SUPPLEMENTUM VII.

AD TOM. I. SECT. II. CAP. V.

DE

COMPARATIONE QUANTITATUM TRANSCENDENTIUM IN FORMA $\int \frac{p\,\partial x}{\sqrt{(A + 2Bx + Cxx)}}$ CONTENTARUM.

Plenior explicatio circa comparationem quantitatum in formula integrali $\int \frac{Z\,\partial z}{\sqrt{(1 + mzz + nz^4)}}$ contentarum, denotante Z functionem quamcunque rationalem ipsius zz. *Acta Academiae Scientiar. Petropolitanae. Tom. V. Pars II. Pag.* 3 — 22.

§. 1. Etsi hoc argumentum jam saepius tractavi atque Illustrissimus *La Grange* plures egregias observationes super hujusmodi formulis cum publico communicavit: id tamen neutiquam adhuc satis exploratum, multo minus exhaustum est censendum, sed plurima adhuc maxime abscondita involvere videtur, quae profundissimam indagationem requirunt atque insignia incrementa Analyseos pollicentur. Imprimis autem ipsae operationes analyticae, quae me primum ad hanc investigationem perduxerunt, ita sunt comparatae, ut non nisi per plures ambages totum negotium conficiant, unde merito etiamnunc methodus directa ad easdem comparationes perducens maxime est desideranda. Praeterea vero universa haec investigatio multo latius patet, quam eas formulas integrales, quas primo sum contemplatus, ubi pro littera Z tantum vel quantitatem constantem vel functionem integram ipsius zz hujus formae $F + Gzz + Hz^4 + Iz^6 + Kz^8$ etc. assumsi, quibus casibus ostendi, propositis duabus quibuscunque quantitatibus hujus generis, semper tertiam ejusdem generis inveniri posse, quae a summa illarum discrepet quantitate algebraica, quae quidem evanescat casu quo Z est tantum quantitas constans.

§. 2. Nunc autem observavi, easdem comparationes institui posse, si pro Z accipiatur functio quaecunque rationalis ipsius zz, quae scilicet habeat hujusmodi formam

$$\frac{F + Gzz + Hz^4 + Iz^6 + Kz^8 + etc.}{\mathfrak{F} + \mathfrak{G}zz + \mathfrak{H}z^4 + \mathfrak{I}z^6 + \mathfrak{K}z^8 + etc.},$$

ubi quidem hoc discrimen occurrit, quod differentia inter summam duarum hujusmodi formularum et tertiam formulam ejusdem generis inveniendam non amplius sit quantitas algebraica, verumtamen per logarithmos et arcus circulares semper exhiberi possit; ita ut nunc ista investigatio multo latius pateat, quam eam adhuc eram complexus. Atque hinc fortasse, si omnes operationes, quae ad hunc scopum manuducunt, debita attentione perpendantur, faciliorem viam apperire poterunt ad methodum directam perveniendi, totumque hoc argumentum maxime abstrusum feliciori successu perscrutandi.

§. 3. Quo autem haec omnia clarius perspici queant, denotet iste character $\Pi : z$ eam quantitatem transcendentem, quae ex integratione formulae propositae $\int \frac{Z \partial z}{\sqrt{(1 + mzz + nz^4)}}$ nascitur, dum integrale ita capi assumitur, ut evanescat posito $z = 0$; unde statim manifestum est, fore quoque $\Pi : 0 = 0$. Deinde cum Z involvat tantum pares potestates ipsius z, cujusmodi etiam in formula radicali insunt, evidens est, si loco z scribatur $-z$, tum valorem quoque istius formulae integralis ideoque etiam characteris $\Pi : z$ in sui negativum abire, ita ut sit $\Pi : (-z) = -\Pi : z$. His praenotatis si proponantur duae quaecunque hujusmodi quantitates $\Pi : p$ et $\Pi : q$, semper invenire licet tertiam quantitatem ejusdem generis $\Pi : r$, quae a summa illarum formularum $\Pi : p + \Pi : q$ differat quantitate vel algebraica vel saltem per logarithmos et arcus circulares assignabili. Regula vero, qua ex datis litteris p et q tertia r elicitur, semper manet eadem, quaecunque functio per litteram Z designetur: semper enim erit

$$r = \frac{p\sqrt{(1 + mqq + nq^4)} + q\sqrt{(1 + mpp + np^4)}}{1 - nppqq}$$

Hinc autem pro sequentibus combinationibus observasse juvabit, fore

$$\sqrt{(1 + mrr + nr^4)} =$$
$$\frac{(mpq + [\sqrt{(1 + mpp + np^4)}]\,[\sqrt{(1 + mqq + nq^4)}]\,(1 + nppqq) + 2npq(pp + qq)}{(1 - nppqq)^2}.$$

§. 4. Non solum autem haec iuvestigatio adstringitur ad hujusmodi formulas $\Pi : p$ et $\Pi : q$ pro arbitrio accipiendas, sed adeo ad quotcunque formulas datas potest extendi, ita ut, quotcunque hujusmodi formulae fuerint propositae, scilicet

$$\Pi : f + \Pi : g + \Pi : h + \Pi : i + \Pi : k + \text{etc.}$$

semper nova hujusmodi formula $\Pi : r$ assignari possit, quae ab illarum summa discrepet quantitate vel algebraica vel saltem per logarithmos et arcus circulares assignabili. Quin etiam formulas illas, quas tanquam datas spectavimus, ita definire licebit, ut discrimen illud, sive algebraicum sive a logarithmis arcubusque circularibus pendens, prorsus evanescat, ita ut futurum sit

$$\Pi : r = \Pi : f + \Pi : g + \Pi : h + \Pi : i + \Pi : k + \text{etc.}$$

Atque haec fere sunt, ad quae hanc investigationem generaliorem, quam hic exponere constitui, mihi quidem extendere licuit; quamobrem singulas operationes, quae me huc perduxerunt, succincte sum propositurus.

Operatio I.

§. 5. Universam hanc investigationem inchoavi a consideratione hujus aequationis algebraicae

$$\alpha + \gamma(xx + yy) + 2\delta xy + \zeta xxyy = 0,$$

ex qua, cum sit quadratica, tam pro x quam pro y radicem extrahendo, colligitur vel

$$y = \frac{-\delta x + \sqrt{[-\alpha\gamma + (\delta\delta - \gamma\gamma - \alpha\zeta)xx - \gamma\zeta x^4]}}{\gamma + \zeta xx}, \text{ vel}$$

$$x = \frac{-\delta y + \sqrt{[-\alpha\gamma + (\delta\delta - \gamma\gamma - \alpha\zeta)yy - \gamma\zeta y^4]}}{\gamma + \zeta yy},$$

ita ut hinc fiat

$$\sqrt{[-\alpha\gamma + (\delta\delta - \gamma\gamma - \alpha\zeta)xx - \gamma\zeta x^4]} = \gamma y + \delta x + \zeta xxy, \text{ et}$$

$$\sqrt{[-\alpha\gamma + (\delta\delta - \gamma\gamma - \alpha\zeta)yy - \gamma\zeta y^4]} = \gamma x + \delta y + \zeta xyy.$$

§. 6. Nunc litteras α, γ, δ, ζ, ita definio, ut ambae formulae radicales ad formam

$$\sqrt{(1 + mxx + nx^4)} \text{ et } \sqrt{(1 + myy + ny^4)}$$

reducantur, quem in finem facio

$$1^\circ.\ -\alpha\gamma = k,$$

$$2^\circ.\ \delta\delta - \gamma\gamma - \alpha\zeta = mk, \text{ et}$$

$$3^\circ.\ -\gamma\zeta = nk,$$

ex quarum aequalitatum prima fit $\alpha = \frac{-k}{\gamma}$, ex tertia $\zeta = \frac{-nk}{\gamma}$, qui valores in secunda substituti praebent

$$\delta\delta = \gamma\gamma + \frac{nkk}{\gamma\gamma} + mk,$$

ideoque

$$\delta = \sqrt{(\gamma\gamma + \frac{nkk}{\gamma\gamma} + mk)} = \frac{1}{\gamma\gamma}\sqrt{(\gamma^4 + m\gamma\gamma k + nkk)},$$

unde aequatio nostra nunc erit

$$-k+\gamma\gamma(xx+yy)+2xy\sqrt{(\gamma^4+m\gamma\gamma k+nkk)}-nkxxyy=0:$$

hinc igitur ambae nostrae formulae irrationales erunt

$$\sqrt{k(1+mxx+nx^4)}=\gamma y+\tfrac{1}{\gamma}x\sqrt{(\gamma^4+m\gamma\gamma k+nkk)}-\tfrac{nk}{\gamma}xxy,$$

$$\sqrt{k(1+myy+ny^4)}=\gamma x+\tfrac{1}{\gamma}y\sqrt{(\gamma^4+m\gamma\gamma k+nkk)}-\tfrac{nk}{\gamma}xyy.$$

§. 7. Cum nunc ambae quantitates x et y ita a se invicem pendeant, quemadmodum aequatio assumta declarat, litteras adhuc indefinitas γ et k ita definiamus, ut posito $x=0$ fiat $y=a$. Oportebit igitur esse $-k+\gamma\gamma aa=0$, ideoque $k=\gamma\gamma aa$, quo valore substituto aequatio nostra erit

$$0=\gamma\gamma(xx+yy-aa)+2\gamma\gamma xy\sqrt{(1+maa+na^4)}$$
$$-n\gamma\gamma aaxxyy,$$

hincque fiet per $\gamma\gamma$ dividendo

$$0=(xx+yy-aa)+2xy\sqrt{(1+maa+na^4)}-naaxxyy.$$

Tum vero ambae nostrae formulae radicales ita exprimentur

$$\sqrt{(1+mxx+nx^4)}=\tfrac{y}{a}+\tfrac{x}{a}\sqrt{(1+maa+na^4)}-naxxy,$$

$$\sqrt{(1+myy+ny^4)}=\tfrac{x}{a}+\tfrac{y}{a}\sqrt{(1+maa+na^4)}-naxyy.$$

§. 8. Quo has formulas tractatu faciliores reddamus, ponamus $\sqrt{(1+maa-na^4)}=\mathfrak{A}$, similique modo

$$\sqrt{(1+mxx+nx^4)}=\mathfrak{X} \text{ et}$$
$$\sqrt{(1+myy+ny^4)}=\mathfrak{Y},$$

et aequatio nostra erit

$$xx+yy-aa+2\mathfrak{A}xy-naaxxyy=0;$$

unde reperitur

$$y=-\frac{\mathfrak{A}x+a\mathfrak{X}}{1-naaxx}, \text{ et } x=-\frac{\mathfrak{A}y+a\mathfrak{Y}}{1-naayy},$$

unde patet si fuerit $y = 0$ fore $x = a$; tum vero erunt formulae radicales

$$\sqrt{(1 + m x x + n x^4)} = \mathfrak{X} = \tfrac{y}{a} + \tfrac{\mathfrak{A} x}{a} - n a x x y,$$
$$\sqrt{(1 + m y y + n y^4)} = \mathfrak{Y} = \tfrac{x}{a} + \tfrac{\mathfrak{A} y}{a} - n a x y y.$$

§. 9. Quemadmodum autem tam y per x quam x per y exprimere licuit, ita etiam $\mathfrak{Y}$ per solum x et $\mathfrak{X}$ per solum y exprimere licebit. Calculo autem instituto reperietur fore

$$\mathfrak{X} = \frac{(-m a y + \mathfrak{A} \mathfrak{Y})(1 + n a a y y) - 2 n a y (a a + y y)}{(1 - n a a y y)^2},$$
$$\mathfrak{Y} = \frac{(-m a x + \mathfrak{A} \mathfrak{X})(1 + n a a x x) - 2 n a x (a a + x x)}{(1 - n a a x x)^2}.$$

§. 10. Praecipue autem circa nostram aequationem

$$x x + y y - a a + 2 \mathfrak{A} x y - n a a x x y y = 0$$

notari meretur, quod ternae quantitates $x x$, $y y$, $a a$ perfecte inter se sint permutabiles. Si enim membrum irrationale ad alteram partem transferatur, ut sit

$$x x + y y - a a - n a a x x y y = - 2 \mathfrak{A} x y,$$

et quadrata sumantur, restituendo pro $\mathfrak{A}^2$ valorem suum $1 + m a a + n a^4$, prodibit ista aequatio

$$\left.\begin{array}{l} + x^4 - 2 x x y y - 4 m a a x x y y - 2 n a^4 x x y y + n n a^4 x^4 y^4 \\ + y^4 - 2 a a x x \qquad\qquad\qquad\qquad - 2 n a a x^4 y y \\ + a^4 - 2 a a y y \qquad\qquad\qquad\qquad - 2 n a a x x y^4 \end{array}\right\} = 0,$$

ubi permutabilitas litterarum a, x, y manifesto in oculos incurrit. In ipsis quidem formulis superioribus, ubi ipsa quantitas a ingreditur, permutabilitas non adeo est manifesta, sed prorsus elucebit, si loco a scribamus $- b$, itemque $\mathfrak{B}$ loco $\mathfrak{A}$; tum enim, quemadmodum erat

$$y = - \frac{x \mathfrak{B} - b \mathfrak{X}}{1 - n b b x x} \text{ et } x = - \frac{y \mathfrak{B} - b \mathfrak{Y}}{1 - n b b y y},$$

ita erit $b = -\frac{x\mathfrak{Y} - y\mathfrak{X}}{1 - nxxyy}$; similique modo pro formulis radicalibus seu litteris majusculis erit

$$\mathfrak{Y} = \frac{(mbx + \mathfrak{B}\mathfrak{X})(1 + nbbxx) + 2nbx(bb + xx)}{(1 - nbbxx)^2},$$

$$\mathfrak{X} = \frac{(mby + \mathfrak{B}\mathfrak{Y})(1 + nbbyy) + 2nby(bb + yy)}{(1 - nbbyy)^2},$$

$$\mathfrak{B} = \frac{(mxy + \mathfrak{X}\mathfrak{Y})(1 + nxxyy) + 2nxy(xx + yy)}{(1 - nxxyy)^2},$$

sicque perfecta permutabilitas perspicitur.

Operatio II.

§. 11. Differentiemus nunc nostram aequationem algebraicam assumtam, quae est

$$xx + yy - aa + 2\mathfrak{A}xy - naaxxyy = 0,$$

et aequatio differentialis erit

$$\partial x(x + \mathfrak{A}y - naaxyy) + \partial y(y + \mathfrak{A}x - naaxxy) = 0,$$

sive

$$\frac{\partial x}{y + \mathfrak{A}x - naaxxy} + \frac{\partial y}{x + \mathfrak{A}y - naaxyy} = 0.$$

Ex superioribus autem constat esse

$$y + \mathfrak{A}x - naaxxy = a\mathfrak{X} \text{ et}$$

$$x + \mathfrak{A}y - naaxyy = a\mathfrak{Y},$$

unde aequatio differentialis hanc induet formam

$$\frac{\partial x}{a\mathfrak{X}} + \frac{\partial y}{a\mathfrak{Y}} = 0, \text{ sive}$$

$$\frac{\partial x}{\sqrt{(1 + mxx + nx^4)}} + \frac{\partial y}{\sqrt{(1 + myy + ny^4)}} = 0.$$

§. 12. Inventa igitur hac aequatione differentiali, denotet iste character $\Gamma : x$ integrale $\int \frac{\partial x}{\mathfrak{X}}$, et character $\Gamma : y$ integrale $\int \frac{\partial y}{\mathfrak{Y}}$, utroque integrali ita sumto, ut evanescat posito vel $x = 0$ vel $y = 0$, atque aequationem illam differentialem integrando fiet $\Gamma : x + \Gamma : y = C$. Cum autem sumto $x = 0$

fiat etiam $\Gamma : x = 0$ et $y = a$, erit constans illa $C = \Gamma : a$, ita ut habeamus hanc aequationem $\Gamma : x + \Gamma : y = \Gamma : a$.

§. 13. Quoniam hic nulla amplius variabilitatis ratio tenetur, patet, sumtis binis litteris x et y pro lubitu, litteram a ita semper definiri posse, ut fiat

$$\Gamma : a = \Gamma : x + \Gamma : y.$$

Si enim in §. 10. loco b scribatur $-a$, sumi debet

$$a = \frac{x\mathfrak{Y} + y\mathfrak{X}}{1 - nxxyy},$$

quae comparatio jam casum constituit specialem investigationis generalis, quam suscepimus. Si enim loco x et y scribamus p et q, at r loco a, tum vero $\mathfrak{P}$, $\mathfrak{Q}$ et $\mathfrak{R}$ loco $\mathfrak{X}$, $\mathfrak{Y}$ et $\mathfrak{A}$, atque si, sumtis pro lubitu quantitatibus p, q, capiatur $r = \frac{p\mathfrak{Q} + q\mathfrak{P}}{1 - nppqq}$, tum utique erit $\Gamma : r = \Gamma : p + \Gamma : q$, ita ut hoc casu discrimen illud inter $\Gamma : r$ et summam $\Gamma : p + \Gamma : q$ plane evanescat. Sicque jam evolvimus casum, quo in nostra forma generali $\int \frac{Z \partial z}{\sqrt{(1 + mzz + nz^4)}}$ pro Z sumitur quantitas constans.

Operatio III.

§. 14. Quo nunc propius ad nostrum institutum accedamus, sint X et Y tales functiones ipsarum x et y, qualem volumus esse Z ipsius z, et quoniam modo invenimus

$$\frac{\partial x}{\sqrt{(1 + mxx + nx^4)}} + \frac{\partial y}{\sqrt{(1 + myy + ny^4)}} = 0,$$

ponamus esse

$$\frac{X \partial x}{\sqrt{(1 + mxx + nx^4)}} + \frac{Y \partial y}{\sqrt{(1 + myy + ny^4)}} = \partial V,$$

ita ut, si X et Y essent quantitates constantes, foret $\partial V = 0$. Hinc ergo si loco $\frac{\partial y}{\sqrt{(1 + myy + ny^4)}}$ scribamus $\frac{-\partial x}{\sqrt{(1 + mxx + nx^4)}}$, fiet

$$\partial V = \frac{(X-Y)\partial x}{\sqrt{(1+mxx+nx^4)}},$$

vel etiam

$$\partial V = \frac{(Y-X)\partial y}{\sqrt{(1+myy+ny^4)}}.$$

At si loco radicalium suos valores rationales scribamus, erit

$$\partial V = \frac{a(X-Y)\partial x}{y+\mathfrak{A}x-naaxxy}, \text{ vel}$$

$$\partial V = \frac{a(Y-X)\partial y}{x+\mathfrak{A}y-naaxyy}.$$

§. 15. Cum autem nulla sit ratio, cur istud differentiale ∂V potius per ∂x quam per ∂y exprimamus; consultum erit novam quantitatem in calculum introducere, quae aeque referatur ad x et ad y. Hunc in finem faciamus productum $xy = u$, ac statuamus

$$\frac{\partial x}{x+\mathfrak{A}x-naaxxy} = -\frac{\partial y}{x+\mathfrak{A}y-naaxyy} = s\,\partial u.$$

Hinc igitur erit

$$\partial x = s\,\partial u\,(y+\mathfrak{A}x-naaxxy) \text{ et}$$

$$\partial y = -s\,\partial u\,(x+\mathfrak{A}y-naaxyy),$$

ex quibus colligimus

$$y\,\partial x + x\,\partial y = s\,\partial u\,(yy-xx) = \partial u,$$

sicque habebimus $s = \frac{1}{yy-xx}$, ita ut habeamus

$$\frac{\partial x}{y+\mathfrak{A}x-naaxxy} = -\frac{\partial y}{x+\mathfrak{A}y-naaxyy} = \frac{\partial u}{yy-xx}.$$

Hoc igitur valore substituto nanciscimur

$$\partial V = \frac{a(X-Y)\partial u}{yy-xx} = \frac{-a\,\partial u\,(X-Y)}{xx-yy}.$$

§. 16. Cum autem X et Y sint functiones rationales pares ipsarum x et y, in quibus tantum insunt potestates pares harum litterarum; facile intelligitur, formulam $X-Y$ semper esse divisibilem per $xx-yy$, et quotum praeter productum $xy = u$ insuper involvere summam quadratorum $xx + yy$; quamobrem statuamus $xx + yy = t$, et cum aequatio nostra fundamentalis fiat

$$t - aa + 2\mathfrak{A}u - naauu = 0,$$

ex ea fit

$$t = aa - 2\mathfrak{A}u + naauu,$$

ita ut t aequetur functioni rationali ipsius u. Quod si ergo hic valor ubique loco t scribatur, differentiale nostrum quaesitum ∂V per solam variabilem u exprimetur, ita ut posito $\partial V = U\partial u$ semper sit U functio rationalis ipsius u, quae ergo si fuerit integra, tum V aequabitur functioni algebraicae ipsius u; sin autem sit functio fracta, tum integrale $V = \int U\partial u$ semper per logarithmos et arcus circulares exhiberi poterit. Hoc ergo integrale si ita capiatur, ut evanescat posito $u = xy = 0$, id etiam evanescet posito $x = 0$ vel $y = 0$. Atque hinc integrando impetrabimus

$$\int \frac{X\partial x}{\sqrt{(1 + mxx + nx^4)}} + \int \frac{Y\partial y}{\sqrt{(1 + myy + ny^4)}} = C + V = C + \int U\partial u.$$

§. 17. Quod si igitur characteres $\Pi : x$ et $\Pi : y$ denotent valores horum integralium, ita ut utrumque evanescat sumto vel $x = 0$ vel $y = 0$, quoniam facto $x = 0$ per hypothesin fit $y = a$, manifestum est constantem hanc fore $\Pi : a$, sicque aequatio finita resultabit ista

$$\Pi : x + \Pi : y = \Pi : a + \int U\partial u.$$

§. 18. Accuratius autem in valores hujus fractionis U pro quovis casu inquiramus. Ac primo quidem, si sumatur

$$Z = \alpha + \beta zz + \gamma z^4 + \delta z^6 + \text{etc.}$$

erit simili modo

$$X = \alpha + \beta xx + \gamma x^4 + \delta x^6 + \text{etc. et}$$
$$Y = \alpha + \beta yy + \gamma y^4 + \delta y^6 + \text{etc.},$$

quare cum invenerimus

$$\partial V = U \partial u = -\frac{a \partial u (X - Y)}{xx - yy}, \text{ erit}$$

$$U = -\frac{a(X - Y)}{xx - yy}, \text{ ideoque}$$

$$U = -\frac{a[\beta(xx - yy) + \gamma(x^4 - y^4) + \delta(x^6 - y^6) + \text{etc.}]}{xx - yy},$$

unde fit

$$U = -a\beta - a\gamma(xx + yy) - a\delta(x^4 + xxyy + y^4) - \text{etc.}$$

Cum igitur sit $xx + yy = t$ et $xy = u$, erit

$$U = -a\beta - a\gamma t - a\delta(tt - uu) - \text{etc.},$$

unde cum sit

$$t = aa - 2\mathfrak{A}u + naauu,$$

calculo subducto altiores potestates omittendo, fiet

$$\begin{aligned} \int U \partial u = & -a\beta u - a\gamma(aau - \mathfrak{A}uu + \tfrac{1}{3}naau^3) \\ & - a\delta(a^4uu - 2aa\mathfrak{A}uu + \tfrac{2}{3}na^4u^3 - n\mathfrak{A}a^2u^4 + \tfrac{1}{5}n^2a^4u^5) \\ & \qquad + \tfrac{4}{3}\mathfrak{A}^2u^5 \\ & \qquad + \tfrac{1}{3}u^3. \end{aligned}$$

Atque hinc intelligitur, si functio Z ad altiores potestates exsurgat, quomodo valor integralis ipsius $\int U \partial u$ inde inveniri queat.

§. 19. Sin autem Z fuerit functio fracta, scilicet

$$Z = \frac{\alpha + \beta zz + \gamma z^4}{\zeta + \eta zz + \theta z^4},$$

hincque

$$X = \frac{\alpha + \beta xx + \gamma x^4}{\zeta + \eta xx + \theta x^4} \text{ et}$$

$$Y = \frac{\alpha + \beta yy + \gamma y^4}{\zeta + \eta yy + \theta y^4}, \text{ erit}$$

$$X - Y = \frac{(\beta\zeta - \alpha\eta)(xx - yy) + (\gamma\zeta - \alpha\theta)(x^4 - y^4) + (\gamma\eta - \beta\theta)x^2y^2(x^2 - y^2)}{\zeta\zeta + \zeta\eta(xx + yy) + \zeta\theta(x^4 + y^4) + \eta^2x^2y^2 + \eta\theta x^2y^2(xx + yy) + \theta\theta x^4y^4}.$$

Hinc igitur introductis litteris t et u erit

$$\frac{X - Y}{xx - yy} = \frac{\beta\zeta - \alpha\eta + (\gamma\zeta - \alpha\theta)t + (\gamma\eta - \beta\theta)uu}{\zeta\zeta + \zeta\eta t + \zeta\theta(tt - 2uu) + \eta\eta uu + \eta\theta tuu + \theta\theta u^4},$$

quam ob rem cum sit

$$U = -\frac{a(X-Y)}{xx-yy}, \text{ ob}$$

$$t = aa - 2\mathfrak{A}u + naauu,$$

manifestum est, integrale formulae $\int U\partial u$ nisi fuerit algebraicum, semper, concessis logarithmis et arcubus circularibus, exhiberi posse. Sicque per has tres operationes omnia praestitimus, quibus opus est ad omnia problemata huc spectantia solvenda

Problema I.

§. 20. *Si* $\Pi : x$ *et* $\Pi : y$ *denotent valores formularum integralium*

$$\int\frac{X\partial x}{\sqrt{(1+mxx+nx^4)}} \text{ et } \int\frac{Y\partial y}{\sqrt{(1+myy+ny^4)}},$$

ubi X et Y sint functiones pares similes ipsarum x et y, atque dentur binae hujusmodi formulae $\Pi : x$ *et* $\Pi : y$; *invenire tertiam formulam ejusdem generis* $\Pi : z$, *ut sit*

$$\Pi : z = \Pi : x + \Pi : y + W,$$

ita ut W sit functio vel algebraica vel per logarithmos et arcus circulares assignabilis.

Solutio.

Cum dentur binae quantitates x et y, ex iis formentur radicales

$$\mathfrak{X} = \sqrt{(1 + mxx + nx^4)} \text{ et}$$

$$\mathfrak{Y} = \sqrt{(1 + nyy + ny^4)},$$

ex quibus definiatur quantitas z, eodem modo quo supra litteram a per x et y definire docuimus, ita ut sit $z = \frac{x\mathfrak{Y} + y\mathfrak{X}}{1-nxxyy}$, ejusque valor irrationalis

$$\mathfrak{Z} = \sqrt{(1+mzz+nz^4)} = \frac{(mxy+\mathfrak{X}\mathfrak{Y})(1+nxxyy)+2nxy(xx+yy)}{(1-nxxyy)^2},$$

tum in superioribus formulis ubique loco a et $\mathfrak{A}$ scribamus z et $\mathfrak{Z}$, et capiatur $= -\frac{z(X-Y)}{xx-yy}$, quam quantitatem vidimus semper reduci posse ad functionem ipsius u, existente $u = xy$, ac ponatur $V = \int U \partial u$, in qua integratione quantitates z et $\mathfrak{Z}$ pro constantibus sunt habendae, ita ut littera V spectari possit tanquam functio ipsius $u = x\,y$, quandoquidem etiam z et $\mathfrak{Z}$ per x et y determinantur. Probe autem teneatur, in ista formula integrali solam quantitatem u ut variabilem esse tractandam. Hac igitur quantitate V inventa erit

$$\Pi : x + \Pi : y = \Pi : z + V,$$

unde cum debeat esse

$$\Pi : z = \Pi : x + \Pi : y + W,$$

patet esse $W = -V$, ideoque quantitatem vel algebraicam, vel per logarithmos et arcus circulares assignabilem.

Corollarium 1.

§. 21. Totum ergo negotium hic redit ad integrationem formulae $U \partial u$, existente

$$u = xy \text{ et } U = -\frac{z(X-Y)}{xx-yy},$$

quam supra vidimus semper per u exprimi posse, siquidem in hac integratione litterae z et $\mathfrak{Z}$ ut quantitates constantes tractentur.

Corollarium 2.

§. 22. Cum igitur pro data indole binarum functionum X et Y haec integratio nulla laboret difficultate, ipsumque integrale per u, hoc est per $x\,y$ exprimatur, cujus valorem ex datis quantitatibus x et y semper exhibere liceat, loco quantitatis V scribemus in posterum characterem $\Phi : x\,y$, unde pro quibusque aliis litteris loco x et y assumtis intelligitur significatus characterum $\Phi : b\,q$, $\Phi : a\,b$ etc.

Corollarium 3.

§. 23. Hoc igitur charactere recepto, si pro datis quantitatibus x et y capiamus $z = \frac{x\mathfrak{Y} - y\mathfrak{X}}{1 - nxxyy}$, unde fit

$$\mathfrak{Z} = \frac{(mxy + \mathfrak{X}\mathfrak{Y})(1 + nxxyy) + 2nxy(xx + yy)}{(1 - nxxyy)^2}, \text{ erit}$$

$$\Pi : z = \Pi : x + \Pi : y - \Phi : xy.$$

Problema 2.

§. 24. *Servatis omnibus characteribus, quos hactenus explicavimus, si dentur ternae formulae,* $\Pi : p, \Pi : q, \Pi : r$, *invenire quartam ejusdem generis* $\Pi : z$, *ut fiat*

$$\Pi : z = \Pi\, p + \Pi : q + \Pi : r + W,$$

ita ut W *sit quantitas algebraica, vel per logarithmos arcusve circulares assignabilis.*

Solutio.

Ex datis binis quantitatibus p et q, ideoque etiam $\mathfrak{P}$ et $\mathfrak{Q}$ inde oriundis, capiatur $x = \frac{p\mathfrak{Q} + q\mathfrak{P}}{1 - nppqq}$, simulque

$$\mathfrak{X} = \frac{(mpq + \mathfrak{P}\mathfrak{Q})(1 + nppqq) + 2npq(pp + qq)}{(1 - nppqq)^2}.$$

Tum vero etiam colligatur valor characteris $\Phi : pq$, eritque per praecedentia

$$\Pi : x = \Pi : p + \Pi : q - \Phi : pq, \text{ sive}$$

$$\Pi : p + \Pi : q = \Pi : x + \Phi : pq,$$

quo valore substituto erit

$$\Pi : z = \Pi : x + \Pi : r + \Phi : pq + W.$$

Ex praecedente autem problemate, si loco y hic scribamus r et capiamus $z = \frac{x\mathfrak{R} + r\mathfrak{X}}{1 - nrrxx}$, unde fit

$$\mathfrak{Z} = \frac{(mrx + \mathfrak{R}\mathfrak{X})(1 + nrrxx) + 2nrx(rr + xx)}{(1 - nrrxx)^2}, \text{ erit}$$

$$\Pi : z = \Pi : x + \Pi : r - \Phi : rx,$$

qua forma cum praecedente collata colligitur

$$W = -\Phi : pq - \Phi : rx,$$

ita ut sit

$$\Pi : z = \Pi : p + \Pi : q + \Pi : r - \Phi : pq - \Phi : rx.$$

Problema 3.

§. 25. *Propositis hujusmodi formulis* $\Pi : p$, $\Pi : q$, $\Pi : r$, $\Pi : s$, *invenire quintam ejusdem generis* $\Pi : z$, *ut fiat*

$$\Pi : z = \Pi : p + \Pi : q + \Pi : r + \Pi : s + W,$$

ita ut W *sit quantitas algebraica, vel per logarithmos arcusve circulares assignabilis.*

Solutio.

Ex datis binis p et q quaeratur x, ut sit $x = \frac{p\mathfrak{Q} + q\mathfrak{P}}{1 - nppqq}$, item

$$\mathfrak{X} = \frac{(mpq + \mathfrak{P}\mathfrak{Q})(1 + nppqq) + 2npq(pp + qq)}{(1 - nppqq)^2},$$

eritque

$$\Pi : x = \Pi : p + \Pi : q - \Phi : pq.$$

Simili modo ex binis datis r et s quaeratur y, ut sit $y = \frac{r\mathfrak{S} + s\mathfrak{R}}{1 - nrrss}$, eritque

$$\mathfrak{Y} = \frac{(mrs + \mathfrak{R}\mathfrak{S})(1 + nrrss) + 2nrs(rr + ss)}{(1 - nrrss)^2},$$

tum vero

$$\Pi : y = \Pi : r + \Pi : s - \Phi : rs.$$

Nunc denique ex inventis x et y sumatur $z = \frac{x\mathfrak{Y} + y\mathfrak{X}}{1 - nxxyy}$, et

$$\mathfrak{Z} = \frac{(mxy + \mathfrak{Y}\mathfrak{X})(1 + nxxyy) + 2nxy(xx + yy)}{(1 - nxxyy)^2},$$

eritque

$$\Pi : z = \Pi : x + \Pi : y - \Phi : xy.$$

Quod si ergo loco $\Pi : x$ et $\Pi : y$ valores modo inventi substituantur, fiet

$$\Pi : z = \Pi : p + \Pi : q + \Pi : r + \Pi : s - \Phi : p\,q - \Phi : r\,s - \Phi : x\,y.$$

Corollarium 1.

§. 26. Hinc jam abunde intelligitur, si proponantur quotcunque hujusmodi formulae, quemadmodum novam ejusdem generis $\Pi : z$ investigari oporteat, quae ab illis junctim sumtis discrepet quantitate algebraica, vel per logarithmos arcusve circulares assignabili.

Corollarium 2.

§. 27. Quod si omnes illae formulae fuerint inter se aequales earumque numerus $= \lambda$, semper nova formula $\Pi : z$ inveniri poterit, ut sit $\Pi : z = \lambda\, \Pi : p + W$, existente W quantitate vel algebraica, vel per logarithmos arcusve circulares assignabili. Quin etiam, duabus hujusmodi formulis $\Pi : p$ et $\Pi : q$ propositis, inveniri poterit $\Pi : z$ ut sit

$$\Pi : z = \lambda : \Pi : p + \mu\, \Pi : q + W.$$

Scholion.

§. 28. Hoc igitur modo non solum principia et fundamenta, quibus hoc argumentum innititur, succincte ac dilucide mihi quidem exposuisse videor: sed hoc argumentum etiam multo latius amplificavi quam adhuc est factum. Semper autem maxime est optandum, ut via magis directa detegatur, quae ad easdem investigationes perducat. Nemo enim certe dubitabit, quin hinc maxima in universam Analysin incrementa essent redundatura.

Applicatio
ad quantitates transcendentes
in forma $\int \frac{\partial z (\alpha + \beta z z)}{\sqrt{(1 + m z z + n z^4)}} = \Pi : z$ contentas.

§. 29. Cum igitur hic sit $Z = \alpha + \beta z z$, propositis duabus formulis hujus generis $\Pi : x$ et $\Pi : y$; sumtoque

$$z = \frac{x\mathfrak{Y} + y\mathfrak{X}}{1 - n x x y y}, \text{ hincque}$$

$$\mathfrak{Z} = \frac{(m x y + \mathfrak{X}\mathfrak{Y})(1 + n x x y y) + 2 n x y (x^2 + y^2)}{(1 - n x x y y)^2},$$

ex §. 18. ubi $u = x y$ et $a = z$, erit

$$\Pi : z = \Pi : x + \Pi : y + \beta x y z,$$

ita ut character ante adhibitus $\Phi : y$ hoc casu accipiat valorem $\beta\, x\, y\, z$. Hujus igitur regulae ope propositis duabus hujusmodi formulis $\Pi : x$ et $\Pi : y$, tertia $\Pi : z$ semper reperiri potest, quae a summa illarum differat quantitate algebraica $\beta x y z$.

§. 30. Ponamus igitur, quotcunque hujusmodi formulas transcendentes proponi

$$\Pi : a, \ \Pi : b, \ \Pi : c, \ \Pi : d, \ \Pi : e, \ \Pi : f, \ \Pi : g, \text{ etc.}$$

et ex singulis quantitatibus a, b, c, d, e, colligi valores irrationales litteris germanicis insignitas

$$\mathfrak{A} = \sqrt{(1 + m a a + n a^4)}; \ \mathfrak{B} = \sqrt{(1 + m b b + n b^4)};$$
$$\mathfrak{C} = \sqrt{(1 + m c c + n c^4)}; \ \mathfrak{D} = \sqrt{(1 + m d d + n d^4)};$$
$$\text{etc.} \qquad\qquad \text{etc}$$

semper nova formula ejusdem generis exhiberi poterit, quae a summa earum discrepet quantitate algebraica, quantuscunque etiam fuerit earum formularum datarum numerus. Operationes autem ad hunc finem perducentes commodissime sequenti modo instituentur.

§. 31. Primo scilicet ex binis datarum a et b quaeratur p, ut sit

$$p = \frac{a\mathfrak{B} + b\mathfrak{A}}{1 - naabb} \text{ et } \mathfrak{P} = \frac{(mab + \mathfrak{A}\mathfrak{B})(1 + naabb) + 2nab(aa + bb)}{(1 - naabb)^2}.$$

Deinde ex hac quantitate p, cum datarum tertia c juncta, definiatur q, ut sit

$$q = \frac{p\mathfrak{C} + c\mathfrak{P}}{1 - nccpp} \text{ et } \mathfrak{Q} = \frac{(mcp + \mathfrak{C}\mathfrak{P})(1 + ccpp) + 2ncp(cc + pp)}{(1 - nccpp)^2}.$$

Tertio ex hac quantitate q cum quarta datarum d juncta, quaeratur r, ut sit

$$r = \frac{q\mathfrak{D} + d\mathfrak{Q}}{1 - nddqq} \text{ et } \mathfrak{R} = \frac{(mdq + \mathfrak{D}\mathfrak{Q})(1 + nddqq) + 2ndq(dd + qq)}{(1 - nddqq)^2}.$$

Quarto ex ista quantitate r cum quinta datarum e definiatur s, ut sit

$$s = \frac{r\mathfrak{E} + e\mathfrak{R}}{1 - neerr} \text{ et } \mathfrak{S} = \frac{(mer - \mathfrak{E}\mathfrak{R})(1 + neerr) + 2ner(ee + rr)}{(1 - neerr)^2}.$$

Haeque operationes continuentur, donec omnes quantitates datae in computum fuerint ductae.

§. 32. His autem omnibus valoribus inventis, sequentes comparationes desideratae ordine ita se habebunt

I. $\Pi : p = \Pi : a + \Pi : b + \beta abp$

II. $\Pi : q = \Pi : a + \Pi : b + \Pi : c + \beta abp + \beta cpq$

III. $\Pi : r = \Pi : a + \Pi : b + \Pi : c + \Pi : d + \beta abp + \beta cpq + \beta dqr$

IV. $\Pi : s = \Pi : a + \Pi : b + \Pi : c + \Pi : d + \Pi : e + \beta abp + \beta cpq + \beta dqr + \beta ers$

V. $\Pi : t = \Pi : a + \Pi : b + \Pi : c + \Pi : d + \Pi : e + \Pi : f + \beta abp + \beta cpq + \beta dqr + \beta ers + \beta fst.$

etc. etc.

§. 33. Cum igitur ista formula transcendens

$$\Pi : z = \int \frac{\partial z(\alpha + \beta zz)}{\sqrt{(1 + mzz + z^4)}},$$

in se contineat arcus omnium sectionum conicarum a vertice sumtos, harum formularum ope, quotcunque proponantur arcus in quavis sectione conica, qui omnes a vertice sint sumti, semper novus in eadem sectione conica arcus pariter a vertice abscindi poterit, qui a summa illorum arcuum datorum discrepet quantitate algebraice assignabili. Quin etiam nihil impedit, quo minus aliqui inter arcus datos capiantur negativi, quandoquidem jam annotavimus esse $\Pi : (-z) = -\Pi : z$, ita ut nostra determinatio etiam accommodari possit ad arcus inter terminos quoscunque interceptos. Hocque modo tractatio, quam nuper circa comparationem talium arcuum dedi, multo generalior reddi poterit.

§. 34. Caeterum, quemadmodum hoc casu, quo sumsimus $Z = \alpha + \beta zz$, character supra usurpatus $\Phi : xy$ abiit in βxyz, dum scilicet ex binis quantitatibus x et y, secundum praecepta data tertia z determinatur: ita etiam, quaecunque alia functio loco Z adhibeatur, quoniam posuimus

$$\Phi : xy = a \int \frac{(X - Y)\partial u}{xx - yy}, \text{ existente } u = xy,$$

integratione absoluta, functio inde resultans tantum quantitatem u cum litteris a et $\mathfrak{A}$ continebit, quandoquidem littera t ita exprimebatur $t = aa - 2\mathfrak{A}u + naauu$, cum invento integrali ubique loco u scribatur xy, at vero loco a et $\mathfrak{A}$ litterae z et $\mathfrak{Z}$; atque hoc modo obtinebitur valor characteris $\Phi : xy$ pro quovis casu proposito, quae functio, nisi fuerit algebraica, semper per logarithmos et arcus circulares exhiberi poterit; siquidem, uti assumsimus, littera Z fuerit functio rationalis par ipsius z.