK}}{kk},$$

cujus ratio ex superioribus est manifesta. Cum enim k denotet quantitatem constantem, erit

$$\partial . \Pi : x \pm \partial . \Pi : y = 0, \text{ sive } \frac{\partial x}{\sqrt{X}} \pm \frac{\partial y}{\sqrt{Y}} = 0,$$

cujus integrale modo ante vidimus ita exprimi

$$\frac{2A + B(x+y) + 2Cxy + Dxy(x+y) + 2Exxyy \mp 2\sqrt{XY}}{(x-y)^2} = \Delta.$$

Quare cum esse debeat $\Pi : x \pm \Pi : y = \Pi : k$, manifestum est posito $y = 0$, fieri debere $\Pi : x = \Pi : k$ ideoque $x = k$ unde constans indefinita Δ eodem prorsus modo definitur, uti est exhibita.

Corollarium 1.

§. 66. Hinc si formula $\Pi : z$ exprimat arcum cujuspiam lineae curvae abscissae sive applicatae Z respondentem, in hac curva omnes arcus eodem modo inter se comparare licebit, quo arcus circulares inter se comparantur; quandoquidem, propositis duobus arcubus $\Pi : x$ et $\Pi : y$, tertius arcus $\Pi : k$ semper exhiberi poterit vel summae vel differentiae eorum arcuum aequalis.

Corollarium 2.

§. 67. Ita si in hac forma $\Pi : k = \Pi : x + \Pi : y$ statuatur $y = x$, prodibit $\Pi\, k = 2\, \Pi : x$; sicque arcus reperitur duplo alterius aequalis. At vero si in nostra forma faciamus $y = x$, tam numerator quam denominator in nihilum abeunt. Ut autem ejus verum valorem eruamus, utamur aequatione primum (§. 38.) inventa

$$\frac{\sqrt{X} - \sqrt{Y}}{x - y} = \sqrt{[\Delta + D(x+y) + E(x+y)^2]},$$

et jam in membro sinistro spectetur y ut constans; ipsi x vero valorem tribuamus infinite parum discrepantem, sive, quod eodem redit, loco numeratoris et denominatoris eorum differentialia substituantur, sumta sola x variabili, hocque modo pro casu $y = x$ membrum sinistrum evadit $\frac{X'}{2\sqrt{X}}$, ubi est

$$X' = B + 2\, C\, x + 3\, D\, x\, x + 4\, E\, x^3.$$

Nunc ergo sumtis quadratis habebitur

$$\frac{X' X'}{4 X} = \Delta + 2\, D\, x + 4\, E\, x\, x,$$

existente Δ ut ante $= \frac{2\, A + B\, k - 2\sqrt{A\, K}}{k\, k}$.

Corollarium 3.

§. 68. Verum sine his ambagibus duplicatio arcus ex altera forma $\Pi : k = \Pi : x - \Pi\, y$ deduci potest, ponendo $y = k$, siquidem hinc fit $\Pi : x = 2\, \Pi : k$, pro quo ergo casu relatio inter x et k hac aequatione exprimetur

$$\frac{2A+B(k+x)+2Ckx+Dkx(k+x)+2Ekkxx+2\sqrt{KX}}{(x-k)^2}$$
$$=\frac{2A+Bk+2\sqrt{AK}}{kk}.$$

Facile autem patet, quomodo hic ad triplicationem, quadruplicationem et quamlibet multiplicationem arcuum progredi debeat, quod argumentum olim fusius sum tractatus.

Theorema 6.

§. 69. Si in formis supra inventis ponatur tam $B=0$ quam $D=0$, ut sit

$$X=A+Cxx+Ex^4$$
$$Y=A+Cyy+Ey^4 \text{ et}$$
$$K=A+Ckk+Ek^4;$$

tum si ista aequatio $\frac{\partial x}{\sqrt{X}}\pm\frac{\partial y}{\sqrt{Y}}=0$ ita integretur, ut posito $y=0$ fiat $x=k$, tum aequatio integralis erit

$$\frac{A+Cxy+Exxyy\mp\sqrt{XY}}{(x-y)^2}=\frac{A\mp\sqrt{AK}}{kk}.$$

Corollarium 1.

§. 70. Hic notari meretur, istum casum adhuc alio modo ex forma generali deduci posse, si scilicet sumatur $A=0$ et $E=0$, tum enim prodit ista aequatio differentialis

$$\frac{\partial x}{\sqrt{(Bx+Cxx+Dx^3)}}\pm\frac{\partial y}{\sqrt{(By+Cyy+Dy^3)}}=0,$$

cujus ergo integrale erit

$$\frac{2B(x+y)+2Cxy+Dxy(x+y)\mp2\sqrt{(Bx+Cxx+Dx^3)(By+Cyy+Dy^3)}}{(x-y)^2}$$
$$=\frac{Bk}{kk}=\frac{B}{k},$$

ubi valor constantis admodum simplex evasit. Nunc in his formulis loco x et y scribamus xx et yy, at vero loco litterarum B et D scribamus A et E, fietque aequatio differentialis

$$\frac{\partial x}{\sqrt{(A+Cxx+Dx^4)}} \pm \frac{\partial y}{\sqrt{(A+Cyy+Dy^4)}} = 0,$$

cujus ergo integrale etiam hoc modo exprimetur

$$\frac{A(xx+yy)+2Cxxyy+Dxxyy(xx+yy)\mp 2xy\sqrt{XY}}{(xx-yy)^2} = \frac{A}{kk}.$$

Corollarium 2.

§. 71. Ecce ergo hac ratione pervenimus ad alium integralis formam non minus notabilem priore, atque adeo nunc ex earum combinatione formula radicalis $\sqrt{XY}$ eliminari poterit, quandoquidem ex posteriore fit

$$\mp 2\sqrt{XY} = \frac{A(xx-yy)^2}{kkxy} - \frac{A(xx+yy)}{xy} - 2Cxy$$
$$- Exy(xx+yy),$$

qui valor in priore substitutus conducit ad hanc aequationem rationalem

$$2A + 2Cxy + 2Exxyy$$
$$+ \frac{A(xx-yy)^2}{kkxy} - \frac{A(xx+yy)}{xy} - 2Cxy - Exy(xx+yy)$$
$$= \frac{2A(x-y)^2}{kk} \mp \frac{2(x-y)^2\sqrt{AK}}{kk},$$

quae porro reducta et per $(x-y)^2$ divisa revocatur ad hanc formam

$$\frac{2A\mp 2\sqrt{AK}}{kk} = \frac{A(x+y)^2}{kkxy} - Exy - \frac{A}{xy},$$

sive ad hanc

$$\frac{A}{kk}\cdot(xx+yy-kk) - Exxyy \pm \frac{2xy\sqrt{AK}}{kk} = 0,$$

quae egregie convenit cum aequatione canonica, qua olim sum usus: scilicet

$$0 = \alpha + \gamma(xx+yy) + 2\delta xy + \zeta xxyy,$$

si quidem est

$$\alpha = -A,\ \gamma = +\frac{A}{kk},\ 2\delta = \pm\frac{2\sqrt{AK}}{kk},\ \zeta = -E.$$

Corollarium 3.

§. 72. Methodo posteriore, qua hic usi sumus ad hanc aequationem integrandam, aequatio multo generalior tractari poterit, ubi in formulis radicalibus potestates usque ad sextam dimensionem assurgunt. Namque si tantum statuamus $A = 0$, ut sit aequatio

$$\frac{\partial x}{\sqrt{x(B + Cx + Dxx + Ex^3)}} \pm \frac{\partial y}{\sqrt{y(B + Cy + Dyy + Ey^3)}} = 0,$$

cujus integrale est

$$\frac{B(x+y) + 2Cxy + Dxy(x+y) + 2Exxyy}{(x-y)^2}$$
$$\frac{\mp 2\sqrt{xy(B + Cx + Dxx + Ex^3)(B + Cy + Dyy + Ey^3)}}{(x-y)^2} = \frac{B}{kk}.$$

Quod si jam hic loco x et y scribamus xx et yy, aequatio differentialis fiet

$$\frac{\partial x}{\sqrt{(B + Cxx + Dx^4 + Ex^6)}} \pm \frac{\partial y}{\sqrt{(B + Cyy + Dy^4 + Ey^6)}} = 0,$$

ejus ergo integrale erit

$$\frac{B(xx+yy) + 2Cxxyy + Dxxyy(xx+yy) + 2Ex^4y^4}{(xx-yy)^2}$$
$$\frac{\mp 2xy\sqrt{(B + Cxx + Dx^4 + Ex^6)(B + Cyy + Dy^4 + Ey^6)}}{(xx-yy)^2} = \frac{B}{kk}.$$

Nunc autem ostendamus, quomodo ope methodi illustris *de la Grange* idem integrale impetrari queat.

Analysis

pro integratione aequationis differentialis

$$\frac{\partial x}{\sqrt{X}} \pm \frac{\partial y}{\sqrt{Y}} = 0,$$

existente

$$X = B + Cxx + Dx^4 + Ex^6 \text{ et}$$
$$Y = B + Cyy + Dy^4 + Ey^6.$$

§. 73. Posito igitur

$$\frac{\partial x}{\sqrt{X}} = \partial t \text{ erit } \frac{\partial y}{\sqrt{Y}} = \mp \partial t,$$

hincque sumtis quadratis

$$\frac{\partial x^2}{\partial t^2} = X \text{ et } \frac{\partial y^2}{\partial t^2} = Y,$$

hinc formentur hae aequationes

$$\frac{xx\partial x^2}{\partial t^2} = xxX \text{ et } \frac{yy\partial y^2}{\partial t^2} = yy$$

Jam introducantur duae novae variabiles p et q, ut sit

$$xx + yy = 2p \text{ et } xx - yy = 2q,$$

ex quo fit $x\partial x + y\partial y = \partial p$, et $x\partial x - y\partial y = \partial q$, hincque $xx\partial x^2 - yy\partial y^2 = \partial p\partial q$; quamobrem habebimus

$$\frac{\partial p\partial q}{\partial t^2} = xxX - yyY,$$

quae aequatio dividatur per $xx - yy = 2q$, prodibitque

$$\frac{\partial p\partial q}{2q\partial t^2} = \frac{xxX - yyY}{xx - yy},$$

quae forma, valoribus pro X et Y substitutis, dabit

$$\frac{\partial p\partial q}{2q\partial t^2} = B + 2Cp + D(3pp + qq) + 4E(p^3 + pqq).$$

§. 74. Nunc porro aequationes $\frac{\partial x^2}{\partial t^2}$ et $\frac{\partial y^2}{\partial t^2}$ differentiatae dabunt

$$\frac{2\partial\partial x}{\partial t^2} = X' \text{ et } \frac{2\partial\partial y}{\partial t^2} = Y'.$$

Ex priore fit $\frac{2x\partial\partial x}{\partial t^2} = xX'$, cui addatur $\frac{2\partial x^2}{\partial t^2} = 2X$, ut prodeat

$$\frac{2(x\partial\partial x + \partial x^2)}{\partial t^2} = \frac{2\partial . x\partial x}{\partial t^2} = xX' + 2X.$$

Simili modo erit $\frac{2\partial . y\partial y}{\partial t^2} = yY' + 2Y$, quae duae aequationes invicem additae dabunt

$$\frac{2\partial . \partial p}{\partial t^2} = \frac{2\partial\partial p}{\partial t^2} = xX' + yY' + 2(X + Y).$$

Substitutis autem valoribus et facta substitutione respectu litterarum

p et q, reperitur

$$2X + 2Y = 4B + 4Cp + 4D(pp+qq) + 4Ep(pp+3qq).$$

Deinde ob

$$xX' = 2Cxx + 4Dx^4 + 6Ex^6 \text{ et}$$
$$yY' = 2Cyy + 4Dy^4 + 6Ey^6 \text{ erit}$$
$$xX' + yY' = 4Cp + 8D(pp+qq) + 12Ep(pp+3qq),$$

ex quibus conjunctis fit

$$\frac{2\partial\partial p}{\partial t^2} = 4B + 8Cp + 12D(pp+qq) + 16Ep(pp+3qq).$$

§. 75. Ab hac formula subtrahatur supra inventa $\frac{\partial p\partial q}{2q\partial t^2}$ quater sumta, ac remanebit

$$\frac{2\partial\partial p}{\partial t^2} - \frac{2\partial p\partial q}{q\partial t^2} = 8Dqq + 32Epqq.$$

Nunc utrinque multiplicetur per $\frac{\partial p}{qq}$, et prodibit

$$\frac{1}{\partial t^2}\cdot\left(\frac{2\partial p\partial\partial p}{qq} - \frac{2\partial p^2\partial q}{q^3}\right) = 8D\partial p + 32Ep\partial p,$$

cujus integrale sponte se offert ita expressum

$$\frac{\partial p^2}{qq\partial t^2} = 4\Delta + 8Dp + 16Epp,$$

ideoque extracta radice

$$\frac{\partial p}{q\partial t} = 2\sqrt{(\Delta + 2Dp + 4Epp)}.$$

§. 76. Cum nunc sit

$$\frac{\partial p}{\partial t} = x\sqrt{X} \mp y\sqrt{Y} \text{ et } 2q = xx - yy,$$

facta substitutione orietur haec aequatio

$$\frac{x\sqrt{X} \mp y\sqrt{Y}}{xx-yy} = \sqrt{[\Delta + D(xx+yy) + E(xx+yy)^2]},$$

quae sumtis quadratis reducetur ad istam formam

$$\frac{xxX + yyY \mp 2xy\sqrt{XY}}{(xx-yy)^2} = \Delta + D(xx+yy) + E(xx+yy)^2.$$

Est vero

$$xx\,X + yy\,Y = B(xx + yy) + C(x^4 + y^4) + D(x^6 + y^6) + E(x^8 + y^8),$$

hincque pervenietur ad hanc aequationem

$$\frac{B(xx+yy) + C(x^4+y^4) + D\,xxyy(xx+yy) + 2E\,x^4y^4 \mp 2xy\sqrt{XY}}{(xx-yy)^2} = \Delta.$$

§. 77. Sumamus nunc ut supra constantem Δ ita, ut posito

$y = 0$ fiat $x = k$ et $X = K = B + C\,kk + D\,k^4 + E\,k^6$,

et aequatio integralis induet hanc formam

$$\frac{B(xx+yy) + C(x^4+y^4) + D\,xxyy(xx+yy) + 2E\,x^4y^4 \mp 2xy\sqrt{XY}}{(xx-yy)^2} = \frac{B + C\,kk}{kk},$$

quae aliquanto simplicior evadit, si utrinque subtrahamus C: erit enim

$$\frac{B(xx+yy) + 2C\,xxyy + D\,xxyy(xx+yy) + 2E\,x^4y^4 \mp 2xy\sqrt{XY}}{(xx-yy)^2} = \frac{B}{kk},$$

quae egregie convenit cum integrali supra §. 72. exhibito.

§. 78. Hic casus notatu dignus se offert, dum $B = 0$, tum autem aequatio differentialis ita se habebit

$$\frac{\partial x}{x\sqrt{(C + D\,xx + E\,x^4)}} \pm \frac{\partial y}{y\sqrt{(C + D\,yy + E\,y^4)}} = 0,$$

cujus ergo integrale per constantem Δ expressum erit

$$\frac{C(x^4+y^4) + D\,xxyy(xx+yy) + 2E\,x^4y^4 \mp 2xy\sqrt{XY}}{(xx-yy)^2} = \Delta.$$

Hoc autem casu integratio non ita determinari potest, ut posito $y = 0$ fiat $x = k$, quia integrale posterioris membri hoc casu manifesto abit in infinitum; quamobrem alio modo integrationem determinari conveniet, veluti ut posito $y = b$ fiat $x = a$, tum autem erit ista constans

$$\Delta = \frac{C(a^4 + b^4) + D\,a^2b^2(aa + bb) + 2E\,a^4b^4 \mp 2ab\sqrt{AB}}{(aa + bb)^2}, \text{ existente}$$

$$A = C + D\,aa + E\,a^4 \text{ et } B = C + D\,bb + E\,b^4.$$

Conclusio.

§. 79. Qui processum analyseos hic usitatae comparare voluerit cum methodo, qua illustris *de la Grange* usus est in *Miscellan. Taur. Tom. IV.* facile perspiciet, eam multo facilius ad scopum desideratum perducere atque multo commodius ad quosvis casus applicari posse. Introduxerat autem Vir Ill. in calculum formulam $\frac{\partial t}{T}$, cujus loco hic simplici elemento ∂t sumus usi, ac deinceps quantitatem T tanquam functionem litterarum p et q spectavit, quae positio satis difficiles calculos postulavit, dum nostra methodo longe concinnius easdem integrationes investigare licuit. Quanquam autem nullum est dubium, quin ista analyseos species insigne incrementum polliceatur, tamen nondum patet, quemadmodum ad alias integrationes ea accommodari queat, praeter hos ipsos casus, quos hic tractavimus et quos olim ex aequatione canonica derivaveram.

2.) Methodus succinctior comparationes quantitatum transcendentium in forma $\int \frac{P \partial z}{\sqrt{(A + 2Bz + Czz + 2Dz^3 + Ez^4)}}$ contentarum inveniendi.

M. S. Academiae exhib. die 3 *Nov.* 1777.

In Capite VI. Sect. II. Institutionum mearum Calculi Integralis Tom. I. insignes tradidi comparationes inter quantitates maxime transcendentes, ad quam deductus eram methodo penitus indirecta. Postquam igitur non ita pridem illustris *de la Grange* methodum maxime ingeniosam excogitasset easdem comparationes inveniendi, totum hoc argumentum multo succinctius et elegantius tractari poterit, quam mihi quidem tum temporis licebat, unde sequentia Supplementa Geometris haud displicebunt.

Hypothesis 1.

§. 80. Denotet hic perpetuo character $\Pi : z$ valorem formulae integralis $\int \frac{\partial z}{\sqrt{(\alpha + \beta z + \gamma zz + \delta z^3 + \varepsilon z^4)}}$, ita sumtae ut evanescat posito $z = 0$. Ponatur autem brevitatis gratia $\alpha +$ Fig. 13. $\beta z + \gamma z z + \delta z^3 + \varepsilon z^4 = Z$, ita ut sit $\Pi : z = \int \frac{\partial z}{\sqrt{Z}}$. Tum vero concipiatur super axe $o\,z$ exstructa ejusmodi curva O Z, cujus singuli arcus O Z abscissis $o\,z = z$ respondentes exprimantur per formulam $\Pi : z = \int \frac{\partial z}{\sqrt{Z}}$; atque haec curva ista insigni proprietate erit praedita, ut sumto in ea pro lubitu arcu quocunque F G, a quovis alio puncto X semper arcus X Y illi arcui F G aequalis geometrice abscindi possit, cujus demonstrationem solutio sequentis problematis suppeditabit.

Problema 1.

Si in curva modo descripta proponatur arcus quicunque F G, innumerabiles alios arcus X Y in eadem curva geometrice assignare, qui singuli eidem arcui F G sint aequales.

Solutio.

§. 81. Ductis ex punctis F et G ad axem oz applicatis F f et G g, vocentur abscissae $of = f$ et $og = g$, eruntque arcus O F $= \Pi : f$ et O G $= \Pi : g$, unde longitudo arcus propositi F G erit $= \Pi : g - \Pi : f$. Simili modo pro quovis arcu quaesito X Y vocentur abscissae $ox = x$ et $oy = y$, eruntque arcus O X $= \Pi : x$ et O Y $= \Pi : y$, ideoque arcus X Y $= \Pi : y - \Pi : x$, qui cum aequalis esse debeat arcui F G, habebitur ista aequatio $\Pi : y - \Pi : x = \Pi : g - \Pi : f$, cui satisfieri oportet.

§. 82. Quoniam puncta F et G considerantur ut fixa, dum puncta X et Y per totam curvam variari possunt, differentiatio nobis praebebit hanc aequationem $\partial \cdot \Pi : y - \partial \cdot \Pi : x = 0$. Quare cum sit per hypothesin

$$\Pi : x = \int \frac{\partial x}{\sqrt{X}} \text{ et } \Pi : y = \int \frac{\partial y}{\sqrt{Y}},$$

existente

$$X = \alpha + \beta x + \gamma x x + \delta x^3 + \varepsilon x^4 \text{ et}$$
$$Y = \alpha + \beta y + \gamma y y + \delta y^3 + \varepsilon y^4,$$

solutio problematis perducta est ad hanc aequationem differentialem $\frac{\partial y}{\sqrt{Y}} - \frac{\partial x}{\sqrt{X}} = 0$.

§. 83. Hic jam methodum ill. *de la Grange* in subsidium vocantes statuamus $\frac{\partial x}{\sqrt{X}} = \partial t$, eritque $\frac{\partial y}{\sqrt{Y}} = \partial t$. Hic scilicet novum elementum ∂t in calculum introducimus, quod in sequentibus differentiationibus ut constans tractetur; tum igitur habebimus

$$\frac{\partial x}{\partial t} = \sqrt{X} \text{ et } \frac{\partial y}{\partial t} = \sqrt{Y}.$$

Quod si ergo porro statuamus $y + x = p$ et $y - x = q$, habebimus hinc

$$\frac{\partial p}{\partial t} = \sqrt{Y} + \sqrt{X} \text{ et } \frac{\partial q}{\partial t} = \sqrt{Y} - \sqrt{X},$$

quarum formularum productum praebet

$$\frac{\partial p \partial q}{\partial t^2} = Y - X.$$

Valoribus ergo loco Y et X substitutis erit

$$\frac{2 p \partial q}{\partial t^2} = \beta (y - x) + \gamma (y^2 - x^2) + \delta (y^3 - x^3) + \varepsilon (y^4 - x^4).$$

Quare cum sit

$$y = \frac{p+q}{2} \text{ et } x = \frac{p-q}{2} \text{ erit}$$

$$y - x = q,\ y^2 - x^2 = p q,\ y^3 - x^3 = \tfrac{1}{4} q (3 p p + q q) \text{ et } y^4 - x^4 = \tfrac{1}{2} p q (p p + q q),$$

quibus substitutis factaque divisione per q habebitur

$$\frac{\partial p \partial q}{q \partial t^2} = \beta + \gamma p + \tfrac{1}{4} \delta (3 p p + q q) + \tfrac{1}{2} \varepsilon p (p p + q q),$$

cujus aequationis plurimus erit usus in sequenti calculo.

§. 84. Jam sumtis quadratis primae aequationes dabunt

$$\frac{\partial x^2}{\partial t^2} = X \text{ et } \frac{\partial y^2}{\partial t^2} = Y,$$

quae denuo differentientur, quem in finem ponamus brevitatis gratia

$$\partial X = X' \partial x \text{ et } \partial Y = Y' \partial y,$$

atque hinc nanciscemur

$$\frac{2\partial\partial x}{\partial t^2} = X' \text{ et } \frac{2\partial\partial y}{\partial t^2} = Y',$$

quibus additis erit

$$\frac{2\partial\partial p}{\partial t^2} = X' + Y'.$$

Cum igitur sit

$$X' = \beta + 2\gamma x + 3\delta x x + 4\varepsilon x^3 \text{ et}$$
$$Y' = \beta + 2\gamma y + 3\delta y y + 4\varepsilon y^3, \text{ erit}$$
$$\frac{2\partial\partial p}{\partial t^2} = 2\beta + 2\gamma(x+y) + 3\delta(x^2+y^2) + 4\varepsilon(x^3+y^3).$$

Introducendo igitur litteras p et q ut ante, fiet

$$x + y = p,\ x^2 + y^2 = \tfrac{1}{2}(pp + qq),$$
$$x^3 + y^3 = \tfrac{1}{4}p(pp + 3qq),$$

sicque ista aequatio hanc induet formam

$$\frac{2\partial\partial p}{\partial t^2} = 2\beta + 2\gamma p + \tfrac{3}{2}\delta(pp+qq) + \varepsilon p(pp+3qq).$$

§. 85. Ab hac jam postrema aequatione subtrahatur praecedens bis sumta, ac remanebit

$$\frac{2\partial\partial p}{\partial t^2} - \frac{2\partial p\partial q}{q\partial t^2} = \delta q q + 2\varepsilon p q q.$$

Hinc per $q q$ dividendo habebimus

$$\frac{1}{\partial t^2}\cdot\left(\frac{2\partial\partial p}{qq} - \frac{2\partial p\partial q}{q^3}\right) = \delta + 2\varepsilon p,$$

cujus utrumque membrum manifesto integrationem admittit, si ducatur in elementum ∂p. Hoc enim facto aequatio integralis erit

$$\frac{\partial p^2}{qq\partial t^2} = C + \delta p + \varepsilon p p.$$

§. 86. Initio autem vidimus esse $\frac{\partial p}{\partial t} = \sqrt{X} + \sqrt{Y}$, hincque statim pervenimus ad aequationem integralem algebraicam hanc

$$\frac{(\sqrt{X}+\sqrt{Y})^2}{qq} = C + \delta p + \varepsilon p p.$$

Quare cum sit $p = x + y$ et $q = y - x$, haec aequatio evoluta fiet

$$\frac{X + Y + 2\sqrt{XY}}{(y-x)^2} = C + \delta(x+y) + \varepsilon(x+y)^2,$$

ubi constantem per integrationem ingressam secundum indolem problematis ita definiri oportet, ut dum punctum X incidit in punctum F, punctum Y in ipsum punctum G cadat, sive ut facto $x = f$ fiat $y = g$.

§. 87. Cum jam sit

$$X + Y = 2\alpha + \beta(x+y) + \gamma(x^2+y^2) + \delta(x^3+y^3) + \varepsilon(x^4+y^4),$$

si terminos $\delta(x+y) + \varepsilon(x+y)^2$ in alteram partem transferimus, perveniemus ad hanc aequationem

$$\frac{2\alpha + \beta(x+y) + \gamma(x^2+y^2) + \delta xy(x+y) + 2\varepsilon xxyy + 2\sqrt{XY}}{(y-x)^2} = C.$$

Subtrahamus autem insuper utrinque γ, et loco $C - \gamma$ scribamus Δ, hocque modo nostra aequatio reducetur ad hanc formam satis concinnam

$$\frac{2\alpha + \beta(x+y) + 2\gamma xy + \delta xy(x+y) + 2\varepsilon xxyy + 2\sqrt{XY}}{(y-x)^2} = \Delta.$$

§. 88. Quia nunc Δ ita determinari debet, ut sumto $x = f$ fiat $y = g$, si secundum analogiam statuamus

$$\alpha + \beta f + \gamma ff + \delta f^3 + \varepsilon f^4 = F \text{ et}$$
$$\alpha + \beta g + \gamma gg + \delta g^3 + \varepsilon g^4 = G,$$

erit ista constans Δ ita expressa

$$\Delta = \frac{2\alpha + \beta(f+g) + 2\gamma fg + \delta fg(f+g) + 2\varepsilon ffgg + 2\sqrt{FG}}{(g-f)^2}.$$

Hac igitur aequatione inventa, si ipsi x pro lubitu tribuatur valor quicunque, inde elici poterit valor ipsius y, ita ut alter terminus X arcus quaesiti X Y pro arbitrio assumi possit. Verum facile patet, istam determinationem in calculos perquam molestos praecipitare, quandoquidem aequatio inventa quadratis sumendis ab irrationalitate $\sqrt{XY}$ liberari deberet. Sequenti autem modo ista investigatio sublevari poterit.

§. 89. Quoniam ista formula

$$2\alpha + \beta(x+y) + 2\gamma xy + \delta xy(x+y) + 2\varepsilon xxyy$$

essentialiter in calculum ingreditur, ejus loco brevitatis gratia scribamus hunc characterem $[x, y]$, cujus ergo valor erit cognitus, etiam si loco x et y aliae litterae accipiantur. Hoc igitur modo aequatio inventa ita referri poterit

$$\frac{[x,y] + 2\sqrt{XY}}{(y-x)^2} = \frac{[f,g] + 2\sqrt{FG}}{(g-f)^2},$$

quae ergo aequatio exprimit relationem inter bina ordinata x et y, ut problemati satisfiat, hoc est, ut fiat

$$\Pi : y - \Pi : x = \Pi : g - \Pi : f.$$

Quare cum hic etiam sequatur

$$\Pi : y - \Pi : g = \Pi : x - \Pi : f,$$

aequatio hinc ista exsurget

$$\frac{[g,y] + 2\sqrt{GY}}{(y-g)^2} = \frac{[f,x] + 2\sqrt{FX}}{(x-f)^2}.$$

§. 90. Ex hac jam aequatione cum priore conjuncta facile eliminari poterit formula radicalis $\sqrt{Y}$, sicque aequatio habebitur tantum litteram y tanquam incognitam involvens, unde ejus valor haud difficulter definiri potest. Calculum autem hunc instituenti patebit, tantum ad aequationem quadraticam perveniri, ita ut bini valores pro puncto Y reperiantur, quemadmodum rei natura postulat, dum sumto puncto X alterum punctum Y tam dextrorsum quam sinistrorsum cadere poterit. Hinc autem calculo fusius non immoramur, quandoquidem hic potissimum est propositum, totam hujus problematis solutionem per methodum directam a priori repetere.

Hypothesis 2.

Fig. 14. §. 91. Constituta super axe $o\,z$ curva O Z in priori hypothesi descripta, concipiatur super eodem axe alia curva insuper descripta $\mathfrak{O}\,\mathfrak{Z}$, ita comparata, ut abscissae $o\,z = z$ respondeat arcus $\mathfrak{O}\,\mathfrak{Z} = \phi : z$, ita ut sit

$$\phi : z = \int \frac{\partial z\,(\mathfrak{A} + \mathfrak{B}\,z + \mathfrak{C}\,z\,z + \mathfrak{D}\,z^3 + etc.)}{\sqrt{Z}},$$

integrali hoc pariter ita sumto ut evanescat posito $z = 0$, existente ut ante

$$Z = \alpha + \beta\,z + \gamma\,z\,z + \delta\,z^3 + \varepsilon\,z^4.$$

Pro numeratore autem ponamus brevitatis gratia

$$\mathfrak{A} + \mathfrak{B}\,z + \mathfrak{C}\,z\,z + \mathfrak{D}\,z^3 + \text{etc.} = \mathfrak{Z},$$

ita ut sit $\phi : z = \int \frac{\mathfrak{Z}\,\partial z}{\sqrt{Z}}$.

§. 92. Ista jam curva hac ratione descripta hac insigni proprietate erit praedita, ut, si in priore curva rescissi fuerint arcus F G et X Y inter se aequales, productis iisdem applicatis in nova curva, arcuum hoc modo rescissorum $\mathfrak{F}\,\mathfrak{G}$ et $\mathfrak{X}\,\mathfrak{Y}$ differentia vel algebraice vel saltem per logarithmos et arcus circulares assignari possit, cujus rei veritatem solutio sequentis problematis demonstrabit.

Problema 2.

Si in curva secundum primam hypothesin descripta abscissi fuerint duo arcus aequales FG et XY, iisque in curva modo descripta respondeant arcus $\mathfrak{F}\,\mathfrak{G}$ et $\mathfrak{X}\,\mathfrak{Y}$, quibus scilicet eaedem abscissae in axe conveniant, differentiam inter hos binos arcus investigare.

Solutio.

§. 93. Quia igitur hic quaeritur differentia inter arcus $\mathfrak{F}\mathfrak{G}$ et $\mathfrak{X}\mathfrak{Y}$, ponatur ea $= V$, quae ergo spectari poterit tanquam certa functio ipsarum x et y, si quidem puncta $\mathfrak{F}$ et $\mathfrak{G}$ tanquam fixa consideramus. Cum igitur sit arcus

$$\mathfrak{F}\mathfrak{G} = \Phi : g - \Phi : f \text{ et arcus}$$

$$\mathfrak{X}\mathfrak{Y} = \Phi : y - \Phi : x,$$

habebimus

$$\Phi : y - \Phi : x = \Phi : g - \Phi : f + V,$$

unde differentiando habebimus

$$\frac{\mathfrak{Y}\,\partial y}{\sqrt{Y}} - \frac{\mathfrak{X}\,\partial x}{\sqrt{X}} = \partial V,$$

quia litteras f et g pro constantibus habemus.

§. 94. Ponamus nunc ut supra factum est

$$\frac{\partial x}{\sqrt{X}} = \frac{\partial y}{\sqrt{Y}} = \partial t,$$

et haec aequatio induet istam formam

$$(\mathfrak{Y} - \mathfrak{X})\,\partial t = \partial V.$$

Verum in solutione primi problematis deducti fuimus ad hanc aequationem finalem

$$\frac{\partial p^2}{qq\,\partial t^2} = C + \delta p + \varepsilon p p,$$

unde fit

$$\frac{\partial p}{\partial t} = \sqrt{(C + \delta p + \varepsilon p p)} = \sqrt{(\Delta + \gamma + \delta p + \varepsilon p p)},$$

atque hinc colligimus

$$\partial t = \frac{\partial p}{q\sqrt{(\Delta + \gamma + \delta p + \varepsilon p p)}},$$

ubi est $p = x + y$ et $q = y - x$. Hoc ergo valore inducto aequatio differentialis resolvenda est

$$\partial V = \frac{(\mathfrak{Y} - \mathfrak{X})\,\partial p}{q\sqrt{(\Delta + \gamma + \delta p + \varepsilon p p)}},$$

ubi est

$$\mathfrak{X} = \mathfrak{A} + \mathfrak{B}x + \mathfrak{C}xx + \mathfrak{D}x^3 + \text{etc.}$$

similique modo

$$\mathfrak{Y} = \mathfrak{A} + \mathfrak{B}y + \mathfrak{C}yy + \mathfrak{D}y^3 + \text{etc.},$$

quousque libuerit continuando.

§. 95. Quod si jam hos valores substituamus, habebimus

$$\mathfrak{Y} - \mathfrak{X} = \mathfrak{B}(y-x) + \mathfrak{C}(y^2 - x^2) + \mathfrak{D}(y^3 - x^3) + \mathfrak{E}(y^4 - x^4) + \text{etc.}$$

unde si loco x et y introducamus quantitates p et q, ob $x = \frac{p-q}{2}$ et $y = \frac{p+q}{2}$, orientur sequentes valores

$$y - x = q,\ y^2 - x^2 = pq,\ y^3 - x^3 = \tfrac{1}{4}q(3pp + qq),$$

$$y^4 - x^4 = \tfrac{1}{2}pq(pp+qq),\ y^5 - x^5 = \tfrac{1}{16}q(5p^4 + 10ppqq + q^4).$$

§. 96. Quantitas ergo V per sequentes formulas integrales secundum numerum litterarum $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, etc. determinatur

$$\mathrm{V} = \mathfrak{B}\int\frac{\partial p}{\sqrt{(\Delta+\gamma+\delta p+\varepsilon pp)}} + \mathfrak{C}\int\frac{p\,\partial p}{\sqrt{(\Delta+\gamma+\delta p+\varepsilon pp)}} + \tfrac{1}{4}\mathfrak{D}\int\frac{(3pp+qq)\,\partial p}{\sqrt{(\Delta+\gamma+\delta p+\varepsilon pp)}} + \tfrac{1}{2}\mathfrak{E}\int\frac{p(pp+qq)\,\partial p}{\sqrt{(\Delta+\gamma+\delta p+\varepsilon pp)}} + \tfrac{1}{16}\mathfrak{F}\int\frac{(5p^4+10ppqq+q^4)\,\partial p}{\sqrt{(\Delta+\gamma+\delta p+\varepsilon pp)}} + \text{etc.}$$

Quarum formularum duae priores jam absolute exhiberi possunt, sive algebraice, quod evenit si $\varepsilon = 0$, sive per logarithmos, si valor ipsius ε fuerit positivus, sive per arcus circulares, si valor ipsius ε fuerint negativus. Reliquae vero formulae exigunt relationem inter p et q, quam deinceps investigabimus. Hic tantum notetur, potestates solas pares ipsius q in has formulas ingredi.

§. 97. Hic autem littera Δ eundem valorem constantem designat, quem supra jam definivimus, qui erat

$$\Delta = \frac{2\alpha + \beta(f+g) + 2\gamma fg + \delta fg(f+g) + 2\varepsilon ffgg + 2\sqrt{FG}}{(g-f)^2}.$$

Praeterea vero cum esse debeat

$$\Phi : y - \Phi : x = \Phi : g - \Phi : f + V,$$

evidens est, casu quo $x = f$ et $y = g$ fieri debere $V = 0$; quamobrem formulae illae integrales pro V inventae ita capi debebunt, ut posito $p = f + g$ et $q = g - f$ valor ipsius V evanescat.

Analysis
pro investiganda relatione inter p et q.

§. 98. Quia jam invenimus aequationem finitam inter x et y, ex ea quoque ponendo $y = \frac{p+q}{2}$ et $x = \frac{p-q}{2}$ relatio inter litteras p et q derivari posset; verum hoc calculos nimis taediosos postularet, quamobrem aliam viam ineamus istam relationem ex formulis differentialibus deducendi. Cum enim sit $\frac{\partial p}{\partial q} = \frac{\partial y + \partial x}{\partial y - \partial x}$, ob proportionem

$$\partial x : \partial y = \sqrt{X} : \sqrt{Y} \text{ erit } \frac{\partial p}{\partial q} = \frac{\sqrt{Y} + \sqrt{X}}{\sqrt{Y} - \sqrt{X}};$$

supra autem invenimus esse

$$\frac{\sqrt{Y} + \sqrt{X}}{q} = \sqrt{(\Delta + \gamma + \delta p + \varepsilon pp)},$$

ubi Δ eandem denotat constantem, quam modo ante definivimus.

§. 99. Nunc igitur fractio pro $\frac{\partial p}{\partial q}$ inventa supra et infra multiplicetur per $\sqrt{Y} + \sqrt{X}$, et cum sit

$$(\sqrt{Y} + \sqrt{X})^2 = qq(\Delta + \gamma + \delta p + \varepsilon pp),$$

habebimus hanc aequationem

$$\frac{\partial p}{\partial q} = \frac{qq(\Delta + \gamma + \delta p + \varepsilon pp)}{Y - X},$$

cujus denominatorem jam supra §. 83. evolvimus, ubi invenimus esse

$$Y - X = \beta q + \gamma p q + \tfrac{1}{4}\delta q(3pp + qq) + \tfrac{1}{2}\varepsilon p q(pp + qq),$$

quo valore substituto erit

$$\frac{\partial p}{\partial q} = \frac{q(\Delta + \gamma + \delta p + \varepsilon p p)}{\beta + \gamma p + \frac{1}{4}\delta(3pp + qq) + \frac{1}{2}\varepsilon p(pp + qq)},$$

quae reducitur ad hanc formam

$$2q\partial q = \frac{[2\beta + 2\gamma p + \frac{1}{2}\delta(3pp + qq) + \varepsilon p(pp + qq)]\partial p}{\Delta + \gamma + \delta p + \varepsilon pp}.$$

100. Transferamus terminos qui continent qq a dextra in sinistram partem ut obtineamus hanc aequationem

$$2q\partial q - \frac{qq\partial p(\frac{1}{2}\delta + \varepsilon p)}{\Delta + \gamma + \delta p + \varepsilon pp} = \frac{(2\beta + 2\gamma p + \frac{3}{2}\delta pp + \varepsilon p^3)\partial p}{\Delta + \gamma + \delta p + \varepsilon pp}.$$

Membrum hujus aequationis sinistrum integrabile reddi potest, si per certam functionem ipsius p, quae sit $= \Pi$, multiplicetur, quando fuerit

$$\frac{\partial \Pi}{\Pi} = -\frac{\partial p(\frac{1}{2}\delta + \varepsilon p)}{\Delta + \gamma + \delta p + \varepsilon pp},$$

quae aequatio integrata dat

$$l\Pi = -\tfrac{1}{2}l(\Delta + \gamma + \delta p + \varepsilon pp).$$

Sicque erit multiplicator iste

$$\Pi = \frac{1}{\sqrt{(\Delta + \gamma + \delta p + \varepsilon pp)}};$$

tum autem integrale quaesitum erit

$$\frac{qq}{\sqrt{(\Delta + \gamma + \delta p + \varepsilon pp)}} = \int \frac{(2\beta + 2\gamma p + \frac{3}{2}\delta pp + \varepsilon p^3)\partial p}{(\Delta + \gamma + \delta p + \varepsilon pp)^{\frac{3}{2}}}.$$

§. 101. Hoc postremum integrale manifesto continet formam $\frac{pp}{\sqrt{(\Delta+\gamma+\delta p+\varepsilon pp)}}$, quippe cujus differentiale est

$$\frac{(2\Delta p+2\gamma p+\frac{3}{2}\delta pp+\varepsilon p^3)\partial p}{(\Delta+\gamma+\delta p+\varepsilon pp)^{\frac{3}{2}}};$$

quare integrale ita potest repraesentari

$$\frac{qq}{\sqrt{(\Delta+\gamma+\delta p+\varepsilon pp)}}=\frac{pp}{\sqrt{(\Delta+\gamma+\delta p+\varepsilon pp)}}$$
$$+\int\frac{(2\beta-2\Delta p)\partial p}{(\Delta+\gamma+\delta p+\varepsilon pp)^{\frac{3}{2}}},$$

quod postremum integrale statuatur $=\frac{m+np}{\sqrt{(\Delta+\gamma+\delta p+\varepsilon pp)}}$, hujus enim differentiale est

$$\frac{[(\Delta+\gamma)n-\frac{1}{2}\delta m+(\frac{1}{2}\delta n-\varepsilon m)p]\partial p}{(\Delta+\gamma+\delta p+\varepsilon pp)^{\frac{3}{2}}},$$

ideoque fieri debet

$$(\Delta+\gamma)n-\tfrac{1}{2}\delta m=2\beta \text{ et}$$
$$\tfrac{1}{2}\delta n-\varepsilon m=-2\Delta,$$

unde deducuntur valores

$$m=\frac{4\beta\delta+8\Delta\Delta+8\Delta\gamma}{4\Delta\varepsilon+4\gamma\varepsilon-\delta\delta} \text{ et } n=\frac{8\beta\varepsilon+4\Delta\delta}{4\Delta\varepsilon+4\gamma\varepsilon-\delta\delta},$$

quarum fractionum loco in calculo retineamus litteras m et n, consequenter adjecta constante aequatio integralis ita se habebit

$$qq=pp+np+m+C\sqrt{(\Delta+\gamma+\delta p+\varepsilon pp)}.$$

§. 102. Ista autem constans ita definiri debet, ut posito $p=f+g$ fiat $q=g-f$, ex quo quantitas illa constans ita determinabitur

$$C=-\frac{4fg-n(f+g)-m}{\sqrt{[\Delta+\gamma+\delta(f+g)+\varepsilon(f+g)^2]}}.$$

Hoc ergo valore invento, facile assignari poterunt valores non solum ipsius $q\,q$ sed etiam ejus potestatum parium q^4, q^6, q^8, etc., quibus indigemus. Atque hinc intelligitur pro inveniendo valore ipsius V alias formulas integrales non occurrere nisi quae involvant quantitatem radicalem $\sqrt{(\Delta + \gamma + \delta p + \varepsilon p p)}$, quarum ergo integratio, nisi algebraice institui queat, semper per logarithmos et arcus circulares expediri poterit. Evidens autem est, casu quo $\varepsilon = 0$ omnia integralia algebraica exprimi posse.

§. 103. Quod si ergo pro priori curva O Z fuerit

$$\Pi : z = \int \frac{\partial z}{\sqrt{(\alpha + \beta z + \gamma z^2 + \delta z^3)}},$$

pro altera vero curva

$$\Phi : z = \int \frac{\partial z\,(\mathfrak{A} + \mathfrak{B} z + \mathfrak{C} z z + \mathfrak{D} z^3 + etc.)}{\sqrt{(\alpha + \beta z + \gamma z z + \delta z^3)}},$$

tum sumtis in priori curva arcubus aequalibus F G et X Y, iis in altera curva respondebunt arcus $\mathfrak{F}\mathfrak{G}$ et $\mathfrak{X}\mathfrak{Y}$, quorum differentia semper geometrice assignari poterit. Interdum etiam fieri potest, ut differentia V in nihilum abeat, id quod quidem semper evenit, sumto $x = f$.

§. 104. Praeterea vero etiam datur alius casus maxime memorabilis, quod differentia illa V algebraice exprimi poterit, qui scilicet semper locum habebit, quando tam in denominatore quam in numeratore tantum potestates pares ipsius z occurrunt, hoc est si fuerit pro curva priore

$$\Pi : z = \int \frac{\partial z}{\sqrt{(\alpha + \gamma z z + \varepsilon z^4)}},$$

pro altera vero curva

$$\Phi : z = \int \frac{\partial z\,(\mathfrak{A} + \mathfrak{C} z z + \mathfrak{E} z^4 + \mathfrak{G} z^6 + etc.)}{\sqrt{(\alpha + \gamma z z + \varepsilon z^4)}}.$$

His enim casibus, si in priore curva arcus aequales F G et X Y abscindantur, tum arcuum in altera curva respondentium

$\mathfrak{F}\,\mathfrak{G}$ et $\mathfrak{X}\,\mathfrak{Y}$ differentia semper algebraice seu geometrice exhiberi poterit, ad quotcunque terminos etiam numerator $\mathfrak{A} + \mathfrak{C}\,z\,z + \mathfrak{E}\,z^4 +$ etc. continuetur, atque hic est casus, quem olim tam in calculo integrali quam alibi fusius pertractavi.

§. 105. Ad hoc ostendendum, quia habemus tam $\delta = 0$ quam $\beta = 0$, primo erit

$$q\,q = p\,p + m + \mathrm{C}\,\surd(\Delta + \gamma + \varepsilon\,p\,p),$$

ita ut hic tantum potestates pares ipsius p occurrant, tum autem pro litteris germanicis $\mathfrak{C}$, $\mathfrak{E}$, $\mathfrak{G}$, etc. formulae integrandae sequenti modo se habebunt:

Pro littera $\mathfrak{C}$ . . . . . $\int \frac{p\,\partial q}{\surd(\Delta + \gamma + \varepsilon\,p\,p)}$,

quae per se est absolute integrabilis.

Pro littera $\mathfrak{E}$ . . . . . $\int \frac{p\,(p\,p + q\,q)\,\partial p}{\surd(\Delta + \gamma + \varepsilon\,p\,p)}$,

quae loco $q\,q$ substituto valore induet hanc formam

$$\int \frac{p\,(2\,p\,p + m)\,\partial p}{\surd(\Delta + \gamma + \varepsilon\,p\,p)} + \mathrm{C}\int p\,\partial\,p,$$

ubi integratio est manifesta, quod etiam usu venit pro sequentibus formulis litteris $\mathfrak{G}$, $\mathfrak{F}$, affectis. Evidens enim est, si ponatur $\surd(\Delta + \gamma + \varepsilon\,p\,p) = s$ fieri

$$p\,p = \frac{s\,s - \Delta - \gamma}{\varepsilon}, \text{ et } p\,\partial\,p = \frac{s\,\partial\,s}{\varepsilon}, \text{ ideoque}$$

$$\frac{p\,\partial p}{\surd(\Delta + \gamma + \varepsilon\,p\,p)} = \frac{\partial s}{\varepsilon},$$

qua substitutione omnes formulae integrandae fiunt rationales et integrae.

§. 106. Cum autem iste posterior casus jam satis prolixe sit tractatus, ac pluribus exemplis a rectificatione Ellipsis et Hyperbolae desumtis illustratus, casus prior quo tantum erat $\varepsilon = 0$ eo majore attentione est dignus, quod quantum equidem scio, a nemine adhuc est observatus, cujus ergo evolutio novae huic methodo unice accepta est referenda. Quemadmodum autem haec deducta sunt ex relatione inter p et q, ita etiam relatio elegantissima erui potest inter has quantitates $p = x + y$ et $u = xy$, quam hic subjungamus.

Analysis
pro investiganda relatione inter p et u.

§. 107. Hic pariter primo in relationem inter ∂p et ∂u inquiramus, et cum sit

$$\frac{\partial p}{\partial u} = \frac{\partial x + \partial y}{y\partial x + x\partial y}, \text{ ob}$$

$$\partial x : \partial y = \sqrt{X} : \sqrt{Y} \text{ erit}$$

$$\frac{\partial p}{\partial u} = \frac{\sqrt{X} + \sqrt{Y}}{y\sqrt{X} + x\sqrt{Y}},$$

et sumtis quadratis

$$\frac{\partial p^2}{\partial u^2} = \frac{X + Y + 2\sqrt{XY}}{yyX + xxY + 2xy\sqrt{XY}}.$$

Supra autem vidimus esse

$$(\sqrt{X} + \sqrt{Y})^2 = qq(\Delta + \gamma + \delta p + \varepsilon pp), \text{ existente } q = y - x.$$

Pro denominatore autem utamur relatione §. 87. inventa

$$\Delta = \frac{2\alpha + \beta(x+y) + 2\gamma xy + \delta xy(x+y) + 2\varepsilon xxyy + 2\sqrt{XY}}{(y-x)^2},$$

unde fit

$$2\sqrt{XY} = \Delta qq - 2\alpha - \beta p - 2\gamma u - \delta pu - 2\varepsilon uu,$$

quo valore substituto aequatio nostra erit

$$\frac{\partial p^2}{\partial u^2} = \frac{qq(\Delta + \gamma + \delta p + \varepsilon pp)}{yyX + xxY + \Delta qqu - 2\alpha u - \beta pu - 2\gamma uu - \delta puu - 2\varepsilon u^3}.$$

§. 108. Hic autem substitutis loco X et Y valoribus, habebimus primo

$$yyX + xxY = \alpha(xx + yy) + \beta xy(x + y) + 2\gamma xxyy$$
$$+ \delta xxyy(x + y) + \varepsilon xxyy(xx + yy),$$

quae ob $x+y=p$, $xy=u$ et $xx+yy=pp-2u$, erit

$$yy\mathrm{X}+xx\mathrm{Y}=\alpha(pp-2u)+\beta pu+2\gamma uu+\delta puu$$
$$+\varepsilon uu(pp-2u),$$

unde totus denominator reperietur fore

$$\alpha(pp+4u)+\varepsilon uu(pp-4u)+\Delta qqu,$$

quare cum sit $pp-4u=qq$, nostra fractio erit

$$\frac{\partial p^2}{\partial u^2}=\frac{\Delta+\gamma+\delta p+\varepsilon pp}{\Delta u+\alpha+\varepsilon uu},$$

unde sequitur haec aequatio separata

$$\frac{\partial p}{\sqrt{(\Delta+\gamma+\delta p+\varepsilon pp)}}=\frac{\partial u}{\sqrt{(\alpha+\Delta u+\varepsilon uu)}}:$$

unde deducitur hoc

Theorema memorabile.

§. 109. Si inter binas variabiles x et y habeatur haec aequatio differentialis

$$\frac{\partial x}{\sqrt{(\alpha+\beta x+\gamma xx+\delta x^3+\varepsilon x^4)}}=\frac{\partial y}{\sqrt{(\alpha+\beta y+\gamma yy+\delta y^3+\varepsilon y^4)}},$$

tum posito $x+y=p$ et $xy=u$, inter has variabiles p et u semper locum habebit haec aequatio differentialis

$$\frac{\partial p}{\sqrt{(\Delta+\gamma+\delta p+\varepsilon pp)}}=\frac{\partial u}{\sqrt{(\alpha+\Delta u+\varepsilon uu)}},$$

ubi Δ quidem est constans arbitraria in aequationem posteriorem ingressa, contra vero etiam prior aequatio continet constantem arbitrariam β in altera non occurrentem.

§. 110. Aequationis autem posterioris integratio in promptu est. Si enim utrinque multiplicemus per $\sqrt{\varepsilon}$, integrale per logarithmos ita exprimitur

$$l[p\sqrt{\varepsilon}+\frac{\delta}{2\sqrt{\varepsilon}}+\sqrt{(\Delta+\gamma+\delta p+\varepsilon pp)}]=$$
$$l[u\sqrt{\varepsilon}+\frac{\Delta}{2\sqrt{\varepsilon}}+\sqrt{(\alpha+\Delta u+\varepsilon uu)}]+l\Gamma,$$

ideoque integrale ita algebraice exprimetur

$$\varepsilon p + \tfrac{1}{2}\delta + \sqrt{\varepsilon}(\Delta + \gamma + \delta p + \varepsilon pp) =$$
$$\Gamma[\varepsilon u + \tfrac{1}{2}\Delta + \sqrt{\varepsilon}(\alpha + \Delta u + \varepsilon uu)].$$

Ubi constans ista Γ facile definitur ex conditione, quod posito $x = f$ fieri debet $y = g$, hoc est ut posito $p = f + g$ fiat $u = fg$, quippe ex qua conditione constans prior Δ jam est definita.

§. 111. Quo hinc jam facilius sive p per u sive u per p definiri possit, notatur esse

$$\frac{1}{\varepsilon p + \frac{1}{2}\delta + \sqrt{[\varepsilon(\Delta + \gamma + \delta p + \varepsilon pp)]}} = \frac{\varepsilon p + \frac{1}{2}\delta - \sqrt{[\varepsilon(\Delta + \gamma + \delta p + \varepsilon pp)]}}{\frac{1}{4}\delta\delta - \varepsilon(\Delta + \gamma)} \text{ et}$$

$$\frac{1}{\varepsilon u + \frac{1}{2}\Delta + \sqrt{[\varepsilon(\alpha + \Delta u + \varepsilon uu)]}} = \frac{\varepsilon u + \frac{1}{2}\Delta - \sqrt{[\varepsilon(\alpha + \Delta u + \varepsilon uu)]}}{\frac{1}{4}\Delta\Delta - \alpha\varepsilon}.$$

Hinc igitur per inversionem sequens aequatio resultabit

$$\frac{\varepsilon p + \frac{1}{2}\delta - \sqrt{\varepsilon}(\Delta + \gamma + \delta p + \varepsilon pp)}{\frac{1}{4}\delta\delta - \varepsilon(\Delta + \gamma)} = \frac{1}{\Gamma} \cdot \frac{\varepsilon u + \frac{1}{2}\Delta - \sqrt{\varepsilon}(\alpha + \Delta u + \varepsilon uu)}{\frac{1}{4}\Delta\Delta - \alpha\varepsilon}, \text{ sive}$$

$$\varepsilon p + \tfrac{1}{2}\delta - \sqrt{\varepsilon}(\Delta + \gamma + \delta p + \varepsilon pp) = \frac{\frac{1}{4}\delta\delta - \varepsilon(\Delta + \gamma)}{\Gamma(\frac{1}{4}\Delta\Delta - \alpha\varepsilon)} \times [\varepsilon u + \tfrac{1}{2}\Delta - \sqrt{\varepsilon}(\alpha + \Delta u + \varepsilon uu)],$$

ex quibus duabus aequationibus sine alio negotio sive p per u sive u per p exprimi poterit.

§. 112. Hoc igitur modo loco variabilis p pro invenienda quantitate V facile introduci posset variabilis u, si quidem loco formulae $\frac{\partial p}{\sqrt{(\Delta+\gamma+\delta p+\varepsilon p p)}}$ substituatur formula ipsi aequalis $\frac{\partial u}{\sqrt{(\alpha+\Delta u+\varepsilon u u)}}$. Verum hoc modo casus illi, quibus quantitas V fieri potest algebraica, non tam facile patescent; interim tamen etiam hoc modo certi erimus, tam casibus quibus $\varepsilon=0$, quam quo $\beta=0$, $\delta=0$ etc. in serie $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, etc. tantum potestates pares occurrunt, omnes integrationes algebraice succedere debere. Coronidis loco adhuc aliam relationem inter quantitates p et u investigemus, cujus contemplatio insigne incrementum in integratione aequationum polliceri videtur.

Alia Analysis
pro investigatione relationis inter p et u.

§. 113. Cum sit ut ante vidimus $\frac{\partial p}{\partial u}=\frac{\sqrt{X}+\sqrt{Y}}{y\sqrt{X}+x\sqrt{Y}}$, multiplicemus supra et infra per $\sqrt{X}+\sqrt{Y}$, ut numerator evadat

$$(\sqrt{X}+\sqrt{Y})^2=qq(\Delta+\gamma+\delta p+\varepsilon pp);$$

tum autem denominatur prodibit

$$yX+xY+(x+y)\sqrt{XY},$$

ubi denominatoris pars rationalis dat

$$\alpha p+2\beta xy+\gamma xy(x+y)+\delta xy(xx+yy)+\varepsilon xy(x^3+y^3),$$

quae expressio, ob $x+y=p$, $y-x=q$, et $xy=u$, abit in

$$\alpha p+2\beta u+\gamma pu+\delta u(pp-2u)+\varepsilon pu(pp-3u).$$

Deinde ante vidimus esse

$$2\sqrt{XY}=\Delta qq-2\alpha-\beta p-2\gamma u-\delta pu-2\varepsilon uu,$$

quod ductum in $\frac{1}{2}p$ et superiori additum praebet

$$\tfrac{1}{2}\Delta pqq - \tfrac{1}{2}\beta(pp-4u) + \tfrac{1}{2}\delta u(pp-4u) + \varepsilon pu(pp-4u),$$

quae denominator ob $pp - 4u = qq$ induet hanc formam

$$\tfrac{1}{2}\Delta pqq - \tfrac{1}{2}\beta qq + \tfrac{1}{2}\delta uqq + \varepsilon puqq:$$

hinc aequatio erit

$$\frac{\partial p}{\partial u} = \frac{\Delta + \gamma + \delta p + \varepsilon pp}{\frac{1}{2}\Delta p - \frac{1}{2}\beta + \frac{1}{2}\delta u + \varepsilon pu},$$

unde deducitur

$$\partial p(\tfrac{1}{2}\Delta p - \tfrac{1}{2}\beta + \tfrac{1}{2}\delta u + \varepsilon pu) = \partial u(\Delta + \gamma + \delta p + \varepsilon pp),$$

quae ergo certe est integrabilis; id quod adeo inde patet, quod altera variabilis u nusquam ultra primam dimensionem exsurgit.

§. 114. Verum adhuc alio modo relatio inter p et u investigari potest; scilicet aequatio primo inventa

$$\frac{\partial p}{\partial u} = \frac{\sqrt{X} + \sqrt{Y}}{y\sqrt{X} + x\sqrt{Y}},$$

si supra et infra multiplicetur per $\sqrt{Y} - \sqrt{X}$ dabit

$$\frac{\partial p}{\partial u} = \frac{Y - X}{-yX + xY + \sqrt{XY}(y-x)}.$$

Nunc igitur pro numeratore habebimus

$$\beta q + \gamma pq + \delta q(pp-u) + \varepsilon pq(pp-2u).$$

Pro denominatore vero pars rationalis erit

$$-\alpha q + \gamma qu + \delta pqu + \varepsilon qu(pp-u),$$

pars vero irrationalis

$$\tfrac{1}{2}\Delta q^3 - \alpha q - \tfrac{1}{2}\beta pq - \gamma qu - \tfrac{1}{2}\delta pqu - \varepsilon quu,$$

unde totus denominator conficitur

$$\tfrac{1}{2}\Delta q^3 - \alpha q - \tfrac{1}{2}\beta pq + \tfrac{1}{2}\delta pqu + \varepsilon qu(pp - 2u),$$

unde sequitur haec aequatio differentialis

$$\frac{\partial p}{\partial u} = \frac{\beta + \gamma p + \delta(pp - u) + \varepsilon p(pp - 2u)}{\tfrac{1}{2}\Delta(pp - 4u) - 2\alpha - \tfrac{1}{2}\beta p + \tfrac{1}{2}\delta pu + \varepsilon u(pp - 2u)},$$

quae in ordinem redacta ita se habebit

$$\partial p[\Delta(pp - 4u) - 4\alpha - \beta p + \delta pu + 2\varepsilon u(pp - 2u)] =$$
$$2\partial[\beta + \gamma p + \delta(pp - u) + \varepsilon p(pp - 2u)],$$

quae jam ita est comparata, ut nulla via ejus integrationem instituenda perspici queat, etiamsi ejus integrale revera exhibere queamus.

§. 115. Alio insuper modo relationem inter p et u definire licet, si aequationis

$$\frac{\partial p}{\partial u} = \frac{\sqrt{X} + \sqrt{Y}}{y\sqrt{X} + x\sqrt{Y}}$$

posterius membrum supra et infra multiplicemus per $y\sqrt{X} - x\sqrt{Y}$ ut prodeat

$$\frac{\partial p}{\partial u} = \frac{yX - xY + (y - x)\sqrt{XY}}{yyX - xxY}.$$

Nunc enim denominator evadet

$$\alpha pq + \beta qu + \delta quu - \varepsilon pquu.$$

Pro numeratore autem pars rationalis praebet

$$\alpha q - \gamma qu - \delta pqu - \varepsilon qu(pp - u),$$

et pars irrationalis

$$\tfrac{1}{2}\Delta q^3 - \alpha q - \tfrac{1}{2}\beta pq - \gamma qu - \tfrac{1}{2}\delta pqu - \varepsilon quu,$$

totus igitur numerator erit

$$\tfrac{1}{2}\Delta q^3 - \tfrac{1}{2}\beta pq - 2\gamma qu - \tfrac{3}{2}\delta pqu - \varepsilon qupp,$$

ideoque

$$\frac{\partial p}{\partial u} = \frac{\frac{1}{2}\Delta(pp-4u)-\frac{1}{2}\beta p-2\gamma u-\frac{3}{2}\delta pu-\varepsilon ppu}{\alpha p+\beta u-\delta uu-\varepsilon puu},$$

sive

$$2\,\partial p\,(\alpha p+\beta u-\delta uu-\varepsilon puu)=$$
$$\partial u\,[\Delta(pp-4u)-\beta p-4\gamma u-3\delta pu-2\varepsilon ppu].$$

Hic autem penitus non patet, quomodo multiplicator hanc aequationem integrabilem reddens investigari debeat, unde nullum est dubium, quin ista contemplatio haud parum ad limites analyseos prolatandos conferre possit.