SUPPLEMENTUM V.

AD TOM. I. CAP. VIII.

DE

VALORIBUS INTEGRALIUM QUOS CERTIS TANTUM CASIBUS RECIPIUNT.

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1) Nova Methodus quantitates integrales determinandi. *Novi Commentarii Academiae Scient. Petropolitanae Tom. XIX. Pag.* 66—102.

§. 1. Cum mihi saepius occurrissent formulae differentiales, quae per logarithmum quantitatis variabilis erant divisae, veluti $\frac{P\partial z}{lz}$, nunquam perspicere potui, ad quodnam genus quantitatum earum integralia sint referenda, quin etiam maxime difficile videbatur eorum valores saltem vero proxime assignare. Quod quidem ad formulam integralem simplicissimam hujus generis $\int\frac{\partial z}{lz}$ attinet, facile patet, si eam ita integrari concipiam, ut evanescat posito $z=0$, tum vero statuatur $z=1$, quantitatem infinite magnam esse proditura; quod si enim variabilis z jam proxime ad unitatem accesserit, ut sit $z=1-u$, existente u quantitate infinite parva, tum ob

$$\partial z=-\partial u \text{ et } lz=l(1-u)=-u,$$

haec formula erit $\int\frac{\partial u}{u}$, cujus valor utique fit infinitus. At vero dantur omnino hujusmodi formulae integrales $\int\frac{P\partial z}{lz}$, quae, etiamsi ponatur $z = 1$, tamen valores finitae magnitudinis sortiuntur: quod determinasse eo magis operae pretium videtur, quod nulla adhuc cognita est via istos valores investigandi.

§. 2. Consideremus exempli gratia hanc formulam satis simplicem $\int \frac{(z-1)\,\partial z}{lz}$, quae memorata lege integrata valorem finitum habere facile ostendi potest. Posito enim $\frac{z-1}{lz} = y$, ut formula nostra fiat $\int y\,\partial z$, ideoque exprimat aream curvae, pro abscissa z applicatam habentis $= y$, ista area a termino $z = 0$ usque ad terminum $z = 1$ extensa utique valorem finitum non multo majorem quam $\frac{1}{2}$ repraesentabit; posita enim abscissa $z = 0$, fiet etiam applicata $y = 0$, at sumta $z = 1$, pro applicata $y = \frac{z-1}{lz}$ tam numerator quam denominator evanescit, ergo eorum loco substitutis suis differentialibus, fiet $y = z = 1$. Pro abscissis autem mediis ponamus $z = e^{-n}$, existente e numero, cujus logarithmus hyperbolicus est unitas, erit

$$y = \frac{e^{-n} - 1}{-n} = \frac{e^n - 1}{n\, e^n},$$

quae, si n fuerit numerus valde magnus, ut abscissa z fiat minima, applicata erit proxime $y = \frac{1}{n}$; qui ergo valor multo major erit quam abscissa z; forma scilicet hujus curvae similis erit figurae adjectae, ubi A P denotat abscissam z et P M applicatam y, abscissae vero A B $= 1$ respondet applicata B C $= 1$, qua curva descripta, ejus area A M C B non multum superabit aream trianguli A B C quae est $= \frac{1}{2}$. Fig. 1.

§. 3. Nuper autem, in aliis investigationibus occupatus, praeter exspectationem inveni, hanc aream aequalem esse logarithmo hyperbolico binarii, ita ut ea per fractiones decimales sit l

$2 = 0{,}6931471805$; sequenti autem ratiocinio huc sum perductus. Cum revera sit $l\,z = \frac{z^0 - 1}{0}$, quia differentiando utrinque prodit $\frac{\partial z}{z} = \frac{\partial z}{z}$, et sumto $z = 1$ utraque expressio evanescit, loco 0 scribo $\frac{1}{i}$, denotante i numerum infinitum, eritque $l\,z = i\,(z^{\frac{1}{i}} - 1)$, hincque applicata

$$y = \frac{z - 1}{i\,(z^{\frac{1}{i}} - 1)} = \frac{1 - z}{i\,(1 - z^{\frac{1}{i}})},$$

et formula integralis

$$\int \frac{(1 - z)\,\partial z}{i\,(1 - z^{\frac{1}{i}})}.$$

Nunc igitur statuo $z^{\frac{1}{i}} = x$, ut fiat $z = x^i$, ubi notetur, pro utroque integrationis termino $z = 0$ et $z = 1$ etiam fore $x = 0$ et $x = 1$; quia igitur hinc fit $\partial z = i x^{i-1} \partial x$, formula integralis evadit

$$\int \frac{x^{i-1}\,\partial x\,(1 - x^i)}{(1 - x)},$$

quam ergo integrari oportet a termino $x = 0$ usque ad terminum $x = 1$.

§. 4. Spectemus nunc i ut numerum valde magnum, et fractio $\frac{1 - x^i}{1 - x}$ resolvitur in hanc progressionem geometricam

$$1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + \ldots x^{i-1},$$

cujus singuli termini in $x^{i-1}\partial x$ ducti et integrati praebent hanc seriem

$$\frac{x^i}{i} + \frac{x^{i+1}}{i+1} + \frac{x^{i+2}}{i+2} + \frac{x^{i+3}}{i+3} + \ldots\ldots + \frac{x^{2i-1}}{2i-1},$$

quae utique evanescit facto $x = 0$. Nunc igitur sumatur $x = 1$, et valor quaesitus nostrae formulae integralis erit

$$\frac{1}{i} + \frac{1}{i+1} + \frac{1}{i+2} + \frac{1}{i+3} + \ldots + \frac{1}{2i-1},$$

ubi quidem littera i denotat numerum infinite magnum, ita ut numerus horum terminorum sit revera infinitus. Nihilo vero minus, quia singuli termini sunt infinite parvi, haec series summam habebit finitam, quam sequenti modo ad seriem ordinariam reducere licet.

§. 5. Series inventa spectari potest tanquam differentia inter binas sequentes progressiones harmonicas

$$A = 1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{5} + \tfrac{1}{6} + \ldots + \frac{1}{2i-1}$$
$$B = 1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{5} + \tfrac{1}{6} + \ldots + \frac{1}{i-1}$$

quandoquidem differentia $A - B$ ipsam seriem inventam exhibet; quia autem numerus terminorum seriei A est $2i - 1$, seriei vero $B = i - 1$, ille duplo major est quam hic, quocirca, ut seriem regularem obtineamus, singulos terminos seriei B per saltum a seriei A termino secundo, quarto, sexto, octavo etc. auferamus, quo pacto simul ad finem utriusque pervenietur, eritque

$$A - B = 1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} - \tfrac{1}{6} + \text{etc.}$$

in infinitum, cujus ergo valor est $l\,2$, ita ut nunc quidem solide sit demonstratum, formulae integralis propositae $\int \frac{(z-1)\partial z}{lz}$, casu $z = 1$, valorem revera esse $= l\,2$.

§. 6. Simile ratiocinium etiam ad formulam integralem generaliorem $\int \frac{(z^m - 1)\partial z}{lz}$ accommodari potest, ac tandem reperietur, casu $z = 1$ ejus valorem fore $l\,(m+1)$; quia igitur pari modo erit

$$\int \frac{(z^n - 1)\partial z}{lz} = l(n+1),$$

si hanc ab illa subtrahamus, prodit sequens integratio

$$\int \frac{(z^m - z^n)\partial z}{lz} = l\frac{m+1}{n+1},$$

si scilicet integratio a termino $z = 0$ usque ad terminum $z = 1$ extendatur.

§. 7. Quia autem haec demonstratio per quantitates infinitas et infinite parvas procedit, merito aliam methodum planam et consuetam desideramus, quae ad easdem summas perducere valeat; quae quidem investigatio maxime ardua videbitur. Interim tamen, cum nuper consideratio functionum duas variabiles involventium me ad integrationem formularum differentialium prorsus singularium perduxisset, quae aliis methodis frustra tentantur, ex eodem principio quoque integrationes hic exhibitas derivandas esse intellexi. Hanc igitur methodum tanquam fontem prorsus novum, ex quo integrationes, aliis methodis inaccessas, haurire liceat, clare et perspicue explicabo, cui negotio istam disquisitionem praecipue destinavi.

Lemma I.

§. 8. Si P fuerit functio quaecunque duarum variabilium z et u, ac ponatur $\int P\partial z = S$, ut etiam S sit functio binarum variabilium z et u, tum erit

$$\int \partial z \left(\frac{\partial P}{\partial u}\right) = \left(\frac{\partial S}{\partial u}\right).$$

Demonstratio.

Cum in integratione formulae $\int P\partial z$ sola z ut variabilis spectetur, erit $\left(\frac{\partial S}{\partial z}\right) = P$, quae formula denuo differentiata, sola u

pro variabili habita, praebet $\left(\frac{\partial \partial S}{\partial u \partial z}\right) = \left(\frac{\partial P}{\partial u}\right)$, quae in ∂z ducta et integrata producit $\left(\frac{\partial S}{\partial u}\right) = \int \partial z \cdot \left(\frac{\partial P}{\partial u}\right)$, quandoquidem ex principiis calculi integralis est

$$\int \partial z \left(\frac{\partial \partial S}{\partial z \partial u}\right) = \left(\frac{\partial S}{\partial u}\right) \text{ q. e. d.}$$

Corollarium I.

§. 9. Eodem modo per hujusmodi differentialia, ubi tantum u pro variabili spectatur, ulterius progredi licet, unde sequentes oriuntur integrationes

$$\left(\frac{\partial \partial S}{\partial u^2}\right) = \int \partial z \left(\frac{\partial \partial P}{\partial u^2}\right) \text{ et}$$

$$\left(\frac{\partial^3 S}{\partial u^3}\right) = \int \partial z \left(\frac{\partial^3 P}{\partial u^3}\right)$$

etc. etc.

Corollarium 2.

§. 10. Quod si ergo formula $\int P \partial z$ fuerit integrabilis, ita ut ejus integrale S exhiberi possit, tum etiam omnes istae formulae integrales

$$\int \partial z \left(\frac{\partial P}{\partial u}\right), \int \partial z \left(\frac{\partial \partial P}{\partial u^2}\right), \int \partial z \left(\frac{\partial^3 P}{\partial u^3}\right) \text{ etc.}$$

integrationem admittent, atque adeo ipsa integralia exhiberi poterunt.

Scholion.

§. 11. Ex his quidem formulis si in genere tractentur, parum utilitatis in calculum integralem redundat. At si functio P ita fuerit comparata, ut integrale $\int P \partial z$, casu saltem particulari, quo post integrationem variabili z certus quidam valor puta $z = a$ tribuitur, commode exhiberi potest, ut hoc casu quantitas S abeat in functionem solius variabilis u satis simplicem, tum integrationes memoratae perinde locum habebunt, si quidem post singulas integrationes ponatur $z = a$, atque hinc ad ejusmodi integrationes plerumque pervenitur, quas aliis methodis vix, ac ne vix quidem perficere liceat: atque hinc oritur

Primum principium integrationum.

§. 12. Si P ejusmodi fuerit functio binarum variabilium z et u, ut valor integralis $\int P \partial z$ saltem casu certo $z = a$ commode exprimi queat, qui valor sit $= S$, functio scilicet ipsius u tantum; tum etiam sequentia integralia, si quidem post integrationem pariter statuatur $z = a$, commode exhiberi poterunt, scilicet

$$\int P \partial z = S$$
$$\int \partial z \left(\frac{\partial P}{\partial u}\right) = \left(\frac{\partial S}{\partial u}\right)$$
$$\int \partial z \left(\frac{\partial \partial P}{\partial u^2}\right) = \left(\frac{\partial \partial S}{\partial u^2}\right)$$
$$\int \partial z \left(\frac{\partial^3 P}{\partial u^3}\right) = \left(\frac{\partial^3 S}{\partial u^3}\right)$$
$$\int \partial z \left(\frac{\partial^4 P}{\partial u^4}\right) = \left(\frac{\partial^4 S}{\partial u^4}\right)$$
etc. etc.

Exemplum I.

§. 13. Si fuerit $P = z^u$, erit quidem in genere

$$\int P \partial z = \frac{z^{u+1}}{u+1};$$

unde casu $z = 1$ hic valor satis simplex nascitur $\frac{1}{u+1}$, ita ut sit $S = \frac{1}{u+1}$; cum deinde per differentiationes continuas, dum sola u pro variabili habetur, prodeat $\left(\frac{\partial P}{\partial u}\right) = z^u\, l z$, tum vero $\left(\frac{\partial \partial P}{\partial u^2}\right) = z^u\, (l z)^2$, porro

$$\left(\frac{\partial^3 P}{\partial u^3}\right) = z^u\, (l z)^3, \quad \left(\frac{\partial^4 P}{\partial u^4}\right) = z^u\, (l z)^4, \text{ etc.}$$

hinc sequentes obtinentur valores integrales, si quidem post singulas integrationes statuatur $z = 1$

$$\int z^u \partial z = + \frac{1}{u+1}$$
$$\int z^u \partial z \, l z = - \frac{1}{(u+1)^2}$$
$$\int z^u \partial z (l z)^2 = + \frac{1.\,2}{(u+1)^3}$$
$$\int z^u \partial z (l z)^3 = - \frac{1.\,2.\,3}{(u+1)^4}$$
$$\int z^u \partial z (l z)^4 = + \frac{1.\,2.\,3.\,4}{(u+1)^5}$$
$$\int z^u \partial z (l z)^5 = - \frac{1.2.3.4.5}{(u+1)^6}$$
$$\int z^u \partial z (l z)^6 = + \frac{1 \ldots\ldots 6}{(u+1)^7}$$
$$\int z^u \partial z (l z)^7 = - \frac{1 \ldots\ldots 7}{(u+1)^8}$$

unde concludimus generaliter fore

$$\int z^u \partial z (l z)^n = \pm \frac{1.\,2.\,3.\,4 \ldots n}{(u+1)^{n+1}},$$

ubi signum $+$ valet si n sit numerus par, alterum vero $-$ si n sit numerus impar. Hae quidem integrationes jam aliunde satis sunt notae, id quod mirum non est, quoniam tam simplicem formulam pro P assumsimus: breviter igitur repetamus eos casus, quos jam nuper expedivi.

Exemplum 2.

§. 14. Si fuerit

$$P = \frac{z^{n-u-1} + z^{n+u-1}}{1 + z^{2n}},$$

jam dudum demonstravi, formulae $\int P \partial z$ valorem integralem casu quo post integrationem ponitur $z = 1$, esse

$$S = \frac{\pi}{2 n \cos. \frac{\pi u}{2 n}}.$$

Hinc ergo cum sit

$$\left(\frac{\partial P}{\partial u}\right) = - \frac{z^{n-u-1} + z^{n+u-1}}{1 + z^{2-n}} \, l z,$$

tum vero

$$\left(\frac{\partial\partial P}{\partial u^2}\right) = + \frac{z^{n-u-1} + z^{n+u-1}}{1 + z^{2n}} (lz)^2 \text{ et}$$

$$\left(\frac{\partial^3 P}{\partial u^3}\right) = - \frac{z^{n-u-1} + z^{n+u-1}}{1 + z^{2n}} (lz)^3$$

etc. etc.

ex cognito valore S sequentes nacti sumus integrationes

$$\text{I.} \int \frac{z^{n-u-1} + z^{n+u-1}}{1 + z^{2n}} \partial z = S = \frac{\pi}{2n \cos. \frac{\pi u}{2u}}$$

$$\text{II.} \int - \frac{z^{n-u-1} + z^{n+u-1}}{1 + z^{2n}} \partial z \, lz = \left(\frac{\partial S}{\partial u}\right)$$

$$\text{III.} \int \frac{z^{n-u-1} + z^{n+u-1}}{1 + z^{2n}} \partial z \, (lz)^2 = \left(\frac{\partial\partial S}{\partial u^2}\right)$$

$$\text{IV.} \int - \frac{z^{n-u-1} + z^{n+u-1}}{1 + z^{2n}} \partial z \, (lz)^3 = \left(\frac{\partial^3 S}{\partial u^3}\right)$$

$$\text{V.} \int \frac{z^{n-u-1} + z^{n+u-1}}{1 + z^{2n}} \partial z \, (lz)^4 = \left(\frac{\partial^4 S}{\partial u^4}\right)$$

etc. etc.

Exemplum 3.

§. 15. Si fuerit

$$P = \frac{z^{n-u-1} - z^{n+u-1}}{1 - z^{2n}},$$

simili modo demonstravi, valorem formulae integralis $\int P \partial z$, casu quo post integrationem ponitur $z = 1$, fore

$$S = \frac{\pi}{2n} \text{ tang. } \frac{\pi u}{2n};$$

atque hinc sequentes integrationes pro eodem casu $z = 1$ fuerunt deductae

$$\text{I. } \int \frac{z^{n-u-1} - z^{n+u-1}}{1 - z^{2n}} \partial z = S = \frac{\pi}{2n} \text{ tang. } \frac{\pi u}{2n}$$

$$\text{II. } \int \frac{- z^{n-u-1} + z^{n+u-1}}{1 - z^{2n}} \partial z \, lz = \left(\frac{\partial S}{\partial u}\right)$$

$$\text{III. } \int \frac{z^{n-u-1} - z^{n+u-1}}{1 - z^{2n}} \partial z \, (lz)^2 = \left(\frac{\partial \partial S}{\partial u^2}\right)$$

$$\text{IV. } \int \frac{- z^{n-u-1} + z^{n+u-1}}{1 - z^{2n}} \partial z \, (lz)^3 = \left(\frac{\partial^3 S}{\partial u^3}\right)$$

$$\text{V. } \int \frac{- z^{n\;\; u-1} - z^{n-u-1}}{1 - z^{2n}} \partial z \, (lz)^4 = \left(\frac{\partial^4 S}{\partial u^4}\right)$$

etc. etc.

Scholion.

§. 16. Quo igitur uberiores fructus ex hoc principio expectare queamus, praecipuum negotium huc redit, ut ejusmodi functiones binarum variabilium z et u pro P investigemus, ita ut valor formulae integralis saltem certo quodam casu puta $z = 1$ succincte assignari possit, quemadmodum in allatis exemplis fieri licuit. Quemadmodum autem hoc principium ex continua differentiatione est deductum, ita eodem modo continua integratio ad usum nostrum accommodari poterit.

Lemma II.

§. 17. Si P fuerit functio duarum variabilium z et u, ac ponatur $\int P \partial z = S$, ut etiam S sit functio duarum variabilium z et u, tum erit $\int S \partial u = \int \partial z \int P \partial u$, ubi in integralibus formulis $\int P \partial u$ et $\int S \partial u$ sola u pro variabili habetur, in formula autem $\int \partial z \int P \partial u$ sola z.

Demonstratio.

Ponatur $\int S \partial u = V$, ut sit $S = \left(\frac{\partial V}{\partial u}\right)$, ideoque

$$\left(\frac{\partial V}{\partial u}\right) = \int P \partial z, \text{ eritque } \left(\frac{\partial \partial V}{\partial z \partial u}\right) = P;$$

unde, per ∂u multiplicando et integrando, erit $\left(\frac{\partial V}{\partial z}\right) = \int P \partial u$, ex quo sequitur

$$V = \int \partial z \int P \partial u = \int S \partial u. \text{ q. e. d.}$$

Corollarium 1.

§. 18. Hoc modo etiam integratio repeti potest, unde orietur talis aequatio

$$\int \partial u \int S \partial u = \int \partial z \int \partial u \int P \partial u;$$

hinc autem plerumque parum utilitatis exspectari potest, nisi forte istae integrationes commode succedant.

Corollarium 2.

§. 19. Quod si ergo formula $\int P \partial z$ fuerit integrabilis, scilicet $= S$, altera hinc deducta $\int \partial z \int P \partial u$ eatenus tantum integrari poterit, quatenus integrale $\int S \partial u$ integrare licet.

Secundum principium integrationum.

§. 20. Si P ejusmodi fuerit functio duarum variabilium z et u, ut formulae integralis $\int P \partial z$ valor certo saltem casu, puta $z = a$, commode exhiberi queat, ita ut hoc casu quantitas S fiat functio solius variabilis u; tum etiam pro eodem casu $z = a$ hujus formulae integralis $\int \partial z \int P \partial u$ valor assignari poterit, si modo formulam $\int S \partial u$ integrare licuerit.

Exemplum I.

§. 21. Sumamus $P = z^u$, eritque $\int P \partial z = \frac{z^{u+1}}{u+1}$; quae formula casu $z = 1$ abit in $\frac{1}{u+1}$, quod ergo loco S scribatur. Tum vero quia est

$$\int P \partial u = \int z^u \partial u = \frac{z^u}{l z},$$

et quia

$$\int S \partial u = l(u+1), \text{ erit}$$

$$\int \frac{z^u \partial z}{l z} = l(u+1);$$

si quidem post illam integrationem ponatur $z = 1$. Quia autem omnis integratio additionem constantis postulat, hic potius statui oportebit

$$\int \frac{z^u \partial z}{l z} = l(u+1) + C;$$

atque hic quidem facile intelligitur, hanc constantem C esse debere infinitam, quoniam in formula integrali fractio $\frac{z^u}{l z}$ posito $z = 1$ fit infinita, ita ut hinc parum pro instituto nostro sequi videatur.

Corollarium 1.

§. 22. Quoniam autem haec constans C non a variabili u pendet, ea retinebit eundem valorem, quicunque numeri determinati pro u accipiantur. Sumamus igitur primo $u = m$, tum vero etiam $u = n$, ut habeamus istos valores

$$\text{I. } \int \frac{z^m \partial z}{l z} = l(m+1) + C \text{ et}$$

$$\text{II.} \int \frac{z^n \partial z}{l z} = l(n+1) + C,$$

quarum altera ab altera subtracta relinquet istam integrationem notatu dignissimam

$$\int \frac{(z^m - z^n)\, \partial z}{l z} = l \frac{m+1}{n+1},$$

quemadmodum jam supra ex longe aliis principiis demonstravimus.

Corollarium 2.

§. 23. Si ad alteram integrationem ascendamus, quia est $\int P \partial u = \frac{z^u}{l z}$, erit $\int \partial u \int P \partial u = \frac{z^u}{(l z)^2}$; tum vero ob

$$\int S \partial u = l(u+1) + C, \text{ erit}$$
$$\int \partial u \int S \partial u = (u+1)\,[l(u+1) - 1] + C u + D,$$

sicque habebimus

$$\int \frac{z^u \partial z}{(l z)^2} = (u+1)[l(u+1) - 1] + C u + D,$$

ubi constantes C et D non ab u pendent: quare ut eas eliminemus tres casus determinatos evolvamus

$$\text{I.} \int \frac{z^m \partial z}{(l z)^2} = (m+1)\, l\,(m+1) - m - 1 + C m + D,$$

$$\text{II.} \int \frac{z^n \partial z}{(l z)^2} = (n+1)\, l(n+1) - n - 1 + C n + D,$$

$$\text{III.} \int \frac{z^k \partial z}{(l z)^2} = (k+1)\, l\,(k+1) - k - 1 + C k + D,$$

eritque

$$\text{I} - \text{III} = (m+1)\, l(m+1) - (k+1)\, l(k+1) + k - m + C(m-k) \text{ et}$$
$$\text{II} - \text{III} = (n+1)\, l(n+1) - (k+1)\, l(k+1) + k - n + C(n-k)$$

hincque deducimus

$$(I-III)(n-k)-(II-III)(m-k)=\begin{cases}+(n+1)(n-k)l(m+1)\\-(k+1)(n-k)l(k+1)+(k-m)(n-k)\\-(n+1)(m-k)l(n+1)-(k-n)(m-k)\\+(k+1)(m-k)l(k+1)\end{cases}$$

atque hinc pervenimus ad sequentem integrationem

$$\int\frac{\partial z\,[(n-k)\,z^m-(m-k)\,z^n+(m-n)\,z^k]}{(l\,z)^2}=$$
$$\begin{aligned}&+(m+1)\;(n-k)\;l\,(m+1)\\&-(n+1)\;(m-k)\;l\,(n+1)\\&+(k+1)\;(m-n)\;l\,(k+1).\end{aligned}$$

Corollarium 3.

§. 24. Operae pretium erit aliquot casus evolvere, ubi quidem numeros m, n et k inter se inaequales accipi convenit, quia aliter omnes termini se destruerent.

I. Sit igitur $m=2$, $n=1$ et $k=0$, erit

$$\int\frac{(z-1)^2\,\partial z}{(l\,z)^2}=3\,l\,3-4\,l\,2=l\,\tfrac{27}{16},$$

II. Sit $m=3$, $n=1$ et $k=0$, eritque

$$\int\frac{(z^3-3\,z+2)\,\partial z}{(l\,z)^2}=\int\frac{\partial z\,(z-1)^2\,(z+2)}{(l\,z)^2}=4\,l\,4-6\,l\,2=2\,l\,2=l\,4,$$

III. Sit $m=3$, $n=2$ et $k=0$, et erit

$$\int\frac{(2\,z^3-3\,z\,z+1)\,\partial z}{(l\,z)^2}=\int\frac{\partial z\,(z-1)^2\,(2\,z+1)}{(l\,z)^2}=8\,l\,4-9\,l\,3=l\,\tfrac{4^8}{3^9},$$

IV. Sit $m=3$, $n=2$ et $k=1$, et prodit

$$\int\frac{(z^3-2\,z\,z+z)\,\partial z}{(l\,z)^2}=\int\frac{z\,\partial z\,(z-1)^2}{(l\,z)^2}=4\,l\,4-6\,l\,3+2\,l\,2=l\,\tfrac{2^{10}}{3^6}.$$

Corollarium 4.

§. 25. In his casibus notatu dignum occurrit, quod numerator in formulis integralibus factorem habet $(z-1)^2$, quod ideo necessario usu venit, ne valores integralium evadant infiniti. Quia enim denominator $(lz)^2$ evanescit casu $z = 1$, si ponamus $z = 1 - \omega$, existente ω infinite parvo, erit

$$lz = -\omega \text{ et } (lz)^2 = +\omega\omega.$$

Necesse ergo est ut in numeratore adsit factor, qui casu $z = 1$ $-\omega$ itidem praebeat $\omega\omega$, quod evenit si ibi factor fuerit $(z-1)^2$.

Scholion.

§. 26. Integratio, quam in corollario primo sumus nacti, ideo omni digna videtur attentione, quod valores integrales inde nati casu $z = 1$ nullo adhuc modo assignare potuerim, etiamsi tam simpliciter per logarithmos exprimantur. At vero integrationes in corollario secundo inventae, etiamsi multo magis arduae, videantur, tamen ex prioribus ope reductionum cognitarum non difficulter derivari possunt; id quod pro unico casu ostendisse sufficiet. Ponamus

$$\int \frac{\partial z (z-1)^2}{(lz)^2} = \frac{p}{lz} + \int \frac{q \partial z}{lz},$$

eritque differentiando

$$\frac{\partial z (z-1)^2}{(lz)^2} = \frac{\partial p}{lz} - \frac{p \partial z}{z (lz)^2} + \frac{q \partial z}{lz},$$

unde aequatis terminis seorsim vel per $(lz)^2$ vel per lz divisis, habebimus has duas aequalitates

$$(z-1)^2 = -\frac{p}{z} \text{ et } \partial p = -g \partial z,$$

ex quarum priore oritur $p = -z(z-1)^2$, hincque

$$\frac{\partial p}{\partial z} = -3zz + 4z - 1,$$

ideoque

$$q = 3zz - 4z + 1,$$

ita ut sit

$$\int \frac{\partial z (z-1)^2}{(lz)^2} = \frac{-z(z-1)^2}{lz} + \int \frac{(3zz - 4z + 1) \partial z}{lz},$$

hic autem prius membrum posito $z = 1$ sponte evanescit; posito enim $z = 1 - \omega$, ut sit $lz = -\omega$, erit

$$p = -\omega\omega\,(1-\omega), \text{ ideoque}$$

$$\frac{p}{lz} = \omega\,(1-\omega) = 0, \text{ ob } \omega = 0:$$

posterius vero membrum in has partes discerpi potest

$$3\int\frac{(zz-z)\,\partial z}{lz} - \int\frac{(z-1)\,\partial z}{lz},$$

cujus prioris partis integrale est $3\,l\frac{3}{2}$, posterioris vero $-1\,l\,2$; sicque totum hoc integrale erit

$$3\,l\tfrac{3}{2} - l\,2 = 3\,l\,3 - 4\,l\,2 = l\tfrac{27}{16},$$

prorsus uti invenimus. Hoc igitur modo si in genere statuamus

$$\int\frac{V\,\partial z}{(lz)^2} = \frac{p}{lz} + \int\frac{q\,\partial z}{lz},$$

erit differentiando

$$\frac{V\,\partial z}{(lz)^2} = \frac{\partial p}{lz} - \frac{p\,\partial z}{z\,(lz)^2} + \frac{q\,\partial z}{lz},$$

unde istae duae fluunt aequalitates

$$p = -Vz \text{ et } q = -\frac{\partial p}{\partial z}.$$

Jam ut terminus $\frac{p}{lz}$ evanescat posito $z = 1$, numerator p factorem habere debet $(z-1)^2$; qui ergo etiam factor esse debet quantitatis V. Sit igitur

$$V = \frac{U\,(z-1)^2}{z}, \text{ eritque } p = -U\,(z-1)^2,$$

unde fit

$$\partial p = -\partial U(z-1)^2 - 2U\partial z(z-1) = (z-1)[\partial U(z-1) - 2U\partial z],$$

hincque

$$q\,\partial z = (z-1)\,[2\,U\,\partial z - \partial\,U\,(z-1)];$$

quia ergo q factorem habet $z-1$, formula $\int\frac{q\,\partial z}{lz}$ semper in partes resolvi potest, quarum integralia per corollarium primum assignare licet, si modo U fuerit aggregatum ex quotcunque potestatibus ipsius z; unde sequens deducitur theorema.

Theorema.

§. 27. Si fuerit

$$P = A z^{\alpha} + B z^{\beta} + C z^{\gamma} + D z^{\delta} + \text{etc.}$$

ita ut summa coëfficientium

$$A + B + C + D + \text{etc.} = 0,$$

tum erit

$$\int \frac{P \partial z}{l z} = A\, l(\alpha + 1) + B\, l(\beta + 1) + C\, l(\gamma + 1) + D\, l(\delta + 1) + \text{etc.}$$

si quidem post integrationem statuatur $z = 1$.

Demonstratio.

Cum hoc ipso casu, quo post integrationem ponitur $z = 1$, sit

$$\int \frac{z^n \partial z}{l z} = l(n + 1) + \Delta,$$

denotante Δ illam constantem infinitam integratione ingressam, erit

$$A \int \frac{z^{\alpha} \partial z}{l z} = A\, l(\alpha + 1) + A \Delta,$$

eodemque modo

$$B \int \frac{z^{\beta} \partial z}{l z} = B\, l(\beta + 1) + B \Delta,$$

etc. etc.

si nunc haec integralia omnia in unam summam colligantur, erit ob

$$(A + B + C + D + \text{etc.}) \Delta = 0$$

integrale quaesitum

$$\int \frac{P \partial z}{l z} = A l(\alpha+1) + B l(\beta+1) + C l(\gamma+1) + D l(\delta+1) \text{ etc.}$$

q. e. d.

Corollarium 1.

§. 28. Quia supponimus

$$A + B + C + D + \text{etc.} = 0,$$

evidens est, formulam

$$P = A z^{\alpha} + B z^{\beta} + C z^{\gamma} + D z^{\delta} + \text{etc.}$$

factorem habere $z - 1$, quemadmodum jam ante notavimus.

Corollarium 2.

§. 29. Quia est

$$(z-1)^n = z^n - \tfrac{n}{1} z^{n-1} + \tfrac{n(n-1)}{1.2} z^{n-2} - \tfrac{n(n-1)(n-2)}{1.2.3} z^{n-3},$$

hoc valore loco P posito, erit $A = 1$ et $\alpha = n$, deinde

$$B = -\tfrac{n}{1} \text{ et } \beta = n - 1,$$

porro

$$C = \tfrac{n(n-1)}{1.2} \text{ et } \gamma = n - 2, \text{ etc.}$$

hinc igitur erit

$$\int \frac{(z-1)^n \partial z}{l z} = l(n+1) - \tfrac{n}{1} l n + \tfrac{n.(n-1)}{1.2} l(n-1) - \tfrac{n.(n-1).(n-2)}{1.2.3} l(n-2)$$
$$+ \tfrac{n(n-1)(n-2)(n-3)}{1.2.3.4} l(n-3) + \text{etc.}$$

si modo exponens n fuerit nihilo major, vel saltem unitate non minor, quia alioquin casu $z = 1$ fractio $\frac{(z-1)^n}{l z}$ fieret infinita; hoc autem non obstante area supra considerata fiet finita, ita ut sufficiat, dummodo sit $n > 0$.

Exemplum 2.

§. 30. Sit

$$=\frac{z^{n-u-1}+z^{n+u-1}}{1+z^{2n}}, \text{ erit } \int P\,\partial z = \frac{\pi}{2\,n\cos.\frac{\pi u}{2n}};$$

si quidem post integrationem ponatur $z = 1$, quem ergo valorem litterae S tribuimus. Nunc spectata z ut constante, erit

$$\int P\,\partial u = \frac{1}{1+z^{2n}}\left(\int z^{n-u-1}\,\partial u + \int z^{n+u-1}\,\partial u\right),$$

ideoque

$$\int P\,\partial u = -\frac{z^{n-u-1}+z^{n+u-1}}{(1+z^{2n})\,l\,z},$$

unde fiet

$$\int S\,\partial u = \int \frac{z^{n-u-1}+z^{n+u-1}}{1+z^{2n}}\cdot\frac{\partial z}{l\,z};$$

cum igitur sit $\cos.\frac{\pi u}{2n} = \sin.\frac{\pi(n-u)}{2n}$, erit

$$\int S\,\partial u = \int \frac{\pi\,\partial u}{2\,n\sin.\frac{\pi(n-u)}{2n}},$$

hinc si ponamus

$$\frac{\pi(n-u)}{2n} = \Phi, \quad \text{erit } \partial\Phi = -\frac{\pi\,\partial u}{2n},$$

ideoque

$$\int S\,\partial u = -\int\frac{\partial\Phi}{\sin.\Phi} = -\,l\,\text{tang.}\,\tfrac{1}{2}\Phi,$$

quocirca habebimus

$$\int S\,\partial u = -\,l\,\text{tang.}\frac{n(n-u)}{4n},$$

ita ut posito post integrationem $z = 1$, assecuti sumus hanc integrationem

$$\int\frac{-z^{n-u-1}+z^{n+u-1}}{1+z^{2n}}\cdot\frac{\partial z}{l\,z} = -\,l\,\text{tang.}\frac{\pi(n-u)}{4n} =$$

$$+\,l\,\text{tang.}\frac{\pi(n+u)}{4n}.$$

Exemplum 3.

§. 31. Sit

$$P = \frac{z^{n-u-1} - z^{n+u-1}}{1 - z^{2n}}, \text{ erit}$$

$$\int P \partial z = \frac{\pi}{2n} \text{ tang. } \frac{\pi u}{2n} = S$$

unde fit

$$\int S \partial u = - l \cos. \frac{\pi u}{2n},$$

hinc cum sit

$$\int P \partial u = - \frac{z^{n-u-1} - z^{n+u-1}}{(1 - z^{2n})\, l z},$$

nanciscimur sequentem integrationem, si quidem integrale a termino $z = 0$ usque ad terminum $z = 1$ extendatur,

$$\int \frac{z^{n-u-1} + z^{n+u-1}}{1 - z^{2n}} \cdot \frac{\partial z}{l z} = + l \cos. \frac{\pi u}{2n}.$$

Haec quidem duo posteriora exempla jam ante fusius expedivi; unde iis magis evolvendis non immoror, sed ad sequens problema progredior.

Problema.

§. 32. *Si proponantur hae duae series infinitae*

$P = z \cos. u + z^2 \cos. 2u + z^3 \cos. 3u + z^4 \cos. 4u + z^5 \cos. 5u +$ etc. et

$Q = z \sin. u + z^2 \sin. 2u + z^3 \sin. 3u + z^4 \sin. 4u + z^5 \sin. 5u +$ etc.

quae binas variabiles z et u involvunt, invenire relationes inter formulas integrales $\int \frac{P \partial z}{z}$, $\int P \partial u$ *et* $\int \frac{Q \partial z}{z}$, $\int Q \partial u$, *aliasque formulas integrales per continuam integrationem inde natas.*

Solutio.

Cum utraque series sit recurrens, reperitur per formulas finitas

$$P = \frac{z\cos.u - zz}{1-2z\cos.u+zz} \text{ et } Q = \frac{z\sin.u}{1-2z\cos.u+zz},$$

unde fit

$$\int\frac{P\partial z}{z} = \int\frac{\partial z\cos.u - z\partial z}{1-2z\cos.u+zz} = -l\sqrt{(1-2z\cos.u+zz)} \text{ et}$$

$$\int Q\partial u = \int\frac{z\partial u\sin.u}{1-2z\cos.u+zz} = +l\sqrt{(1-2z\cos.u+zz)},$$

ita ut sit

$$\int\frac{P\partial z}{z} = -\int Q\partial u;$$

tum vero etiam erit

$$\int\frac{Q\partial z}{z} = \int\frac{\partial z\sin.u}{1-2z\cos.u+zz} = \text{arc. tang. } \frac{z\sin.u}{1-z\cos.u};$$

at si iste arcus differentietur sumto solo angulo u variabili, erit

$$\frac{1}{\partial u}\partial.\text{arc. tang. }\frac{z\sin.u}{1-z\cos.u} = \frac{z\cos.u - zz}{1-2z\cos.u+zz},$$

ita ut sit

$$\int\frac{Q\partial z}{z} = \int P\partial u$$

§. 33. Verum eaedem relationes facilius ex ipsis seriebus derivantur: cum enim sit

$$P = z\cos.u + z^2\cos.2u + z^3\cos.3u + z^4\cos.4u + \text{etc.}$$

erit

$$\int\frac{P\partial z}{z} = \frac{z\cos.u}{1} + \frac{zz\cos.2u}{2} + \frac{z^3\cos.3u}{3} + \text{etc. et}$$

$$\int P\partial u = \frac{z\sin.u}{1} + \frac{zz\sin.2u}{2} + \frac{z^3\sin.3u}{3} + \text{etc.}$$

et quia est

$$Q = z\sin.u + zz\sin.2u + z^3\sin.3u + \text{etc. erit}$$

$$\int\frac{Q\partial z}{z} = \frac{z\sin.u}{1} + \frac{zz\sin 2u}{2} + \frac{z^3\sin.3u}{3} + \text{etc. et}$$

$$\int Q\partial u = -\frac{z\cos.u}{1} - \frac{zz\cos.2u}{2} - \frac{z^3\cos.3u}{3} - \text{etc.}$$

unde manifestum est fore

$$\int\frac{P\partial z}{z}=-\int Q\partial u \text{ et } \int\frac{Q\partial z}{z}=\int P\partial u.$$

§. 34. Quo hoc modo ulterius progredi liceat, statuamus brevitatis gratia

$$P'=\frac{z\cos.u}{1}+\frac{zz\cos.2u}{2}+\frac{z^3\cos.3u}{3}+\text{etc. et } Q'=\frac{z\sin.u}{1}+\frac{zz\sin.2u}{2}+\frac{z^3\sin.3u}{3}+\text{etc.}$$
$$P''=\frac{z\cos.u}{1^2}+\frac{zz\cos.2u}{2^2}+\frac{z^3\cos.3u}{3^2}+\text{etc. et } Q''=\frac{z\sin.u}{1^2}+\frac{zz\sin.2u}{2^2}+\frac{z^3\sin.3u}{3^2}+\text{etc.}$$
$$P'''=\frac{z\cos.u}{1^3}+\frac{zz\cos.2u}{2^3}+\frac{z^3\cos.3u}{3^3}+\text{etc. et } Q'''=\frac{z\sin.u}{1^3}+\frac{zz\sin.2u}{2^3}+\frac{z^3\sin.3u}{3^3}+\text{etc.}$$
$$P''''=\frac{z\cos.u}{1^4}+\frac{zz\cos.2u}{2^4}+\frac{z^3\cos.3u}{3^4}+\text{etc. et } Q''''=\frac{z\sin.u}{1^4}+\frac{zz\sin.2u}{2^4}+\frac{z^3\sin.3u}{3^4}+\text{etc.}$$

etc. etc. etc. etc.

et hinc comparationes ante inventae continuabuntur

$$P'=\int\frac{P\partial z}{z}=-\int Q\partial u,\quad Q'=\int\frac{Q\partial z}{z}=\int P\partial u,$$
$$P''=\int\frac{P'\partial z}{z}=-\int Q'\partial u,\quad Q''=\int\frac{Q'\partial z}{z}=\int P'\partial u,$$
$$P'''=\int\frac{P''\partial z}{z}=-\int Q''\partial u,\quad Q'''=\int\frac{Q''\partial z}{z}=\int P''\partial u,$$
$$P''''=\int\frac{P'''\partial z}{z}=-\int Q'''\partial u,\quad Q''''=\int\frac{Q'''\partial z}{z}=\int P'''\partial u,$$

etc. etc. etc. etc.

unde plures insignes relationes deduci possunt.

§. 35. Maxime autem notatu dignae et ad nostrum institutum accommodatae sunt eae relationes, ubi formulae integrales, in quibus sola z est variabilis, reducuntur ad alias formulas integrales, in quibus sola u est variabilis; cujusmodi sunt, quae sequuntur

$$P'=\int\frac{P\partial z}{z}=-\int Q\partial u,$$
$$P''=\int\frac{\partial z}{z}\int\frac{P\partial z}{z}=-\int\partial u\int P\partial u,$$
$$P'''=\int\frac{\partial z}{z}\int\frac{\partial z}{z}\int\frac{P\partial z}{z}=+\int\partial u\int\partial u\int Q\partial u,$$

$$P'''' = \int\frac{\partial z}{z}\int\frac{\partial z}{z}\int\frac{\partial z}{z}\int\frac{P\partial z}{z} = +\int\partial u\int\partial u\int\partial u\int P\partial u,$$
$$P^V = \int\frac{\partial z}{z}\int\frac{\partial z}{z}\int\frac{\partial z}{z}\int\frac{\partial z}{z}\int\frac{P\partial z}{z} = -\int\partial u\int u\int\partial u\int\partial u\int Q\partial u,$$
etc. etc. etc.

Similique modo pro altero genere

$$Q' = \int\frac{Q\partial z}{z} = +\int P\partial u,$$
$$Q'' = \int\frac{\partial z}{z}\int\frac{Q\partial z}{z} = -\int\partial u\int Q\partial u,$$
$$Q''' = \int\frac{\partial z}{z}\int\frac{\partial z}{z}\int\frac{Q\partial z}{z} = -\int\partial u\int\partial u\int\partial P u,$$
$$Q'''' = \int\frac{\partial z}{z}\int\frac{\partial z}{z}\int\frac{\partial z}{z}\int\frac{Q\partial z}{z} = +\int\partial u\int\partial u\int\partial u\int P\partial u,$$
$$Q^V = \int\frac{\partial z}{z}\int\frac{\partial z}{z}\int\frac{\partial z}{z}\int\frac{\partial z}{z}\int\frac{Q\partial z}{z} = +\int\partial u\int\partial u\int\partial u\int\partial u\int P\partial u,$$
etc. etc. etc.

§. 36. Quod si jam nostrarum serierum, sive quod eodem redit, quantitatum

P, P', P'', P''', P'''', etc. et Q, Q', Q'', Q''', Q'''', etc. eos tantum valores desideremus, quos adipiscuntur posito $z = 1$, hoc commodi assequimur, ut in formulis integralibus, ubi solus angulus u pro variabili habetur, statim ante integrationes ponere liceat $z = 1$, hoc autem facto erit

$$P = \frac{\cos. u - 1}{2 - 2\cos. u} = -\tfrac{1}{2} \text{ et } Q = \frac{\sin. u}{2 - 2\cos. u} = \tfrac{1}{2}\cot.\tfrac{1}{2}u,$$

tum vero porro

$$\int P\partial u = A - \tfrac{1}{2}u,$$
$$\int\partial u\int P\partial u = B + Au - \tfrac{1}{4}uu,$$
$$\int\partial u\int\partial u\int P\partial u = C + Bu + \tfrac{1}{2}Auu - \tfrac{1}{12}u^3,$$
$$\int\partial u\int\partial u\int\partial u\int P\partial u = D + Cu + \tfrac{1}{2}Buu + \tfrac{1}{6}Au^3 - \tfrac{1}{48}u^4,$$

at pro formulis, ubi est Q, calculus non tam concinne succedit; erit enim

$$Q = \tfrac{1}{2}\cot.\tfrac{1}{2}u,$$
$$\int Q\partial u = l\sin.\tfrac{1}{2}u,$$
$$\int\partial u\int Q\partial u = \int\partial u\, l\sin.\tfrac{1}{2}u,$$

quae formula cum omnem integrationem respuat, vix ulterius progredi licet; interim tamen erit

$$\int\partial u\int\partial u\int Q\partial u = \int\partial u\int\partial u\, l\sin.\tfrac{1}{2}u,$$
$$\int\partial u\int\partial u\int\partial u\int\partial Q\, u = \int\partial u\int\partial u\int\partial u\, l\sin.\tfrac{1}{2}u.$$

§. 37. Quod ad priores formulas variabilem z involventes attinet, per notas reductiones elicitur

$$\int\frac{P'\partial z}{z} = \int\frac{\partial z}{z}\int\frac{P\partial z}{z} = lz\int\frac{P\partial z}{z} - \int\frac{P\partial z}{z}\,lz,$$

ubi prius membrum $lz\int P\partial z$ evanescit posito $z = 1$, tum vero

$$\int\frac{\partial z}{z}\int\frac{P'\partial z}{z} = \int\frac{\partial z}{z}\int\frac{\partial z}{z}\int\frac{P\partial z}{z} = +\int\frac{P\partial z}{z}\cdot\frac{(lz)^2}{2},$$

quibus expressionibus ulterius exhibitis colligimus fore

$P' = \int\frac{P\partial z}{z},$	$Q' = \int\frac{Q\partial z}{z},$
$P'' = -\int\frac{P\partial z}{z}\,lz,$	$Q'' = -\int\frac{Q\partial z}{z}\,lz,$
$P''' = +\int\frac{P\partial z}{z}\cdot\frac{(lz)^2}{1\cdot 2},$	$Q''' = +\int\frac{Q\partial z}{z}\cdot\frac{(lz)^2}{1\cdot 2},$
$P^{IV} = -\int\frac{P\partial z}{z}\cdot\frac{(lz)^3}{1\cdot 2\cdot 3}$	$Q^{IV} = -\int\frac{Q\partial z}{z}\cdot\frac{(lz)^3}{1\cdot 2\cdot 3}.$

§. 38. Ex his igitur sequentium formularum integralium valores assignare possumus, casu quo $z = 1$,

$$P = -\tfrac{1}{2},$$
$$P' = \int\frac{P\partial z}{z} = -l\sin.\tfrac{1}{8}u,$$
$$P'' = -\int\frac{P\partial z}{z}\,lz = -B - Au + \tfrac{1}{4}uu,$$
$$P''' = +\int\frac{P\partial z}{z}\cdot\frac{(lz)^2}{1\cdot 2} = \int\partial u\int\partial u\, l\sin.\tfrac{1}{2}u,$$

$$P'''' = -\int \frac{P\partial z}{z} \cdot \frac{(lz)^3}{1\cdot 2\cdot 3} = D + Cu + \tfrac{1}{2}Buu + \tfrac{1}{6}Au^3 - \tfrac{1}{48}u^4,$$
$$P^V = +\int \frac{P\partial z}{z} \cdot \frac{(lz^4)}{1\cdot 2\cdot 3\cdot 4} = \int \partial u \int \partial u \int \partial u \int \partial u\, l\, \sin. \tfrac{1}{2}u,$$
etc. etc.

Eodem modo

$$Q = \tfrac{1}{2}\cot. \tfrac{1}{2}u,$$
$$Q' = \int \frac{Q\partial z}{z} = A - \tfrac{1}{2}u,$$
$$Q'' = -\int \frac{Q\partial z}{z} \cdot \frac{lz}{1} = -\int \partial u l \sin. \tfrac{1}{2}u,$$
$$Q''' = +\int \frac{Q\partial z}{z} \cdot \frac{(lz)^2}{2} = -C - Bu - \tfrac{1}{2}Auu + \tfrac{1}{12}u^3,$$
$$Q'''' = -\int \frac{Q\partial z}{z} \cdot \frac{(lz)^3}{6} = \int \partial u \int \partial u \int \partial u\, l\, \sin. \tfrac{1}{2}u,$$
$$Q^V = +\int \frac{Q\partial z}{z} \cdot \frac{(lz)^4}{24} = E + Du + \tfrac{1}{2}Cuu + \tfrac{1}{6}Bu^3 + \tfrac{1}{24}Au^4 - \tfrac{1}{240}u^5,$$
etc. etc.

§. 39. Cum igitur sit

$$P = \frac{z\cos. u - zz}{1 - 2z\cos. u + zz} \text{ et } Q = \frac{z\sin. u}{1 - 2z\cos. u + zz},$$

hactenus id sumus assecuti, ut harum duarum formularum integralium

$$\int \frac{\partial z(\cos. u - z)}{1 - 2z\cos. u + zz}(lz)^n \text{ et } \int \frac{\partial z \sin. u}{1 - 2z\cos. u + zz}(lz)^n$$

valores casu $z = 1$ commode per angulum u assignare valeamus, si modo constaret, quo facto quantitates A, B, C, D, etc. determinari oporteat, id quod vix alio modo nisi per ipsas series, unde hae quantitates sunt natae, fieri posse videtur.

§. 40. Omissis igitur formulis integralibus, quae quantitatem Q involvunt, quippe quarum integratio minus succedit, alteras tantum consideremus, et posito statim $z = 1$ ubi sit $P = -\tfrac{1}{2}$, ita ut sit

$$\cos. u + \cos. 2u + \cos. 3u + \cos. 4u + \text{etc.} = -\tfrac{1}{2},$$

si per ∂u multiplicemus et integremus, habebimus

$$Q' = \frac{sin.\,u}{1} + \frac{sin.\,2u}{2} + \frac{sin.\,3u}{3} + \frac{sin.\,4u}{4} + \frac{sin.\,5u}{5} + \text{etc.} = A - \tfrac{1}{2}u,$$

quae constans nihilo aequalis videri potest, quia posito $u = 0$ summa seriei evanescere videtur; at sumto angulo u infinite parvo series praebebit

$$u + u + u + u + u + u + \text{etc. et infinitum};$$

notum autem est, talem seriem summam finitam habere posse, unde hoc casu omisso statuamus $u = \pi$, seu potius $u = \pi + \omega$, prodibitque haec series existente ω angulo infinite parvo,

$$-\omega + \omega - \omega + \omega - \omega + \omega - \omega + \text{etc.}$$

ubi, quia signa alternantur, nullum est dubium, quin summa seriei evanescat, quae cum esse debeat $A - \frac{\pi}{2}$, evidens est, fieri constantem $A = \frac{1}{2}\pi$, ita, ut jam habeamus

$$Q' = \frac{sin.\,u}{1} + \frac{sin.\,2u}{2} + \frac{sin.\,3u}{3} + \frac{sin.\,4u}{4} + \frac{sin.\,5u}{5} + \text{etc.} = \frac{\pi - u}{2}.$$

Hoc modo constantem determinandi Illustr. *Daniel Bernoulli* primus est usus, qui praeterea multa praeclara circa indolem harum serierum annotavit.

§. 41. Multiplicemus porro hanc ultimam seriem per $-\partial u$, et integratio dabit

$$P'' = \frac{cos.\,u}{1^2} + \frac{cos.\,2u}{2^2} + \frac{cos.\,3u}{3^2} + \frac{cos.\,4u}{4^2} + \text{etc.} = B - \frac{\pi u}{2} + \frac{uu}{4},$$

ad quam constantem inveniendam ponamus primo $u = 0$, fietque

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \text{etc.} = B.$$

Cujus seriei summam jam pridem primus demonstravi esse $= \frac{\pi\pi}{6}$; verum si haec veritas nobis esset ignota, egregia illa methodo a magno *Bernoullio* adhibita utamur, ac ponamus $u = \pi$ eritque

$$-\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \frac{1}{5^2} + \frac{1}{6^2} - \text{etc.} = B - \frac{\pi\pi}{2} + \frac{\pi\pi}{4} = B - \frac{\pi\pi}{4};$$

ambae hae series additae dabunt

$$\frac{2}{2^2}+\frac{2}{4^2}+\frac{2}{6^2}+\frac{2}{8^2}+\text{etc.}=2B-\frac{\pi\pi}{4},$$

cujus duplum praebet

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\frac{1}{6^2}=4B-\frac{\pi\pi}{2}=B;$$

unde colligitur $B=\frac{\pi\pi}{6}$, ita ut sit

$$P''=\frac{\cos.u}{1^2}+\frac{\cos.2u}{2^2}+\frac{\cos.3u}{3^2}+\frac{\cos.4u}{4^2}+\text{etc.}=\frac{\pi\pi}{6}-\frac{\pi u}{2}+\frac{uu}{4}.$$

§. 42. Eodem modo ulterius progrediamur, et denuo per ∂u multiplicando et integrando adipiscimur

$$Q'''=\frac{\sin.u}{1^3}+\frac{\sin.2u}{2^3}+\frac{\sin.3u}{3^3}+\frac{\sin.4u}{4^3}+\text{etc.}$$
$$=C+\frac{\pi\pi u}{6}-\frac{\pi uu}{4}+\frac{u^3}{12},$$

ubi si statuatur $u=0$, summa seriei manifesto evanescit, prodiret enim posito $u=\omega$

$$\frac{\omega}{1^2}+\frac{\omega}{2^2}+\frac{\omega}{3^2}+\frac{\omega}{4^2}+\text{etc.}=\frac{\omega\pi\pi}{6},$$

quae ob $\omega=0$ fit $=0$, sicque erit $C=0$, ideoque

$$Q'''=\frac{\sin.u}{1^3}+\frac{\sin.2u}{2^3}+\frac{\sin.3u}{3^3}+\frac{\sin.4u}{4^3}+\text{etc.}=\frac{\pi\pi u}{6}-\frac{\pi uu}{4}+\frac{u^3}{12}$$

§. 43. Ducatur haec series in $-\partial u$, et integratio praebebit

$$P^{IV}=\frac{\cos.u}{1^4}+\frac{\cos.2u}{2^4}+\frac{\cos.3u}{3^4}+\frac{\cos.4u}{4^4}+\text{etc.}$$
$$=D-\frac{\pi\pi uu}{12}+\frac{\pi u^3}{12}+\frac{u^4}{48},$$

hinc sumto $u=0$ fiet

$$\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\frac{1}{5^4}+\text{etc.}=D,$$

nunc vero fiat etiam $u=\pi$, fietque

$$-\frac{1}{1^4}+\frac{1}{2^4}-\frac{1}{3^4}+\frac{1}{4^4}-\frac{1}{5^4}+\text{etc.}=D-\frac{\pi^4}{48},$$

hae autem ambae series additae dant

$$\frac{2}{2^4}+\frac{2}{4^4}+\frac{2}{6^4}+\frac{2}{8^4}+\text{etc.}=2D-\frac{\pi^4}{48},$$

quae octies sumta ut numeratores fiant $=2^4$, praebebit

$$\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\text{etc.}=16D-\frac{\pi^4}{6},$$

unde oritur $D=\frac{\pi^4}{96}$, quae est eadem summa seriei

$$\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\text{etc.}$$

quam jam dudum inveneram, habebimus jam

$$P'''' = \frac{\cos. u}{1^4}+\frac{\cos. 2u}{2^4}+\frac{\cos. 3u}{3^4}+\frac{\cos. 4u}{4^4}+\text{etc.}$$
$$=\frac{\pi^4}{90}-\frac{\pi^2 u^2}{12}+\frac{\pi u^3}{12}-\frac{u^4}{48}.$$

§. 44. Multiplicando iterum per ∂u et integrando consequimur

$$Q^{V}=\frac{\sin. u}{1^5}+\frac{\sin. 2u}{2^5}+\frac{\sin. 3u}{3^5}+\frac{\sin. 4u}{4^5}+\text{etc.}$$
$$=E+\frac{\pi^4 u}{90}-\frac{\pi^2 u^3}{36}+\frac{\pi u^4}{48}-\frac{u^5}{240};$$

ubi uti in casu penultimo constans E iterum fit $=0$, ita ut habeamus

$$Q^{V}=\frac{\sin. u}{1^5}+\frac{\sin. 2u}{2^5}+\frac{\sin. 3u}{3^5}+\frac{\sin. 4u}{4^5}+\text{etc.}$$
$$=\frac{\pi^4 u}{90}-\frac{\pi^2 u^3}{36}+\frac{\pi u^4}{48}-\frac{u^5}{240}.$$

§. 45. Multiplicemus denuo per $-\partial u$, prodibitque integrando

$$P^{VI}=\frac{\cos. u}{1^6}+\frac{\cos. 2u}{2^6}+\frac{\cos. 3u}{3^6}+\frac{\cos. 4u}{4^6}+\text{etc.}$$
$$=F-\frac{\pi^4}{90}\cdot\frac{uu}{2}+\frac{\pi\pi}{6}\cdot\frac{u^4}{24}-\frac{\pi}{2}\cdot\frac{u^5}{120}+\frac{1}{2}\cdot\frac{u^6}{720};$$

ubi ad constantem determinandam ponatur $u=0$, eritque

$$\frac{1}{1^6}+\frac{1}{2^6}+\frac{1}{3^6}+\frac{1}{4^6}+\text{etc.}=F,$$

tum vero sumatur $u=\pi$, et fiet

$$-\frac{1}{1^6}+\frac{1}{2^6}-\frac{1}{3^6}+\frac{1}{4^6}-\text{etc.}=F-\frac{\pi^6}{480},$$

quae additae dant

$$\frac{2}{2^6}+\frac{2}{4^6}+\frac{2}{6^6}+\frac{2}{8^6}+\text{etc.}=2F-\frac{\pi^6}{480},$$

quae multiplicetur per 32, ut omnes numeratores fiant $64=2^6$, et orietur

$$\frac{1}{1^6}+\frac{1}{2^6}+\frac{1}{3^6}+\frac{1}{4^6}+\text{etc.}=64F-\frac{\pi^6}{15}=F;$$

unde colligitur $F=\frac{\pi^6}{945}$, ita ut sit

$$P^{VI}=\frac{\cos.u}{1^6}+\frac{\cos.2u}{2^6}+\frac{\cos.3u}{3^6}+\frac{\cos.4u}{4^6}+\text{etc.}$$
$$=\frac{\pi^6}{945}-\frac{\pi^4}{90}\cdot\frac{u^2}{2}+\frac{\pi^2}{6}\cdot\frac{u^4}{24}-\frac{\pi}{2}\cdot\frac{u^5}{120}+\frac{1}{2}\cdot\frac{u^6}{720}.$$

§. 46. Has series ulterius continuare superfluum foret, cum lex progressionis jam satis sit manifesta, praecipue si in subsidium vocentur summationes potestatum reciprocarum parium, quas olim usque ad potestatem trigesimam supputatas dedi. Quod quo clarius perspiciatur, istas summas sequenti modo repraesentemus

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\frac{1}{6^2}+\text{etc.}=\alpha\pi\pi,\ \text{ut sit}\ \alpha=\frac{1}{8}$$
$$\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\frac{1}{5^4}+\frac{1}{6^4}+\text{etc.}=\beta\pi^4,\ \text{ut sit}\ \beta=\frac{1}{90}$$
$$\frac{1}{1^6}+\frac{1}{2^6}+\frac{1}{3^6}+\frac{1}{4^6}+\frac{1}{5^6}+\frac{1}{6^6}+\text{etc.}=\gamma\pi^6,\ \text{ut sit}\ \gamma=\frac{1}{945}$$
$$\frac{1}{1^8}+\frac{1}{2^8}+\frac{1}{3^8}+\frac{1}{4^8}+\frac{1}{5^8}+\frac{1}{6^8}+\text{etc.}=\delta\pi^8,\ \text{ut sit}\ \delta=\frac{1}{9450}$$
etc. etc. etc.

atque his positis, sequentes habebimus integrationes, pro casu scilicet $z=1$,

$$Q'=+\int\frac{\partial z\,\sin.u}{1-2z\cos.u+zz}=\tfrac{1}{2}\pi-\tfrac{1}{2}u=\text{Arc. tang.}\frac{\sin.u}{1-\cos.u}$$
$$P''=-\int\frac{\partial z(\cos.u-z)}{1-2z\cos.u+zz}\cdot\frac{lz}{1}=\alpha\pi\pi-\tfrac{1}{2}\pi u+\tfrac{1}{2}\cdot\frac{uu}{2}$$
$$Q'''=+\int\frac{\partial z\,\sin.u}{1-2z\cos.u+zz}\cdot\frac{(lz)^2}{2}=\alpha\pi\pi\frac{u}{1}-\tfrac{1}{2}\pi\cdot\frac{uu}{2}+\tfrac{1}{2}\cdot\frac{u^3}{6}$$
$$P^{IV}=-\int\frac{\partial z(\cos.u-z)}{1-2z\cos.u+zz}\cdot\frac{(lz)^3}{6}=\beta\pi^4-\alpha\pi\pi\cdot\frac{uu}{2}+\tfrac{1}{2}\pi\cdot\frac{u^3}{6}-\tfrac{1}{2}\cdot\frac{u^4}{24}$$

$$Q^{V} = +\int \frac{\partial z \sin. u}{1 - 2z \cos. u + zz} \cdot \frac{(lz)^4}{24} = \beta\pi^4 . \frac{u}{1} - \alpha\pi\pi \cdot \frac{u^3}{6} + \frac{1}{2}\pi \cdot \frac{u^4}{24} - \frac{1}{2} \cdot \frac{u^5}{120}$$

$$P^{VI} = -\int \frac{\partial z (\cos. u - z)}{1 - 2z \cos. u + zz} \cdot \frac{(lz)^5}{120} = \gamma\pi^6 - \beta\pi^4 \cdot \frac{uu}{2} + \alpha\pi\pi \cdot \frac{u^4}{24} - \frac{1}{2}\pi \cdot \frac{u^5}{120} + \frac{1}{2} \cdot \frac{u^6}{720}$$

$$Q^{VII} = +\int \frac{\partial z \sin. u}{1 - 2z \cos. u + zz} \cdot \frac{(lz)^6}{720} = \gamma\pi^6 . \frac{u}{1} - \beta\pi^4 \cdot \frac{u^3}{6} + \alpha\pi\pi \cdot \frac{u^5}{120} - \frac{1}{2}\pi \cdot \frac{u^6}{720} + \frac{1}{2} \cdot \frac{u^7}{5040}$$

etc. etc. etc.

§. 47. Operae pretium erit, aliquos casus, quibus angulo u datus valor tribuitur, ob oculos exponere. Ponamus igitur $u = 0$, quo casu formulae nostrae alternatim evanescunt, reliquae vero praebebunt

$$-\int \frac{\partial z}{1 - z}\, lz = \alpha\pi\pi = \frac{\pi\pi}{6}$$

$$-\int \frac{\partial z}{1 - z} \cdot \frac{(lz)^3}{6} = \beta\pi^4 = \frac{\pi^4}{90}$$

$$-\int \frac{\partial z}{1 - z} \cdot \frac{(lz)^5}{120} = \gamma\pi^6 = \frac{\pi^6}{945}$$

his affines sunt formulae, quae oriuntur ex positione $u = \pi$, ubi iterum abeunt alternae sinum u involventes, et remanebunt sequentes

$$\int \frac{\partial z}{1 + z}\, lz = -\frac{\pi\pi}{12} = -\tfrac{1}{2}\alpha\pi\pi$$

$$\int \frac{\partial z}{1 + z} \cdot \frac{(lz)^3}{6} = -\frac{7\pi^4}{720} = -\tfrac{7}{8}\beta\pi^6$$

$$\int \frac{\partial z}{1 + z} \cdot \frac{(lz)^5}{120} = -\tfrac{31}{32}\gamma\pi^6$$

$$\int \frac{\partial z}{1 + z} \cdot \frac{(lz)^7}{720} = -\tfrac{127}{128}\delta\pi^8.$$

§. 48. Hic notatu dignum occurrit, quod valores alterni, quos hic omisimus, etiam evanescant posito $u = \pi$; deinde non minus notatu dignum est, easdem formulas quoque evanescere posito $u = 2\pi$, sola prima excepta, quippe quae etiam non evanescit posito $u = 0$; reliquae vero, scilicet tertia, quinta, septima etc. certe evanescunt casibus $u = 0$ et $u = \pi$, quin etiam $u = 2\pi$. Quod quo clarius appareat, has formulas per factores repraesentemus, eritque tertiae valor

$$= \tfrac{1}{12} u\, (\pi - u)\, (2\pi - u),$$

quintae vero valor reperitur

$$\tfrac{u}{720} (\pi - u)\, (2\pi - u)\, (4\pi\pi + 6\pi u - 3uu),$$

quod etiam in sequentibus usu venit. In genere autem observari meretur, omnes nostras formulas sola prima excepta eosdem sortiri valores, sive ponatur $u = 0$ sine $u = 2\pi$, quippe quibus tam idem sinus quam cosinus respondet. Videtur quidem eundem consensum locum habere debere, si ponatur $u = 4\pi$ et $u = 6\pi$, verum Illustr. *Bernoullius* jam luculenter ostendit, angulum u in his valoribus non ultra quatuor rectos augeri posse. Hujusmodi autem anomalia etiam in omnibus vulgaribus seriebus quibus arcus exprimuntur occurrit; atque adeo in *Leibniziana*, inqua est

$$u = \frac{tang.\, u}{1} - \frac{(tang.\, u)^3}{3} + \frac{(tang.\, u)^5}{5} - \frac{(tang.\, u)^7}{7} + \frac{(tang.\, u)^9}{9} - \text{etc.}$$

angulum u non ultra 180 gr. augere licet. Si enim poneremus $u = 180^\circ + u$, foret utique tang. u = tang. u, neque tamen series illa exprimeret arcum $\pi + u$ sed tantum arcum u, cujusmodi phaenomena etiam in aliis similibus seriebus locum habent. Quod autem prima series hinc plerumque excipi debeat, ratio in eo est sita, quod in formula integrali posito $u = 0$ denominator fiat $(1 - z)$, qui casu $z = 1$ evanescit, ideoque formula in infinitum excrescit, id quod in sequentibus, quae per lz sunt multiplicatae, non amplius evenit, quia $\frac{lz}{1-z}$ casu $z = 1$ non amplius fit infinitus sed tantum $= -1$, et si major potestas logarithmi adsit, fit adeo $= 0$.

§. 49. Ponamus nunc etiam $u = 90^\circ$, seu $u = \frac{\pi}{2}$, ut sit cos. $u = 0$ et sin. $u = 1$, hocque casu omnes formulae generales sequentes obtinebunt valores

$$\int \frac{\partial z}{1+zz} = \frac{\pi}{4}$$

$$\int \frac{z\partial z}{1+zz} lz = -\frac{\pi\pi}{48}$$
$$\int \frac{\partial z}{1+zz} \cdot \frac{(lz)^2}{2} = \frac{\pi^3}{32}$$
$$\int \frac{z\partial z}{1+zz} \cdot \frac{(lz)^3}{6} = -\frac{7\pi^4}{90\cdot 128}.$$
etc.

§. 50. Consideremus etiam casum $u = 60^\circ$, sive $u = \frac{\pi}{3}$, ut sit cos. $u = \frac{1}{2}$ et sin. $u = \frac{\sqrt{3}}{2}$, et formulae generales perducent ad sequentia integralia

$$\frac{\sqrt{3}}{2}\int \frac{\partial z}{1-z+zz} = \frac{\pi}{3}$$
$$\frac{1}{2}\int \frac{\partial z\,(1-2z)}{1-z+zz} lz = -\frac{\pi\pi}{36}$$
$$\frac{\sqrt{3}}{2}\int \frac{\partial z}{1-z+zz} \cdot \frac{(lz)^2}{2} = \frac{5\pi^3}{162}.$$

Simili modo si ponamus $u = 120^\circ = \frac{2\pi}{3}$, ut sit cos. $u = -\frac{1}{2}$ et sin. $u = \frac{\sqrt{3}}{2}$, sequentes integrationes istis affines prodibunt

$$\frac{\sqrt{3}}{2}\int \frac{\partial z}{1+z+zz} = \frac{\pi}{6}$$
$$\frac{1}{2}\int \frac{\partial z\,(1+2z)}{1+z+zz} lz = -\frac{\pi\pi}{18}$$
$$\frac{\sqrt{3}}{2}\int \frac{\partial z}{1+z+zz} \cdot \frac{(lz)^2}{2} = \frac{2\pi^3}{81},$$

sicque pro lubitu numerus hujusmodi integrationum specialium augeri poterit.

§. 51. Quemadmodum istae integrationes memorabiles ex priore serie nostra P posito $z = 1$ sunt deductae, ita eodem modo alteram seriem Q pertractemus. Cum igitur sit

$$Q = \sin. u + \sin. 2u + \sin. 3u + \sin. 4u + \text{etc.} = \tfrac{1}{2}\cot. \tfrac{1}{2}u,$$

si per $-\partial u$ multiplicemus et integremus, reperitur series

$$P' = \frac{\cos.u}{1} + \frac{\cos.2u}{2} + \frac{\cos.3u}{3} + \frac{\cos.4u}{4} + \text{etc.} = \tfrac{1}{2} - l\sin.\tfrac{1}{2}u + A,$$

pro qua constante determinanda ponatur $u = \pi$, ut sit

$$-1 + \tfrac{1}{2} - \tfrac{1}{3} + \tfrac{1}{4} - \tfrac{1}{5} + \tfrac{1}{6} - \text{etc.} = A,$$

quocirca fit $A = -l2$, ita ut habeamus

$$P' = \frac{\cos. u}{1} + \frac{\cos. 2u}{2} + \frac{\cos. 3u}{3} + \frac{\cos. 4u}{4} + \text{etc.} = -l2 \sin. \tfrac{1}{2}u,$$

pro quo valore scribamus brevitatis gratia $\Delta : u$, si quidem eum spectamus tanquam certam ipsius u functionem, ita ut sit $P' = \Delta : u$.

§. 52. Multiplicando porro per ∂u et integrando, nanciscimur hanc seriem

$$Q'' = \frac{\sin. u}{1^2} + \frac{\sin. 2u}{2^2} + \frac{\sin. 3u}{3^2} + \frac{\sin. 4u}{4^2} + \text{etc.} = \int \partial u \Delta : u = \Delta' : u;$$

ubi haec formula integralis involvet certam constantem, quam facile definire licet ex casu $u = 0$, quia enim series evanescit, fieri debet $\Delta' : 0 = 0$, sicque integratio plene determinatur.

§. 53. Si eodem modo ulterius progrediamur, multiplicando per $-\partial u$, prodibit haec series

$$P''' = \frac{\cos. u}{1^3} + \frac{\cos. 2u}{2^3} + \frac{\cos. 3u}{3^3} + \frac{\cos. 4u}{4^3} + \text{etc.} = -\int \partial u \Delta' : u = \Delta'' : u.$$

Jam ad constantem, quae in hac expressione continetur, definiendam, sit 1^0 $u = 0$, eritque

$$\frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \frac{1}{5^3} + \text{etc.} = \Delta'' : 0.$$

Sit 2^0. $u = \pi$, et fiet

$$-\frac{1}{1^3} + \frac{1}{2^3} - \frac{1}{3^3} + \frac{1}{4^3} - \frac{1}{5^3} + \text{etc.} = \Delta'' : \pi,$$

quibus additis prodit

$$\frac{2}{2^3} + \frac{2}{4^3} + \frac{2}{6^3} + \frac{2}{8^3} + \text{etc.} = \Delta'' 0 + \Delta'' : \pi,$$

hacque quatuor sumta erit

$$\frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \text{etc.} = 4\Delta'' : 0 + 4\Delta'' : \pi = \Delta'' : 0,$$

unde oritur

$$3\,\Delta'' : 0 + 4\Delta'' : \pi = 0;$$

ex qua constans in formulam nostram integralem

$$\Delta'': u = -\int \partial u \Delta' u$$

ingressa determinari debet.

§. 54. Multiplicemus denuo per ∂u, et integremus, prodibitque

$$Q^{IV} = \frac{sin.\, u}{1^4} + \frac{sin.\, 2u}{2^4} + \frac{sin.\, 3u}{3^4} + \frac{sin.\, 4u}{4^4} + \text{etc.} = \int \partial u \Delta'': u = \Delta''': u,$$

atque haec functio $\Delta''': u$ ita debet determinari, ut evanescat sumto $u = 0$, sive ut fiat $\Delta''': 0 = 0$. Eodem modo ulterius progrediendo fiet

$$P^{V} = \frac{cos.\, u}{1^5} + \frac{cos.\, 2u}{2^5} + \frac{cos.\, 3u}{3^5} + \frac{cos.\, 4u}{4^5} + \text{etc.} = -\int \partial u \Delta''': u = \Delta'''': u,$$

hujusque functionis indoles sequenti modo determinabitur: ponatur scilicet ut hactenus $u = 0$, et $u = \pi$, eritque

$$\frac{1}{1^5} + \frac{1}{2^5} + \frac{1}{3^5} + \frac{1}{4^5} + \frac{1}{5^5} + \text{etc.} = \Delta^{IV}: 0, \text{ et}$$

$$-\frac{1}{1^5} + \frac{1}{2^5} - \frac{1}{3^5} + \frac{1}{4^5} - \frac{1}{5^5} + \text{etc.} = \Delta^{IV}: \pi;$$

hinc addendo

$$\frac{2}{2^5} + \frac{2}{4^5} + \frac{2}{6^5} + \frac{2}{8^5} + \text{etc.} = \Delta^{IV}: 0 + \Delta^{IV}: \pi,$$

et multiplicando per 16

$$\frac{1}{1^5} + \frac{1}{2^5} + \frac{1}{3^5} + \frac{1}{4^5} + \text{etc.} = 16\Delta^{IV}: 0 + 16\Delta^{IV}: \pi = \Delta^{IV}: 0,$$

sicque fieri debebit

$$15\Delta^{IV}: 0 + 16\Delta^{IV} \pi = 0 \text{ etc.}$$

§. 55. Hinc igitur sequentes adipiscemur integrationes pro casu $z = 1$

$$\text{I. } -\int \frac{\partial z\,(cos.\, u - z)}{1 - 2z\, cos.\, u + zz} = -\, l2\, \sin.\, \tfrac{1}{2} u = \Delta : u$$

$$\text{II. } \int \frac{\partial z\, sin.\, u}{1 - 2z\, cos.\, u + zz}\, lz = \int \partial u \Delta u = \Delta' : u$$

$$\text{III. } -\int \frac{\partial z\,(cos.\, u - z)}{1 - 2z\, cos.\, u + zz} \cdot \frac{(lz)^2}{2} = -\int \partial u \Delta' u = \Delta'': u$$

$$\text{IV.}\quad \int \frac{\partial z \sin. u}{1 - 2z\cos. u + zz} \cdot \frac{(lz)^3}{6} = \int \partial u \Delta'' u = \Delta''' u$$

$$\text{V.} - \int \frac{\partial z(\cos. u - z)}{1 - 2z\cos. u + zz} \cdot \frac{(lz)^4}{24} = - \int \partial u \Delta''' u = \Delta^{IV} u$$

$$\text{VI.}\quad \int \frac{\partial z \sin. u}{1 - 2z\cos. u + zz} \cdot \frac{(lz)^5}{120} = \int \partial u \Delta^{IV} u = \Delta^{V} u$$

etc. etc. etc. etc.

Has autem expressiones facile quousque libuerit continuare licet, si modo integratio cujusque integralis rite instituatur; conditiones autem, quas impleri oporteat, sequenti modo referri possunt

$\Delta' : 0 = 0$	$3\Delta'' : 0 + 4\Delta'' : \pi = 0$
$\Delta''' : 0 = 0$	$15\Delta^{IV} : 0 + 16\Delta^{IV} : \pi = 0$
$\Delta^{V} : 0 = 0$	$63\Delta^{VI} : 0 + 64\Delta^{VI} : \pi = 0$
$\Delta^{VII} : 0 = 0$	$255\Delta^{VIII} : 0 + 256\Delta^{VIII} : \pi = 0$
etc. etc.	etc. etc.

caeterum quia posteriores integrationes absolvere non licet, hinc parum utilitatis exspectare possumus.

§. 56. Caeterum methodus, qua hic sumus usi, ad constantes per quamque integrationem ingressas determinandas, a celeberrimo *Bernoullio* primum est adhibita, atque eo majori attentione digna est aestimanda, quod ejus ope summationes meae serierum reciprocarum potestatum obtineri possunt, quandoquidem credideram, eas non aliter nisi ex consideratione infinitorum arcuum, qui vel eodem sinu vel cosinu gaudent, demonstrari posse.

2). Comparatio valorum formulae integralis

$$\int \frac{x^{p-1}\,\partial x}{\sqrt[n]{(1-x^n)^{n-q}}},$$

a termino $x = 0$ usque ad $x = 1$ extensae. *Nova Acta Acad. Imp. Scient. Petropolitanae. Tom. V. Pag.* 86 — 117.

§. 57. In hac formula litterae n, p et q perpetuo designant numeros integros positivos, et pro quolibet numero n binis litteris p et q omnes valores tribui concipiuntur, ita ut hinc pro quovis numero n innumerae nascantur hujusmodi formulae integrales, quarum valores plurimas egregias relationes inter se servant; unde si eorum aliquot fuerint cogniti, reliquae omnes ex iis definiri queant. Jam dudum equidem plures hujusmodi relationes demonstravi; cum autem hoc argumentum tum temporis neutiquam exhausissem, nunc accuratius in istas relationes inquirere constitui, et ejusmodi methodum adhibebo, quae omnes plane hujus generis relationes sit exhibitura; his enim inventis innumerabilia theoremata condi poterunt, quibus universa analysis non mediocriter locupletari erit censenda.

§. 58. Quoniam igitur hoc modo pro quolibet numero n ambae litterae p et q infinitos valores recipere possunt, ante omnia hic observari convenit, omnes hos innumerabiles casus semper ad numerum finitum revocari posse. Quantumvis enim magni numeri pro litteris p et q accipiantur, eos casus semper ad alios reducere licet, in quibus numeri p et q quantitate n futuri sint diminuti. Hoc igitur modo omnes hujusmodi casus tandem eo redigi poterunt, ut ambo numeri p et q infra exponentem n deprimantur; unde pro quolibet numero n eos tantum casus considerasse sufficiet, quibus litterae p et q minores valores recipiant quam n, vel saltem hunc limitem non superent. Hoc igitur modo pro quovis numero n multitudo casuum, qui in computum veniunt, et quos inter se comparari oportet, prorsus erit determinata.

§. 59. Quemadmodum autem ista reductio litterarum p et q ad numeros continuo minores institui debeat, quamquam id satis in vulgus est notum, tamen ad formulam praesentem accommodasse juvabit. Statuatur scilicet haec formula algebraica $x^p (1-x^n)^{\frac{q}{n}} = V$, eritque

$$lV = plx + \tfrac{q}{n}\, l(1-x^n),$$

hinc differentiando

$$\frac{\partial V}{V} = \frac{p\,\partial x}{x} - \frac{q x^{n-1}\partial x}{1-x^n} = \frac{p\,\partial x - (p+q)\,x^n\,\partial x}{x(1-x^n)},$$

ubi si per V multiplicemus, ac per partes integremus, orietur ista aequatio

$$V = p\int x^{p-1}\,\partial x\,(1-x^n)^{\frac{q-n}{n}}$$
$$-(p+q)\int x^{p+n-1}\,\partial x\,(1-a^n)^{\frac{q-n}{n}}.$$

Quoniam igitur quantitas V pro utroque integrationis termino evanescit, hinc adipiscimur istam reductionem

$$\int x^{p+n-1}\,\partial x\,(1-x^n)^{\frac{q-n}{n}} = \frac{p}{p+q}\int x^{p-1}\,\partial x\,(1-x^n)^{\frac{q-n}{n}},$$

cujus ergo reductionis ope exponens ipsius x continuo quantitate n diminui poterit, donec tandem infra n deprimatur.

§. 60. Deinde formula pro

$$\frac{\partial V}{V} = \frac{p\,\partial x - (p+q)\,x^n\,\partial x}{x(1-x^n)}$$

inventa hoc modo referri poterit

$$\frac{\partial V}{V} = \frac{(p+q)\,\partial x\,(1-x^n) - q\,\partial x}{x\,(1-x^n)},$$

quae forma per V multiplicata ac denuo per partes integrata dabit

$$V = (p+q)\int x^{p-1}\,\partial x\,(1-x^n)^{\frac{q}{n}} - q\int x^{p-1}\,\partial x\,(1-x^n)^{\frac{q-n}{n}},$$

unde quia posito $x = 1$ fit $V = 0$, oritur haec reductio

$$\int x^{p-1}\,\partial x\,(1-x^n)^{\frac{q}{n}} = \frac{q}{p+q}\int x^{p-1}\,\partial x\,(1-x^n)^{\frac{q-n}{n}},$$

cujus reductionis ope exponens Binomii $1 - x^n$ unitate minuitur, sive quod eodem redit, numerus q numero n imminuitur. Tali igitur reductione, quoties opus fuerit, repetita, exponens q tandem infra n deprimi poterit.

§. 61. Quoniam igitur pro quovis numero n ambos exponentes p et q tanquam minores quam n spectare licet, formulam propositam hoc modo expresssam repraesentemus

$$\int \frac{x^{p-1}\,\partial x}{\sqrt[n]{(1-x^n)^{n-q}}}.$$

Hic scilicet pro quovis numero n sufficiet litteris p et q omnes valores ipso n minores tribuisse, quo pacto multitudo omnium casuum ad quemlibet exponentem n pertinentium ad numerum satis modicum reducetur, qui tamen eo major evadit, quo major fuerit exponens n.

§. 62. Multo magis autem numerus casuum diversorum diminuetur, si perpendamus, ambas litteras p et q inter se permutari posse, ita ut hujus formulae

$$\frac{x^{q-1}\,\partial x}{\sqrt[n]{(1-x^n)^{n-p}}}$$

valor ab illo prorsus non discrepet. Ad quod ostendendum ponamus

$$\int \frac{x^{p-1}\partial x}{\sqrt[n]{(1-x^n)^{n-q}}} = S,$$

si scilicet ista formula integralis ab $x = 0$ usque ad $x = 1$ extendatur. Jam faciamus $1 - x^n = y^n$, ut formula sit

$$S = \int \frac{x^{p-1}\partial x}{y^{n-q}};$$

tum vero quia $x^n = 1 - y^n$, erit $x = (1 - y^n)^{\frac{1}{n}}$, hincque $x^p = (1 - y^n)^{\frac{p}{n}}$, unde differentiando fit

$$px^{p-1}\partial x = - py^{n-1}\partial y\,(1 - y^n)^{\frac{p-n}{n}},$$

quo valore substituto erit

$$S = - \int y^{q-1}\partial y\,(1 - y^n)^{\frac{p-n}{n}},$$

quam formulam ab $x = 0$ usque ad $x = 1$, hoc est ab $y = 1$ usque ad $y = 0$, extendi oportet; permutatis igitur his terminis erit

$$S = \int \frac{y^{q-1}\partial y}{\sqrt[n]{(1-y^n)^{n-p}}} \begin{bmatrix}\text{ab } y = 0\\ \text{ad } y = 1\end{bmatrix}.$$

Sicque demonstratum est ambas litteras p et q semper inter se esse permutabiles.

§. 63. His praemissis, quo calculos sequentes magis in compendium redigere liceat, loco formulae hujus integralis

$$\int \frac{x^{p-1}\partial x}{\sqrt[n]{(1-x^n)^{n-q}}} = \int \frac{x^{q-1}\partial x}{\sqrt[n]{(1-x^n)^{n-p}}}$$

scribamus hunc characterem (p, q), ubi perinde est, sive p ante q, sive q ante p collocetur; semper autem hic certus exponens n subintelligi debet. Hic autem duo casus prae reliquis maxime memorabiles occurrunt. Prior casus est, quo numerorum p et q alteruter ipsi exponenti n est aequalis; si enim fuerit $q = n$, erit ex priore formula $(p, n) = \int x^{p-1}\partial x = \frac{1}{p}$, sicque perpetuo habebimus $(p, n) = \frac{1}{p}$, hincque etiam $(n, q) = \frac{1}{q}$. Alter casus notatu dignissimus locum habet, quando $p + q = n$, quo casu semper est

$$(p, q) = \frac{\pi}{n \sin. \frac{p\pi}{n}} = \frac{\pi}{n \sin. \frac{q\pi}{n}}.$$

Ad hoc ostendendum sit $q = n - p$, hincque formula proposita $\int \frac{x^{p-1}\partial x}{\sqrt[n]{(1-x^n)^p}}$, tum ponatur $\frac{x}{\sqrt[n]{(1-x^n)}} = z$, et quia $\frac{x^p}{\sqrt[n]{(1-x^n)^p}} = z^p$, erit $S = \int \frac{z^p \partial x}{x}$. Ex facta autem positione sequitur $x^n = \frac{z^n}{1+z^n}$, hincque

$$n\,l\,x = n\,l\,z - l\,(1+z^n),$$

ergo differentiando

$$\frac{\partial x}{x} = \frac{\partial z}{z} - \frac{z^{n-1}\partial z}{1+z^n} = \frac{\partial z}{z\,(1+z^n)},$$

ita ut jam sit

$$S = \int \frac{z^{p-1}\partial z}{1+z^n}.$$

Quia autem sumto $x = 0$ fit etiam $z = 0$, at vero sumto $x = 1$ prodit $z = \infty$, hoc integrale a termino $z = 0$ usque ad $z = \infty$ extendi debet. Notum autem est valorem hoc modo resultantem esse $\frac{\pi}{n \sin. \frac{\pi p}{n}}$.

§. 64. Progrediamur nunc ad ipsum fundamentum, unde omnes relationes, quas quaerimus, derivari convenit, et quod reductioni priori innititur; unde fit

$$\int \frac{x^{p-1}\,\partial x}{\sqrt[n]{(1-x^n)^{n-q}}} = \frac{p+q}{p} \cdot \int \frac{x^{n+p-1}\,\partial x}{\sqrt[n]{(1-x^n)^{n-q}}},$$

ubi loco $\sqrt[n]{(1-x^n)^{n-q}}$ scribamus X, ut sit

$$\int \frac{x^{p-1}\,\partial x}{X} = \frac{p+q}{p} \cdot \int \frac{x^{n+p-1}\,\partial x}{X}:$$

hinc jam simili modo, si loco p scribamus $n+p$, erit

$$\int \frac{x^{n+p-1}\,\partial x}{X} = \frac{n+p+q}{n+p} \cdot \int \frac{x^{2n+p-1}\,\partial x}{X},$$

hincque sequitur fore

$$\int \frac{x^{p-1}\,\partial x}{X} = \frac{p+q}{p} \cdot \frac{n+p+q}{n+p} \int \frac{x^{2n+p-1}\,\partial x}{X}.$$

Quodsi simili modo ulterius progrediamur, perveniemus ad hanc aequationem

$$\int \frac{x^{p-1}\,\partial x}{X} = \frac{p+q}{p} \cdot \frac{n+p+q}{n+p} \cdot \frac{2n+p+q}{2n+p} \int \frac{x^{3n+p-1}\,\partial x}{X}.$$

Quare si hoc modo in infinitum progrediamur, habebimus

$$\int \frac{x^{p-1}\,\partial x}{X} = \frac{p+q}{p} \cdot \frac{n+p+q}{n+p} \cdot \frac{2n+p+q}{2n+p} \times \ldots \times \frac{in+p+q}{in+p} \int \frac{x^{(i+1)n+p-1}\,\partial x}{X},$$

ubi i denotat numerum infinite magnum.

§. 65. Quodsi jam loco p alium quemcunque numerum r, pariter ipso n minorem, assumamus, erit simili modo

$$\int \frac{x^{r-1}\,\partial x}{X} = \frac{r+q}{r} \cdot \frac{n+r+q}{n+r} \cdot \frac{2n+r+q}{2n+r} \times$$
$$\times \ldots \ldots \frac{in+r+q}{in+r} \int \frac{x^{(i+1)n+r-1}\,\partial x}{X},$$

ubi littera i eundem numerum infinitum designat, ita ut utrinque idem factorum numerus adsit. Dividamus jam priorem expressionem per istam, et quoniam extremae formulae integrales, ob litteras p et r prae $(i+1)\,n$ evanescentes, pro aequalibus inter se sunt habendae, facta divisione per singulos factores reperiemus hanc aequationem

$$\frac{\int x^{p-1}\,\partial x : X}{\int x^{r-1}\,\partial x : X} = \frac{r\,(p+q)}{p\,(r+q)} \cdot \frac{(n+r)\,(n+p+q)}{(n+p)\,(n+r+q)} \times$$
$$\times \frac{(2n+r)\,(2n+p+q)}{(2n+p)\,(2n+r+q)} \cdot \frac{(3n+r)\,(3n+p+q)}{(3n+p)\,(3n+r+q)} \times \text{etc.}$$

Restituamus jam loco harum formularum integralium characteres ante stabilitos, atque adipiscemur istam relationem notatu dignissimam

$$\frac{(p,q)}{(r,q)} = \frac{r\,(p+q)}{p\,(r+q)} \cdot \frac{(n+r)\,(n+p+q)}{(n+p)\,(n+r+q)} \cdot \frac{(2n+r)\,(2n+p+q)}{(2n+p)\,(2n+r+q)} \cdot \text{etc.}$$

quod productum ex infinitis membris componitur, quorum singula sunt fractiones, quarum tam numeratores quam denominatores ex binis factoribus constant. Hos factores singulos eodem numero n augeri oportet, dum a quovis membro ad sequens progredimur, unde sufficiet solum primum productum nosse, quod ergo ita repraesentabimus

$$\frac{(p,q)}{(r,q)} = \frac{r\,(p+q)}{p\,(r+q)} \text{ etc.}$$

§. 66. Quoniam litterae p et q nobis numeros quasi indefinitos significant, utamur litteris alphabeti initialibus ad numeros determinatos designandos, eritque eodem modo

$$\frac{(\alpha, b)}{(a, b)} = \frac{\alpha(\alpha+b)}{\alpha(\alpha+b)} \cdot \frac{(n+\alpha)(n+\alpha+b)}{(n+\alpha)(n+\alpha+b)} \text{ etc.}$$

Hic jam loco α scribamus $a + c$, et productum infinitum hanc induet formam

$$\frac{(a, b)}{(a+c, b)} = \frac{(a+c)(a+b)}{a(a+c+b)} \cdot \frac{(n+a+c)(n+a+b)}{(n+a)(n+a+c+b)} \cdot \text{etc.}$$

in quo producto ambae litterae b et c manifesto permutari possunt, unde idem productum infinitum etiam exprimet valorem hujus formae $\frac{(a, c)}{(a+b, c)}$, unde sequitur ista aequalitas maxime memorabilis $\frac{(a, b)}{(a+c, b)} = \frac{(a, c)}{(a+b, c)}$; fractionibus igitur sublatis habebimus istud insigne theorema

$$(a, b)\ (a + b, c) = (a, c)\ (a + c, b);$$

huicque theorematí universa analysis, qua utemur, erit superstructa.

§. 67. Cum ob rationes supra allegatas numeri p et q exponentem n superare non debeant, etiam in forma theorematis modo allati singuli termini ibi occurrentes, qui sunt a, b, c, $a + b$ et $a + c$, quovis casu exponentem n superare non debent, sicque nec $a + b$, neque $a + c$ major capi poterit quam n. Hic autem primo observo litteras b et c, inter se inaequales statui debere: si enim esset $c = b$, aequalitas in theoremate expressa foret identica; hanc ob rem perpetuo assumemus $b > c$, ita ut maximus terminus in theoremate sit $a + b$, quem ergo exponentem n quovis casu excedere non oportet, quamobrem evolutionem formae generalis in theoremate contentae ita in classes distribuamus, quae inter se per maximum valorem termini $a + b$ distinguantur. Cum igitur nulla litterarum a, b, c nihilo aequalis sumi queat, ac esse debeat $b > c$, minimus valor, quem terminus $a+b$ recipere potest, erit 3, in quo ergo primam classem constituemus; sequentes vero classes constituentur, dum termino $a+b$ valores 4, 5, 6, 7, etc. tribuantur.

I. Evolutio classis qua $a+b=3$.

§. 68. Hic ergo necessario erit $a=1$, $b=2$ et $c=1$, ita ut hic nulla varietas locum inveniat, unde theorema nostrum suppeditat hanc unicam relationem $(1, 2)(3, 1)=(1, 1)(2, 2)$. Dummodo igitur exponens n non fuerit minor quam 3, semper haec insignis relatio locum habet

$$\int\frac{\partial x}{\sqrt[n]{(1-x^n)^{n-2}}}\cdot\int\frac{xx\partial x}{\sqrt[n]{(1-x^n)^{n-1}}}=\int\frac{\partial x}{\sqrt[n]{(1-x^n)^{n-1}}}\cdot\int\frac{x\partial x}{\sqrt[n]{(1-x^n)^{n-2}}},$$

quae forma, quia in quolibet charactere terminos inter se permutare licet, etiam hoc modo repraesentari poterit

$$\int\frac{x\partial x}{\sqrt[n]{(1-x^n)^{n-1}}}\cdot\int\frac{\partial x}{\sqrt[n]{(1-x^n)^{n-3}}}=\int\frac{\partial x}{\sqrt[n]{(1-x^n)^{n-1}}}\cdot\int\frac{x\partial x}{\sqrt[n]{(1-x^n)^{n-2}}}.$$

II. Evolutio classis qua $a+b=4$.

§. 69. Quoniam b binario minor esse nequit, hic erit vel $b=2$, vel $b=3$. Sit igitur primo $b=2$, eritque $a=2$ et $c=1$; unde ex nostro theoremate sequitur haec relatio $(2, 2)(4, 1)=(2, 1)(3, 2)$, quae forma manifesto oritur ex classe prima, si ibi termini priores cujusque characteris unitate augeantur; id quod etiam inde intelligere licet, quod omnes termini priores litteram a continent, qua unitate aucta processus semper fit ad classem sequentem.

§. 70. Deinde vero hic quoque statui potest $b = 3$, unde fit $a = 1$; at vero littera c jam duos valores, vel 1, vel 2 sortiri poterit; priore casu, quo $c = 1$, prodibit ista aequatio $(1, 3)(4, 1) = (1, 1)(2, 3)$; alter vero casus, quo $c = 2$, praebet hanc aequationem $(1, 3)(4, 2) = (1, 2)(3, 3)$. Sicque haec classis omnino sequentes tres relationes continebit

1°. $(2, 2)(4, 1) = (2, 1)(3, 2)$,
2°. $(1, 3)(4, 1) = (1, 1)(2, 3)$,
3°. $(1, 3)(4, 2) = (1, 2)(3, 3)$.

III. Evolutio classis qua $a + b = 5$.

§. 71. In hac igitur classe primo occurrent tres relationes praecedentes, si modo termini priores cujusque characteris unitate augeantur: hinc enim casus exsurgent, quibus est vel $b = 2$, vel $b = 3$. De novo igitur hic accedent casus, quibus $b = 4$ et $a = 1$, ubi ergo erit vel $c = 1$, vel $c = 2$, vel $c = 3$, quibus ergo tribus casibus evolutis omnino in hac classe sex continebuntur relationes, quae erunt

1°. $(3, 2)(5, 1) = (3, 1)(4, 2)$,
2°. $(2, 3)(5, 1) = (2, 1)(3, 3)$,
3°. $(2, 3)(5, 2) = (2, 2)(4, 3)$,
4°. $(1, 4)(5, 1) = (1, 1)(2, 4)$,
5°. $(1, 4)(5, 3) = (1, 2)(3, 4)$,
6°. $(1, 4)(5, 3) = (1, 3)(4, 4)$.

IV. Evolutio classis quo $a + b = 6$.

§. 72. Hic igitur primum occurrent omnes relationes proxime praecedentes, si modo termini priores cujusque characteris unitate augeantur: hi scilicet nascuntur, si fuerit vel $b = 2$, vel $b = 3$, vel $b = 4$. Praeterea vero insuper accedent casus $b = 5$ et $a = 1$, ubi littera c recipere poterit valores 1, 2, 3, 4, sicque, omnino in hac classe occurrent decem relationes sequentes

1°. (4, 2) (6, 1) = (4, 1) (5, 2),
2°. (3, 3) (6, 1) = (3, 1) (4, 3),
3°. (3, 3) (6, 2) = (3, 2) (5, 2),
4°. (2, 4) (6, 1) = (2, 1) (3, 4),
5°. (2, 4) (6, 2) = (2, 2) (4, 4),
6°. (2, 4) (6, 3) = (2, 3) (5, 4),
7°. (1, 5) (6, 1) = (1, 1) (2, 5),
8°. (1, 5) (6, 2) = (1, 2) (3, 5),
9°. (1, 5) (6, 3) = (1, 3) (4, 5),
10°. (1, 5) (6, 4) = (1, 4) (5, 5),

V. Evolutio classis qua $a + b = 7$.

§. 73. Hic igitur primo occurrent omnes relationes classis IV. postquam scilicet omnes terminos priores singulorum characterum unitate auxerimus, quos igitur hic apposuisse non erit necesse, ac sufficiet eas tantum relationes hic exponere, quae de novo accedunt et ex valore $b = 6$ oriuntur, existente $a = 1$; ubi pro c sumi poterunt numeri 1, 2, 3, 4, 5, ita ut harum numerus sit quinque. Haec ergo relationes sunt

(1, 6) (7, 1) = (1, 1) (2, 6)
(1, 6) (7, 2) = (1, 2) (3, 6)
(1, 6) (7, 3) = (1, 3) (4, 6)
(1, 6) (7, 4) = (1, 4) (5, 6)
(1, 6) (7, 5) = (1, 5) (6, 6).

VI. Evolutio classis qua $a + b = 8$.

§. 74. In hac jam classe primo occurrent omnes decem relationes classis IV., dum scilicet omnes termini priores binario augentur; praeterea quoque accedent quinque relationes in classe V allatae, dum partes priores unitate augebuntur; praeter has vero de novo accedent 6 sequentes relationes ex valoribus $a = 1$ et $b = 7$ oriundae, dum litterae c valores 1, 2, 3, 4, 5, 6 ordine tribuuntur, quae ergo erunt

$$(1, 7)(8, 1) = (1, 1)(2, 7)$$
$$(1, 7)(8, 2) = (1, 2)(3, 7)$$
$$(1, 7)(8, 3) = (1, 3)(4, 7)$$
$$(1, 7)(8, 4) = (1, 4)(5, 7)$$
$$(1, 7)(8, 5) = (1, 5)(6, 7)$$
$$(1, 7)(8, 6) = (1, 6)(7, 7).$$

VII. Evolutio classis qua $a + b = 9$.

§. 75. Ut omnes relationes ad hanc classem pertinentes adipiscamur, notandum est primo hic occurrere decem relationes classis IV, dum partes priores ternario augentur. Secundo adjici oportet quinque relationes in classe V exhibitas, ubi partes priores binario augeri debent. Tertio huc referri debent sex relationes classis VI, partes priores unitate augendo. Insuper vero de novo accedent septem relationes ex valoribus $a = 1$ et $b = 8$ natae, dum litterae c tribuuntur ordine valores 1, 2, 3, 4, 5, 6, 7. Hae relationes sunt

$$(1, 8)(9, 1) = (1, 1)(2, 8)$$
$$(1, 8)(9, 2) = (1, 2)(3, 8)$$
$$(1, 8)(9, 3) = (1, 3)(4, 8)$$

(1, 8) (9, 4) = (1, 4) (5, 8)
(1, 8) (9, 5) = (1, 5) (6, 8)
(1, 8) (9, 6) = (1, 6) (7, 8)
(1, 8) (9, 7) = (1, 7) (8, 8).

§. 76. Hinc jam ordo progressionis tam clare perspicitur, ut superfluum foret has evolutiones ulterius prosequi; quandoquidem ob ingentem multitudinem relationum, quae in sequentibus classibus occurrerent, nimis molestum foret omnes percurrere. Quin etiam nostrum institutum vix permittere videtur, ut in nostra formula generali exponentem n ultra sex vel septem augeamus, si quidem omnes relationes ad eum pertinentes enumerare voluerimus. Sin autem animus sit aliquas tantum expendere, classes allatae abunde sufficiunt, dum termini priores cujusque classis quovis numero augebuntur.

§. 77. His jam classibus expeditis, formulam integralem propositam $\int \frac{x^{p-1}\partial x}{\sqrt[n]{(1-x^n)^{n-q}}}$ secundum diversos valores exponentis n pertractemus, dum scilicet successive assumemus $n = 3$, $n = 4$, $n = 5$, etc. et pro quolibet ordine omnes relationes, quae in eo occurrere possunt, expendamus. Evidens autem est, quicunque numerus exponenti n tributaur, formulas omnium classium inferiorum, in quibus scilicet terminus $a + b$ non superet n, in usum vocari posse. Ex quo intelligitur, si fuerit $n = 3$ unicam relationem locum invenire; statim autem ac n magis augetur, numerus omnium relationum mox ita increscit, ut nimis molestum foret omnes recensere. Hos igitur diversos ordines, ex exponente n constituendos, a primo incipiendo, ordine involvamus.

Ordo I.

quo $n = 3$ et formula

$$(p, q) = \int \frac{x^{p-1} \partial x}{\sqrt[3]{(1-x^3)^{3-q}}} = \int \frac{x^{q-1} \partial x}{\sqrt[3]{(1-x^3)^{3-p}}}.$$

§. 78. Cum hic sit $n = 3$, erit $(3, 1) = 1$; formulae autem integrales hujus ordinis erunt tres; scilicet 1°. $(1, 1)$, 2°. $(1, 2)$, 3°. $(2, 2)$, quarum media, ob $1 + 2 = 3$, a circulo pendet, quae ergo, quia est cognita, ponatur

$$(1, 2) = \frac{\pi}{3 \cdot \sin. \frac{\pi}{3}} = \frac{2\pi}{3\sqrt{3}} = A.$$

Hic igitur tantum classis prima locum habet, quae nobis hanc unicam aequationem suppeditat $A = (1, 1)\,(2, 2)$.

§. 79. Hinc ergo patet, productum ex binis formulis transcendentibus $(1, 1)$ et $(2, 2)$ aequari quantitati circulari $A = \frac{2\pi}{3\sqrt{3}}$, ita ut pro ipsis formulis integralibus habeamus hanc relationem

$$\int \frac{\partial x}{\sqrt[3]{(1-x^3)^2}} \cdot \int \frac{x \partial x}{\sqrt[3]{(1-x^3)}} = \frac{2\pi}{3\sqrt{3}};$$

unde si altera harum duarum formularum fuerit cognita, etiam valor alterius assignari potest. Spectemus ergo priorem quasi nobis esset cognita, etiamsi sit transcendens, eamque ponamus

$$(1, 1) = \int \frac{\partial x}{\sqrt[3]{(1-x^3)^2}} = P,$$

eritque $(2, 2) = \frac{A}{P}$. Sicque nihil praeterea in hoc ordine notandum relinquitur.

Ordo II.

quo $n = 4$ et formula

$$(p, q) = \int \frac{x^{p-1}\partial x}{\sqrt[4]{(1-x^4)^{4-q}}} = \frac{x^{q-1}\partial x}{\sqrt[4]{(1-x^4)^{4-q}}}.$$

§. 80. Cum igitur hic sit $n = 4$, erit $(4, 1) = 1$ et $(4, 2) = \frac{1}{2}$; formulae autem integrales ad hunc ordinem pertinentes erunt sex sequentes: 1°. $(1, 1)$, 2°. $(1, 2)$, 3°. $(1, 3)$, 4°. $(2, 2)$, 5°. $(2, 3)$, 6°. $(3, 3)$, inter quas ergo reperiuntur duae formulae circulares $(1, 3)$ et $(2, 2)$, quas propterea litteris A et B designemus, ponendo

$$(1, 3) = \frac{\pi}{4 \sin. \frac{\pi}{4}} = \frac{\pi}{2\sqrt{2}} = A, \text{ et}$$

$$(2, 2) = \frac{\pi}{4 \sin. \frac{2\pi}{4}} = \frac{\pi}{4} = B,$$

ita ut sit $\frac{A}{B} = \sqrt{2}$.

§. 81. In hoc ergo ordine aequationes tam primae quam secundae classis locum habere possunt; secunda autem classis nobis has tres praebet aequationes

$$1°.\ B = (2,1)(3,2),\quad 2°.\ A = (1,1)(2,3),\quad 3°.\ A = 2(1,2)(3,3),$$

classis vero prima insuper dat hanc aequationem $A(1,2) = (1,1)B$, sive $\frac{A}{B} = \frac{(1,1)}{(1,2)}$ quae autem aequatio jam ex duabus prioribus deducitur; namque ob $(3, 2) = (2, 3)$, secunda per primam divisa dabit $\frac{A}{B} = \frac{(1,1)}{(1,2)} = \sqrt{2}$, ita ut ratio inter has duas formulas sit algebraica, quae ergo imprimis notari meretur

$$\int \frac{\partial x}{\sqrt[4]{(1-x^4)^3}} : \int \frac{\partial x}{\sqrt{(1-x^4)}} = \sqrt{2}.$$

§. 82. Jam in hoc ordine, praeter binas formulas circulares, $(1, 3) = A$ et $(2, 2) = B$, tanquam cognitam etiam introducamus formulam $(1, 2)$, quae in ordine praecedente erat circularis, nunc autem est transcendens, eamque ponamus $(1, 2) = \int \frac{\partial x}{\sqrt{(1-x^4)}} = P$; ubi caveatur, ne litterae A et P cum iis confundantur, quibus in formulis praecedentibus sumus usi, id quod etiam de ordinibus sequentibus est tenendum. His igitur litteris introductis aequationes nostrae erunt sequentes tres

$$1^\circ.\ B = P(3, 2),\quad 2^\circ.\ A = (1, 1)(2, 3),\quad 3^\circ.\ A = 2P(3, 3),$$

quandoquidem vidimus, quartam in praecedentibus jam contineri.

§. 83. Ope harum trium aequationum ergo ternas formulas integrales etiamnunc incognitas per ternas A, B et P, quas ut datas spectamus, determinare licebit. Ex prima enim fit $(3,2) = \frac{B}{P}$; ex tertia autem fit $(3, 3) = \frac{A}{2P}$; tum vero ex secunda colligitur $(1, 1) = \frac{A}{(3, 2)} = \frac{AP}{B}$. Cum igitur in hoc ordine omnino sint sex formulae integrales, earum ternae per tres reliquas definiri possunt, quas determinationes igitur ob oculos posuisse juvabit

$$(1, 3) = A = \frac{\pi}{2\sqrt{2}};$$
$$(2, 2) = B = \frac{\pi}{4};$$
$$(1, 2) = P = \int \frac{\partial x}{\sqrt{(1-x^4)}};$$
$$(1, 1) = \frac{AP}{B};$$
$$(2, 3) = \frac{B}{P};$$
$$(3, 3) = \frac{A}{2P}.$$

Ex postremis ergo erit

$$(2, 3) : (3, 3) = 2B : A = \sqrt{2} : 1,$$

ita ut etiam hae duae formulae inter se habeant rationem algebraicam, qua est

$$\int \frac{x x \partial x}{\sqrt{(1 - x^4)}} = \sqrt{2} \int \frac{x x \partial x}{\sqrt[4]{(1 - x^4)}}.$$

Aliis insignibus relationibus, utpote satis cognitis, hic non immoramur.

Ordo III.

quo $n = 5$ et formula

$$(p, q) = \int \frac{x^{p-1} \partial x}{\sqrt[5]{(1 - x^5)^{n-q}}} = \int \frac{x^{q-1} \partial x}{\sqrt[5]{(1 - x^5)^{n-p}}}.$$

§. 84. Hic igitur ob $n = 5$ ante omnia erit

$$(5, 1) = 1, \; (5, 2) = \tfrac{1}{2}, \; (5, 3) = \tfrac{1}{3},$$

formulae autem integrales hujus ordinis erunt hae decem

1°. $(1, 1)$, 2°. $(1, 2)$, 3°. $(1, 3)$, 4°. $(1, 4)$, 5°. $(2, 2)$,
6°. $(2, 3)$, 7°. $(2, 4)$, 8°. $(3, 3)$, 9°. $(3, 4)$, 10°. $(4, 4)$,

inter quas quarta et sexta sunt circulares, quas ergo ita designemus

$$(1, 4) = \frac{\pi}{5 \sin. \frac{1}{5}\pi} = A \text{ et}$$

$$(2, 3) = \frac{\pi}{5 \sin. \frac{2\pi}{5}} = B.$$

Praeterea vero binas formulas, quae in ordine praecedenti erant circulares, nunc autem sunt transcendentes, etiam peculiaribus litteris notemus, scilicet $(1, 3) = P$ et $(2, 2) = Q$. Mox enim patebit, dummodo etiam istae formulae tanquam cognitae spectentur reliquas sex omnes per has quatuor determinari posse.

§. 85. Quoniam hic tres classes priores locum habere possunt, consideremus primo aequationes, quas tertia classis suppeditat, et quae introductis his valoribus erunt

1°. $B = P(4, 2)$,
2°. $B = (2, 1)(3, 3)$,
3°. $B = 2\,Q(4, 3)$,
4°. $A = (1, 1)(2, 4)$,
5°. $A = 2(1, 2)(3, 4)$,
6°. $A = 3\,P(4, 4)$.

Quas hoc modo succinctius repraesentare licet

$$A = (1, 1)(2, 4) = 2(1, 2)(3, 4) = 3\,P(4, 4),$$
$$B = P(4, 2) = (2, 1)(3, 3) = 2\,Q(4, 3);$$

ubi sex occurrunt producta ex binis formulis integralibus, quae singula quantitati circulari aequantur, unde totidem egregia theoremata formari possent, nisi hinc jam clare in oculos incurrerent.

§. 86. Jam videamus, quot formulas integrales incognitas ex quatuor cognitis A, B, P et Q definire queamus, at vero prima dat $(4, 2) = \frac{B}{P}$, tertia praebet $(4, 3) = \frac{B}{2Q}$, sexta dat $(4, 4) = \frac{A}{3P}$; hinc autem porro ex quarta deducimus

$$(1, 1) = \frac{A}{(2, 4)} = \frac{AP}{B},$$

ex quinta vero deducimus

$$(1, 2) = \frac{A}{2(3, 4)} = \frac{AQ}{B}.$$

Denique ex secunda elicimus.

$$(3, 3) = \frac{B}{(2, 1)} = \frac{BB}{AQ},$$

sicque ex his sex aequationibus sex determinationes sumus adepti; atque adeo per litteras A, B, P et Q valores omnium reliquarum litterarum assignavimus.

§. 87. Quoniam igitur hactenus tantum classe tertia sumus usi, consideremus etiam aequationes secundae classis, quae sunt

1°. $AQ = B(2, 1)$,
2°. $AP = B(1, 1)$, et
3°. $P(4, 2) = (1, 2)(3, 3)$;

verum si hic valores modo inventos substituamus, aequationes mere identicae resultant, ita ut hinc nulla nova determinatio sequatur. Idem usu venit ex aequatione primae classis, quae erat $(2, 1)(3, 1) = (1, 1)(2, 2)$, quae facta substitutione quoque fit identica, ita ut duae priores classes nihil novi involvant. Neque tamen hinc concludere licet, etiam in sequentibus ordinibus classes praecedentes praetermitti posse, siquidem in ordine sequente statim contrarium se manifestabit.

§. 88. Cum igitur hic ordo complectatur decem formulas integrales, earum valores per quatuor litteras A, B, P et Q ordine ita aspectui exponamus

1°. $(1, 1) = \frac{AP}{B}$
2°. $(1, 2) = \frac{AQ}{B}$
3°. $(1, 3) = P$
4°. $(1, 4) = A$
5°. $(2, 2) = Q$
6°. $(2, 3) = B$
7°. $(2, 4) = \frac{B}{P}$
8°. $(3, 3) = \frac{BB}{AQ}$
9°. $(3, 4) = \frac{B}{2Q}$
10°. $(4, 4) = \frac{A}{3P}$.

§. 89. Cum sit

$$\frac{A}{B} = \frac{\sin.\frac{2}{5}\pi}{\sin.\frac{1}{5}\pi} = 2\cos.\tfrac{1}{5}\pi,$$

tum vero

$$\cos.\tfrac{1}{5}\pi = \frac{1+\sqrt{5}}{4}, \text{ erit } \frac{A}{B} = \frac{1+\sqrt{5}}{2},$$

ideoque quantitas algebraica. Hinc igitur aliquot paria formularum integralium exhiberi poterunt, quae inter se teneant rationem algebraicam; erit enim

$$\frac{(1,1)}{(1,3)} = \frac{1+\sqrt{5}}{2}, \frac{(1,2)}{(2,2)} = \frac{1+\sqrt{5}}{2}, \frac{(3,4)}{(3,3)} = \frac{1+\sqrt{5}}{4}, \frac{(4,4)}{(2,4)} = \frac{1+\sqrt{5}}{6};$$

unde totidem egregia theoremata condi possent, nisi ex his formulis manifesto elucerent.

Ordo IV.

quo $n = 6$ et formula

$$(p,q) = \int \frac{x^{p-1}\partial x}{\sqrt[6]{(1-x^6)^{6-q}}} = \int \frac{x^{q-1}\partial x}{\sqrt[6]{(1-x^6)^{6-p}}}.$$

§. 90. Quoniam hic est $n = 6$, habebimus ante omnia

$$(6,1) = 1,\ (6,2) = \tfrac{1}{2},\ (6,3) = \tfrac{1}{3},\ (6,4) = \tfrac{1}{4};$$

formularum autem integralium in hoc ordine occurrentium numerus est 15, quae sunt

1°. (1, 1), 2°. (1, 2), 3°. (1, 3), 4°. (1, 4), 5°. (1, 5),
6°. (2, 2), 7°. (2, 3), 8°. (2, 4), 9°. (2, 5), 10°. (3, 3),
11°. (3, 4), 12°. (3, 5), 13°. (4, 4), 14°. (4, 5), 15°. (5, 5);

inter quas reperiuntur tres circulares, quas singulari modo designemus, scilicet

$$1^\circ.\ (1,5) = \frac{\pi}{6\sin.\frac{1}{6}\pi} = \frac{\pi}{3} = A,$$

$$2^{\circ}. \ (2, 4) = \frac{\pi}{6 \sin. \frac{2\pi}{6}} = \frac{\pi}{3\sqrt{3}} = B, \text{ et}$$

$$3^{\circ}. \ (3, 3) = \frac{\pi}{6 \sin. \frac{3\pi}{6}} = \frac{\pi}{6} = C;$$

ita ut sit $A = 2\,C$. Praeterea vero ambas formulas, quae in ordine praecedente erant circulares, nunc vero sunt transcendentes, statuamus $(1, 4) = P$ et $(2, 3) = Q$. His factis denominationibus evolvamus decem aequationes classis quartae, quae sunt

1°. $B = P\,(5, 2)$,
2°. $C = (3, 1)\,(4, 3)$,
3°. $C = 2\,Q\,(5, 3)$,
4°. $B = (2, 1)\,(3, 4)$,
5°. $B = 2\,(2, 2)\,(4, 4)$,
6°. $B = 3\,Q\,(5, 4)$,
7°. $A = (1, 1)\,(5, 2)$,
8°. $A = 2\,(1, 2)\,(3, 5)$,
9°. $A = 3\,(1, 3)\,(4, 5)$,
10°. $A = 4\,P\,(5, 5)$,

quas ita succinctius referre licet

$$A = (1,1)(5,2) = 2\,(1,2)(3,5) = 3\,(1,3)(4,5) = 4P\,(5,5),$$
$$B = P(5,2) = (2,1)(3,4) = 2\,(2,2)(4,4) = 3\,Q\,(4,5),$$
$$C = (3,1)(5,2) = 2\,Q\,(5,3).$$

Ecce ergo decem producta ex binis formulis integralibus, quorum singula quantitati circulari aequantur.

§. 91. Cum deinde sit $\frac{A}{B} = \sqrt{3}$ et $\frac{A}{C} = 2$, tum vero etiam $\frac{B}{C} = \frac{2}{\sqrt{3}}$, plura paria binarum formularum integralium exhiberi possunt, quae inter se teneant rationem algebraicam; erit enim

$$\frac{A}{B} = \sqrt{3} = \frac{(1,1)}{(1,4)} = \frac{2\,(3,5)}{(3,4)} = \frac{(1,3)}{(2,3)} = \frac{4\,(5,5)}{(5,2)},$$
$$\frac{A}{C} = 2 = \frac{(1,1)}{(1,3)} = \frac{(1,2)}{(2,3)} = \frac{3\,(4,5)}{(2,5)},$$
$$\frac{B}{C} = \frac{2}{\sqrt{3}} = \frac{(1,4)}{(1,3)} = \frac{3\,(4,5)}{2\,(3,5)}.$$

§. 92. Quodsi jam quinque formulas litteris A, B, C, P et Q designatas tanquam cognitas spectemus, videamus, quomodo reliquae formulae per eas definire queant. Ac primo quidem percurramus decem aequationes classis quartae supra allatas, quarum prima dabit $(5,2) = \frac{B}{P}$, tertia dat $(5,3) = \frac{C}{2\,Q}$, sexta praebet $(5,4) = \frac{B}{3\,Q}$, decima dat $(5,5) = \frac{A}{4\,P}$. Quodsi jam hos valores in reliquis surrogemus, secunda dabit $(3,1) = \frac{C}{(4,3)} = \frac{A\,Q}{B}$, septima praebet $(1,1) = \frac{A}{(5,2)} = \frac{A\,P}{B}$, octava dat $(1,2) = \frac{A}{2\,(3,5)} = \frac{A\,Q}{C}$, nona dat $(3,1) = \frac{A}{3\,(4,5)} = \frac{A\,Q}{B}$, quem valorem etiam secunda praebuit. Porro vero quarta dat $(3,4) = \frac{B}{(2,1)} = \frac{B\,C}{A\,Q}$. At vero ex aequatione quinta nullum valorem elicere possumus, quia neque formula $(2,2)$ nec $(4,4)$ etiamnunc constat. Causa est quia duae reliquarum aequationum eandem determinationem produxerunt.

§. 93. Coacti igitur sumus, ad aequationes praecedentium classium confugere, atque adeo ex prima classe

$$(1,2)\,(3,1) = (1,1)\,(2,2)$$

statim colligimus

$$(2,2) = \frac{(1,2)\,(3,1)}{(1,1)} = \frac{A\,Q\,Q}{C\,P},$$

qui valor in quinta aequatione substitutus suppeditat postremam aequationem, nempe

$$(4,4) = \frac{B}{2\,(2,2)} = \frac{B\,C\,P}{2\,A\,Q\,Q}.$$

Omnes igitur hos valores hic ordine referemus

1°. $(1, 1) = \frac{AP}{B}$.	9°. $(2, 5) = \frac{B}{P}$.
2°. $(1, 2) = \frac{AQ}{C}$.	10°. $(3, 3) = C$.
3°. $(1, 3) = \frac{AQ}{B}$.	11°. $(3, 4) = \frac{BC}{AQ}$.
4°. $(1, 4) = P$.	12°. $(3, 5) = \frac{C}{2Q}$.
5°. $(1, 5) = A$.	13°. $(4, 4) = \frac{BCP}{2AQQ}$.
6°. $(2, 2) = \frac{AQQ}{CP}$.	14°. $(4, 5) = \frac{B}{3Q}$.
7°. $(2, 3) = Q$.	15°. $(5, 5) = \frac{A}{4P}$.
8°. $(2, 4) = B$.	

§. 94. Cum autem in hoc ordine etiam aequationes tam classis secundae quam tertiae valere debeant, videamus utrum valores inventi his classibus conveniant, an vero forte novam determinationem suppeditent? Facta autem substitutione in tribus aequationibus secundae classis, ad identitatem pervenitur, quod idem quoque in aequationibus tertiae classis contingere debet, id quod evolventi mox patebit. Unde memorabile est omnes aequationes in quatuor primis classibus contentas, quarum numerus est 20, tantum decem determinationes in se complecti.

Ordo V.

quo $n = 7$ et formula

$$(p, q) = \int \frac{x^{p-1}\partial x}{\sqrt[7]{(1-x^7)^{7-q}}} = \int \frac{x^{q-1}\partial x}{\sqrt[7]{(1-x^7)^{7-p}}}.$$

§. 95. Quia hic $n = 7$, ante omnia habebimus valores absolutos $(7, 1) = 1$, $(7, 2) = \frac{1}{2}$, $(7, 3) = \frac{1}{3}$, $(7, 4) = \frac{1}{4}$, et $(7, 5) = \frac{1}{5}$; deinde inter formulas integrales hujus ordinis imprimis notari debent circulares, quas hoc modo designemus

$$(1, 6) = \frac{\pi}{7 \sin. \frac{\pi}{7}} = A,$$

$$(2, 5) = \frac{\pi}{7 \sin. \frac{2\pi}{7}} = B,$$

$$(3, 4) = \frac{\pi}{7 \sin. \frac{3\pi}{7}} = C.$$

Praeterea vero peculiaribus litteris notentur eae formulae, quae in ordine praecedenti erant circulares, hic autem valores transcendentes sortiuntur, qui sint $(1, 5) = P$, $(2, 4) = Q$, et $(3, 3) = R$: per has enim sex litteras videbimus omnes reliquas formulas hujus ordinis determinari posse.

§. 96. Quoniam supra non omnes aequationes quintae classis expressimus, eas hic conjunctim exhibeamus, et ad nostrum casum accommodemus

I°.	$(1, 6)(7, 1) = (1, 1)(2, 6)$	$A = (1, 1)(2, 6)$,
II°.	$(1, 6)(7, 2) = (1, 2)(3, 6)$	$A = 2(1, 2)(3, 6)$,
III°.	$(1, 6)(7, 3) = (1, 3)(4, 6)$	$A = 3(1, 3)(4, 6)$,
IV°.	$(1, 6)(7, 4) = (1, 4)(5, 6)$	$A = 4(1, 4)(5, 6)$,
V°.	$(1, 6)(7, 5) = (1, 5)(6, 6)$	$A = 5\ P\ (6, 6)$,
VI°.	$(2, 5)(7, 1) = (2, 1)(3, 5)$	$B = (2, 1)(3, 5)$,
VII°.	$(2, 5)(7, 2) = (2, 2)(4, 5)$	$B = 2(2, 2)(4, 5)$,
VIII°.	$(2, 5)(7, 3) = (2, 3)(5, 5)$	$B = 3(2, 3)(5, 5)$,
IX°.	$(2, 5)(7, 4) = (2, 4)(6, 5)$	$B = 4\ Q\ (6, 5)$,
X°.	$(3, 4)(7, 1) = (3, 1)(4, 4)$	$C = (3, 1)(4, 4)$,
XI°.	$(3, 4)(7, 2) = (3, 2)(5, 4)$	$C = 2(3, 2)(5, 4)$,
XII°.	$(3, 4)(7, 3) = (3, 3)(6, 4)$	$C = 3\ R\ (6, 4)$,
XIII°.	$(4, 3)(7, 1) = (4, 1)(5, 3)$	$C = (4, 1)(5, 3)$,
XIV°.	$(4, 3)(7, 2) = (4, 2)(6, 3)$	$C = 2\ Q\ (6, 3)$,
XV°.	$(5, 2)(7, 1) = (5, 1)(6, 2)$	$B = P\ )(6, 2)$.

Hic igitur habemus quina producta formulae A aequalia, totidemque formulis B et C aequalia.

§. 97. Omnino autem in hoc ordine occurrunt 21 formulae integrales, ex quibus sex litteris A, B, C, P, Q et R designavimus, per quas igitur reliquas quindecim formulas integrales definiri oportet, quae sunt: 1°. (1, 1), 2°. (1, 2), 3°. (1, 3), 4°. (2, 2), 5°. (1, 4), 6°. (2, 3), 7°. (2, 6), 8°. (3, 5), 9°. (4, 4), 10°. (3, 6), 11°. (4, 5), 12°. (4, 6), 13°. (5, 5), 14°. (5, 6), 15°. (6, 6).

§. 98. Videamus igitur, quot harum formularum ex superioribus quindecim aequationibus determinare liceat, ac primo quidem ex aequationibus V, IX, XII, XIV et XV, immediate deducuntur sequentes formulae $(6, 6) = \frac{A}{5P}$, $(6, 5) = \frac{B}{4Q}$, $(6, 4) = \frac{C}{3R}$, $(6, 3) = \frac{C}{2Q}$, $(6, 2) = \frac{B}{P}$. His jam inventis ex aequationibus I, II, III et IV, derivamus has formulas $(1, 1) = \frac{AP}{B}$, $(1, 2) = \frac{AQ}{C}$, $(1, 3) = \frac{AR}{C}$, $(1, 4) = \frac{AQ}{B}$. Ex his vero valoribus per aequationes VI, X et XIII, colligimus $(3, 5) = \frac{B, C}{A, Q}$, $(4, 4) = \frac{CC}{AR}$, et $(5, 3) = \frac{BC}{AQ}$; ubi notasse juvabit eundem valorem pro (5, 3) prodiisse ex aequationibus VI, et XIII. Ex reliquis autem aequationibus VII, VIII et IX, nihil concludere licet, unde istae quatuor formulae (2, 2), (2, 3), (5, 4), et (5, 5), nobis etiamnunc manent incognitae.

§. 99. Recurrere ergo coacti sumus ad aequationes praecedentium classium, quippe quae aeque ad nostrum ordinem pertinent atque aequationes classis quintae; hanc ob rem simili modo aequationes classis quartae hic opponamus et ad nostrum casum applicemus:

I°.	$(1, 5)(6, 1) = (1, 1)(2, 5)$	$P\,A = (1, 1)\,B$
II°.	$(1, 5)(6, 2) = (1, 2)(3, 5)$	$P\,(6, 2) = (1, 2)(3, 5)$
III°.	$(1, 5)(6, 3) = (1, 3)(4, 5)$	$P\,(6, 3) = (1, 3)(4, 5)$
IV°.	$(1, 5)(6, 4) = (1, 4)(5, 5)$	$P\,(6, 4) = (1, 4)(5, 5)$
V°.	$(2, 4)(6, 1) = (2, 1)(3, 4)$	$Q\,A = (2, 1)\,C$
VI°.	$(2, 4)(6, 2) = (2, 2)(4, 4)$	$Q\,(6, 2) = (2, 2)(4, 4)$
VII°.	$(2, 4)(6, 3) = (2, 3)(5, 4)$	$Q\,(6, 3) = (2, 3)(5, 4)$
VIII°.	$(3, 3)(6, 1) = (3, 1)(4, 3)$	$R\,A = (3, 1)\,C$
IX°.	$(3, 3)(6, 2) = (3, 2)(5, 3)$	$R\,(6, 2) = (3, 2)(5, 3)$
X°.	$(4, 2)(6, 1) = (4, 1)(5, 2)$	$Q\,A = (4, 1)\,B.$

§. 100. Ex aequationibus I, V, VIII, et X immediate concludimus has formulas $(1, 1) = \frac{PA}{B}$, $(2, 1) = \frac{QA}{C}$, $(3, 1) = \frac{AR}{C}$, $(4, 1) = \frac{AQ}{B}$, quos autem valores jam ante adepti sumus. Secunda aequatio, si formulae jam inventae substituantur, praebet aequationem identicam. Ex tertia autem poterimus definire formulam $(4, 5)$, cujus valor hinc colligitur $(4, 5) = \frac{CCP}{2AQA}$. Simili modo ex IV elicimus $(5, 5) = \frac{BCP}{3AQR}$. Porro ex aequatione VI concludimus fore $(2, 2) = \frac{ABQR}{CCP}$. Deinde septima aequatio dat $(2, 3) = \frac{AQR}{CP}$. Nona vero aequatio etiam praebet $(3, 2) = \frac{AQR}{CP}$. Sicque omnes quindecim formulas incognitas determinavimus per sex litteras cognitas A, B, C, P, Q et R.

§. 101. Valores igitur omnium formularum hujus ordinis hic aspectui conjunctim exponamus

$(1, 6) = A$	$(6, 2) = \frac{B}{P}$	$(1, 1) = \frac{AP}{B}$	$(3, 5) = \frac{BC}{AQ}$	$(2, 3) = \frac{AQR}{CP}$
$(2, 5) = B$	$(6, 3) = \frac{C}{2Q}$	$(1, 2) = \frac{AQ}{C}$	$(4, 4) = \frac{CC}{AR}$	$(4, 5) = \frac{CCP}{2AQR}$
$(3, 4) = C$	$(6, 4) = \frac{C}{3R}$	$(1, 3) = \frac{AR}{C}$		$(5, 5) = \frac{BCP}{3AQR}$
$(1, 5) = P$	$(6, 5) = \frac{B}{4Q}$	$(1, 4) = \frac{AQ}{B}$		$(2, 2) = \frac{ABQR}{CCP}$
$(2, 4) = Q$	$(6, 6) = \frac{A}{5P}$			
$(3, 3) = R$				

§. 102. Quoniam autem aequationes primae, secundae ac tertiae classis etiam in hoc ordine valent, si in iis valores hic inventos substituamus, perpetuo in aequationes identicas incidemus. Ita cum aequatio primae classis sit $(1, 2)\ (3, 1) = (1, 1)\ (2, 2)$, facta substitutione reperietur $(1, 2)\ (3, 1) = \frac{AAQR}{CC}$; at vero $(1, 1)\ (2, 2)$ fit $= \frac{AAQR}{CC}$, haecque identitas etiam deprehendetur, in tribus aequationibus secundae classis, atque etiam in sex aequationibus tertiae classis, quemadmodum calculum instituenti mox patebit.

§. 103. Simili modo haud difficile erit hanc investigationem ad ordines superiores extendere, neque tamen legem observare licet, secundum quam determinationes singularum formularum cujusque ordinis progrediuntur. Interim tamen observasse juvabit, in ordine sequente sexto, ubi $n = 8$ et formulae occurrunt 28, eas omnes primo per quatuor formulas circulares $(1, 7) = A$, $(2, 6) = B$, $(3, 5) = C$, $(4, 4) = D$, praeterea vero per has tres transcendentes $(1, 6) = P$, $(2, 5) = Q$, et $(3, 4) = R$, determinari posse. Cum igitur quovis ordine determinatio singularum formularum, praeter formulas circulares, quae utique pro cognitis haberi possunt, etiam aliquot formulas transcendentes postulat, si saltem valores harum formularum vero proxime cognoscere voluerimus, methodus adhuc desideratur, istos valores proxime, veluti in fractionibus decimalibus, definiendi. Talem igitur methodum hic coronidis loco subjungemus.

Problema.

Proposita formula integrali cujusque ordinis

$$S = \int \frac{x^{p-1}\,\partial x}{\sqrt[n]{(1-x^n)^{n-q}}},$$

a termino $x = 0$ usque ad $x = 1$ extendenda, investigare seriem convergentem, quae istum valorem S exprimat.

Solutio.

§. 104. Cum sit

$$\frac{1}{\sqrt[n]{(1-x^n)^{n-q}}} = (1-x^n)^{-\frac{(n-q)}{n}},$$

facta evolutione hujus potestatis binomii more solito, reperietur

$$\frac{1}{\sqrt[n]{(1-x^n)^{n-q}}} = 1 + \frac{n-q}{n} x^n + \frac{n-q}{n} \cdot \frac{2n-q}{2n} x^{2n}$$
$$+ \frac{n-q}{n} \cdot \frac{2n-q}{2n} \cdot \frac{3n-q}{3n} x^{3n} + \text{etc.}$$

Si haec series ducatur in $x^{p-1}\,\partial x$ et integretur, prodibit

$$S = \frac{x^p}{p} + \frac{n-q}{n} \cdot \frac{x^{n+p}}{n+p} + \frac{n-q}{n} \cdot \frac{2n-q}{2n} \cdot \frac{x^{2n+p}}{2n+p}$$
$$+ \frac{n-q}{n} \cdot \frac{2n-q}{2n} \cdot \frac{3n-q}{3n} \cdot \frac{x^{3n+p}}{3n+p} + \text{etc.}$$

quae series jam evanescit posito $x = 0$; unde si ponamus $x = 1$, valor quaesitus nostrae formulae fiet

$$S = \frac{1}{p} + \frac{n-q}{n} \cdot \frac{1}{n+p} + \frac{n-q}{n} \cdot \frac{2n-q}{2n} \cdot \frac{1}{2n+p}$$
$$+ \frac{n-q}{n} \cdot \frac{2n-q}{2n} \cdot \frac{3n-q}{3n} \cdot \frac{1}{3n+p} + \text{etc.}$$

§. 105. Verum ista series, quicunque numeri pro litteris n, p et q accipiantur, nimis lente convergit, quam ut ex ea valores ipsius S saltem ad tres quatuorve figuras decimales satis exacte definiri queant; quamobrem aliam evolutionem institui conveniet, dum scilicet valorem quaesitum in duas partes resolvemus. Statuamus igitur

$$\int \frac{x^{p-1}\,\partial x}{\sqrt[n]{(1-x^n)^{n-q}}} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x^n = \frac{1}{2} \end{bmatrix} = P \text{ et}$$

$$\int \frac{x^{p-1}\,\partial x}{\sqrt[n]{(1-x^n)^{n-q}}} \begin{bmatrix} \text{ab } x^n = \frac{1}{2} \\ \text{ad } x = 1 \end{bmatrix} = Q,$$

atque evidens est fore $S = P + Q$. Nunc autem tam pro P quam pro Q haud difficulter series satis convergentes exhiberi poterunt.

§. 106. Quod primum ad valorem P attinet, eum ex valore generali, quem supra pro S invenimus, facile derivabimus ponendo $x^n = \frac{1}{2}$, ita ut sit $x = \sqrt[n]{\frac{1}{2}}$ et $x^p = \frac{1}{\sqrt[n]{2^p}}$, quo facto pro P obtinebimus hanc seriem

$$P = \frac{1}{\sqrt[n]{2^p}}\Big(\frac{1}{p} + \frac{n-q}{2n} \cdot \frac{1}{n+p} + \frac{n-q}{2n} \cdot \frac{2n-q}{4n} \cdot \frac{1}{2n+q}$$
$$+ \frac{n-q}{2n} \cdot \frac{2n-q}{4n} \cdot \frac{3n-q}{6n} \cdot \frac{1}{3n+p} + \text{etc.}\Big).$$

In qua serie singuli termini plus quam in ratione dupla decrescunt; ita ut verbi gratia terminus decimus jam multo minor futurus sit quam $\frac{1}{1024}$, unde si ad partes millionesimas certi esse velimus, sufficeret calculum nequidem ad vigesimum usque terminum extendere.

§. 107. Cum deinde posuerimus

$$Q = \int \frac{x^{p-1}\partial x}{\sqrt[n]{(1-x^n)^{n-q}}} \begin{bmatrix} \text{ab } x^n = \frac{1}{2} \\ \text{ad } x = 1 \end{bmatrix},$$

statuamus $1 - x^n = y^n$, ut sit $Q = \int \frac{x^{p-1}\partial x}{y^{n-q}}$; tum vero erit $x^n = 1 - y^n$, ideoque $x^p = \sqrt[n]{(1-y^n)^p}$, unde differentiando colligitur

$$x^{p-1}\partial x = -\, y^{n-1}\partial y\,(1-y^n)^{\frac{p-n}{n}},$$

quo valore substituto erit

$$Q = -\int y^{q-1}\partial y\,(1-y^n)^{\frac{p-n}{n}} \begin{bmatrix} \text{ab } y^n = \frac{1}{2} \\ \text{ad } y = 0 \end{bmatrix}.$$

Quando enim fit $x^n = \frac{1}{2}$, tum etiam erit $y^n = \frac{1}{2}$, at facto $x = 1$, manifesto fit $y = 0$; quare si terminos integrationis permutemus, etiam signum ipsius formulae immutari debet, sicque fiet

$$Q = \int y^{q-1}\partial y\,(1-y^n)^{\frac{p-n}{n}} \begin{bmatrix} \text{ab } y = 0 \\ \text{ad } y^n = \frac{1}{2} \end{bmatrix}.$$

§. 108. Haec autem formula pro Q inventa omnino similis est illi, quam pro P invenimus, hoc tantum discrimine, quod litterae p et q inter se sunt permutatae; quocirca, si integratio per seriem instituatur, proveniet sequens

$$Q = \frac{1}{\sqrt{2^q}} \left(\frac{1}{q} + \frac{n-p}{2n} \cdot \frac{1}{n+q} + \frac{n-p}{2n} \cdot \frac{2n-p}{4n} \cdot \frac{1}{2n+q} \right.$$
$$\left. + \frac{n-p}{2n} \cdot \frac{2n-p}{4n} \cdot \frac{3n-p}{6n} \cdot \frac{1}{3n+q} + \text{etc.} \right),$$

quae series aeque converget, ac praecedens pro P inventa. His autem duabus seriebus ad calculum revocatis semper erit valor quaesitus $S = P + Q$.

Corollarium 1.

§. 109. Iste calculus plurimum contrahetur iis casibus, quibus est $p = q$, tum enim fiet $P = Q$, hisque casibus, quibus $S = \int \frac{x^{p-1} \partial x}{\sqrt[n]{(1-x^n)^{n-p}}}$, valor istius formulae ab $x = 0$ ad $x = 1$ extensae erit

$$S = \frac{2}{\sqrt[n]{2^p}} \left(\frac{1}{p} + \frac{n-p}{2n} \cdot \frac{1}{n+p} + \frac{n-p}{2n} \cdot \frac{2n-p}{4n} \cdot \frac{1}{2n+p} \right.$$
$$\left. + \frac{n-p}{2n} \cdot \frac{2n-p}{4n} \cdot \frac{3n-p}{6n} \cdot \frac{1}{3n+p} + \text{etc.} \right).$$

Corollarium 2.

§. 110. Quoniam igitur in singulis ordinibus nonnullae hujusmodi formulae (p, p) occurrunt, statim atque valores aliquot hujusmodi formularum fuerint ad calculum decimalem revocati, quoniam formulae circulares per se sunt notae, ex iis valores omnium reliquarum formularum ejusdem ordinis assignare licebit.

Exemplum.

§. 111. Proposita sit formula ordinis primi, ubi $p = q = 2$ et $S = \int \frac{x \partial x}{\sqrt[3]{(1-x^3)}}$. Series igitur pro S inventa erit

$$S = \sqrt[3]{2}\left(\tfrac{1}{2} + \tfrac{1}{6}\cdot\tfrac{1}{5} + \tfrac{1}{6}\cdot\tfrac{4}{12}\cdot\tfrac{1}{8} + \tfrac{1}{6}\cdot\tfrac{4}{12}\cdot\tfrac{7}{18}\cdot\tfrac{1}{11} + \tfrac{1}{6}\cdot\tfrac{4}{12}\cdot\tfrac{7}{18}\cdot\tfrac{10}{24}\cdot\tfrac{1}{14} + \text{etc.}\right).$$

Subducta autem calculo reperitur

$$S = 0,54325 \times \sqrt[3]{2} = 0,68445,$$

qui ergo est valor formulae (2, 2) in ordine 1^{mo} §. 22. ubi invenimus $(2, 2) = \frac{A}{P}$, ita ut jam sit $P = \frac{A}{(2,2)}$. Est vero $A = \frac{2\pi}{3\sqrt{3}} = 1,20918$, hinc erit $P = 2,22582 = (1, 1)$: unde in fractionibus decimalibus ternae formulae ordinis primi erunt $(1, 1) = 2,22582$, $(1, 2) = 1,20918$, $(2, 2) = 0,68445$. Hocque modo etiam omnes formulas sequentium ordinum evolvere licebit.

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3) Additamentum ad Dissertationem praecedentem, de valoribus formulae integralis

$$\int \frac{x^{p-1} \partial x}{\sqrt[n]{(1-x^n)^{n-1}}},$$

ab $x = 0$ ad $x = 1$ extensae. *Nova Acta Acad. Imp. Scient. Petropolitanae. Tom. V. Pag.* 118 — 129.

§. 112. Si methodum in praecedente dissertatione traditam ad altiores ordines quam $n = 7$ transferre vellemus, ob ingentem aequationum considerandarum numerum labor fieret nimis molestus. Quoniam autem vidimus, non omnes istas aequationes concurrere ad valores singularum formularum determinandos, opus non mediocriter sublevabitur, si quovis casu eas tantum aequationes in computum ducamus, quae immediate ad determinationes formularum perducant, quemadmodum hic pro casu $n = 10$ sum ostensurus.

Determinatio

harum formularum pro casu $n = 10$, ubi formula

$$(p, q) = \int \frac{x^{p-1}\partial x}{\sqrt[10]{(1-x^{10})^{10-q}}} = \int \frac{x^{q-1}\partial x}{\sqrt[10]{(1-x^{10})^{10-p}}}.$$

§. 113. Hoc casu ergo formulae valorem absolutum recipientes sunt $(10, 1) = 1$, $(10, 2) = \frac{1}{2}$, $(10, 3) = \frac{1}{3}$ et in genere $(10, \alpha) = \frac{1}{\alpha}$. Deinde omnes formulae, in quibus est $p + q = 10$, a circulo pendent, ideoque pro cognitis haberi possunt, quas ergo propriis litteris designemus

$$(1, 9) = \frac{\pi}{10 \sin. \frac{1}{10}\pi} = A, \qquad (6, 4) = \frac{\pi}{10 \sin. \frac{6}{10}\pi} = D,$$

$$(2, 8) = \frac{\pi}{10 \sin. \frac{2}{10}\pi} = B, \qquad (7, 3) = \frac{\pi}{10 \sin. \frac{7}{10}\pi} = C,$$

$$(3, 7) = \frac{\pi}{10 \sin. \frac{3}{10}\pi} = C, \qquad (8, 2) = \frac{\pi}{10 \sin. \frac{8}{10}\pi} = B,$$

$$(4, 6) = \frac{\pi}{10 \sin. \frac{4}{10}\pi} = D, \qquad (9, 1) = \frac{\pi}{10 \sin. \frac{9}{10}\pi} = A,$$

$$(5, 5) = \frac{\pi}{10 \sin. \frac{5}{10}\pi} = E,$$

§. 114. Per has autem formulas circulares reliquas in forma generali contentas neutiquam determinare licet; sed insuper aliquot formulas transcendentes in subsidium vocari oportet, ex quibus cum circularibus illis conjunctis reliquarum omnium valores assignare licebit. Nostro autem casu, quo $n = 10$, sequentes formulas tanquam cognitas spectari conveniet, quae in ordine praecedenti, ubi $n = 9$, erant circulares, nunc autem in ordinem transcendentium transeunt. Eas igitur sequenti modo designemus

$$(1, 8) = P, \ (2, 7) = Q, \ (3, 6) = R, \ (4, 5) = S,$$
$$(5, 4) = S, \ (6, 3) = R, \ (7, 2) = Q, \ (8, 1) = P.$$

Scilicet si valores harum litterarum quoque tanquam cognitos spectemus, per eos cum circularibus junctos reliquas formulas omnes in hoc ordine contentas determinare poterimus. Cum igitur numerus omnium formularum integralium in hoc ordine $n = 10$ contentarum sit 45, ex iis autem novem ut cognitae spectentur, reliquae 36 per has litteras majusculas determinari debebunt.

§. 115. Istas autem determinationes ex aequatione generali supra demonstrata peti oportet, quae hac forma continetur

$$(a, b)\,(a + b, c) = (a, c)\,(a + c, b),$$

ubi assumere licebit, semper esse $b > c$, quoniam, si foret $c = b$, aequatio foret identica. Primo igitur ut hinc aequationes, quae immediate determinationes praebeant, nanciscamur, sumamus $a + b = 10$, ut sit $(10, c) = \frac{1}{c}$; tum vero capiatur $c = b - 1$, quo facto pro a ordine scribendo numeros 1, 2, 3, etc. sequentes prodibunt determinationes

$$(1, 9)\,(10, 8) = (1, 8)\,(9, 9), \text{ sive } \tfrac{1}{8}A = P\,(9, 9), \text{ ergo}$$
$$(9, 9) = \frac{A}{8P}.$$
$$(2, 8)\,(10, 7) = (2, 7)\,(9, 8), \text{ sive } \tfrac{1}{7}B = Q\,(9, 8), \text{ ergo}$$
$$(9, 8) = \frac{B}{7Q}.$$

$(3, 7)\,(10, 6) = (3, 6)\,(9, 7)$, sive $\frac{1}{6}C = R\,(9, 7)$, ergo
$$(9, 7) = \frac{C}{6R}.$$
$(4, 6)\,(10, 5) = (4, 5)\,(9, 6)$, sive $\frac{1}{5}D = S\,(9, 6)$, ergo
$$(9, 6) = \frac{D}{5S}.$$
$(5, 5)\,(10, 4) = (5, 4)\,(9, 5)$, sive $\frac{1}{4}E = S\,(9, 5)$, ergo
$$(9, 5) = \frac{E}{4S}.$$
$(6, 4)\,(10, 3) = (6, 3)\,(9, 4)$, sive $\frac{1}{3}D = R\,(9, 4)$, ergo
$$(9, 4) = \frac{D}{3R}.$$
$(7, 3)\,(10, 2) = (7, 2)\,(9, 3)$, sive $\frac{1}{2}C = Q\,(9, 3)$, ergo
$$(9, 3) = \frac{C}{2Q}.$$
$(8, 2)\,(10, 1) = (8, 1)\,(9, 2)$, sive $B = P\,(9, 2)$, ergo
$$(9, 2) = \frac{B}{P}.$$

§. 116. Ex formulis igitur incognitis illis numero 36 jam octo determinavimus, quae nobis viam sternent ad novas determinationes, quas primo derivabimus ex aequatione generali sumendo $a = 1$, $b = 9$, et pro c scribendo ordine numeros 1, 2, 3 . . . . . 8, unde calculus ita se habebit

$(1, 9)\,(10, 1) = (1, 1)\,(2, 9)$	$A = (1, 1)\frac{B}{P}$, ergo $(1, 1) = \frac{AP}{B}$
$(1, 9)\,(10, 2) = (1, 2)\,(3, 9)$	$\frac{1}{2}A = (1, 2)\frac{C}{2Q}$, ergo $(1, 2) = \frac{AQ}{C}$
$(1, 9)\,(10, 3) = (1, 3)\,(4, 9)$	$\frac{1}{3}A = (1, 3)\frac{D}{3R}$, ergo $(1, 3) = \frac{AR}{D}$
$(1, 9)\,(10, 4) = (1, 4)\,(5, 9)$	$\frac{1}{4}A = (1, 4)\frac{E}{4S}$, ergo $(1, 4) = \frac{AS}{E}$
$(1, 9)\,(10, 5) = (1, 5)\,(6, 9)$	$\frac{1}{5}A = (1, 5)\frac{D}{5S}$, ergo $(1, 5) = \frac{AS}{D}$
$(1, 9)\,(10, 6) = (1, 6)\,(7, 6)$	$\frac{1}{6}A = (1, 6)\frac{C}{6R}$, ergo $(1, 6) = \frac{AR}{C}$
$(1, 9)\,(10, 7) = (1, 7)\,(8, 9)$	$\frac{1}{7}A = (1, 7)\frac{B}{7Q}$, ergo $(1, 7) = \frac{AQ}{B}$
$(1, 9)\,(10, 8) = (1, 8)\,(9, 9)$	$\frac{1}{8}A = (1, 8)\frac{A}{8P}$, ergo $(1, 8) = \frac{AP}{A}$

hocque modo septem novas determinationes sumus adepti.

§. 117. His autem inventis consideremus aequationes ex valoribus $a = 1$, $b = 8$, $c = 1, 2, \ldots\ldots 7$ ortas, eritque

$(1, 8)(9, 1) = (1, 1)(2, 8)$	$AP = (1, 1) B$	Identica.
$(1, 8)(9, 2) = (1, 2)(3, 8)$	$B = (3, 8) \frac{AQ}{C}$	$(3, 8) = \frac{BC}{AQ}$
$(1, 8)(9, 3) = (1, 3)(4, 8)$	$\frac{CP}{2Q} = (4, 8) \frac{AR}{D}$	$(4, 8) = \frac{CDP}{2AQR}$
$(1, 8)(9, 4) = (1, 4)(5, 8)$	$\frac{DP}{3R} = (5, 8) \frac{AS}{E}$	$(5, 8) = \frac{DEP}{3ARS}$
$(1, 8)(9, 5) = (1, 5)(6, 8)$	$\frac{EP}{4S} = (6, 8) \frac{AS}{D}$	$(6, 8) = \frac{DEP}{4ASS}$
$(1, 8)(9, 6) = (1, 6)(7, 8)$	$\frac{DP}{5S} = (7, 8) \frac{AR}{C}$	$(7, 8) = \frac{CDP}{5ARS}$
$(1, 8)(9, 7) = (1, 7)(8, 8)$	$\frac{CP}{6R} = (8, 8) \frac{AQ}{B}$	$(8, 8) = \frac{BCP}{6AQR}$

§. 118. Novas determinationes reperiemus ponendo $a = 1$, $b = 7$, $c = 3, 4, 5, 6$; hinc enim nanciscimur sequentes determinationes

$(1, 7)(8, 3) = (1, 3)(4, 7)$	$C = (4,7) \frac{AR}{D}$	$(4,7) = \frac{CD}{AR}$
$(1, 7)(8, 4) = (1, 4)(5, 7)$	$\frac{CDP}{2BR} = (5,7) \frac{AS}{E}$	$(5,7) = \frac{CDEP}{2ABRS}$
$(1, 7)(8, 5) = (1, 5)(6, 7)$	$\frac{DEPQ}{3BRS} = (6,7) \frac{AS}{D}$	$(6,7) = \frac{DDEPQ}{3ABRSS}$
$(1, 7)(8, 6) = (1, 6)(7, 7)$	$\frac{DEPQ}{4BSS} = (7,8) \frac{AR}{C}$	$(7,7) = \frac{CDEPQ}{4ABRSS}$

§. 119. Sumamus nunc $a = 1$, $b = 6$, $c = 4, 5$, eritque

$(1, 6)(7, 4) = (1, 4)(5, 6)$	$D = (5,6) \frac{AS}{E}$	$(5, 6) = \frac{DE}{AS}$
$(1, 6)(7, 5) = (1, 5)(6, 6)$	$\frac{DEP}{2BS} = (6,6) \frac{AS}{D}$	$(6, 6) = \frac{DDEP}{2ABSS}$

Haetenus igitur omnes formulas (p, q) determinavimus, in quibus $p + q > 10$. Ex reliquis autem, ubi $p + q > 9$, jam nacti sumus istas

$(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (1, 7),$

ita ut adhuc determinandae relinquantur istae

$(2, 2), (2, 3), (2, 4), (2, 5), (2, 6),$
$(3, 3), (3, 4), (3, 5),$
$(4, 4).$

§. 120. Pro his inveniendis sumamus $a = 1$ et $c = 1$, pro b autem ordine capiamus numeros 2, 3, etc. atque consequemur has aequationes

$(1, 2)\ (3,1) = (1, 1)\ (2, 2)$	$\frac{AAQR}{CD} = (2,2)\frac{AP}{B}$	$(2, 2) = \frac{ABQR}{CDP}$
$(1, 3)\ (4,1) = (1, 1)\ (2, 3)$	$\frac{AARS}{DE} = (2,3)\frac{AP}{B}$	$(2, 3) = \frac{ABRS}{DEP}$
$(1, 4)\ (5,1) = (1, 1)\ (2, 4)$	$\frac{AASS}{DE} = (2,4)\frac{AP}{B}$	$(2, 4) = \frac{ABSS}{DEP}$
$(1, 5)\ (6,1) = (1, 1)\ (2, 5)$	$\frac{AARS}{CD} = (2,5)\frac{AP}{B}$	$(2, 5) = \frac{ABRS}{CDP}$
$(1, 6)\ (7,1) = (1, 1)\ (2, 6)$	$\frac{AAQR}{BC} = (2,6)\frac{AP}{B}$	$(2, 6) = \frac{ABQR}{BCP}$

sicque etiamnunc determinandae restant formulae $(3, 3)$, $(3, 4)$, $(3, 5)$ et $(4, 4)$.

§. 121. Pro his sumatur $a = 1$, $c = 2$, et $b = 3, 4, 5$, etc. tum enim prodibunt hae aequationes

$(1, 3)\ (4,2) = (1, 2)\ (3, 3)$	$\frac{AABRSS}{DDEP} = (3,3)\frac{AQ}{C}$	$(3, 3) = \frac{ABCRSS}{DDEPQ}$
$(1, 4)\ (5,2) = (1, 2)\ (3, 4)$	$\frac{AABRSS}{CDEP} = (3,4)\frac{AQ}{C}$	$(3, 4) = \frac{ABRSS}{DEPQ}$
$(1, 5)\ (6,2) = (1, 2)\ (3, 5)$	$\frac{AAQRS}{CDP} = (3,5)\frac{AQ}{C}$	$(3, 5) = \frac{ARS}{DP}$

Unica ergo formula restat determinanda, scilicet $(4, 4)$, quae ex hac aequatione $(1, 4)\ (5, 3) = (1, 3)\ (4, 4)$ definietur; erit enim $\frac{AARSS}{DEP} = (4, 4)\frac{AR}{D}$, ideoque $(4, 4) = \frac{ASS}{EP}$.

§. 122. Ut nunc omnes has determinationes simul aspectui exponamus, quoniam in hoc ordine $n = 10$ omnino 45 formulae integrales occurrunt, si ex iis ut cognitae spectentur novem sequentes

$$(1,9) = A,\ (2,8) = B,\ (3,7) = C,\ (4,6) = D,\ (5,5) = E,$$
$$(1,8) = P,\ (2,7) = Q,\ (3,6) = R,\ (4,5) = S,$$

reliquae triginta sex ex his sequenti modo determinabuntur

1. $(9,9) = \frac{A}{8P}$	19. $(2,6) = \frac{AQR}{CP}$
2. $(9,8) = \frac{B}{7Q}$	20. $(3,5) = \frac{ARS}{DP}$
3. $(9,7) = \frac{C}{6R}$	21. $(4,4) = \frac{ASS}{EP}$
4. $(9,6) = \frac{D}{5S}$	22. $(4,8) = \frac{CDP}{2AQR}$
5. $(9,5) = \frac{E}{4S}$	23. $(5,8) = \frac{DEP}{3ARS}$
6. $(9,4) = \frac{D}{3R}$	24. $(6,8) = \frac{DEP}{4ASS}$
7. $(9,3) = \frac{C}{2Q}$	25. $(7,8) = \frac{CDP}{5ARS}$
8. $(9,2) = \frac{B}{P}$	26. $(8,8) = \frac{BCP}{6AQR}$
9. $(1,1) = \frac{AP}{B}$	27. $(2,2) = \frac{ABQR}{CDP}$
10. $(1,2) = \frac{AQ}{C}$	28. $(2,3) = \frac{ABRS}{DEP}$
11. $(1,3) = \frac{AR}{D}$	29. $(2,4) = \frac{ABSS}{DEP}$
12. $(1,4) = \frac{AS}{E}$	30. $(2,5) = \frac{ABRS}{CDP}$
13. $(1,5) = \frac{AS}{D}$	31. $(5,7) = \frac{CDEP}{2ABRS}$
14. $(1,6) = \frac{AR}{C}$	32. $(6,6) = \frac{DDEP}{2ABSS}$
15. $(1,7) = \frac{AQ}{B}$	33. $(3,4) = \frac{ABRSS}{DEPQ}$
16. $(3,8) = \frac{BC}{AQ}$	34. $(6,7) = \frac{DDEPQ}{3ABRSS}$
17. $(4,7) = \frac{CD}{AR}$	35. $(7,7) = \frac{CDEPQ}{4ABRSS}$
18. $(5,6) = \frac{DE}{AS}$	36. $(3,3) = \frac{ABCRSS}{DDEPQ}$

§. 123. Eadem methodo, qua hic usi sumus pro casu $n = 10$, haud difficile erit ordines altiores evolvere; neque tamen hinc adhuc elucet, quanam lege omnes determinationes progrediantur, quandoquidem valores certarum formularum continuo magis evadunt complicati. Ceterum valores, quos hic invenimus, omnibus aequationibus in forma generali

$$(a, b)\,(a + b, c) = (a, c)\,(a + c, b)$$

contentis satisfacere deprehenduntur, ita ut perpetuo aequatio identica resultet, neque idcirco inde ulla nova relatio inter litteras nostras majusculas deduci queat. Tandem probe hic notasse juvabit, quod in omnibus ordinibus, praeter formulas a circulo pendentes, commodissime eae formulae, quae in ordine proxime praecedente erant circulares, hic etiam tanquam cognitae accipi queant, quippe quibus determinationes omnes optimo successu perfici possunt.

Methodus generalis determinandi valores formulae

$$(p, q) = \int \frac{x^{p-1}\,\partial x}{\sqrt[n]{(1-x^n)^{n-q}}} = \int \frac{x^{q-1}\,\partial x}{\sqrt[n]{(1-x^n)^{n-p}}},$$

a termino $x = 0$ usque ad $x = 1$ extensa: ubi praeter formulas circulum involventes, in quibus est $p + q = n$, etiam illae pro cognitis accipiuntur, in quibus est $p + q = n - 1$.

I. Cum aequatio generalis, unde omnes hae determinationes sunt petendae, sit

$$(a, b)\,(a + b, c) = (a, c)\,(a + c, b),$$

sumatur primo $a = n - \alpha$, $b = \alpha$, et $c = \alpha - 1$, eritque aequatio

$$(n - \alpha, \alpha)\ (n, \alpha - 1) = (n - \alpha, \alpha - 1)\ (n - 1, \alpha),$$

ubi est $(n, \alpha - 1) = \frac{1}{\alpha - 1}$. In primo autem factore, ob $p = n - \alpha$ et $q = \alpha$, est $p + q = n$, ideoque datur. In tertio porro factore, ubi $p = n - \alpha$ et $q = \alpha - 1$, est $p + q = n - 1$, ideoque pariter datur. Hinc ergo colligimus

$$(n - 1, \alpha) = \frac{1}{\alpha - 1} \cdot \frac{(n - \alpha, \alpha)}{(n - \alpha, \alpha - 1)},$$

ubi esse debet $\alpha > 1$, ita ut pro α accipi queant omnes numeri a 2 usque ad $n - 1$; at vero casu $\alpha = 1$ valor formulae per se est notus.

II. In aequatione generali jam sumatur $a = \beta$, $b = n - \beta - 1$, et $c = 1$, eritque nostra aequatio

$$(\beta, n - \beta - 1)\ (n - 1, 1) = (\beta, 1)\ (\beta + 1, n - \beta - 1),$$

ex qua aequatione colligitur

$$(\beta, 1) = \frac{(\beta, n - \beta - 1)\ (n - 1, 1)}{(\beta + 1, n - \beta - 1)},$$

ubi esse debet $\beta < n - 1$, ita ut hinc omnes formulae $(\beta, 1)$ definiantur, a valore $\beta = 1$ usque ad $\beta = n - 1$, quo posteriore casu formula $(n - 1, 1)$ per se cognoscitur.

III. Ut hinc etiam alias formas eliciamus, sumamus $a = 1$, $b = n - 2$, $c = \gamma$, ut oriatur haec aequatio

$$(1, n - 2)\ (n - 1, \gamma) = (1, \gamma)\ (1 + \gamma, n - 2),$$

ubi primus factor ac tertius dantur per N°. II. secundus vero per N°. I. unde quartus derivatur, scilicet

$$(1 + \gamma, n - 2) = \frac{(1, n - 2)\ (n - 1, \gamma)}{(1, \gamma)},$$

ubi valores ipsius $1 + \gamma$ a 2 usque ad $n - 2$ augeri possunt.

Cum igitur per N°. I. sit

$$(n-1,\gamma) = \frac{1}{\gamma-1}\cdot\frac{(n-\gamma,\gamma)}{(n-\gamma,\gamma-1)},$$

tum vero per N°. II. fit

$$(\gamma,1) = \frac{(\gamma,n-\gamma-1)\,(n-1,1)}{(\gamma+1,n-\gamma-1)},$$

his valoribus substitutis fiet

$$(n-2,1+\gamma) = \frac{1}{\gamma-1}\cdot\frac{(1,n-2)\,(n-\gamma,\gamma)\,(\gamma+1,n-\gamma-1)}{(n-\gamma,\gamma-1)\,(\gamma,n-\gamma-1)\,(n-1,1)}.$$

IV. Sumamus nunc $a = 1$, $b = n-3$, $c = \delta$, prodibitque haec aequatio

$$(1,n-3)\,(n-2,\delta) = (1,\delta)\,(1+\delta,n-3),$$

unde colligitur

$$(n-3,1+\delta) = \frac{(n-3,1)\,(n-2,\delta)}{(\delta,1)},$$

ubi ergo $1+\delta$ continet numeros $2, 3, 4 \ldots n-3$, ita ut hinc excludatur $n-3, 1$, quae autem per N°. I. datur. At si valores ante reperti substituantur, fiet

$$(n-3,1+\delta) = \frac{1}{\delta-2}\cdot\frac{(n-3,2)\,(n-2,1)\,(n-\delta+1,\delta-1)(\delta,n-\delta)(\delta+1,n-\delta-1)}{(n-2,2)\,(n-\delta+1,\delta-2)(\delta-1,n-\delta)(n-1,1)(\delta,n-\delta-1)},$$

unde patet esse debere $\delta > 2$, eodemque modo pro praecedente formula $\gamma > 1$, ita ut hic excludantur casus $(n-3,1)$, $(n-3,2)$, quorum quidem prior per N°. I. datur, alter vero per se.

V. Statuamus nunc $a = 1$, $b = n-4$ et $c = \varepsilon$, prodibitque haec aequatio

$$(1,n-4)\,(n-3,\varepsilon) = (1,\varepsilon)\,(1+\varepsilon,n-4),$$

unde concluditur

$$(n-4,1+\varepsilon) = \frac{(n-4,1)\,(n-3,\varepsilon)}{(1,\varepsilon)};$$

ubi si loco $(n-3,\varepsilon)$ valor ante inventus substitueretur, factor absolutus ingredeformer $\frac{1}{\varepsilon - 3}$, ita ut esse debeat $\varepsilon > 3$, ideoque $1 + \varepsilon > 4$, unde hic excluduntur casus $(n-4, 1)$, $(n-4, 2)$, $(n-4, 3)$, quorum quidem primus ex N°. II. tertius autem per se datur, medius vero revera manet incognitus.

VI. Statuamus porro $a = 1$, $b = n - 5$, $c = \zeta$, et aequatio erit

$$(1, n-5)\,(n-4, \zeta) = (1, \zeta)\,(1+\zeta, n-5),$$

unde fit

$$(n-5, 1+\zeta) = \frac{(n-5, 1)\,(n-4, \zeta)}{(1, \zeta)},$$

ubi ob formulam $(n-4, \zeta)$ debet esse $\zeta > 4$, ideoque $1+\zeta > 5$, unde hinc excluduntur casus $(n-5, 1)$, $(n-5, 2)$, $(n-5, 3)$, $(n-5, 4)$, quorum quidem primus ex N°. II. constat, quartus vero per se datur, ita ut hic occurrant duo casus etiamnunc incogniti $(n-5, 2)$ et $(n-5, 3)$.

VII. Simili modo si ulterius sumamus $a = 1$, $b = n - 6$ et $c = \eta$, prodibit

$$(n-6, 1+\eta) = \frac{(n-6, 1)\,(n-5, \eta)}{(1, \eta)},$$

ubi revera occurrunt tres sequentes casus $(n-6, 2)$, $(n-6, 3)$, $(n-6, 4)$, qui adhuc manent incogniti, atque hoc modo progredi licebit, quousque necesse fuerit; unde patet numerum casuum incognitorum continuo augeri, ita ut terminorum p et q alter futurus sit vel 2, vel 3, vel 4, etc. qui igitur casus adhuc definiendi restant.

VIII. Sumamus nunc primo $a = 1$, $b = \theta$, $c = 1$, ut aequatio nostra fiat

$$(1, \theta)\,(1+\theta, 1) = (1, 1)\,(2, \theta),$$

unde concludimus

$$(2, \theta) = \frac{(1, \theta)\,(1+\theta, 1)}{(1, 1)},$$

quae formula jam omnes casus exclusos suppeditat, in quibus alter terminus erat 2.

IX. Deinde sumamus $a = 2$, $b = \varkappa$ et $c = 1$, ut aequatio prodeat $(2, \varkappa)\,(2+\varkappa, 1) = (2, 1)\,(3, \varkappa)$, unde fit

$$(3, \varkappa) = \frac{(2, \varkappa)\,(2+\varkappa, 1)}{(2, 1)},$$

ubi cum $(2, \varkappa)$ per praecedentem N^{rum} detur, nunc etiam ii casus innotescunt, ubi alter terminus erat 3.

X. Sumatur porro $a = 3$, $b = \varkappa$, $c = 1$, eritque $(3, \varkappa)$ $(3+\varkappa, 1) = (3, 1)\,(4, \varkappa)$, unde fit

$$(4, \varkappa) = \frac{(3, \varkappa)\,(3+\varkappa, 1)}{(3, 1)},$$

unde igitur ii casus eliciuntur, ubi alter terminus erat 4. Eodem modo pro reliquis proceditur; sicque omnes plane casus in formula proposita contenti plene sunt determinati.

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4) De valoribus integralium a termino variabilis $x = 0$ usque ad $x = \infty$ extensorum. *M. S. Academiae exhib. d.* 30 *Aprilis* 1781.

§. 124. Talium formularum, quae a termino $x = 0$ usque ad terminum $x = \infty$ extensae finitum sortiuntur valorem, simplicissima est circularis $\int \frac{\partial x}{1+xx}$, cujus valor est $\frac{\pi}{2}$ denotante π peripheriam pro diametro $= 1$. Deinde etiam methodo prorsus singulari inveni esse

$$\int \frac{x^{m-1}\,\partial x}{(1+x)^n} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = \infty \end{bmatrix} = \frac{\pi}{n \sin. \frac{m\pi}{n}}.$$

Praeterea vero hoc modo plures alias formulas hujus generis expedivi, in quarum differentialia non solum functiones algebraicae ipsius x sed etiam lx ingrediatur.

§. 125. Obtulerunt se mihi autem quondam aliae hujusmodi formulae etiam functiones transcendentes involventes, quarum valores desiderati omnes methodos adhuc cognitas respuere videantur. Quaesiveram scilicet eam lineam curvam in qua radius osculi ubique reciproce esset proportionalis arcui curvae, ita ut posito arcu $= s$ et radio osculi $= r$, esset $rs = aa$. Hinc enim haud difficile est, figuram curvae libero quasi manus ductu describere, quandoquidem ea talem habere

Fig. 2. debet figuram. Initio nimirum curvae in A constituto inde curva continuo magis incurvabitur et tandem post infinitas spiras in certum punctum O glomerabitur, quod polum hujus curvae appellare licebit. Propositum igitur mihi fuerat locum hujus poli accuratius investigare, pro eoque quantitatem coordinatarum AC et CO perscrutari.

§. 126. Hunc in finem, introducta in calculum portionis cujusvis AM $= s$ amplitudine $= \phi$, ut sit $r = \frac{\partial s}{\partial \phi}$, fit $s \partial s = aa \partial \phi$, hincque

$$ss = 2aa\phi, \text{ et } s = a\sqrt{2\phi} = 2c\sqrt{\phi}.$$

Hinc jam prodit $\partial s = \frac{c \partial \phi}{\sqrt{\phi}}$, unde posita abscissa pro hoc arcu AP $= x$ et applicata PM $= y$, colligitur fore

$$x = c\int \frac{\partial \phi \cos. \phi}{\sqrt{\phi}} \text{ et } y = c\int \frac{\partial \phi \sin. \phi}{\sqrt{\phi}}.$$

§. 127. Hinc ergo pro polo O determinando requiruntur valores harum duarum formularum integralium, postquam a termino $\phi = 0$ usque ad $\phi = \infty$ fuerint extensae. Initio quidem sum arbitratus, hos valores aliter obtineri non posse nisi approximando, dum utraque formula successive per partes evolvatur; primo scilicet a $\phi = 0$ usque ad $\phi = \pi$; deinde a $\phi = \pi$ usque ad $\phi = 2\pi$; porro a $\phi = 2\pi$ usque ad $\phi = 3\pi$; etc. quippe quo pacto series prodibunt satis prompte convergentes. Verum evidens est hanc operationem longos calculos satis taediosos requirere, quos quidem evolvere non sum ausus. Nuper autem forte fortuna per methodum prorsus singularem perspexi esse tam

$$\int \frac{\partial\phi \cos.\phi}{\sqrt{\phi}} \left[\begin{smallmatrix} a\ \phi = 0 \\ ad\ \phi = \infty \end{smallmatrix}\right] = \sqrt{\tfrac{\pi}{2}} \text{ quam}$$
$$\int \frac{\partial\phi \sin.\phi}{\sqrt{\phi}} \left[\begin{smallmatrix} a\ \phi = 0 \\ ad\ \phi = \infty \end{smallmatrix}\right] = \sqrt{\tfrac{\pi}{2}};$$

ita ut pro loco poli quaesito O sit

$$AC = c\sqrt{\tfrac{\pi}{2}} \text{ et } CO = c\sqrt{\tfrac{\pi}{2}}.$$

§. 128. Quoniam igitur methodus, qua huc sum perductus, non parum polliceri videtur, Geometris haud ingratum fore arbitror, si eam omni cura hic exposuero. Et quia multo latius quam ad istas formulas patet, eam etiam omni extensione sum propositurus, quae omnia ex consideratione hujus formulae satis simplicis $\int x^{n-1}\partial x e^{-x}$ deduxi, cujus ergo integrale pro variis valoribus exponentis n investigare convenit.

§. 129. Ac primo quidem, pro casu $n = 1$ hujus formulae $\int \partial x e^{-x}$ integrale manifestum est $1 - e^{-x}$, quod casu $x = 0$ evanescit, facto autem $x = \infty$ abit in unitatem. Praeterea, cum hujus formulae $x^\lambda . e^{-x}$ differentiale sit

$$\lambda x^{\lambda-1}\partial x . e^{-x} - x^\lambda \partial x . e^{-x},$$

erit vicissim

$$\int x^\lambda \partial x . e^{-x} = \lambda \int x^{\lambda-1}\partial x . e^{-x} - x^\lambda . e^{-x},$$

quod postremum membrum tam pro casu $x = 0$ quam $x = \infty$ evanescit, si modo fuerit $\lambda > 0$. Tum igitur pro nostris terminis integrationis erit

$$\int x^{\lambda}\,\partial x\,.\,e^{-x} = \lambda \int x^{\lambda-1}\,\partial x\,.\,e^{-x},$$

cujus formulae ope, ob $\int \partial x e^{-x} = 1$, sequentes integralium valores deducuntur

$$\int x\,\partial x\, e^{-x} = 1$$
$$\int x^2\,\partial x\,.\,e^{-x} = 1.\,2$$
$$\int x^3\,\partial x\,.\,e^{-x} = 1.\,3\,.\,3$$
$$\int x^4\,\partial x\,.\,e^{-x} = 1.\,2\,.\,3\,.\,4$$

sicque in genere

$$\int x^{n-1}\,\partial x\, e^{-x} = 1.\,2\,.\,3\,.\,4\,.\,.\,.\,.\,.\,.\,.\,.\,(n-1),$$

cujus producti valores quoties n fuerit numerus integer positivus sponte se produnt; quando autem n est numerus fractus olim ostendi, quomodo valores per quadraturas curvarum algebraicarum exhiberi queant. Sic pro casu $n = \frac{1}{2}$ constat, istum valorem esse $= \sqrt{\pi}$.

§. 130. Cum igitur omnes valores hujus producti infiniti $1\,.\,2\,.\,3\,.\,4\,.\,.\,.\,.\,(n-1)$ tanquam cogniti spectari queant, eos littera Δ designabo, ita ut sit $\Delta = 1.\,2\,.\,3\,.\,4\,.\,.\,.\,.\,(n-1)$, sicque jam adepti sumus hanc insignem formulam integralem

$$\int x^{n-1}\,\partial x\,.\,e^{-x} = \Delta,$$

integrali scilicet ab $x = 0$ ad $x = \infty$ extenso; atque ex hac ipsa formula omnia deduxi, quae ad casum ante memoratum pertinent, ubi quidem ratiocinia penitus singularia adhiberi debent, quae igitur hic diligentius sum expositurus.

§. 131. Posui autem primo $x = ky$, et quoniam ambo termini integralis iidem manent, erit etiam

$$k^n \int y^{n-1}\,\partial y\,.\,e^{-ky} = \Delta,$$

quandoquidem haec formula etiam ab $y = 0$ ad $y = \infty$ usque extenditur; quamobrem per k^n dividendo habebimus

$$\int y^{n-1}\,\partial y \cdot e^{-ky} = \frac{\Delta}{k^n},$$

ubi autem notari oportet, pro k nullos numeros negativos accipi posse, quia alioquin formula e^{-ky} non amplius evanesceret casu $x = 0$, atque hic isti soli valores sunt excludendi, ita ut etiam valores imaginarii loco k adhiberi queant, atque hinc illas arduas integrationes sum assecutus.

§. 132. Ponamus ergo $k = p + q\sqrt{-1}$, et cum sit

$$e^{-qy\sqrt{-1}} = \cos. qy - \sqrt{-1}\sin. qy, \text{ et}$$
$$e^{+qy\sqrt{-1}} = \cos. qy + \sqrt{-1}\sin. qy,$$

nostra formula nunc induet hanc formam

$$\int y^{n-1}\,\partial y \cdot e^{-py}(\cos. qy - \sqrt{-1}\sin. qy) = \frac{\Delta}{(p+q\sqrt{-1})^n}.$$

Quamobrem si formulae imaginariae signum mutemus, erit simili modo

$$\int y^{n-1}\,\partial y \cdot e^{-py}(\cos. qy + \sqrt{-1}\sin. qy) = \frac{\Delta}{(p-q\sqrt{-1})^n}.$$

§. 133. Quo valores inventos commodius exprimere liceat, ponamus $p = f\cos.\theta$ et $q = f\sin.\theta$, eritque

$$(p + q\sqrt{-1})^n = f^n(\cos. n\theta + \sqrt{-1}\sin. n\theta) \text{ et}$$
$$(p - q\sqrt{-1})^n = f^n(\cos. n\theta - \sqrt{-1}\sin. n\theta);$$

ubi notasse juvabit fore tang. $\theta = \frac{q}{p}$, unde ex valoribus p et q assumtis erit etiam $f = \sqrt{(pp + qq)}$. Hoc ergo modo fit priore casu

$$\frac{\Delta}{(p+q\sqrt{-1})^n} = \frac{\Delta}{f^n(\cos. n\theta + \sqrt{-1}\sin. n\theta)},$$

pro altero

$$\frac{\Delta}{(p-q\sqrt{-1})^n} = \frac{\Delta}{f^n\,(\cos.\, n\theta - \sqrt{-1}\,\sin.\, n\theta)}.$$

Quamobrem si hae duae formulae addantur prodibit

$$\frac{2\,\Delta\,\cos.\, n\theta}{f^n}.$$

Differentia autem harum formularum dat

$$\frac{2\,\Delta\sqrt{-1}\,\sin.\, n\theta}{f^n}.$$

§. 134. Addamus igitur quoque ipsas formulas integrales, et habebimus

$$\int y^{n-1}\,\partial y\,.\,e^{-py}\cos.\, qy = \frac{\Delta\,\cos.\, n\theta}{f^n}.$$

Sin autem subtrahamus et per $2\sqrt{-1}$ dividamus, oritur

$$\int y^{n-1}\,\partial y\,.\,e^{-py}\sin.\, qy = \frac{\Delta\,\sin.\, n\theta}{f^n}.$$

quae jam duae formulae integrales latissime patent, cum numeri p et q prorsus arbitrio nostro relinquuntur, id tantum observando, ne pro p numeri negativi accipiantur. Operae igitur pretium erit, has duas formulas integrales sequentibus binis theorematibus complecti.

Theorema I.

Posito $\Delta = 1\,.\,2\,.\,2\,\ldots\,(n-1)$, et pro litteris p et q numeros quoscunque positivos accipiendo, fiat inde $\sqrt{(pp+qq)} = f$, et quaeratur angulus θ, ut fit tang. $\theta = \frac{q}{p}$, et habebitur ista integratio memorabilis

$$\int x^{n-1}\,\partial x\,.\,e^{-px}\cos.\, qx \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = \infty \end{bmatrix} = \frac{\Delta\,\cos.\, n\theta}{f^n}.$$

Theorema II.

Posito $\Delta = 1.2.3 \ldots\ldots (n-1)$, et pro litteris p et q numeros quoscunque positivos accipiendo, fiat inde $\sqrt{(pp+qq)} = f$, et quaeratur angulus θ, ut sit tang. $\theta = \frac{q}{p}$, atque habebitur ista integratio memorabilis

$$\int x^{n-1}\partial x \,.\, e^{-px} \sin. qx \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = \infty \end{bmatrix} = \frac{\Delta \sin. n\theta}{f^n}.$$

§. 135. Cum igitur pro casu curvae supra consideratae pervenerimus ad has formulas integrales

$$\int \frac{\partial\Phi \cos.\Phi}{\sqrt{\Phi}} \text{ et } \int \frac{\partial\Phi \sin.\Phi}{\sqrt{\Phi}},$$

facta applicatione erit $n = \frac{1}{2}$, ideoque $\Delta = \sqrt{\pi}$, tum vero erit $p = 0$ et $q = 1$, unde fit $f = 1$ et tang. $\theta = \frac{q}{p} = \infty$; ideoque $\theta = \frac{\pi}{2}$, ergo cos. $n\theta = \frac{1}{\sqrt{2}} = \sin. n\theta$. Hinc igitur fiet

$$\int \frac{\partial\Phi \cos.\Phi}{\sqrt{\Phi}} \begin{bmatrix} a\ \Phi = 0 \\ ad\ \Phi = \infty \end{bmatrix} = \sqrt{\tfrac{\pi}{2}}, \text{ simulque}$$

$$\int \frac{\partial\Phi \sin.\Phi}{\sqrt{\Phi}} \begin{bmatrix} a\ \Phi = 0 \\ ad\ \Phi = \infty \end{bmatrix} = \sqrt{\tfrac{\pi}{2}}.$$

§. 136. Operae autem pretium erit, hunc casum quo $n = \frac{1}{2}$ et $\Delta = \sqrt{\pi}$ in genere evolvere, et cum posuerimus

$$\sqrt{(pp+qq)} = f \text{ et } \frac{q}{p} = \text{tang. } \theta, \text{ erit}$$

$$\sin. \theta = \frac{q}{f} \text{ et } \cos. \theta = \frac{p}{f}.$$

Hinc ergo primo

$$\sin. \tfrac{1}{2}\theta = \sqrt{\frac{1-\cos.\theta}{2}} = \sqrt{\frac{f-p}{2f}} \text{ et}$$

$$\cos. \tfrac{1}{2}\theta = \sqrt{\frac{1+\cos.\theta}{2}} = \sqrt{\frac{f+p}{2f}};$$

unde fit pro valoribus integralibus

$$\frac{\Delta \sin. \frac{1}{2}\theta}{\sqrt{f}} = \frac{\sqrt{\pi}}{f}\sqrt{\frac{f-p}{2}} \text{ et}$$

$$\frac{\Delta \cos. \frac{1}{2}\theta}{\sqrt{f}} = \frac{\sqrt{\pi}}{f} \cdot \sqrt{\frac{f+p}{2}}.$$

Quamobrem habebimus binas sequentes formulas integrales

$$\int \frac{\partial x}{\sqrt{x}} e^{-px} \sin. qx = \frac{\sqrt{\pi}}{f} \cdot \sqrt{\frac{f-p}{2}}$$

$$\int \frac{\partial x}{\sqrt{x}} e^{-px} \cos. qx = \frac{\sqrt{\pi}}{f} \cdot \sqrt{\frac{f+p}{2}}.$$

§. 137. Casus autem, quibus pro n sumitur numerus integer positivus, ideoque Δ absolute per numeros integros exhiberi potest, ita sunt comparati, ut etiam per methodos cognitas, ope scilicet formularum integralium reductionis satis notae expediri queant, atque adeo integralia in genere exhiberi. Haec autem operatio postulat calculos non parum prolixos, quamobrem formulae nostrae satis simplices pro casu scilicet $x = \infty$ nihilo minus omni attentione sunt dignae. Quando autem exponenti n valores negativos tribuere voluerimus, hi casus statim in initio integrationis additionem constantis infinitae postulant, ut scilicet integralia evanescant casu $x = 0$, sicque adeo valores integralium, quae hic quaerimus, manebunt infiniti, ideoque ad institutum nostrum non sunt referendi.

§. 138. Casus autem maxime memorabilis hic occurrit, quo $n = 0$, et qui prorsus singularem sollertiam postulat, quem igitur accuratius evolvamus. Quoniam posuimus

$$\Delta = 1 . 2 . 3 . 4 \ldots . (n-1),$$

statuamus simili modo

$$\Delta' = 1 . 2 . 3 \ldots . n, \text{ et } \Delta'' = 1 . 2 . 3 \ldots \ldots (n+1),$$

eritque manifesto

$$\Delta = \frac{\Delta'}{n}, \text{ et } \Delta' = \frac{\Delta''}{n+1}, \text{ ideoque } \Delta = \frac{\Delta''}{n(n+1)}.$$

Sumamus nunc $n = \omega$, existente ω infinite parvo, et cum sit

$\Delta'' = 1$, inde fit $\Delta = \frac{1}{\omega}$, ideoque ejus valor erit infinitus. Cum autem pro formula integrali priore sit sin. $n\theta = \omega\theta$, evidens est fore Δ sin. $n\theta = \theta$; quamobrem ista prior formula integralis erit $\int \frac{\partial x}{x} e^{-px}$ sin. $qx = \theta$, dum nempe integrale a termino $x = 0$ usque ad terminum $x = \infty$ extenditur. Alterius autem formulae nostrae integralis $\int \frac{\partial x}{x} e^{-px}$ cos. qx valor erit infinite magnus. Ille autem casus omnino meretur ut eum singulari theoremate complectamur.

Theorema III.

§. 139. Si litterae p et q denotent numeros positivos quoscunque, atque hinc quaeratur angulus θ, ut sit tang. $\theta = \frac{q}{p}$, habebitur sequens integratio maxime memorabilis

$$\int \frac{\partial x}{x} e^{-px} \text{ sin. } qx \begin{bmatrix} ab\ x = 0 \\ ad\ x = \infty \end{bmatrix} = \theta$$

cujus theorematis demonstratio dubito quin alio modo quam per approximationes investigari queat.

§. 140. Casus autem simplicissimus quo $p = 0$ et $q = 1$ jam omnia calculi artificia adhuc cognita superare videtur, quia autem hoc casu fit tang. $\theta = \frac{1}{0} = \infty$, erit $\theta = \frac{\pi}{2}$, unde oritur haec integratio $\int \frac{\partial x}{x}$ sin. $x = \frac{\pi}{2}$. Interim tamen de ejus veritate eo minus dubitare licet, quod approximationes adhibitae ad eundem valorem propemodum perducant. Quodsi hunc casum cum initio memorato $\int \frac{\partial x}{\sqrt{x}}$ sin. $x = \sqrt{\frac{\pi}{2}}$ comparemus, ingens similitudo summam attentionem meretur, cum hujus integrale sit praecise radix quadrata illius.

5) Investigatio formulae integralis $\int \frac{x^{m-1} \partial x}{(1+x^k)^n}$, casu quo post integrationem statuitur $x = \infty$. *Opuscula Analytica. Tom. II. Pag.* 42 — 54.

§. 141. Jam satis notum est, hujus formulae integrale partim logarithmos, partim arcus circulares complecti, et partes logarithmicas hanc progressionem constituere

$$- \frac{2}{k} \cos. \frac{m\pi}{k} \; l\sqrt{(1 - 2x\cos. \frac{\pi}{k} + xx)}$$
$$- \frac{2}{k} \cos. \frac{3m\pi}{k} \; l\sqrt{(1 - 2x\cos. \frac{3\pi}{k} + xx)}$$
$$- \frac{2}{k} \cos. \frac{5m\pi}{k} \; l\sqrt{(1 - 2x\cos. \frac{5\pi}{k} + xx)}$$
$$- \frac{2}{k} \cos. \frac{7m\pi}{k} \; l\sqrt{(1 - 2x\cos. \frac{7\pi}{k} + xx)}$$
$$- \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot$$
$$- \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot$$
$$- \frac{2}{k} \cos. \frac{im\pi}{k} \; l\sqrt{(1 - 2x\cos. \frac{i\pi}{k} + xx)}$$

ubi i denotat numerum imparem non majorem quam k. Hinc si k fuerit numerus par, erit $i = k - 1$; ac si k fuerit numerus impar, hanc progressionem continuari oportet usque ad $i = k$, ejus vero coëfficiens duplo minor capi debet, seu loco $-\frac{2}{k}$ tantum scribi debet $-\frac{1}{k}$, cujus irregularitatis ratio in Tomo I est exposita.

§. 142. Cum hae partes sponte jam evanescant posito $x = 0$, statuamus statim $x = \infty$, et cum in genere sit

$$\sqrt{(1 - 2x\cos. \omega + xx)} = x - \cos. \omega, \text{ erit}$$
$$l\sqrt{(1 - 2x\cos. \omega + xx)} = l(x - \cos. \omega)$$
$$= lx - \frac{\cos. \omega}{x} = lx, \text{ ob } \frac{\cos. \omega}{x} = 0;$$

omnes ergo illi logarithmi reducuntur ad eandem formam lx, quae multiplicanda est per hanc seriem

$$-\frac{2}{k}\cos.\frac{m\pi}{k}-\frac{2}{k}\cos.\frac{3m\pi}{k}-\frac{2}{k}\cos.\frac{5m\pi}{k}\ldots-\frac{2}{k}\cos.\frac{im\pi}{k},$$

ubi, ut diximus, i denotat maximum numerum imparem ipso k non majorem, hac tamen restrictione, ut, si k fuerit impar, ideoque $i=k$, ultimum membrum ad dimidium reduci debeat. Quamobrem, si hujus progressionis summam investigare velimus, duo casus erunt constituendi: alter quo k est numerus par et $i=k-1$, alter vero quo k est impar et $i=k$.

Evolutio casus prioris, quo k est numerus par et $i=k-1$.

§. 143. Hoc ergo casu, posito $x=\infty$, formula $-\frac{2}{klx}$ multiplicatur per hanc cosinuum seriem

$$\cos.\frac{m\pi}{k}+\cos.\frac{3m\pi}{k}+\cos.\frac{5m\pi}{k}+\cos.\frac{7m\pi}{k}+\ldots+\cos.\frac{(k-1)m\pi}{k},$$

cujus summam statuamus $=S$. Ducamus hanc seriem in $\sin.\frac{m\pi}{k}$, et cum in genere sit

$$\sin.\frac{m\pi}{k}\cos.\frac{im\pi}{k}=\tfrac{1}{2}\sin.\frac{(i+1)m\pi}{k}-\tfrac{1}{2}\sin.\frac{(i-1)m\pi}{k},$$

facta hac reductione habebimus

$$S\sin.\frac{m\pi}{k}=\tfrac{1}{2}\sin.\frac{2m\pi}{k}+\tfrac{1}{2}\sin.\frac{4m\pi}{k}+\tfrac{1}{2}\sin.\frac{6m\pi}{k}\ldots+\tfrac{1}{2}\sin.\frac{(k-2)m\pi}{k}+\tfrac{1}{2}\sin.m\pi$$
$$-\tfrac{1}{2}\sin.\frac{2m\pi}{k}-\tfrac{1}{2}\sin.\frac{4m\pi}{k}-\tfrac{1}{2}\sin.\frac{6m\pi}{k}\ldots-\tfrac{1}{2}\sin.\frac{(k-2)m\pi}{k},$$

ubi omnes termini praeter ultimum manifesto se destruunt, ita ut sit

$$S\sin.\frac{m\pi}{k}=\tfrac{1}{2}\sin.m\pi.$$

Jam vero quia nostri coëfficientes m et k supponuntur integri, utique erit $\sin.m\pi=0$, ideoque etiam $S=0$, nisi forte etiam fuerit $\sin.\frac{m\pi}{k}=0$, qui autem casus locum habere nequit, quoniam in integratione formulae propositae $\frac{x^{m-1}\partial x}{1+x^k}$, semper assumi solet esse $m < k$. Hoc igitur modo evictum est, casu quo post integrationem statuitur $x = \infty$, omnes partes logarithmicas integralis se destruere.

Evolutio casus alterius, quo est k numerus impar et $i = k$.

§. 144. Hoc ergo casu, sumto $x = \infty$, formula $l x$ multiplicatur per hanc seriem

$$-\frac{2}{k}\cos.\frac{m\pi}{k} - \frac{2}{k}\cos.\frac{3m\pi}{k} - \frac{2}{k}\cos.\frac{5m\pi}{k} \ldots\ldots - \frac{1}{k}\cos.\frac{km\pi}{k},$$

ubi terminus penultimus est $-\frac{2}{k}\cos.\frac{(k-2)m\pi}{k}$, pro ultimo vero termino erit $\cos. m\pi = \pm 1$, signo superiore valente si m sit numerus par, inferiore si impar; quare remoto termino ultimo pro reliquis ponamus

$$\cos.\frac{m\pi}{k} + \cos.\frac{3m\pi}{k} + \cos.\frac{5m\pi}{k} + \ldots\ldots + \cos.\frac{(k-2)m\pi}{k} = S,$$

ita ut multiplicator ipsius logarithmi x sit

$$-\frac{2S}{k} - \frac{1}{k}\cos. m\pi.$$

Hinc procedendo ut ante fiet

$$S\sin.\frac{m\pi}{k} = \tfrac{1}{2}\sin.\frac{2m\pi}{k} + \tfrac{1}{2}\sin.\frac{4m\pi}{k} + \tfrac{1}{2}\sin.\frac{6m\pi}{k} \ldots\ldots + \tfrac{1}{2}\sin.\frac{(k-3)m\pi}{k} + \tfrac{1}{2}\sin.\frac{(k-1)m\pi}{k}$$
$$-\tfrac{1}{2}\sin.\frac{2m\pi}{k} - \tfrac{1}{2}\sin.\frac{4m\pi}{k} - \tfrac{1}{2}\sin.\frac{6m\pi}{k} \ldots\ldots - \tfrac{1}{2}\sin.\frac{(k-3)m\pi}{k};$$

ubi iterum omnes termini praeter ultimum se mutuo tollunt, ita ut hinc prodeat

$$S\sin.\frac{m\pi}{k} = \tfrac{1}{2}\sin.\frac{(k-1)m\pi}{k} = \tfrac{1}{2}\sin.\left(m\pi - \frac{m\pi}{k}\right);$$

at vero est

$$\sin.\left(m\pi - \frac{m\pi}{k}\right) = \sin. m\pi\cos.\frac{m\pi}{k} - \cos. m\pi\sin.\frac{m\pi}{k},$$

ubi notetur esse $\sin. m\pi = 0$, ob m numerum integrum; habebimus ergo

$$S\sin.\frac{m\pi}{k} = -\tfrac{1}{2}\cos. m\pi\sin.\frac{m\pi}{k}, \text{ sive } S = -\tfrac{1}{2}\cos. m\pi,$$

consequenter multiplicator ipsius lx erit

$$= \tfrac{1}{k}\cos. m\pi - \tfrac{1}{k}\cos. m\pi = 0,$$

sicque manifestum est, sive k sit numerus par sive impar, omnia membra logarithmica in nostro integrali se mutuo destruere, siquidem post integrationem statuamus $x = \infty$, quemadmodum hic semper supponimus.

§. 145. Consideremus nunc etiam partes a circulo pendentes, ex quibus integrale nostrae formulae componitur. Hae autem partes sequentem progressionem constituere sunt compertae

$$\frac{2}{k}\sin.\frac{m\pi}{k}\text{Arc. tang.}\frac{x\sin.\frac{\pi}{k}}{1-x\cos.\frac{\pi}{k}} + \frac{2}{k}\sin.\frac{3m\pi}{k}\text{Arc. tang.}\frac{x\sin.\frac{3\pi}{k}}{1-x\cos.\frac{3\pi}{k}}$$

$$+\frac{2}{k}\sin.\frac{5m\pi}{k}\text{Arc. tang.}\frac{x\sin.\frac{5\pi}{k}}{1-x\cos.\frac{5\pi}{k}} + \frac{2}{k}\sin.\frac{7m\pi}{k}\text{Arc. tang.}\frac{x\sin.\frac{7\pi}{k}}{1-x\cos.\frac{7\pi}{k}}$$

$$+ \ldots\ldots\ldots + \frac{2}{k}\sin.\frac{im\pi}{k}\text{Arc. tang.}\frac{x\sin.\frac{i\pi}{k}}{1-x\cos.\frac{i\pi}{k}}$$

ubi in ultimo membro est vel $i = k-1$, vel $i = k$; prius scilicet valet si i est numerus par, posterius si impar.

§. 146. Cum etiam omnia haec membra evanescant posito $x = 0$, faciamus pro instituto nostro $x = \infty$. In genere igitur fiet

$$\text{Arc. tang.}\frac{x\sin.\frac{i\pi}{k}}{1-x\cos.\frac{i\pi}{k}} = \text{Arc. tang.}\left(-\text{tang.}\frac{i\pi}{k}\right).$$

Est vero

$$-\text{tang.}\tfrac{i\pi}{k} = +\text{tang.}\tfrac{(k-i)\pi}{k},$$

ex quo hic arcus fit $= \frac{(k-i)\pi}{k}$. Hinc ergo loco i scribendo successive numeros 1, 3, 5, 7 etc. istae partes nostri integralis quaesiti erunt

$$\frac{2(k-1)\pi}{kk}\sin.\frac{2m\pi}{k}+\frac{2(k-3)\pi}{kk}\sin.\frac{3m\pi}{k}+\frac{2(k-5)\pi}{kk}\sin.\frac{5m\pi}{k}$$
$$+\frac{2(k-7)\pi}{kk}\sin.\frac{7m\pi}{k}+\frac{2(k-9)\pi}{kk}\sin.\frac{9m\pi}{k}+\dots\frac{2(k-i)\pi}{kk}\sin.\frac{7m\pi}{k}$$

ubi casu, quo k est numerus par, progredi oportet usque ad $i=k-1$: ac si k sit numerus impar, usque ad $i=k$.

§. 147. Statuamus brevitatis gratia

$$(k-1)\sin.\frac{m\pi}{k}+(k-3)\sin.\frac{3m\pi}{k}+(k-5)\sin.\frac{5m\pi}{k}+\dots$$
$$+(k-i)\sin.\frac{im\pi}{k}=S$$

ita ut integrale quaesitum sit $\frac{2\pi S}{kk}$, quandoquidem partes logarithmicae se mutuo destruxerunt. Multiplicemus nunc utrinque per $2\sin.\frac{m\pi}{k}$, et cum in genere sit

$$2\sin.\frac{m\pi}{k}\sin.\frac{im\pi}{k}=\cos.\frac{(i-1)m\pi}{k}-\cos.\frac{(i+1)m\pi}{k},$$

facta substitutione erit

$$2S\sin.\frac{m\pi}{k}=(k-1)\cos.\frac{0m\pi}{k}+(k-3)\cos.\frac{2m\pi}{k}+(k-5)\cos.\frac{4m\pi}{k}\dots$$
$$-(k-1)\cos.\frac{2m\pi}{k}-(k-3)\cos.\frac{4m\pi}{k}-(k-5)\cos.\frac{6m\pi}{k}\dots$$
$$\dots+(k-i)\cos.\frac{(i-1)m\pi}{k}.$$
$$-(k-i)\cos.\frac{(i+1)m\pi}{k}$$

quae series manifesto contrahitur in sequentem

$$2S\sin.\frac{m\pi}{k}=(k-1)-2\cos.\frac{2m\pi}{k}-2\cos.\frac{4m\pi}{k}-2\cos.\frac{6m\pi}{k}\dots$$
$$-2\cos.\frac{(i-1)m\pi}{k}-(k-i)\cos.\frac{(i+1)m\pi}{k}$$

ubi, primo et ultimo membro sublatis, regularem termini intermedii constituunt seriem, pro cujus valore investigando ponamus

$$T=\cos.\frac{2m\pi}{k}+\cos.\frac{4m\pi}{k}+\cos.\frac{6m\pi}{k}+\dots+\cos.\frac{(i-1)m\pi}{k},$$

ita ut sit

$$2\,S \sin.\frac{m\pi}{k} = k - 1 - 2\,T - (k-i)\cos.\frac{(i+1)m\pi}{k}.$$

Hic autem iterum convenit duos casus perpendere, prout k fuerit par vel impar.

Evolutio casus prioris, quo k est numerus par et $i = k - 1$.

§. 148. Hoc ergo casu habebimus

$$T = \cos.\frac{2m\pi}{k} + \cos.\frac{4m\pi}{k} + \cos.\frac{6m\pi}{k} + \ldots\ldots \cos.\frac{(k-2)m\pi}{k}.$$

Multiplicemus denuo per $2\sin.\frac{m\pi}{k}$, et per reductiones supra indicatas habebimus

$$2\,T \sin.\frac{m\pi}{k} = \begin{array}{l} \quad\quad\quad + \sin.\frac{3m\pi}{k} + \sin.\frac{5m\pi}{k} \ldots + \sin.\frac{(k-3)m\pi}{k} + \sin.\frac{(k-1)m\pi}{k} \\ - \sin.\frac{m\pi}{k} - \sin.\frac{3m\pi}{k} - \sin.\frac{5m\pi}{k} \ldots - \sin.\frac{(k-3)m\pi}{k} \end{array}$$

deletis igitur terminis se mutuo tollentibus erit

$$2\,T \sin.\frac{m\pi}{k} = -\sin.\frac{m\pi}{k} + \sin.\frac{(k-1)m\pi}{k}.$$

Est vero

$$\sin.\frac{(k-1)m\pi}{k} = \sin.\left(m\pi - \frac{m\pi}{k}\right) = \sin.m\pi\cos.\frac{m\pi}{k} - \cos.m\pi\sin.\frac{m\pi}{k},$$

ubi $\sin.\,m\pi = 0$, quamobrem fiet $2\,T = -1 - \cos.\,m\pi$.

§. 149. Invento valore pro T colligitur fore

$$2\,S \sin.\frac{m\pi}{k} = k, \text{ ideoque } S = \frac{k}{2\sin.\frac{m\pi}{k}}.$$

Denique vero ipse valor formulae nostrae integralis, quem quaerimus, erit $\frac{2\pi S}{kk}$, et nunc manifestum est, integrale nostrae formulae, casu quo S est numerus par, fore $\frac{\pi}{k\sin.\frac{m\pi}{k}}$, siquidem post integrationem statuatur $x = \infty$.

Evolutio alterius casus, quo k est numerus impar et $i = k$.

§. 150. Hoc ergo casu est

$$T = \cos.\frac{2m\pi}{k} + \cos.\frac{4m\pi}{k} + \cos.\frac{6m\pi}{k} + \ldots + \cos.\frac{(k-1)m\pi}{k},$$

quae series multiplicata per 2 sin. $\frac{m\pi}{k}$ producet ut ante

$$2T\sin.\frac{m\pi}{k} = \quad + \sin.\frac{3m\pi}{k} + \sin.\frac{5m\pi}{k} \ldots + \sin.\frac{(k-2)m\pi}{k} + \sin.\frac{km\pi}{k}$$
$$- \sin.\frac{m\pi}{k} - \sin\frac{3m\pi}{k} - \sin.\frac{5m\pi}{k} \ldots - \sin.\frac{(k-2)m\pi}{k}$$

unde deletis terminis se mutuo tollentibus reperietur

$$2T\sin.\frac{m\pi}{k} = -\sin.\frac{m\pi}{k} + \sin.m\pi$$

ideoque

$$2T = -1 + \frac{\sin.m\pi}{\sin.\frac{m\pi}{k}} = 1, \text{ ob } \sin.m\pi = 0,$$

hincque porro fiet

$$2S\sin.\frac{m\pi}{k} = k;$$

quare cum valor integralis quaesitus sit $\frac{2\pi S}{kk}$, erit etiam hoc casu integrale nostrum $= \frac{\pi}{k\sin.\frac{m\pi}{k}}$, prorsus uti praecedente casu. Hinc ergo deducimus sequens theorema.

Theorema.

§. 151. Si haec formula differentialis $\frac{x^{m-1}\partial x}{1+x^k}$ ita integretur, ut, posito $x = 0$, integrale evanescat, tum vero statuatur $x = \infty$, valor inde resultans semper erit $\frac{\pi}{k\sin.\frac{m\pi}{k}}$, sive k sit numerus par, sive impar. Hujus theorematis demonstratio ex praecedentibus est manifesta.

§. 152. In evolutione hujus formulae assumsimus esse $m < k$, quia alioquin membra logarithmica se non destruissent; at vero ne hac quidem limitatione nunc amplius est opus. Casu enim quo foret $m = k$, integrale formulae $\frac{x^{m-1}\partial x}{1+x^k}$ esset $\frac{1}{k} l(1+x^k)$, quod facto $x = \infty$ fieret etiam ∞; verum hoc idem indicat, nostrum integrale esse $\frac{\pi}{k \sin. \pi} = \infty$. Dummodo ergo m non fuerit majus quam k, nostra formula veritati semper est consentanea

§. 153. Quin etiam ne quidem necesse est ut exponentes m et k sint numeri integri, dummodo non fuerit $m > k$; si enim fuerit $m = \frac{\mu}{\lambda}$ et $k = \frac{\varkappa}{\lambda}$, erit valor per nostram formulam $\frac{\lambda\pi}{\varkappa \sin. \frac{\mu\pi}{\varkappa}}$, cujus veritas ita ostenditur. Quia hoc casu formula integranda est $\int \frac{x^{\frac{\mu}{\lambda}}}{1+x^{\frac{\varkappa}{\lambda}}} \cdot \frac{\partial x}{x}$, statuatur $x = y^{\lambda}$, erit $\frac{\partial x}{x} = \frac{\lambda \partial y}{y}$, et formula fiet

$$\int \frac{y^{\mu}}{1+y^{\varkappa}} \cdot \frac{\lambda \partial y}{y} = \lambda \int \frac{y^{\mu-1}\partial y}{1+y^{\varkappa}},$$

cujus valor utique erit $\frac{\lambda\pi}{\varkappa \sin. \frac{\mu\pi}{\varkappa}}$.

Alia demonstratio theorematis.

§. 154. Denotet P valorem integralis $\int \frac{x^m}{1+x^k} \cdot \frac{\partial x}{x}$ a termino $x = 0$ usque ad $x = 1$; at Q valorem ejusdem integralis a termino $x = 1$ usque ad $x = \infty$, ita ut $P + Q$ praebeat eum ipsum valorem, qui in theoremate continetur.

Nunc pro valore Q inveniendo statuatur $x = \frac{1}{y}$, unde fit $\frac{\partial x}{x} = -\frac{\partial y}{y}$, fietque

$$Q = \int \frac{y^{-m}}{1 + y^{-k}} \cdot \frac{-\partial y}{y} = -\int \frac{y^{k-m}\,\partial y}{1 + y^k} \cdot \frac{\partial y}{y}$$

a termino $y = 1$ usque ad $y = 0$. Hinc igitur, commutatis terminis erit

$$Q = +\int \frac{y^{k-m}}{1 + y^k} \cdot \frac{\partial y}{y}$$

a termino $y = 0$ usque ad $y = 1$. Jam quia hoc integrali expedito littera y ex calculo egreditur, loco y scribere licebit x, ita ut sit

$$Q = \int \frac{x^{k-m}}{1 + x^k} \cdot \frac{\partial x}{x},$$

quo facto habebimus

$$P + Q = \int \frac{x^m + x^{k-m}}{1 + x^k} \cdot \frac{\partial x}{x}$$

a termino $x = 0$ usque ad terminum $x = 1$. Verum non ita pridem demonstravi, valorem hujus formulae integralis intra terminos $x = 0$ et $x = 1$ contentum esse $= \frac{\pi}{k \sin. \frac{m\pi}{k}}$. Hinc igitur nascitur sequens theorema non minus notatu dignum.

Theorema.

§. 155. Valor hujus formulae integralis

$$\int \frac{x^m + x^{k-m}}{1 + x^k} \cdot \frac{\partial x}{x}$$

intra terminos $x = 0$ et $x = 1$ contentus, aequalis est valori istius integralis $\int \frac{x^m}{1+x^k} \cdot \frac{\partial x}{x}$, intra terminos $x = 0$ et $x = \infty$ contento.

§. 156. His expensis formulam integralem in titulo propositam aggrediamur, et quo eam ad formam hactenus tractatam reducamus, in subsidium vocemus sequentem reductionem

$$\int \frac{x^{m-1}\,\partial x}{(1+x^k)^{\lambda+1}} = \frac{A x^m}{(1+x^k)^\lambda} + B \int \frac{x^{m-1}\,\partial x}{(1+x^k)^\lambda},$$

unde facta differentiatione prodit sequens aequatio

$$\frac{x^{m-1}\,\partial x}{(1+x^k)^{\lambda+1}} = \frac{m A x^{m-1}\,\partial x}{(1+x^k)^\lambda} - \frac{\lambda k A x^{m+k-1}\,\partial x}{(1+x^k)^{\lambda+1}} + \frac{B x^{m-1}\,\partial x}{(1+x^k)^\lambda},$$

quae aequatio per $x^{m-1}\,\partial x$ divisa ac per $(1+x^k)^\lambda$ multiplicata, terminum negativum a dextra ad sinistram transponendo, erit

$$\frac{1 + \lambda k A x^k}{1+x^k} = m A + B,$$

quae aequatio manifesto subsistere nequit, nisi sit $\lambda k A = 1$, sive $A = \frac{1}{\lambda k}$, unde erit $1 = mA + B = \frac{m}{\lambda k} + B$, sicque $B = 1 - \frac{m}{\lambda k}$.

§. 157. Inventis his valoribus pro litteris A et B, primum assumimus, integralia ita capi, ut evanescant posito $x = 0$; tum vero posito $x = \infty$, quia exponens n minor supponitur quam k, membrum absolutum littera A affectum sponte evanescit, ita ut hoc casu $x = \infty$ fiat

$$\int \frac{x^{m-1}\,\partial x}{(1+x^k)^{\lambda+1}} = \left(1 - \frac{m}{\lambda k}\right) \int \frac{x^{m-1}\,\partial x}{(1+x^k)^\lambda}.$$

Quod si jam primo capiamus $\lambda = 1$, quia ante invenimus pro eodem casu $x = \infty$ esse

$$\int \frac{x^{m-1}\partial x}{1+x^k} = \frac{\pi}{k \sin.\frac{m\pi}{k}},$$

habebimus valorem istius integralis

$$\int \frac{x^{m-1}\partial x}{(1+x^k)^2} = \left(1-\frac{m}{k}\right)\frac{\pi}{k \sin.\frac{m\pi}{k}},$$

si quidem integrale etiam a termino $x = 0$ usque ad terminum $x = \infty$ extendatur.

§. 158. Quod si jam simili modo ponamus $\lambda = 2$, reperietur pro iisdem terminis integrationis

$$\int \frac{x^{m-1}\partial x}{(1+x^k)^3} = \left(1-\frac{m}{k}\right)\left(1-\frac{m}{2k}\right)\frac{\pi}{k \sin.\frac{m\pi}{k}};$$

eodem modo si litterae λ continuo majores valores tribuantur, reperientur sequentes integralium formae omni attentione dignae

$$\int \frac{x^{m-1}\partial x}{(1+x^k)^4} = \left(1-\frac{m}{k}\right)\left(1-\frac{m}{2k}\right)\left(1-\frac{m}{3k}\right)\frac{\pi}{k \sin.\frac{m\pi}{k}}$$

$$\int \frac{x^{m-1}\partial x}{(1+x^k)^5} = \left(1-\frac{m}{k}\right)\left(1-\frac{m}{2k}\right)\left(1-\frac{m}{3k}\right)\left(1-\frac{m}{4k}\right)\frac{\pi}{k \sin.\frac{m\pi}{k}}$$

$$\int \frac{x^{m-1}\partial x}{(1+x^k)^6} = \left(1-\frac{m}{k}\right)\left(1-\frac{m}{2k}\right)\left(1-\frac{m}{3k}\right)\left(1-\frac{m}{4k}\right)\left(1-\frac{m}{5k}\right)\frac{\pi}{k \sin.\frac{m\pi}{k}}$$

etc. etc.

§. 159. Quare si littera n denotet numerum quemcunque integrum, pro formula in titulo expressa, si ejus integrale a termino $x = 0$ usque ad $x = \infty$ extendatur, ejus valor sequenti modo se habebit

$$\left(1-\frac{m}{k}\right)\left(1-\frac{m}{2k}\right)\left(1-\frac{m}{3k}\right)\left(1-\frac{m}{4k}\right)\ldots\left(1-\frac{m}{(n-1)k}\right)\frac{\pi}{k\sin.\frac{m\pi}{k}}$$

qui ergo conveniet huic formulae integrali $\int\frac{x^{m-1}\partial x}{(1+x^k)^n}$.

§. 160. Hic quidem necessario pro n alii numeri praeter integros accipi non licet: at vero per methodum interpolationum, quae fusius jam passim est explicata, hanc integrationem etiam ad casus, quibus exponens n est numerus fractus, extendere licet. Quod si enim sequentes formulae integrales a termino $y=0$ usque ad $y=1$ extendantur, in genere valor nostrae formulae propositae ita repraesentari poterit

$$\int\frac{x^{m-1}\partial x}{(1+x^k)^n}=\frac{\pi}{k\sin.\frac{m\pi}{k}}\cdot\frac{\int y^{nk-m-1}\partial y\,(1-y^k)^{\frac{m}{k}-1}}{\int y^{k-m-1}\partial y\,(1-y^k)^{\frac{m}{k}-1}}.$$

Unde si fuerit $m=1$ et $k=2$, sequitur fore

$$\int\frac{\partial x}{(1+xx)^n}=\frac{\pi}{2}\int\frac{y^{2(n-1)}\partial y}{\sqrt{(1-yy)}}:\int\frac{\partial y}{\sqrt{(1-yy)}}=\int\frac{y^{2(n-1)}\partial y}{\sqrt{(1-yy)}}.$$

Ita si $n=\frac{3}{2}$ erit

$$\int\frac{\partial x}{(1+xx)^{\frac{3}{2}}}=\int\frac{y\partial y}{\sqrt{(1-yy)}}$$

cujus veritas sponte elucet, quia integrale prius generatim est $\frac{x}{\sqrt{(1+xx)}}$, posterius vero $=1-\sqrt{(1-yy)}$, quae facto $x=\infty$ et $y=1$, utique fiunt aequalia. Caeterum pro hac integratione generali notasse juvabit, exponentem unitate minorem accipi non posse, quia alioquin valores amborum integralium in infinitum excrescerent.

6) Investigatio valoris integralis

$$\int \frac{x^{m-1}\,\partial x}{1 - 2x^k \cos.\theta + x^{2k}}$$

a termino $x = 0$ usque ad $x = \infty$ extensi. *Opuscula analytica. Tom. II. Pag.* 55 — 75.

§. 161. Quaeramus primo integrale formulae propositae indefinitum, atque adeo omnes operationes ex primis analyseos principiis repetamus. Ac primo quidem, quoniam denominator in factores reales simplices resolvi nequit, sit in genere ejus factor duplicatus quicunque $1 - 2x\cos.\omega + xx$; evidens enim est, denominatorem fore productum ex k hujusmodi factoribus duplicatis. Cum igitur, posito hoc factore $= 0$, fiat $x = \cos.\omega \pm \sqrt{-1}\sin.\omega$, etiam ipse denominator duplici modo evanescere debebit, sive si ponatur

$$x = \cos.\omega + \sqrt{-1}\sin.\omega, \text{ sive}$$
$$x = \cos.\omega + \sqrt{-1}\sin.\omega.$$

Constat autem omnes potestates harum formularum ita commode exprimi posse, ut sit

$$(\cos.\omega \pm \sqrt{-1}\sin.\omega)^\lambda = \cos.\lambda\omega \pm \sqrt{-1}\sin.\lambda\omega,$$

hinc igitur erit

$$x^k = \cos.k\omega \pm \sqrt{-1}\sin.k\omega \text{ et}$$
$$x^{2k} = \cos.2k\omega \pm \sqrt{-1}\sin.2k\omega.$$

Substituamus ergo hos valores, et denominator noster evadet

$$1 - 2\cos.\theta\cos.k\omega + \cos.2k\omega$$
$$\pm 2\sqrt{-1}\cos.\theta\sin.k\omega \pm \sqrt{-1}\sin.2k\omega.$$

§. 162. Perspicuum igitur est hujus aequationis tam terminos reales quam imaginarios seorsim se mutuo tollere debere, unde nascuntur hae duae aequationes

$$\text{I.}\quad 1 - 2\cos.\theta\cos.k\omega + \cos.2k\omega = 0,$$

$$\text{II.}\quad -2\cos.\theta\sin.k\omega + \sin.2k\omega = 0.$$

Cum igitur sit

$$\sin.2k\omega = 2\sin k\omega\cos.k\omega,$$

posterior aequatio induet hanc formam

$$-2\cos.\theta\sin.k\omega + 2\sin.k\omega\cos.k\omega = 0,$$

quae per $2\sin.k\omega$ divisa dat $+\cos.k\omega = \cos.\theta$, ideoque

$$\cos.2k\omega = \cos.2\theta = \cos.\theta^2 - \sin.\theta^2 = 2\cos.\theta^2 - 1,$$

qui valores in aequatione priore substituti praebent aequationem identicam, ita ut utrique aequationi satisfiat sumendo $\cos.k\omega = \cos.\theta$.

§. 163. Pro ω igitur ejusmodi angulum assumi oportet, ut fiat $\cos.k\omega = \cos.\theta$, unde quidem statim deducitur $k\omega = \theta$, ideoque $\omega = \frac{\theta}{k}$. Verum quia infiniti dantur anguli eundem cosinum habentes, qui praeter ipsum angulum θ sunt $2\pi \pm \theta$, $4\pi \pm \theta$, $6\pi \pm \theta$, etc. atque adeo in genere $2i\pi \pm \theta$, denotante i omnes numeros integros, quaesito nostro satisfiet, faciendo $k\omega = 2i\pi \pm \theta$, unde colligitur angulus $\omega = \frac{2i\pi \pm \theta}{k}$, sicque pro ω nancisceremur innumerabiles angulos satisfacientes, quorum autem sufficiet tot assumsisse, quot exponens k continet unitates; successive igitur angulo ω sequentes tribuamus valores

$$\frac{\theta}{k},\ \frac{2\pi+\theta}{k},\ \frac{4\pi+\theta}{k},\ \frac{6\pi+\theta}{k},\ \frac{8\pi+\theta}{k},\ \ldots\ldots\ \frac{2(k-1)\pi+\theta}{k}.$$

Quodsi ergo angulo ω successive singulos istos valores, quorum numerus est $= k$, tribuamus, formula $1 - 2x\cos.\omega + xx$ omnes suppeditabit factores duplicatos nostri denominatoris $1 - 2x^k\cos.\theta + x^{2k}$, quorum numerus erit $= k$.

§. 164. Inventis jam omnibus factoribus duplicatis nostri denominatoris, fractio $\frac{x^{m-1}}{1-2x^k \cos.\theta + x^{2k}}$ resolvi debet in tot fractiones partiales, quarum denominatores sint ipsi isti factores duplicati, quorum numerus est k, ita ut in genere talis fractio partialis habitura sit talem formam $\frac{A+Bx}{1-2x\cos.\omega+xx}$, quam insuper resolvamus in binas simplices, etsi imaginarias, et cum sit

$$xx-2x\cos.\omega+1=(x-\cos.\omega+\sqrt{-1}\sin.\omega)(x-\cos.\omega-\sqrt{-1}\sin.\omega),$$

statuantur ambae istae fractiones partiales

$$\frac{f}{x-\cos.\omega-\sqrt{-1}\sin.\omega}+\frac{g}{x-\cos.\omega+\sqrt{-1}\sin.\omega},$$

ita ut totum resolutionis negotium huc redeat, ut ambo numeratores f et g determinentur; iis enim inventis habebitur summa ambarum fractionum

$$=\frac{fx+gx-(f+g)\cos.\omega+\sqrt{-1}(f-g)\sin.\omega}{xx-2x\cos.\omega+1},$$

ubi igitur erit

$$B=f+g \text{ et } A=(f-g)\sqrt{-1}\sin.\omega-(f+g)\cos.\omega.$$

§. 165. Per methodum igitur fractiones quascunque in fractiones simplices resolvendi statuamus

$$\frac{x^{m-1}}{1-2x^k\cos.\theta+x^{2k}}=\frac{f}{x-\cos.\omega-\sqrt{-1}\sin.\omega}+R,$$

ubi R complectatur omnes reliquas fractiones partiales. Hinc per $x-\cos.\omega-\sqrt{-1}\sin.\omega$ multiplicando habebitur

$$\frac{x^m-x^{m-1}(\cos.\omega+\sqrt{-1}\sin.\omega)}{1-2x^k\cos.\theta+x^{2k}}=f+R(x-\cos.\omega-\sqrt{-1}\sin.\omega),$$

quae aequatio cum vera esse debeat, quicunque valor ipsi x tribuatur, statuamus $x=\cos.\omega+\sqrt{-1}\sin.\omega$, ut membrum postremum prorsus e calculo tollatur; tum vero in parte sinistra, quia formula $x - \cos.\omega - \sqrt{-1}\sin.\omega$ simul est factor denominatoris, facta hac substitutione tam numerator quam denominator in nihilum abibunt, ita ut hinc nihil concludi posse videatur.

§. 166. Hinc igitur utamur regula notissima, et loco tam numeratoris quam denominatoris eorum differentialia scribamus, unde nostra aequatio accipiet sequentem formam

$$\frac{m x^{m-1} - (m-1) x^{m-2} (\cos.\omega + \sqrt{-1}\sin.\omega)}{-2 k x^{k-1} \cos.\theta + 2 k x^{2k-1}} = \frac{m x^{m} - (m-1) x^{m-1} (\cos.\omega + \sqrt{-1}\sin.\omega)}{-2 k x^{k} \cos.\theta + 2 k x^{2k}} = f,$$

posito scilicet $x = \cos.\omega + \sqrt{-1}\sin.\omega$. Tum autem erit

$$x^m = \cos.m\omega + \sqrt{-1}\sin.m\omega \text{ et}$$

$$x^{m-1}(\cos.\omega + \sqrt{-1}\sin.\omega) = x^m = \cos.m\omega + \sqrt{-1}\sin.m\omega,$$

et pro denominatore

$$x^k = \cos.k\omega + \sqrt{-1}\sin.k\omega \text{ et}$$

$$x^{2k} = \cos.2k\omega + \sqrt{-1}\sin.2k\omega;$$

unde fit numerator

$$x^m = \cos.m\omega + \sqrt{-1}\sin.m\omega$$

et denominator

$$-2k\cos.\theta\cos.k\omega + 2k\cos.2k\omega$$
$$-2k\sqrt{-1}\cos.\theta\sin.k\omega + 2k\sqrt{-1}\sin.2k\omega.$$

§. 167. Pro denominatore reducendo recordemur, jam supra inventum esse $\cos.k\omega = \cos.\theta$, unde fit $\sin.k\omega = \sin.\theta$, tum vero

$$\cos.2k\omega = \cos.2\theta = 2\cos.\theta^2 - 1 \text{ et}$$

$$\sin.2k\omega = \sin.2\theta = 2\sin.\theta\cos.\theta,$$

quibus valoribus adhibitis denominator noster erit

$$2k\cos.\theta^2 - 2k + 2k\sqrt{-1}\sin.\theta\cos.\theta = -2k\sin.\theta^2 + 2k\sqrt{-1}\sin.\theta\cos.\theta$$
$$= -2k\sin.\theta(\sin.\theta - \sqrt{-1}\cos.\theta),$$

quamobrem hoc valore adhibito habebimus

$$f = \frac{\cos.m\omega + \sqrt{-1}\sin.m\omega}{2k\sin.\theta(\sqrt{-1}\cos.\theta - \sin.\theta)}.$$

Simul vero hinc sine novo calculo deducemus valorem g, quippe qui ab f ratione signi $\sqrt{-1}$ tantum discrepat, sicque erit

$$g = \frac{\cos.m\omega - \sqrt{-1}\sin.m\omega}{-2k\sin.\theta(\sin.\theta + \sqrt{-1}\cos.\theta)}.$$

§. 168. Inventis autem his litteris f et g, pro litteris A et B colligemus primo

$$f + g = \frac{\cos.\theta\sin.m\omega - \sin.\theta\cos.m\omega}{k\sin.\theta} = \frac{\sin.(m\omega - \theta)}{k\sin.\theta}$$

tum vero erit

$$f - g = -\frac{\sqrt{-1}\cos.(m\omega - \theta)}{k\sin.\theta}.$$

Ex his igitur reperiemus

$$B = \frac{\sin.(m\omega - \theta)}{k\sin.\theta} \text{ et}$$

$$A = \frac{\sin.\omega\cos.(m\omega - \theta) - \cos.\omega\sin.(m\omega - \theta)}{k\sin.\theta} = -\frac{\sin.[(m\omega - \theta) - \omega]}{k\sin.\theta},$$

ubi ergo imaginaria sponte se mutuo destruxerunt.

§. 169. Inventis his valoribus A et B, investigari oportet integrale partiale $\int\frac{(A + Bx)\partial x}{1 - 2x\cos.\omega + xx}$, ubi, cum denominatoris differentiale sit

$$2x\partial x - 2\partial x\cos.\omega = 2\partial x(x - \cos.\omega),$$

statuamus,

$$A + Bx = B(x - \cos.\omega) + C, \text{ eritque}$$
$$C = A + B\cos.\omega,$$

hinc igitur erit

$$C = \frac{\cos.\omega \sin.(m\omega - \theta) - \sin.[(m\omega - \theta) - \omega]}{k \sin.\theta}.$$

Quia vero

$$-\sin.(m\omega - \theta - \omega) = -\sin.(m\omega - \theta)\cos.\omega + \cos.(m\omega - \theta)\sin.\omega, \text{ erit}$$

$$C = \frac{\sin.\omega \cos.(m\omega - \theta)}{k \sin.\theta}.$$

Hac ergo forma adhibita, formula integranda $\frac{(A + Bx)\partial x}{1 - 2x\cos.\omega + xx}$ discerpatur in has duas partes

$$\frac{B(x - \cos.\omega)\partial x}{1 - 2x\cos.\omega + xx} + \frac{C\partial x}{1 - 2x\cos.\omega + xx}.$$

Hic igitur prioris partis integrale manifesto est

$$B l \sqrt{(1 - 2x\cos.\omega + xx)},$$

alterius vero partis facile patet integrale per arcum circuli expressum iri, cujus tangens sit $\frac{x \sin.\omega}{1 - x\cos.\omega}$. Ad hoc integrale inveniendum ponamus

$$\int \frac{C\partial x}{1 - 2x\cos.\omega + xx} = D \cdot \text{Arc. tang.} \frac{x\sin.\omega}{1 - x\cos.\omega},$$

et sumtis differentialibus, quia ∂. Arc. tang. t aequale est $\frac{\partial t}{1 + tt}$, habebimus

$$\frac{C\partial x}{1 - 2x\cos.\omega + xx} = D \cdot \frac{\partial x \sin.\omega}{1 - 2x\cos.\omega + xx},$$

unde manifesto fit

$$D = \frac{C}{\sin.\omega} = \frac{\cos.(m\omega - \theta)}{k\sin.\theta}.$$

§. 170. Substituamus igitur loco B et D valores modo inventos, et ex singulis factoribus denominatoris

$$1 - 2x^k \cos.\theta + x^{2k},$$

quorum forma est $1 - 2x\cos.\omega + xx$, oritur pars integralis constans ex membro logarithmico et arcu circulari, quae erit

$$\frac{\sin.(m\omega - \theta)}{k\sin.\theta} l \sqrt{(1 - 2x\cos.\omega + xx)} + \frac{\cos.(m\omega - \theta)}{k\sin.\theta} \text{Arc. tang.} \frac{x\sin.\omega}{1 - x\cos.\omega}$$

quae evanescit sumto $x = 0$. In hac igitur forma tantum opus est, ut loco ω successive scribamus valores supra indicatos, scilicet

$$\omega = \frac{\theta}{k}, \frac{2\pi + \theta}{k}, \frac{4\pi + \theta}{k}, \frac{6\pi + \theta}{k}, \text{ etc.}$$

donec perveniatur ad $\frac{2(k-1)\pi + \theta}{k}$; tum enim summa omnium harum formarum praebebit totum integrale indefinitum formulae propositae.

§. 171. Postquam igitur integrale indefinitum elicuimus, nihil aliud superest, nisi ut in eo faciamus $x = \infty$, quo facto pars logarithmica, ob

$$\sqrt{(1 - 2x\cos.\omega + xx)} = x - \cos.\omega,$$

erit $B l(x - \cos.\omega)$. Est vero

$$l(x - \cos.\omega) = lx - \frac{\cos.\omega}{x} = lx, \text{ ob } \frac{\cos.\omega}{k} = 0,$$

quamobrem facto $x = \infty$ quaelibet pars logarithmica habebit hanc formam $\frac{\sin.(m\omega - \theta)}{k\sin.\theta} lx$. Deinde pro partibus a circulo pendentibus, facto $x = \infty$ fit

$$\frac{x\sin.\omega}{1 - x\cos.\omega} = -\text{tang.}\,\omega = \text{tang.}(\pi - \omega);$$

sicque arcus, cujus haec est tangens, erit $= \pi - \omega$, hincque pars circularis quaecunque fiet $\frac{\cos.(m\omega - \theta)}{k\sin.\theta}(\pi - \omega)$.

§. 172. Cum quilibet valor anguli ω in genere hanc habeat formam $\frac{2i\pi + \theta}{k}$, erit angulus

$$m\omega - \theta = \frac{2im\pi - \theta(k-m)}{k} \text{ et } \pi - \omega = \frac{\pi(k-2i) - \theta}{k}.$$

Ponamus brevitatis gratia

$$\frac{\theta(k-m)}{k} = \zeta \text{ et } \frac{m\pi}{k} = \alpha, \text{ ut sit } m\omega - \theta = 2i\alpha - \zeta,$$

ubi loco i scribi debent successive numeri 0, 1, 2, 3, etc. usque ad $k-1$. Hinc igitur si omnes partes logarithmicas in unam summam colligamus, ea ita repraesentari poterit

$$\frac{1x}{k\,\sin.\theta}\,[-\sin.\zeta+\sin.(2\alpha-\zeta)+\sin.(4\alpha-\zeta)+\sin.(6\alpha-\zeta)$$
$$+\sin.(8\alpha-\zeta)\ldots\ldots+\sin.[2(k-1)\alpha-\zeta]];$$

ubi quidem ex iis, quae hactenus sunt tradita, facile suspicari licet, totam hanc progressionem ad nihilum redigi. Verum hoc ipsum firma demonstratione muniri necesse est.

§. 173. Ad hoc ostendendum ponamus

$$S=-\sin.\zeta+\sin.(2\alpha-\zeta)+\sin.(4\alpha-\zeta)+\ldots+\sin.[2(k-1)\alpha-\zeta],$$

multiplicemus utrinque per 2 sin. α, et cum sit

$$2\sin.\alpha\sin.\phi=\cos.(\alpha-\phi)-\cos.(\alpha+\phi),$$

hujus reductionis ope obtinebimus sequentem expressionem

$$2S\sin.\alpha=\cos.(\alpha+\zeta)+\cos.(\alpha-\zeta)+\cos.(3\alpha-\zeta)+\cos.(5\alpha-\zeta)\ldots$$
$$-\cos.(\alpha-\zeta)-\cos.(3\alpha-\zeta)-\cos.(5\alpha-\zeta)\ldots$$
$$\ldots+\cos.[(2k-3)\alpha-\zeta]-\cos.[(2k-1)\alpha-\zeta]$$
$$\ldots-\cos.[(2k-3)\alpha-\zeta]$$

unde deletis terminis se mutuo destruentibus habebitur

$$2S\sin.\alpha=\cos.(\alpha+\zeta)-\cos.[(2k-1)\alpha-\zeta].$$

§. 174. Ponamus hos duos angulos, qui sunt relicti, $\alpha+\zeta=p$ et $(2k-1)\alpha-\zeta=q$, eritque eorum summa $p+q=2\alpha k$. Quia porro est $\alpha=\frac{m\pi}{k}$, erit $p+q=2m\pi$; hoc est multiplo totius circuli peripheriae, ob m numerum integrum. Quare cum sit $q=2m\pi-p$, erit $\cos.q=\cos.p$; unde patet summam inventam nihilo esse aequalem, sicque manifestum est, omnes partes logarithmicas, quae in integrale formulae nostrae ingrediuntur, casu $x=\infty$ se mutuo destruere.

§. 175. Progrediamur igitur ad partes circulares, quarum forma generalis, ut vidimus, est $\frac{\cos.(m\omega-\theta)}{k\sin.\theta}(\pi-\omega)$, quae $\frac{\cos.(2i\alpha-\zeta)}{k\sin.\theta}(\pi-\frac{2i\pi-\theta}{k})=\frac{\cos.(2i\alpha-\zeta)}{k\sin.\theta}(\pi-\frac{2i\pi}{k}-\frac{\theta}{k})$.
Hic ponatur porro $\frac{\pi}{k}=\beta$ et $\pi-\frac{\theta}{k}=\gamma$, ut forma generalis sit $\frac{\cos.(2i\alpha-\zeta)}{k\sin.\theta}(\gamma-2i\beta)$. Quare si loco i scribamus ordine valores, 0, 1, 2, 3, 4, usque ad $k-1$, omnes partes circulares hanc progressionem constituent

$$\frac{1}{k\sin.\theta}[\gamma\cos.\zeta+(\gamma-2\beta)\cos.(2\alpha-\zeta)+(\gamma-4\beta)\cos.(4\alpha-\zeta)$$
$$\ldots\ldots+[\gamma-2(k-1)\beta]\cos.[2(k-1)\alpha-\zeta]].$$

Ponamus igitur

$$S=\gamma\cos.\zeta+(\gamma-2\beta)\cos.(2\alpha-\zeta)+(\gamma-4\beta)\cos.(4\alpha-\zeta)$$
$$\ldots\ldots+[\gamma-2(k-1)\beta]\cos.[2(k-1)\alpha-\zeta]$$

ut summa omnium partium circularium sit $\frac{S}{k\sin.\theta}$, quae ergo praebebit valorem quaesitum formulae integralis propositae, casu quo post integrationem statuitur $x=\infty$, ita ut totum negotium in investigando valore ipsius S versetur.

§. 176. Hunc in finem multiplicemus utrinque per $2\sin.\alpha$, et cum in genere sit

$$2\sin.\alpha\cos.\Phi=\sin.(\alpha+\Phi)-\sin.(\Phi-\alpha),$$

hac reductione in singulis terminis facta, perveniemus ad hanc aequationem

$$2S\sin.\alpha=\gamma\sin.(\alpha+\zeta)+\gamma\sin.(\alpha-\zeta)+(\gamma-2\beta)\sin.(3\alpha-\zeta)$$
$$-(\gamma-2\beta)\sin.(\alpha-\zeta)-(\gamma-4\beta)\sin.(3\alpha-\zeta)$$
$$+(\gamma-4\beta)\sin.(5\alpha-\zeta)\ldots\ldots+[\gamma-2(k-1)\beta]\sin.[(2k-1)\alpha-\zeta]$$
$$-(\gamma-6\beta)\sin.(5\alpha-\zeta)$$

ubi praeter primum et ultimum terminum omnes reliqui contrahi possunt, ita ut prodeat

$$2S \sin.\alpha = \gamma \sin.(\alpha+\zeta) + 2\beta \sin.(\alpha-\zeta) + 2\beta \sin.(3\alpha-\zeta) + 2\beta \sin.(5\alpha-\zeta) \ldots + 2\beta \sin.[(2k-3)\alpha-\zeta] + [\gamma-\zeta(k-1)\beta]\sin.[(2k-1)\alpha-\zeta].$$

§. 177. Jam pro hac serie summanda ponamus porro

$$T = 2\sin.(\alpha-\zeta) + 2\sin.(3\alpha-\zeta) + 2\sin.(5\alpha-\zeta) + \ldots\ldots + 2\sin.[(2k-3)\alpha-\zeta]$$

ut habeamus

$$2S\sin.\alpha = \gamma\sin.(\alpha+\zeta) + [\gamma-2(k-1)\beta]\sin.[(2k-1)\alpha-\zeta] + \beta T.$$

Jam multiplicemus, ut hactenus, per sin. α, et cum sit

$$2\sin.\alpha\sin.\Phi = \cos.(\Phi-\alpha) - \cos.(\Phi+\alpha),$$

facta hac reductione nanciscimur

$$T\sin.\alpha = +\cos.\zeta + \cos.(2\alpha-\zeta) + \cos.(4\alpha-\zeta) + \ldots + \cos.[2(k-2)\alpha-\zeta] - \cos.(2\alpha-\zeta) - \cos.(4\alpha-\zeta) - \ldots - \cos.[2(k-2)\alpha-\zeta] - \cos.[2(k-1)\alpha-\zeta]$$

unde deletis terminis, quae se mutuo destruunt, remanebit tantum ista expressio

$$T\sin.\alpha = \cos.\zeta - \cos.[2(k-1)\alpha-\zeta].$$

Cum igitur sit $\alpha = \frac{m\pi}{k}$ erit

$$2(k-1)\alpha = 2m\pi - \frac{2m\pi}{k},$$

cujus loco scribere licet $-\frac{2m\pi}{k}$, unde ob $\zeta = \frac{\theta(k-m)}{k}$, erit

$$T\sin.\alpha = \cos.\frac{\theta(k-m)}{k} - \cos.\left(\frac{2m\pi+\theta(k-m)}{k}\right).$$

§. 178. Nunc vero notetur in genere esse

$$\cos.p - \cos.q = 2\sin.\frac{q+p}{2}\sin.\frac{q-p}{2},$$

quare cum sit

$$p = \frac{\theta(k-m)}{k} \text{ et } q = \frac{2m\pi+\theta(k-m)}{k}, \text{ erit}$$

$$\frac{q+p}{2} = \frac{m\pi+\theta(k-m)}{k} \text{ et } \frac{q-p}{2} = \frac{m\pi}{k},$$

unde sequitur fore

$$T \sin. \alpha = 2 \sin. \left(\frac{m\pi + \theta(k-m)}{k}\right) \sin. \frac{m\pi}{k},$$

ideoque

$$T = 2 \sin. \left(\frac{m\pi + \theta(k-m)}{k}\right), \text{ ob } \alpha = \frac{m\pi}{k}.$$

§. 179. Hoc igitur valore T invento reperiemus porro

$$2 S \sin. \alpha = \gamma \sin. (\alpha + \zeta) + [\gamma - 2(k-1)\beta] \sin. [(2k-1)\alpha - \zeta] + 2\beta \sin. \left(\frac{m\pi + \theta(k-m)}{k}\right),$$

quae ob $\frac{m\pi + \theta(k-m)}{k} = \alpha + \zeta$ reducitur ad hanc formam

$$2 S \sin. \alpha = (\gamma + 2\beta) \sin. (\alpha + \zeta) + [\gamma - 2(k-1)\beta] \sin. [(2k-1)\alpha - \zeta],$$

quae ita repraesentari potest

$$2 S \sin. \alpha = (\gamma + 2\beta) [\sin. (\alpha + \zeta) + \sin. [(2k-1)\alpha - \zeta]] - 2\beta k \sin. [(2k-1)\alpha - \zeta],$$

ubi pro parte priore, ob

$$\sin. p + \sin. q = 2 \sin. \frac{p+q}{2} \cos. \frac{p-q}{2}, \text{ erit}$$

$$\frac{p+q}{2} = \alpha k \text{ et } \frac{p-q}{2} = (k-1)\alpha - \zeta,$$

unde pars ipsa prior fit

$$2(\gamma + 2\beta) \sin. \alpha k \cos. [(k-1)\alpha - \zeta],$$

ubi cum sit $\alpha k = m\pi$, erit $\sin. \alpha k = 0$, ita ut tantum supersit

$$2 S \sin. \alpha = -2\beta k \sin. [(2k-1)\alpha - \zeta],$$

hincque

$$S = -\frac{\beta k \sin. [(2k-1)\alpha - \zeta]}{\sin. \alpha}.$$

Est vero

$$(2k-1)\alpha - \zeta = 2m\pi - \frac{m\pi}{k} - \frac{\theta(k-m)}{k};$$

omisso igitur termino $2m\pi$, erit

$$S = + \frac{\pi \sin. \left[\frac{m\pi + \theta(k-m)}{k}\right]}{\sin. \frac{m\pi}{k}},$$

ideoque valor quaesitus

$$\frac{S}{k \sin. \theta} = + \frac{\pi \sin. \left[\frac{m\pi + \theta(k-m)}{k}\right]}{k \sin. \theta \sin. \frac{m\pi}{k}},$$

quae forma reducitur ad hanc

$$\frac{\pi \sin. \left[\frac{m(\pi-\theta) + \theta k}{k}\right]}{k \sin. \theta \sin. \frac{m\pi}{k}}.$$

§. 180. Contemplemur hic ante omnia casum quo $\theta = \frac{\pi}{2}$, et formula integralis proposita abit in hanc $\int \frac{x^{m-1} \partial x}{1+x^{2k}}$, cujus ergo valor, si post integrationem ponatur $x = \infty$, evadet

$$\frac{\pi \sin. \left(\frac{\pi}{2} + \frac{m\pi}{2k}\right)}{k \sin. \frac{m\pi}{k}} = \frac{\pi \cos. \frac{m\pi}{2k}}{k \sin. \frac{m\pi}{k}}.$$

Quia igitur est

$$\sin. \frac{m\pi}{k} = 2 \sin. \frac{m\pi}{2k} \cos. \frac{m\pi}{2k},$$

prodibit iste valor $= \frac{\pi}{2k \sin. \frac{m\pi}{k}}$, qui valor egregie convenit cum eo, quem non ita pridem pro formula $\int \frac{x^{m-1} \partial x}{1+x^k}$ assignavimus, si quidem loco k scribatur $2k$.

§. 181. Evolvamus etiam casum quo $\theta = \pi$, et formula nostra integralis $\int \frac{x^{m-1} \partial x}{(1+x^k)^2}$, cujus ergo, facto $x = \infty$, valor erit

$$\frac{\pi \sin. \left[\frac{m(\pi-\theta)}{k}+\theta\right]}{k \sin. \theta \sin. \frac{m\pi}{k}} = \frac{\pi}{k \sin. \frac{m\pi}{k}} \cdot \frac{\sin. \left[\frac{m(\pi-\theta)}{k}+\theta\right]}{\sin. \theta}.$$

Hujus autem posterioris fractionis, casu $\theta = \pi$, tam numerator quam denominator evanescit; quare, ut ejus verus valor eruatur, loco utriusque ejus differentiale scribamus, quo facto ista fractio abibit in hanc

$$\frac{\partial \theta \left(1-\frac{m}{k}\right) \cos. \left[\frac{m(\pi-\theta)}{k}+\theta\right]}{\partial \theta \cos. \theta},$$

cujus valor facto $\theta = \pi$ nunc manifesto est $1-\frac{m}{k}$; sicque valor integralis quaesitus erit $\left(1-\frac{m}{k}\right)\frac{\pi}{k \sin. \frac{m\pi}{k}}$, prorsus uti in superiore dissertatione invenimus.

§. 182. Quo autem valorem generalem inventum commodiorem reddamus, ponamus $\pi - \theta = \eta$, fietque

$$\sin. \theta = \sin. \eta \text{ et } \cos. \theta = -\cos. \eta;$$

tum vero erit angulus

$$\frac{m(\pi-\theta)}{k} + \theta = \frac{m\eta}{k} + \pi - \eta,$$

cujus sinus est $\sin. \left(1-\frac{m}{k}\right)\eta$, unde valor quaesitus nostrae formulae erit $\frac{\pi \sin. \left(1-\frac{m}{k}\right)\eta}{k \sin. \eta \sin. \frac{m\pi}{k}}$, atque hinc tandem sequens adepti sumus theorema.

Theorema.

§. 183. Si haec formula integralis

$$\int \frac{x^{m-1}\partial x}{1+2x^k \cos. \eta + x^{2k}}$$

a termino $x = 0$ usque ad terminum $x = \infty$ extendatur, ejus valor erit $= \frac{\pi \sin. \left(1 - \frac{m}{k}\right) \eta}{k \sin. \eta \sin. \frac{m\pi}{k}}$, sive cum sit

$$\sin. \left(1 - \frac{m}{k}\right) \eta = \sin. \eta \cos. \frac{m\eta}{k} - \cos. \eta \sin. \frac{m\eta}{k},$$

iste valor etiam hoc modo exprimi potest

$$\frac{\pi \cos. \frac{m\eta}{k}}{k \sin. \frac{m\pi}{k}} - \frac{\pi \sin. \frac{m\eta}{k}}{k \operatorname{tang}. \eta \sin. \frac{m\pi}{k}}.$$

§. 184. Consideremus nunc alio modo hanc formulam integralem

$$\int \frac{x^{m-1} \partial x}{1 + 2x^k \cos. \eta + x^{2k}},$$

cujus valor a termino $x = 0$ usque ad $x = 1$ ponatur $= P$, ejusdem vero valor ab $x = 1$ usque ad $x = \infty$ ponatur $= Q$, ita ut $P + Q$ exhibere debeat ipsum valorem ante inventum. Nunc vero pro valore Q inveniendo ponamus $x = \frac{1}{y}$, et formula nostra ita repraesentata

$$\frac{x^m}{1 + 2x^k \cos. \eta + x^{2k}} \cdot \frac{\partial x}{x},$$

ob $\frac{\partial x}{x} = -\frac{\partial y}{y}$ induet hanc formam

$$-\int \frac{y^{-m}}{1 + 2y^{-k} \cos. \eta + y^{-2k}} \cdot \frac{\partial y}{y} = -\int \frac{y^{2k-m-1} \partial y}{y^{2k} + 2y^k \cos. \eta + 1},$$

cujus valor a termino $y = 1$ usque ad $y = 0$ extendi debet. Commutatis igitur his terminis habebimus

$$Q = + \int \frac{y^{2k-m-1} \partial y}{y^{2k} + 2y^k \cos. \eta + 1}$$

a termino $y = 0$ usque ad $y = 1$.

§. 185. Quia in utraque forma pro P et Q eadem conditio integrationis praescribitur, a termino 0 usque ad 1, nihil impedit quo minus in posteriore loco y scribamus x, unde pro $P + Q$ habebimus hanc formam integralem

$$\int \frac{x^{m-1} + x^{2k-m-1}}{1 + 2x^k \cos.\eta + x^{2k}} \partial x,$$

cujus valor, a termino $x = 0$ usque ad $x = 1$ extensus, aequabitur huic expressioni $\frac{\pi \sin.\left(1 - \frac{m}{k}\right)\eta}{k \sin.\eta \sin.\frac{m\pi}{k}}$. Comparatis igitur his binis formulis integralibus nanciscemur sequens theorema notatu maxime dignum.

T h e o r e m a.

§. 186. Haec formula integralis

$$\int \frac{x^{m-1} + x^{2k-m-1}}{1 + 2x^k \cos.\eta + x^{2k}} \partial x,$$

a termino $x = 0$ usque ad terminum $x = 1$ extensa, aequalis est huic formulae integrali

$$\int \frac{x^{m-1} \partial x}{1 + 2x^k \cos.\eta + x^{2k}},$$

a termino $x = 0$ usque ad terminum $x = \infty$ extensae: utriusque enim valor erit $\frac{\pi \sin.\left(1 - \frac{m}{k}\right)\eta}{k \sin.\eta \sin.\frac{m\pi}{k}}$.

§. 187. Quod si hanc fractionem

$$\frac{\sin.\eta}{1 + 2x^k \cos.\eta + x^{2k}}$$

in seriem infinitam evolvamus, quae sit

$$\sin.\eta + Ax^k + Bx^{2k} + Cx^{3k} + Dx^{4k} + Ex^{5k} + \text{etc.}$$

per denominatorem multiplicando perveniemus ad hanc expressionem infinitam

$$\begin{array}{llllllll} \sin.\eta = \sin.\eta + A x^k & + B x^{2k} & + C x^{3k} & + D x^{4k} & + E x^{5k} & + F x^{6k} & + \text{etc.} \\ + 2\sin.\eta\cos.\eta & + 2A\cos.\eta & + 2B\cos.\eta & + 2C\cos.\eta & + 2D\cos.\eta & + 2E\cos.\eta & + \text{etc.} \\ & + \sin.\eta & + A & + B & + C & + D & + \text{etc.} \end{array}$$

unde singulis terminis ad nihilum reductis reperiemus

1°. $A + 2\sin.\eta\cos.\eta = 0$, hincque $A = -\sin.2\eta$
2°. $B + 2A\cos.\eta + \sin.\eta = 0$, unde fit $B = \sin.3\eta$
3°. $C + 2B\cos.\eta + A = 0$, unde fit $C = -\sin.4\eta$
4°. $D + 2C\cos.\eta + B = 0$, unde fit $D = \sin.5\eta$
etc. etc.

ita ut nostra fractio $\frac{\sin.\eta}{1 + 2x^k\cos.\eta + x^{2k}}$ resolvatur in hanc seriem

$$\sin.\eta - x^k\sin.2\eta + x^{2k}\sin.3\eta - x^{3k}\sin.4\eta + x^{4k}\sin.5\eta - \text{etc.}$$

§. 188. Multiplicemus nunc hanc seriem per

$$x^{m-1}\partial x + x^{2k-m-1}\partial x,$$

et post integrationem faciamus $x = 1$, ut obtineamus valorem hujus formulae

$$\sin.\eta\int\frac{x^{m-1} + x^{2k-m-1}}{1 + 2x^k\cos.\eta + x^{2k}}\partial x$$

pro casu $x = 1$, hocque modo perveniemus ad geminas sequentes series

$$\frac{\sin.\eta}{m} - \frac{\sin.2\eta}{m+k} + \frac{\sin.3\eta}{m+2k} - \frac{\sin.4\eta}{m+3k} + \frac{\sin.5\eta}{m+4k} - \text{etc.}$$
$$\frac{\sin.\eta}{2k-m} - \frac{\sin.2\eta}{3k-m} + \frac{\sin.3\eta}{4k-m} - \frac{\sin.4\eta}{5k-m} + \frac{\sin.5\eta}{6k-m} - \text{etc.}$$

Aggregatum igitur harum duarum serierum junctim - sumtarum aequabitur huic valori $\frac{\pi \sin.\left(1-\frac{m}{k}\right)\eta}{k \sin.\frac{m\pi}{k}}$, unde subjungamus adhuc istud theorema.

Theorema.

§. 189. Si η denotet angulum quemcunque, litterae vero m et k pro lubitu accipiantur, ex iisque binae sequentes series formentur

$$P = \frac{\sin.\eta}{m} - \frac{\sin.2\eta}{m+k} + \frac{\sin.3\eta}{m+2k} - \frac{\sin.4\eta}{m+3k} + \frac{\sin.5\eta}{m+4k} - \text{etc.}$$

$$Q = \frac{\sin.\eta}{2k-m} - \frac{\sin.2\eta}{3k-m} + \frac{\sin.3\eta}{4k-m} - \frac{\sin.4\eta}{5k-m} + \frac{\sin.5\eta}{6k-m} - \text{etc.}$$

neutrius quidem summa exhiberi potest, utriusque autem junctim sumtae summa erit

$$P + Q = \frac{\pi \sin.\left(1-\frac{m}{k}\right)\eta}{k \sin.\frac{m\pi}{k}}.$$

Corollarium.

§. 190. Quod-si ergo angulum η infinite parvum capiamus, ut fiat

$$\sin.\eta = \eta,\quad \sin.2\eta = 2\eta,\quad \sin.3\eta = 3\eta,\ \text{etc.}$$

quia in formula summae fiet

$$\sin.\left(1-\frac{m}{k}\right)\eta = \left(1-\frac{m}{k}\right)\eta;$$

si utrinque per η dividamus, obtinebimus sequentem seriem geminatam

$$\frac{1}{m} - \frac{2}{m+k} + \frac{3}{m+2k} - \frac{4}{m+3k} + \frac{5}{m+4k} - \text{etc.}$$

$$\frac{1}{2k-m} - \frac{2}{3k-m} + \frac{3}{4k-m} - \frac{4}{5k-m} + \frac{5}{6k-m} - \text{etc.}$$

cujus ergo summa erit $\left(1-\frac{m}{k}\right)\frac{\pi}{k \sin.\frac{m\pi}{k}}$, ubi notetur, ambas istas series non incongrue in hanc simplicem contrahi posse

$$\frac{2k}{m(2k-m)} - \frac{8k}{(m+k)(3k-m)} + \frac{18k}{(m+2k)(4k-m)} - \frac{32k}{(m+3k)(5k-m)} + \text{etc.}$$

ubi numeratores sunt numeri quadrati duplicati.

§. 191. Formulae autem, quarum valores hactenus invenimus, multo concinnius et elegantius exprimi possunt, si loco exponentis m scribamus $k-n$, tum enim in valore integrali invento fiet $(1-\frac{m}{k})\eta = \frac{n\eta}{k}$; at vero pro denominatore fiet $\frac{m\pi}{k} = \pi - \frac{n\pi}{k}$, cujus sinus erit sin. $\frac{n\pi}{k}$; sicque nostra formula inventa hanc induet formam $\dfrac{\pi \sin. \frac{n\eta}{k}}{k \sin. \eta \sin. \frac{n\pi}{k}}$, quae ergo exprimet valorem hujus formulae integralis

$$\int \frac{x^{k-n-1}\,\partial x}{1+2x^k \cos. \eta + x^{2k}},$$

ab $x=0$ usque ad $x=\infty$, ut et hujus formulae

$$\int \frac{x^{k-n-1}+x^{k+n-1}}{1+2x^k \cos. \eta + x^{2k}}\,\partial x,$$

a termino $x=0$ usque ad terminum $x=1$; et quia utriusque valor est $\dfrac{\pi \sin. \frac{n\eta}{k}}{k \sin. \eta \sin. \frac{n\pi}{k}}$, perspicuum est eum manere eundem, etsi loco n scribatur $-n$, ex quo prior formula ita repraesentari poterit

$$\int \frac{x^{k\pm n-1}}{1+x^k \cos. \eta + x^{2k}}\,\partial x;$$

at posterior formula ob hanc ambiguitatem nullam plane mutationem patitur.

§. 192. Ponendo $m = k - n$ etiam series nostra geminata pulchriorem accipiet faciem; habebitur enim

$$\frac{\sin.\eta}{k-n} - \frac{\sin.2\eta}{2k-n} + \frac{\sin.3\eta}{3k-n} - \frac{\sin.4\eta}{4k-n} + \text{etc.}$$
$$\frac{\sin.\eta}{k+n} - \frac{\sin.2\eta}{2k+n} + \frac{\sin.3\eta}{3k+n} - \frac{\sin.4\eta}{4k+n} + \text{etc.}$$

cujus ergo summa erit $\frac{\pi \sin.\frac{n\eta}{k}}{k \sin.\frac{n\pi}{k}}$. Tum vero si hae geminae series in unam contrahantur, et utrinque per $2k$ dividatur, obtinebitur sequens summatio memoratu digna

$$\frac{\pi \sin.\frac{n\eta}{k}}{2kk \sin.\frac{n\pi}{k}} = \frac{\sin.\eta}{kk-nn} - \frac{2\sin.2\eta}{4kk-nn} + \frac{3\sin.3\eta}{9kk-nn} - \frac{4\sin.4\eta}{16kk-nn} + \text{etc.}$$

§. 193. Quodsi haec postrema series differentietur, sumendo solum angulum η variabilem, ob

$$\partial.\sin.\frac{n\eta}{k} = \frac{n\partial\eta}{k}\cos.\frac{n\eta}{k}$$

habebimus

$$\frac{\pi n \cos.\frac{n\eta}{k}}{2k^3 \sin.\frac{n\pi}{k}} = \frac{\cos.\eta}{kk-nn} - \frac{4\cos.2\eta}{4kk-nn} + \frac{9\cos.3\eta}{9kk-nn} - \frac{16\cos.4\eta}{16kk-nn} + \text{etc.}$$

Unde si sumatur $\eta = 0$, orietur ista summatio

$$\frac{\pi n}{2k^3 \sin.\frac{n\pi}{k}} = \frac{1}{kk-nn} - \frac{4}{4kk-nn} + \frac{9}{9kk-nn} - \frac{16}{16kk-nn} + \text{etc.}$$

Sin autem sumatur $\eta = 90^\circ = \frac{\pi}{2}$, erit

$$\cos.\eta = 0,\ \cos.2\eta = -1,\ \cos.3\eta = 0,\ \cos.4\eta = +1 \text{ etc.}$$

unde nascitur sequens series

$$\frac{n\pi \cos.\frac{n\pi}{2k}}{2k^3 \sin.\frac{n\pi}{k}} = \frac{4}{4kk-nn} - \frac{16}{16kk-nn} + \frac{36}{36kk-nn} - \frac{64}{64kk-nn} + \text{etc.}$$

Quia autem $\sin.\frac{n\pi}{k} = 2\sin.\frac{n\pi}{2k}\cos.\frac{n\pi}{2k}$, erit ejusdem seriei summa $\frac{n\pi}{4k^3\sin.\frac{n\pi}{2k}}$.

§. 194. At si series illa §. 192. exhibita in $\partial\eta$ ducatur et integretur, ob

$$\int\partial\eta\sin.\frac{n\eta}{k} = -\frac{k}{n}\cos.\frac{n\eta}{k}, \text{ erit}$$

$$C - \frac{\pi\cos.\frac{n\eta}{k}}{2nk\sin.\frac{n\pi}{k}} = -\frac{\cos.\eta}{kk-nn} + \frac{\cos.2\eta}{4kk-nn} - \frac{\cos.3\eta}{9kk-nn} + \frac{\cos.4\eta}{16kk-nn} + \text{etc.}$$

Ut autem hic constantem addendam C definiamus, sumamus $\eta = 0$, fietque

$$C - \frac{\pi}{2nk\sin.\frac{n\pi}{k}} = -\frac{1}{kk-nn} + \frac{1}{4kk-nn} - \frac{1}{9kk-nn} + \text{etc.}$$

quare si hujus seriei summa aliunde pateat, constans C definiri poterit. Series autem haec in sequentem geminatam resolvi potest

$$2nC - \frac{\pi}{k\sin.\frac{n\pi}{k}} = \frac{1}{k+n} - \frac{1}{2k+n} + \frac{1}{3k+n} - \frac{1}{4k+n} + \text{etc.}$$
$$- \frac{1}{k-1} + \frac{1}{2k-n} - \frac{1}{3k-n} + \frac{1}{4k-n} - \text{etc.}$$

§. 195. Cum igitur in *Introductione in Analysin Infinitorum* pag. 142. ad hanc pervenissem seriem

$$\frac{1}{kk-nn} - \frac{1}{4kk-nn} + \frac{1}{9kk-nn} - \frac{1}{16kk-nn} + \text{etc.}$$
$$= \frac{\pi}{2kn\sin.\frac{n\pi}{k}} - \frac{1}{2nn},$$

(hic scilicet loco litterarum ibi adhibitarum m et n scripsi n et k), hoc valore adhibito, nostra aequatio erit

$$C = \frac{\pi}{2nk \sin. \frac{n\pi}{k}} = \frac{1}{2nn} - \frac{\pi}{2nk \sin. \frac{n\pi}{k}},$$

unde fit $C = \frac{1}{2nn}$. Hinc ergo habebimus istam summationem

$$\frac{\pi \cos. \frac{n\eta}{k}}{2nk \sin. \frac{n\pi}{k}} - \frac{1}{2nn} = \frac{\cos. \eta}{kk - nn} - \frac{\cos. 2\eta}{4kk - nn}$$
$$+ \frac{\cos. 3\eta}{9kk - nn} - \frac{\cos. 4\eta}{16kk - nn} + \text{etc.}$$

quae series utique omni attentione digna videtur.

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7) Methodus inveniendi formulas integrales quae certis casibus datam inter se teneant rationem. *Opuscula Analytica. Tom. II. Pag. 178—216.*

§. 196. Quemadmodum in seriebus recurrentibus quilibet terminus ex uno pluribusve praecedentibus secundum legem quandam constantem determinatur, ita hic ejusmodi series sum consideraturus, in quibus quilibet terminus ex uno pluribusve praecedentibus secundum quampiam legem variabilem determinatur. Quoniam autem in talibus seriebus formula generalis singulos terminos exprimens plerumque non est algebraica, sed transcendens, singulos terminos per formulas integrales exhiberi conveniet, quae ut valores determinatos praebeant, post integrationem quantitati variabili valorem determinatum tribui assumo, ita ut singuli termini prodeant quantitates determinatae; atque nunc quaestio principalis huc redit, quemadmodum istae formulae integrales debeant esse comparatae, ut quilibet terminus secundum datam legem ex uno pluribusve praecedentibus determinetur.

§. 197. Quod quo clarius perspiciatur, contemplemur seriem notissimam harum formularum integralium

$$\int \frac{\partial x}{\surd(1-xx)}, \int \frac{xx\partial x}{\surd(1-xx)}, \int \frac{x^4\partial x}{\surd(1-xx)}, \int \frac{x^6\partial x}{\surd(1-xx)}, \text{ etc.}$$

quae si singulae ita integrentur, ut evanescant posito $x = 0$, tum vero variabili x tribuatur valor $= 1$, quilibet terminus a praecedente ita pendet, ut sit

$$\int \frac{xx\partial x}{\surd(1-xx)} = \frac{1}{2} \int \frac{\partial x}{\surd(1-xx)},$$

$$\int \frac{x^4\partial x}{\surd(1-xx)} = \frac{3}{4} \cdot \int \frac{xx\partial x}{\surd(1-xx)},$$

$$\int \frac{x^6\partial x}{\surd(1-xx)} = \frac{5}{6} \cdot \int \frac{x^4\partial x}{\surd(1-xx)},$$

atque in genere

$$\int \frac{x^n\partial x}{\surd(1-xx)} = \frac{n-1}{n} \int \frac{x^{n-2}\partial x}{\surd(1-xx)}.$$

Unde patet, hanc formulam generalem spectari posse tanquam terminum generalem illius seriei, atque quemlibet terminum ex praecedente oriri, si iste multiplicetur per $\frac{n-1}{n}$.

§. 198. Ad similitudinem igitur hujus casus seriem formularum integralium ita in genere constituamus,

$$\int \partial v, \int x\partial v, \int xx\partial v, \int x^3\partial v, \int x^4\partial v, \text{ etc.}$$

ita ut terminus indici n respondens sit $\int x^{n-1}\partial v$, quae singula integralia ita accipi sumamus, ut evanescant posito $x = 0$, post integrationem autem quantitati variabili x tribuamus quempiam valorem constantem, veluti $x = 1$, vel alio cuipiam numero. Quibus positis quaestio huc redit, qualis pro v assumi debeat functio ipsius x, ut quilibet terminus per unum, vel duos pluresve praecedentes, secundum legem quandam datam utcunque variabilem, sive ab indice n pendentem, determinetur; ubi quidem imprimis eo erit respiciendum, ad quot dimensiones index n in scala relationis proposita ascendat: plerumque autem non ultra primam dimensionem assurgere erit opus. Hinc igitur sequentia problemata pertractemus.

Problema I.

§. 199. *Invenire functionem v, ut ista relatio inter binos terminos sibi succedentes locum habeat:*

$$\int x^n \partial v = \frac{\alpha n + a}{\beta n + b} \int x^{n-1} \partial v.$$

Solutio.

Requiritur igitur hic, ut sit

$$(\alpha n + a) \int x^{n-1} \partial v = (\beta n + b) \int x^n \partial v,$$

si scilicet post integrationem variabili x certus valor tribuatur. Quoniam igitur ista conditio tum demum locum habere debet, postquam variabili x iste valor constans fuerit datus, ponamus in genere, dum x est variabilis, hanc aequationem locum habere.

$$(\alpha n + a) \int x^{n-1} \partial v = (\beta n + b) \int x^n \partial v + V,$$

quantitatem autem V ita esse comparatam, ut evanescat postquam variabili ille valor determinatus fuerit assignatus. Praeterea vero, quia ambo integralia ita capi assumimus, ut evanescant posito $x = 0$, necesse est ut etiam ista quantitas V eodem quoque casu evanescat.

§. 200. Quoniam haec aequalitas subsistere debet pro omnibus indicibus n, quos quidem semper ut positivos spectamus, facile intelligitur, quantitatem istam V factorem habere debere x^n;

quo pacto jam isti conditioni satisfit, ut posito $x = 0$ etiam fiat $V = 0$. Quamobrem statuamus $V = x^n Q$, ubi Q denotet functionem ipsius x proposito accommodatam, et quam simul ita comparatam esse desideramus, ut evanescat si ipsi x certus quidem valor tribuatur.

§. 201. Cum igitur esse debeat

$$(\alpha n + a) \int x^{n-1} \partial v = (\beta n + b) \int x^n \partial v + x^n Q,$$

differentietur ista aequatio, ac differentiali per x^{n-1} diviso pervenietur ad hanc aequationem differentialem

$$(\alpha n + a) \partial v = (\beta n + b) x \partial v + n Q \partial x + x \partial Q,$$

quae cum subsistere debeat pro omnibus valoribus ipsius n, termini ista littera affecti seorsim se tollere debent, unde nanciscimur has duas aequalitates

$$\text{I. } (\alpha - \beta x) \partial v = Q \partial x \text{ et}$$

$$\text{II. } (a - b x) \partial v = x \partial Q.$$

Ex priore fit $\partial v = \frac{Q \partial x}{\alpha - \beta x}$, ex altera vero $\partial v = \frac{x \partial Q}{a - bx}$, qui duo valores inter se aequati suppeditant hanc aequationem $\frac{\partial Q}{Q} = \frac{\partial x}{x} \cdot \frac{a - bx}{\alpha - \beta x}$, quae aequatio resolvitur in has partes

$$\frac{\partial Q}{Q} = \frac{a}{\alpha} \cdot \frac{\partial x}{x} + \frac{a\beta - b\alpha}{\alpha} \cdot \frac{\partial x}{\alpha - \beta x},$$

cujus ergo integrale erit

$$lQ = \frac{a}{\alpha}\, lx - \frac{a\beta - b\alpha}{\alpha\beta}\, l(\alpha - \beta x);$$

unde deducitur

$$Q = C x^{\frac{a}{\alpha}} \cdot (\alpha - \beta x)^{\frac{b\alpha - a\beta}{\alpha\beta}}.$$

§. 202. Ex hoc valore pro Q invento statim patet, eum evanescere casu $x = \frac{\alpha}{\beta}$, si modo fuerit $\frac{b\alpha - a\beta}{\alpha\beta} > 0$; sin autem secus eveniat, non patet quomodo haec quantitas ullo casu evanescere queat. Invento autem hoc valore Q, inde reperietur

$$\partial v = C x^{\frac{a}{\alpha}} \partial x (\alpha - \beta x)^{\frac{b\alpha - a\beta}{\alpha\beta} - 1}$$

hincque nostrae seriei terminus indici n respondens erit

$$\int x^{n-1} \partial v = C \int x^{n + \frac{a}{\alpha} - 1} \partial x (\alpha - \beta x)^{\frac{b\alpha - a\beta}{\alpha\beta} - 1},$$

tum vero erit

$$V = C x^{n + \frac{a}{\alpha}} (\alpha - \beta x)^{\frac{b\alpha - a\beta}{\alpha\beta}}.$$

Ubi res imprimis eo redit, ut ista quantitas praeter casum $x = 0$ insuper alio casu evanescat.

Corollarium 1.

§. 203. Hic duo casus occurrunt, qui peculiarem evolutionem postulant; prior est, quo $\alpha = 0$; tum autem inchoandum erit ab aequatione $\frac{\partial Q}{Q} = -\frac{(a - bx)\partial x}{\beta x x}$, unde integrando elicitur $l Q = \frac{a}{\beta x} + \frac{b}{\beta} l x$, hincque sumendo e pro numero cujus logarithmus hyperbolicus $= 1$, colligitur

$$Q = e^{\frac{a}{\beta x}} \cdot x^{\frac{b}{\beta}}$$

quae formula in nihilum abire nequit, nisi fiat $\frac{a}{\beta x} = -\infty$, ideoque $x = 0$, sicque non duo haberentur casus, quibus fieret $V = 0$, cum tamen duo desiderentur. Interim autem hinc fiet

$$\partial v = \frac{e^{\frac{a}{\beta x}} x^{\frac{b}{\beta}} \partial x}{\alpha - \beta x}.$$

Corollarium 2.

§. 204. Alter casus peculiarem integrationem postulans erit quo $\beta = 0$; tum autem erit $\frac{\partial Q}{Q} = \frac{\partial x(a - bx)}{\alpha x}$, unde fit $lQ = \frac{a}{\alpha} lx - \frac{bx}{\alpha}$; ideoque $Q = x^{\frac{a}{\alpha}} \cdot e^{\frac{-bx}{\alpha}}$, quae formula casu $x = \infty$ evanescit, si modo fuerit $\frac{b}{\alpha}$ numerus positivus, sin autem $\frac{b}{\alpha}$ fuerit numerus negativus, tum Q evanescit casu $x = -\infty$. Porro vero hoc casu fiet

$$\partial v = \frac{x^{\frac{a}{\alpha}} \cdot e^{\frac{-bx}{\alpha}} \partial x}{a - \beta x}.$$

Scholion.

§. 205. His in genere observatis aliquot casus speciales evolvamus, quibus litteris α, β et a, b certos valores tribuemus, qui ad casus jam satis cognitos perducant.

Exemplum 1.

§. 206. Quaerantur formulae integrales, ut fiat

$$\int x^n \partial v = \frac{(2n-1)}{2n} \int x^{n-1} \partial v.$$

Cum igitur hic esse debeat

$$(2n-1) \int x^{n-1} \partial v = 2n \int x^n \partial v,$$

erit hoc casu $\alpha = 2$ et $a = -1$, tum vero $\beta = 2$ et $b = 0$; hinc fit

$$\frac{\partial Q}{Q} = -\frac{\partial x}{2x(1-x)} = -\frac{\partial x}{2x} - \frac{\partial x}{2(1-x)},$$

unde integrando

$$lQ = -\tfrac{1}{2} lx + \tfrac{1}{2} l(1-x),$$

ideoque

$$Q = C\sqrt{\frac{1-x}{x}}, \text{ ergo } V = Cx^n\sqrt{\frac{1-x}{x}}.$$

Porro cum hic sit $\partial v = \frac{Q\partial x}{2(1-x)}$, erit

$$\partial v = \frac{C\partial x\sqrt{\frac{1-x}{x}}}{2(1-x)} = \frac{C\partial x}{2\sqrt{(x-xx)}},$$

sumto ergo $C=2$ erit $\partial v = \frac{\partial x}{\sqrt{(x-xx)}}$, et formula nostra generalis

$$\int x^{n-1}\partial v = \int\frac{x^{n-1}\partial x}{\sqrt{(x-xx)}},$$

unde cum sit $V = x^n\sqrt{\frac{1-x}{x}}$, haec quantitas manifesto evanescit sumto $x=1$, ita ut nostra formula, si post integrationem statuatur $x=1$, quaesito satisfaciat. Quod si jam ponamus $x=yy$, ista formula induet hanc formam $2\int\frac{y^{2n-2}\partial y}{\sqrt{(1-yy)}}$, quae, posito post integrationem $y=1$, praebet hanc relationem

$$\int\frac{y^{2n}\partial y}{\sqrt{(1-yy)}} = \frac{2n-1}{2n}\int\frac{y^{2n-2}\partial y}{\sqrt{(1-yy)}},$$

quae continet relationes supra §. 197. commemoratas; hinc enim fiet

$$\int\frac{yy\partial y}{\sqrt{(1-yy)}} = \tfrac{1}{2}\int\frac{\partial y}{\sqrt{(1-yy)}},$$
$$\int\frac{y^4\partial y}{\sqrt{(1-yy)}} = \tfrac{3}{4}\int\frac{yy\partial y}{\sqrt{(1-yy)}},$$
$$\int\frac{y^6\partial y}{\sqrt{(1-yy)}} = \tfrac{5}{6}\int\frac{y^4\partial y}{\sqrt{(1-yy)}} \text{ etc.}$$

Exemplum 2.

§. 207. Quaerantur formulae integrales, ut fiat

$$\int x^n\partial v = \tfrac{\alpha n-1}{\alpha n}\int x^{n-1}\partial v.$$

Cum igitur hic esse debeat

$$(\alpha n-1)\int x^{n-1}\partial v = \alpha n\int x^n\partial v,$$

erit hoc casu $a = -1$, $\beta = \alpha$ et $b = 0$, unde per formulas supra datas colligitur

$$Q = Cx^{\frac{-1}{\alpha}}(\alpha - \alpha x)^{\frac{-a}{\alpha}} = Cx^{\frac{-1}{\alpha}}(1-x)^{\frac{+1}{\alpha}}$$

quae quantitas manifesto evanescit posito $x = 1$. Tum autem erit

$$\partial v = \frac{x^{\frac{-1}{\alpha}}(1-x)^{\frac{+1}{\alpha}}\partial x}{(1-x)},$$

unde formula nostra generalis erit

$$\int x^{n-1}\partial v = \int x^{n-\frac{1}{\alpha}-1}(1-x)^{+\frac{1}{\alpha}-1}\partial x = \int \frac{x^{n-\frac{1}{\alpha}-1}\partial x}{(1-x)^{1-\frac{1}{\alpha}}},$$

quae concinnior redditur, faciendo $x = y^\alpha$, tum enim ea induet hanc formam $\int \frac{y^{\alpha n-2}\partial y}{(1-y^\alpha)^{\frac{\alpha-1}{\alpha}}}$, ubi iterum post integrationem statui debet $y = 1$. Erit hinc

$$\int \frac{y^{\alpha n+\alpha-2}\partial y}{(1-y^\alpha)^{\frac{\alpha-1}{\alpha}}} = \frac{\alpha n-1}{\alpha n}\int \frac{y^{\alpha n-2}\partial y}{(1-y^\alpha)^{\frac{\alpha-1}{\alpha}}},$$

atque hinc oriuntur sequentes casus speciales

$$\int \frac{y^{2\alpha-2}\partial y}{(1-y^\alpha)^{\frac{\alpha-1}{\alpha}}} = \frac{\alpha-1}{\alpha}\int \frac{y^{\alpha-2}\partial y}{(1-y^\alpha)^{\frac{\alpha-1}{\alpha}}} \quad \text{et}$$

$$\int \frac{y^{3\alpha-2}\partial y}{(1-y^\alpha)^{\frac{\alpha-1}{\alpha}}} = \frac{2\alpha-1}{2\alpha}\int \frac{y^{2\alpha-2}\partial y}{(1-y^\alpha)^{\frac{\alpha-1}{\alpha}}}.$$

§. 208. Hinc igitur si sumatur $a = 1$, ut fieri debeat

$$\int x^n \partial v = \tfrac{n-1}{n}\int x^{n-1}\partial v,$$

formula nostra generalis jam in y expressa erit $\int y^{n-2}\partial y$, cujus ergo valor est $\frac{1}{n-1}y^{n-1}=\frac{1}{n-1}$, unde tota series nostrarum formularum integralium abibit in hanc

$$\tfrac{1}{0},\ \tfrac{1}{1},\ \tfrac{1}{2},\ \tfrac{1}{3},\ \tfrac{1}{4},\ \tfrac{1}{5},\ \tfrac{1}{6},\ \tfrac{1}{7},\ \text{etc.}$$

§. 209. Sumamus etiam $\alpha=\frac{1}{2}$, et jam non amplius opus erit ad y procedere. Hoc igitur casu erit

$$Q=\frac{(1-x)^2}{xx} \text{ et } \partial v=\frac{(1-x)\partial x}{xx},$$

unde formula nostra generalis fit

$$\int x^{n-1}\partial v=\int x^{n-3}(1-x)\partial x,$$

cujus ergo valor algebraice expressus erit

$$\frac{1}{n-2}x^{n-2}-\frac{1}{n-1}x^{n-1}=\frac{1}{(n-1)(n-2)};$$

unde series nostrarum formularum evadet

$$\frac{1}{0\cdot-1},\ \frac{1}{0\cdot1},\ \frac{1}{1\cdot2},\ \frac{1}{2\cdot3},\ \frac{1}{3\cdot4},\ \frac{1}{4\cdot5},\ \text{etc.}$$

Exemplum 3.

§. 210. Quaerantur formulae integrales, ut sit

$$\int x^n\partial v=n\int x^{n-1}\partial v.$$

Cum igitur esse debeat

$$n\int x^{n-1}\partial v=1\,.\int x^n\partial v, \text{ erit}$$

$$\alpha=1,\ a=0,\ b=1,\ \beta=0.$$

Cum igitur sit $\beta=0$, casus Coroll. 2. hic locum habet, indeque erit $Q=e^{-x}$, ideoque $V=e^{-x}.x^n$, quae quantitas his duobus casibus evanescit $x=0$ et $x=\infty$. Porro vero erit $\partial v=e^{-x}\partial x$, hincque formula nostra generalis fiet $\int x^{n-1}\partial x\,.\,e^{-x}$, unde ipsi seriei termini ab initio sequenti modo se habebunt

$$\int e^{-x}\partial x,\ \int e^{-x}x\partial x,\ \int e^{-x}xx\partial x,\ \int e^{-x}x^3\partial x \text{ etc.}$$

quibus integratis ita ut evanescant posito $x = 0$, tum vero posito $x = \infty$, orietur sequens series satis simplex

$$1. 1, \quad 1. 2, \quad 1. 2. 3, \quad 1. 2. 3. 4, \quad 1. 2. 3. 4. 5, \text{ etc.}$$

quae est series hypergeometrica Wallisii, cujus ergo terminus generalis est

$$\int x^{n-1} e^{-x} \partial x = 1 . 2 . 3 . 4 \ldots\ldots (n-1).$$

§. 211. Ope ergo hujus termini generalis hanc seriem interpolare licebit. Ita si quaeratur terminus medius inter duos primos, poni debet $n = \frac{3}{2}$, ac valor hujus termini erit $\int e^{-x} \partial x \sqrt{x}$, cujus autem valor nullo modo algebraice exprimi potest. Inventi autem singulari modo hunc ipsum terminum aequari $\frac{1}{2}\sqrt{\pi}$, denotante π peripheriam circuli cujus diameter $= 1$, unde hic vicissim cognoscimus esse $\int e^{-x} \partial x \sqrt{x} = \frac{\sqrt{\pi}}{2}$, posito scilicet post integrationem $x = \infty$. Terminus autem hunc praecedens, indici $\frac{1}{2}$ respondens, erit $= \sqrt{\pi}$, cui ergo aequatur formula $\int \frac{e^{-x} \partial x}{\sqrt{x}}$. Quod si hic ponamus $e^x = y$, ita ut posito $x = 0$ sit $y = 1$, at posito $x = \infty$ fiat $y = \infty$, tum ergo ista formula $\int \frac{e^{-x} \partial x}{\sqrt{x}}$ abit in hanc $\int \frac{\partial y}{yy\sqrt{ly}}$, quae formula si ita integretur ut evanescat posito $y = 1$, tum vero fiat $y = \infty$, praebet valorem ipsius $\sqrt{\pi}$. Si porro fiat $y = \frac{1}{z}$, erunt termini integrationis $z = 1$, et $z = 0$, et formula integralis erit

$$-\int \frac{\partial z}{\sqrt{-lz}} \begin{bmatrix} \text{a } z = 1 \\ \text{ad } z = 0 \end{bmatrix} = \sqrt{\pi},$$

sive permutatis terminis integrationis erit

$$\int \frac{\partial z}{\sqrt{-lz}} \begin{bmatrix} \text{a } z = 0 \\ \text{ad } z = 1 \end{bmatrix} = \sqrt{\pi},$$

quemadmodum jam olim observavi.

Exemplum 4.

§. 212. Quaerantur formulae integrales, ut sit

$\int x^n \partial v = \frac{1}{n}\int x^{n-1}\partial v$, sive

$\int x^{n-1}\partial v = n\int x^n \partial v$.

Hic est $\alpha = 0$ et $a = 1$, $\beta = 1$ et $b = 0$; qui ergo est casus in Coroll. 1. tractatus, unde colligitur fore $Q = e^{\frac{1}{x}}$, ideoque $V = x^n e^{\frac{1}{x}}$, quae formula nequidem evanescit sumto $x = 0$, quandoquidem formula $e^{\frac{1}{0}}$ aequivalet infinito infinitesimae potestatis. Hic autem miro modo evenit, ut casus $x = -0$ reddat formulam $e^{\frac{-1}{0}}$ subito evanescentem. Scilicet, si ω denotet quantitatem infinite parvam, erit $e^{\frac{1}{\omega}} = \infty^\infty$, tum vero repente fiet $e^{\frac{-1}{\omega}} = \frac{1}{\infty^\infty} = 0$, quam ob causam formulam hinc exhibere non licet scopo nostro respondentem. Reperietur quidem $\partial v = -e^{\frac{1}{x}}\frac{\partial x}{x}$, ita ut formula nostra generalis futura sit $-\int x^{n-2}\partial x\, e^{\frac{1}{x}}$, quae autem nobis nullum usum praestare potest.

§. 213. Quod si hic ponamus $\frac{1}{x} = y$, formula ista generalis transit in hanc $+\int\frac{e^y \partial y}{y^n}$. At vero nunc erit $V = \frac{e^y}{y^n}$, quae formula evanescit posito $y = -\infty$. Quomodocunque autem hanc expressionem transformemus, semper idem incommodum occurret. Interim tamen etiam hunc casum sequenti modo resolvere licebit. Sit enim seriei, quam quaerimus, primus terminus $= \omega$, ex quo per regulam praescriptam sequentes ordine ita procedent

$$\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & \quad n \\ \omega, & \frac{\omega}{1}, & \frac{\omega}{1\cdot 2}, & \frac{\omega}{1\cdot 2\cdot 3}, & \frac{\omega}{1\cdot 2\cdot 3\cdot 4}, \ldots\ldots & \frac{\omega}{1\cdot 2\cdot 3\cdots\cdots(n-1)}. \end{array}$$

Supra autem vidimus, hujus formulae $1 . 2 . 3 . 4 . 5 \ldots\ldots (n-1)$ valorem exprimi per hoc integrale $\int x^{n-1} e^{-x} \partial x$, integratione ab $x = 0$ ad $x = \infty$ extensa; tantum igitur opus est ut hanc formulam integralem in denominatorem transferamus, et seriei quam quaerimus terminus generalis erit

$$\frac{1}{\int x^{n-1} e^{-x} \partial x},$$

unde satis intelligitur, negotium non per simplicem formulam integralem expediri posse, quod idem quoque tenendum est de aliis casibus, quibus quantitas V non duobus casibus evanescere potest; tum enim tantum opus est fractionem $\frac{\alpha n + a}{\beta n + b}$ invertere, atque formulam integralem in denominatorem transferre.

Scholion.

§. 214. Nisi sit vel $\alpha = 0$ vel $\beta = 0$, quos casus jam expedivimus, resolutio nostri problematis semper reduci potest ad casum, quo ambae litterae α et β sunt aequales unitati. Cum enim esse debeat

$$\int x^n \partial v = \frac{\alpha n + a}{\beta n + b} \int x^{n-1} \partial v,$$

ponatur $x = \frac{\alpha y}{\beta}$, fietque

$$\frac{\alpha}{\beta} \int y^n \partial v = \frac{\alpha n + a}{\beta n + b} \int y^{n-1} \partial v,$$

quae aequatio reducitur ad hanc formam

$$\int y^n \partial v = \frac{n + a : \alpha}{n + b : \beta} \int y^{n-1} \partial v.$$

Quod si jam nunc loco $\frac{a}{\alpha}$ scribamus a, et b loco $\frac{b}{\beta}$, resolvenda erit haec formula

$$\int y^n \partial v = \tfrac{n+a}{n+b} \int y^{n-1} \partial v,$$

cujus resolutio, si loco x scribamus y et loco litterarum α et β unitatem, ex superiori solutione praebet primo

$$Q = C y^a (1-y)^{b-a},$$

quod ergo evanescit posito $y = 1$, si modo fuerit $b > a$, tum autem erit ipsa formula

$$\int y^{n-1} \partial v = C \int y^{n+a-1} \partial y \; (1-y)^{b-a-1};$$

sin autem fuerit $b < a$, haec solutio, uti vidimus, locum habere nequit; verum hoc casu pro termino nostrae seriei assumi debet haec forma $\frac{1}{\int y^{n-1} \partial v}$, ita ut tum esse debeat

$$\frac{1}{\int y^n \partial v} = \frac{n+a}{n+b} \cdot \frac{1}{\int y^{n-1} \partial v}, \text{ sive}$$

$$\int y^n \partial v = \tfrac{n+b}{n+a} \int y^{n-1} \partial v,$$

cujus resolutio permutatis litteris a et b praebet

$$Q = C y^b (1-y)^{a-b},$$

quae jam casu $y = 1$ evanescit, si fuerit $a > b$, atque tum erit formula generalis

$$\int y^{n-1} \partial v = C \int y^{n+b-1} \partial y \; (1-y)^{a-b-1}.$$

Sive igitur sit $b > a$ sive $a > b$, solutio nulla amplius laborat difficultate.

§. 215. Sin autem fuerit vel $\alpha = 0$ vel $\beta = 0$, loco alterius etiam scribi poterit unitas; unde si esse debeat

$$\int x^n \partial v = \tfrac{n+a}{b} \int x^{n-1} \partial v,$$

ob $\alpha = 1$ et $\beta = 0$, solutio nostra generalis dat

$$\frac{\partial Q}{Q} = \frac{\partial x}{x}(a - bx);$$

unde colligitur $Q = Cx^a \cdot e^{-bx}$; quae formula evanescit posito $x = \infty$, si modo b fuerit numerus positivus; tum autem fit terminus generalis

$$\int x^{n-1} \partial v = C \int x^{n+a-1} \partial x \cdot e^{-x}.$$

At vero numerus b negativus esse nequit, quia alioquin conditio praescripta esset incongrua.

§. 216. Consideremus etiam alterum casum, quo $\alpha = 0$ et $\beta = 1$, ideoque conditio praescripta

$$\int x^n \partial v = \frac{a}{n+b} \int x^{n-1} \partial v,$$

unde fit

$$\frac{\partial Q}{Q} = -\frac{\partial x}{xx}(a - bx).$$

Hinc autem pro Q orietur valor, qui praeter casum $x = 0$ evanescere non posset; quam ob causam formula generalis statui debet $\frac{1}{\int x^{n-1} \partial v}$, ita ut esse debeat

$$\int x^n \partial v = \frac{n+b}{a} \int x^{n-1} \partial v,$$

unde prodit

$$\frac{\partial Q}{Q} = \frac{\partial x}{x}(b - ax), \text{ ideoque } Q = C e^{-ax} \cdot x^b,$$

quae expressio evanescit posito $x = \infty$, quoniam a necessario debet esse numerus positivus; tum autem erit

$$\partial v = C e^{-ax} \cdot x^b \partial x,$$

unde formula generalis seriei erit

$$\frac{1}{C \int x^{a+b-1} \partial x \cdot e^{-ax}}.$$

Problema 2.

Denotet T terminum indici n respondentem in serie quam considerandam suscepimus, at vero T' terminum sequentem, atque proponatur haec conditio adimplenda

$$T' = \frac{(\alpha n + a)(\alpha' n + a')}{(\beta n + b)(\beta' n + b')} T.$$

Solutio.

§. 217. Quoniam hic valores geminati occurrunt, huic conditioni commodissime satisfiet, si terminus generalis T tanquam productum ex duobus factoribus spectetur. Statuatur igitur $T = RS$, sitque terminus sequens $= R'S'$, et quaerantur formulae R et S, ut fiat

$$R' = \frac{\alpha n + a}{\beta n + b} R \text{ et } S' = \frac{\alpha' n + a'}{\beta' n + b'} S,$$

tum enim sumendo $T = RS$ conditioni praescriptae manifesto satisfiet. Hoc igitur modo pro R et S vel hujusmodi formulae $\int x^{n-1} \partial v$, vel inversae $\frac{1}{\int x^{n-1} \partial v}$ reperientur, id quod pro solutione generali sufficit, unde rem exemplo illustremus.

Exemplum.

§. 218. Quaeratur formula generalis T, ut fiat

Resolvamus igitur T in duos factores R et S, ac statuamus

$$T' = \frac{nn - cc}{nn} T.$$

$$R' = \frac{n - c}{n} R \text{ et } S' = \frac{n + c}{n} S.$$

Pro priore forma si statuamus $R = \int x^{n-1} \partial v$, ex solutione generali, ubi erit $\alpha = 1$, $a = -c$, $\beta = 1$ et $b = 0$, fiet

$$Q = Cx^{-c}(1 - x)^c,$$

quae forma manifesto evanescit posito $x = 1$, hincque quia fit

$$V = Cx^{n-c}(1 - x)^c,$$

haec forma etiam casu $x = 0$ evanescit, si modo n fuerit $> c$, id quod tuto assumi potest, quia exponentem n successive in infinitum crescere assumimus, ac plerumque pro c fractiones tantum accipi solent. Hinc ergo erit

$$R = C \int x^{n-c-1} (1-x)^{c-1} \partial x.$$

§. 219. Hinc jam alter valor litterae S deduci posset, scribendo tantum $-c$ loco c, tum autem non amplius fieret $Q = 0$ posito $x = 1$, quamobrem pro S formulam inversam $\frac{1}{\int x^{n-1} \partial v}$ assumi oportet, ut esse debeat

$$\int x^n \partial v = \frac{n}{n+c} \int x^{n-1} \partial v,$$

ubi cum sit $\alpha = 1$, $a = 0$, $\beta = 1$ et $b = c$, reperitur $Q = C (1-x)^c$, quae forma manifesto fit $= 0$ posito $x = 1$, hinc autem prodit

$$\partial v = C (1-x)^{c-1} \partial x,$$

ergo habebimus

$$S = \frac{1}{C \int x^{n-1} (1-x)^{c-1} \partial x},$$

consequenter formula nostra generalis quaesita erit

$$T = \frac{\int x^{n-c-1} (1-x)^{c-1} \partial x}{\int x^{n-1} (1-x)^{c-1} \partial x}.$$

§. 220. Quod si ergo nostrae seriei per factores procedentes primum terminum ponamus $= A$, ipsa series erit

I.	II.	III.	IV.
A,	$\frac{1-cc}{1} A$,	$\frac{1-cc}{1} \cdot \frac{4-cc}{4} A$,	$\frac{1-cc}{1} \cdot \frac{4-cc}{4} \cdot \frac{9-cc}{9} A$, etc.

unde si sumamus $c = \frac{1}{2}$, erit haec series

$$A, \frac{1 \cdot 3}{2 \cdot 2} A, \frac{1 \cdot 3}{2 \cdot 2} \cdot \frac{3 \cdot 5}{4 \cdot 4} A, \frac{1 \cdot 3}{2 \cdot 2} \cdot \frac{3 \cdot 5}{4 \cdot 4} \cdot \frac{5 \cdot 7}{6 \cdot 6} A, \text{ etc.}$$

cujus ergo terminus indici n respondens est

$$\frac{\int x^{n-\frac{3}{2}}(1-x)^{-\frac{1}{2}}\partial x}{\int x^{n-1}(1-x)^{-\frac{1}{2}}\partial x},$$

qui posito $x = yy$ transit in hanc formam

$$\frac{\int y^{2n-2}(1-yy)^{-\frac{1}{2}}\partial y}{\int y^{2n-1}(1-yy)^{-\frac{1}{2}}\partial y};$$

unde patet, terminum primum fore

$$A = \int \frac{\partial y}{\sqrt{(1-yy)}} : \int \frac{y\partial y}{\sqrt{(1-yy)}} = \frac{\pi}{2},$$

posito scilicet post integrationem $y = 1$.

Problema 3.

Denotet T terminum seriei indici n respondentem, sintque T' et T'' termini sequentes pro indicibus $n+1$ et $n+2$, si proponatur inter ternos terminos se insequentes talis relatio, ut sit

$$(\alpha n + a)\, T = (\beta n + b)\, T' + (\gamma n + c)\, T'',$$

investigare formulam pro T, qua terminus generalis hujus seriei exprimatur.

Solutio.

§. 221. Assumatur pro T formula integralis $\int x^{n-1}\partial v$, hujusque integrale ita capiatur, ut evanescat posito $x = 0$, eruntque termini sequentes

$$T' = \int x^n \partial v \text{ et } T'' = \int x^{n+1}\partial v,$$

siquidem post integrationem variabili x certus valor determinatus tribuatur. Quamdiu autem haec quantitas x ut variabilis spectatur, ponamus esse

$$(\alpha n + a)\, T = (\beta n + b)\, T' + (\gamma n + c)\, T'' + x^n Q,$$

ac perspicuum est Q ejusmodi functionem esse debere ipsius x, quae evanescat, si loco x valor ille determinatus substituatur, quem autem a cifra diversum esse oportet, quoniam jam assumsimus, omnes istas formulas in nihilum abire posito $x = 0$. Quodsi vero, absoluto calculo, huic conditioni nullo modo satisfieri poterit, id erit indicio, problema nostrum hac ratione resolvi non posse, ut scilicet ejus terminus generalis T per talem formulam differentialem simplicem $\int x^{n-1}\,\partial v$ exhibeatur.

§. 222. Differentiemus nunc aequationem modo stabilitam, ac divisione facta per x^{n-1} sequens prodibit aequatio

$$(\alpha n + a)\,\partial v = (\beta n + b)\,x\,\partial v + (\gamma n + c)\,xx\,\partial v + nQ\,\partial x + x\,\partial Q,$$

quae, quia termini littera n affecti seorsim se destruere debent, discerpetur in binas sequentes aequationes

$$1^{\circ}.\ \alpha\,\partial v = \beta x\,\partial v + \gamma xx\,\partial v + Q\,\partial x,$$

$$2^{\circ}.\ a\,\partial v = bx\,\partial v + cxx\,\partial v + x\,\partial Q,$$

ex quarum priore fit

$$\partial v = \frac{Q\,\partial x}{\alpha - \beta x - \gamma xx},$$

ex altera vero fit

$$\partial v = \frac{x\,\partial Q}{a - bx - cxx},$$

quorum valorum posterior per priorem divisus praebet

$$\frac{\partial Q}{Q} = \frac{\partial x\,(a - bx - cxx)}{x\,(\alpha - \beta x - \gamma xx)},$$

ex cujus ergo integratione valor ipsius Q elici debet, quo facto facile patebit, utrum is certo quodam casu praeter $x = 0$ evanescere possit. Imprimis autem hic notari convenit, si hoc integrale involvat hujusmodi factorem $e^{\frac{\lambda}{x}}$, tum solutionem quoque successu esse cariturам, quandoquidem posito $x = 0$ iste factor tantam involvet infiniti potestatem, ut, etiamsi per x^n multiplicetur, productum etiamnum infinitum maneat.

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§. 223. Quodsi igitur his conditionibus praescriptis satisfacere licuerit, tum invento valore litterae Q, quem ponamus fieri $= 0$ posito $x = f$, habebitur

$$\partial v = \frac{Q \partial x}{\alpha - \beta x - \gamma x x},$$

et formula generalis naturam seriei complectens erit

$$T = \int x^{n-1} \partial v = \int \frac{x^{n-1} Q \partial x}{\alpha - \beta x - \gamma x x},$$

quippe cujus integrale, a termino $x = 0$ usque ad terminum $x = f$ extensum, praebebit valorem termini T, indici cuicunque n respondentis.

Scholion.

§. 224. Inventa autem tali relatione inter ternos terminos cujuspiam seriei sibi invicem succedentes, inde more solito formari poterit fractio continua, cujus valorem assignare licebit. Si enim characteres T', T'', T''', T^{IV}, etc. denotent ordine omnes terminos post T sequentes in infinitum, ex relationibus, quas inter se tenent, sequentes formulae deducentur. Ex relatione

$$(\alpha n + a) T = (\beta n + b) T' + (\gamma n + c) T''$$

deducitur

$$(\alpha n + a) \frac{T}{T'} = \beta n + b + \frac{(\gamma n + c)(\alpha n + \alpha + a)}{(\alpha n + \alpha + a) T' : T''}.$$

Ex relatione sequente

$$(\alpha n + \alpha + a) T' = (\beta n + \beta + b) T'' + (\gamma n + \gamma + c) T'''$$

deducitur

$$(\alpha n + \alpha + a) \frac{T'}{T''} = \beta n + \beta + b + \frac{(\gamma n + \gamma + c)(\alpha n + 2\alpha + a)}{(\alpha n + 2\alpha + a) T'' : T'''}.$$

Simili modo sequentes relationes suppeditabunt

$$(\alpha n + 2\alpha + a) \frac{T''}{T'''} = \beta n + 2\beta + b + \frac{(\gamma n + 2\gamma + c)(\alpha n + 3\alpha + a)}{(\alpha n + 3\alpha + a) T''' : T''''};$$

$$(\alpha n+3\alpha+a)\frac{T'''}{T''''}=(\beta n+3\beta+b)+\frac{(\gamma n+3\gamma+c)(\alpha n+4\alpha+a)}{(\alpha n+4\alpha+a)T''''':T''''''}, \text{ etc.}$$

unde manifestum est, si in prima formula continuo sequentes valores ordine substituantur, prodituram esse fractionem continuam, cujus valor aequalis erit formulae $(\alpha n+a)\frac{T}{T'}$.

§. 225. Quod si ergo loco n successive scribamus numeros 1, 2, 3, 4, etc. sequens problema circa fractiones continuas resolvere poterimus.

Problema 4.

Proposita fractione continua hujus formae

$$\beta+b+\cfrac{(\gamma+c)(2\alpha+a)}{2\beta+b+\cfrac{(2\gamma+c)(3\alpha+a)}{3\beta+b+\cfrac{(3\gamma+c)(4\alpha+a)}{4\beta+b+\cfrac{(4\gamma+c)(5\alpha+a)}{5\beta+b+\cfrac{(5\gamma+c)(6\alpha+a)}{6\beta+b+\text{etc.}}}}}}$$

ejus valorem investigare.

Solutio.

§. 226. Consideretur in genere ista relatio inter ternas quantitates sibi succedentes T, T', T'', quae sit

$$(\alpha n+a)T=(\beta n+b)T'+(\gamma n+c)T'',$$

atque ex praecedente problemate quaeratur valor ipsius T, siquidem fieri potest, hoc modo expressus

$$T=\int x^{n-1}\partial v=\int\frac{x^{n-1}Q\partial x}{\alpha-\beta x-\gamma xx},$$

cujus integrale ab $x=0$ usque ad $x=f$ extendatur, qua formula inventa ponatur

$$\int \frac{Q\partial x}{\alpha - \beta x - \gamma xx} = A \text{ et } \int \frac{xQ\partial x}{\alpha - \beta x - \gamma xx} = B,$$

ita ut A et B sint valores ipsius T, pro casibus $n = 1$ et $n = 2$, quibus definitis fractionis continuae propositae valor per praecedentia erit $= \frac{(\alpha + a)A}{B}$. Hanc igitur investigationem ad sequentia exempla accommodemus.

Exemplum 1.

§. 227. Investigare valorem fractionis continuae notissimae, quam olim *Brouncherus* pro quadratura circuli protulit, quae est

$$2 + \cfrac{1.1}{2 + \cfrac{3.3}{2 + \cfrac{5.5}{2 + \text{etc.}}}}$$

Quia omnes partes integrae laevam respicientes sunt constantes $= 2$, pro nostra forma generali fiet

$$\beta + b = 2, \; 2\beta + b = 2, \; 3\beta + b = 2, \text{ etc.}$$

ergo $\beta = 0$ et $b = 2$; at pro numeratoribus sequentium fractionum, quandoquidem constant binis factoribus, erit pro factoribus prioribus

$$\gamma + c = 1, \; 2\gamma + c = 3, \; 3\gamma + c = 5, \; 4\gamma + c = 7, \text{ etc.}$$

unde concluditur $\gamma = 2$ et $c = -1$, pro alteris vero erit

$$2\alpha + a = 1, \; 3\alpha + a = 3, \; 4\alpha + a = 5, \text{ etc.}$$

unde $\alpha = 2$ et $a = -3$. Ex his autem valoribus colligimus hanc aequationem

$$\frac{\partial Q}{Q} = -\frac{\partial x(3 + 2x - xx)}{2x(1 - xx)},$$

quae per $1 + x$ depressa praebet

$$\frac{\partial Q}{Q} = -\frac{\partial x(3 - x)}{2x(1 - x)},$$

unde integrando fit

$$lQ = -\tfrac{3}{2}\, lx + l(1-x) \text{ et hinc } Q = \frac{1-x}{x^{\frac{3}{2}}},$$

ex quo valore porro sequitur

$$A = \int \frac{(1-x)\,\partial x}{2x^{\frac{3}{2}}(1-xx)} = \int \frac{\partial x}{2x(1+x)\sqrt{x}}$$

$$B = \int \frac{(1-x)\,\partial x}{2x^{\frac{1}{2}}(1-xx)} = \int \frac{\partial x}{2(1+x)\sqrt{x}}.$$

§. 228. In his autem valoribus istud incommodum deprehenditur, quod prius integrale evanescens reddi nequit posito $x = 0$. Hoc autem incommodum facile removeri potest, si fractionem continuam supremo membro truncemus et quaeramus valorem istius fractionis

$$2 + \cfrac{3 \cdot 3}{2 + \cfrac{5 \cdot 5}{2 + \text{etc.}}}$$

qui si repertus fuerit $= s$, erit ipsius propositae valor $= b + \frac{1}{s}$. Nunc vero, comparatione instituta, fit quidem ut ante $\beta = 0$ et $b = 2$, tum vero $\gamma = 2$ et $c = +1$, $\alpha = 2$ et $a = -1$, unde sequitur

$$\frac{\partial Q}{Q} = -\frac{\partial x(1+2x+xx)}{x(1-xx)} = -\frac{\partial x(1+x)}{2x(1-x)},$$

unde integrando fit

$$lQ = -\tfrac{1}{2}\, lx + l(1-x), \text{ ideoque } Q = \frac{1-x}{\sqrt{x}};$$

ex quo valore jam habebimus

$$A = \int \frac{(1-x)\,\partial x}{2(1-xx)\sqrt{x}} = \tfrac{1}{2}\int \frac{\partial x}{(1+x)\sqrt{x}}, \text{ et}$$

$$B = \tfrac{1}{2}\int \frac{\partial x\sqrt{x}}{1+x}$$

ubi cum sit $Q = \frac{1-x}{\sqrt{x}}$, ejus valor manifesto evanescit posito $x = 1$, quamobrem illa integralia a termino $x = 0$ usque ad $x = 1$ sunt extendenda.

§. 229. Quo nunc haec integralia facilius eruamus, statuamus $x = zz$, ita ut termini integrationis etiamnunc sint $z = 0$ et $z = 1$, eritque

$$A = \int \frac{\partial z}{1+zz} = \text{Arc. tang. } z = \frac{\pi}{4}, \text{ et}$$

$$B = \int \frac{zz\partial z}{1+zz} = 1 - \frac{\pi}{4},$$

sicque habebimus $s = \frac{\pi}{4-\pi}$, quocirca ipsius fractionis *Brounckerianae* valor est $1 + \frac{4}{\pi}$, omnino uti olim *Brounckerus* jam invenerat.

Exemplum 2.

§. 230. Investigare valorem hujus fractionis continuae *Brouncherianae* latius patentis

$$b + \cfrac{1.1}{b + \cfrac{3.3}{b + \cfrac{5.5}{b + \text{etc.}}}}$$

Ut hic incommodum superius evitemus, omittamus membrum supremum et quaeramus

$$s = b + \cfrac{3.3}{b + \cfrac{5.5}{b + \text{etc.}}}$$

quandoquidem tum erit valor quaesitus $= b + \frac{1}{s}$. Nunc igitur erit $\beta = 0$ et $b = b$, $\gamma = 2$, $c = 1$, $\alpha = 2$ et $a = -1$, unde fit

$$\frac{\partial Q}{Q} = -\frac{\partial x\,(1+bx+xx)}{2x\,(1-xx)},$$

ac proinde

$$lQ = -\tfrac{1}{2}lx - \frac{b-2}{4}l(1+x) + \frac{b+2}{4}l(1-x),$$

hincque

$$Q = \frac{(1-x)^{\frac{b+2}{4}}}{(1+x)^{\frac{b-2}{4}}\sqrt{x}},$$

quae formula manifesto fit $=0$ ponendo $x=1$, siquidem $b+2$ fuerit numerus positivus, unde fit

$$\partial v = \frac{(1-x)^{\frac{b-2}{4}}\partial x}{2\,(1+x)^{\frac{b+2}{4}}\sqrt{x}}.$$

Hinc autem colligetur

$$A = \tfrac{1}{2}\int\frac{(1-x)^{\frac{b-2}{4}}\partial x}{(1+x)^{\frac{b+2}{4}}\sqrt{x}} \quad \text{et}$$

$$B = \tfrac{1}{2}\int\frac{(1-x)^{\frac{b-2}{4}}\partial x\sqrt{x}}{(1+x)^{\frac{b+2}{4}}},$$

sive ponendo $x = zz$ habebimus

$$A = \int\frac{(1-zz)^{\frac{b-2}{4}}\partial z}{(1+zz)^{\frac{b+2}{4}}} \quad \text{et}$$

$$B = \int\frac{(1-zz)^{\frac{b-2}{4}}zz\,\partial z}{(1+zz)^{\frac{b+2}{4}}},$$

quae ambo integralia a $z = 0$ usque ad $z = 1$ sunt extendenda. Ex his autem valoribus A et B erit $s = \frac{A}{B}$; ipsius igitur fractionis propositae valor erit $= b + \frac{1}{s} = b + \frac{B}{A}$.

§. 231. Quod si hic ponatur $b = 2$, prodit casus ante expositus a quadratura circuli pendens, quippe quo casu formula fit rationalis. Quando autem exponentes $\frac{b-2}{4}$ et $\frac{b+2}{4}$ non sunt numeri integri, tum litteras A et B neque per arcus circulares, neque per logarithmos exprimere licet. Veluti si fuerit $b = 4$, erit

$$A = \int \frac{\partial z \sqrt{(1 - zz)}}{(1 + zz)^{\frac{3}{2}}},$$

cujus valor per arcus ellipticos exhiberi posset. At si b fuerit numerus impar, hi valores multo magis evadunt transcendentes, ita ut his ipsis litteris A et B debeamus esse contenti. Contra autem si exponentes illi fiant numeri integri, totum negotium per arcus circulares expedire licebit.

§. 232. Exponentes autem illi $\frac{b-2}{4}$ et $\frac{b+2}{4}$ erunt numeri integri, quoties fuerit b numerus hujus formae $b = 4i + 2$, tum enim erit

$$A = \int \frac{(1 - zz)^i \partial z}{(1 + zz)^{i+1}} \text{ et}$$

$$B = \int \frac{(1 - zz)^i zz \partial z}{(1 + zz)^{i+1}};$$

quos ergo casus quomodo evolvi oporteat, operae pretium erit docere, quoniam *Wallisius* eos jam est contemplatus.

§. 233. Quoniam hoc negotium totum redit ad reductionem hujusmodi formularum integralium ad formas simpliciores, consideremus in genere formam

$$P = \frac{z^m}{(1+zz)^n},$$

cujus differentiale sub sequentibus formis exhiberi potest

$$1^\circ).\ \partial P = \frac{m z^{m-1}\partial z}{(1+zz)^n} - \frac{2n z^{m+1}\partial z}{(1+zz)^{n+1}}$$

$$2^\circ).\ \partial P = \frac{m z^{m-1}\partial z}{(1+zz)^{n+1}} - \frac{(2n-m) z^{m+1}\partial z}{(1+zz)^{n+1}}$$

$$3^\circ).\ \partial P = -\frac{(2n-m) z^{m-1}\partial z}{(1+zz)^n} - \frac{2n z^{m-1}\partial z}{(1+zz)^{n+1}}$$

unde hanc triplicem reductionem integralium deducimus.

$$\text{I.}\quad \int\frac{z^{m+1}\partial z}{(1+zz)^{n+1}} = \frac{m}{2n}\int\frac{z^{m-1}\partial z}{(1+zz)^n} - \frac{1}{2n}\cdot\frac{z^m}{(1+zz)^n}$$

$$\text{II.}\quad \int\frac{z^{m+1}\partial z}{(1+zz)^{n+1}} = \frac{m}{2n-m}\int\frac{z^{m-1}\partial z}{(1+zz)^{n+1}} - \frac{1}{2n-m}\cdot\frac{z^m}{(1+zz)^n}$$

$$\text{III.}\quad \int\frac{z^{m-1}\partial z}{(1+zz)^{n+1}} = \frac{2n-m}{2n}\int\frac{z^{m-1}\partial z}{(1+zz)^n} + \frac{1}{2n}\cdot\frac{z^m}{(1+zz)^n}$$

quarum reductionum ope casibus $b = 4i+2$ totum negotium absolvi et ad formulam $\frac{\pi}{4}$ reduci poterit, siquidem post integrationem sumatur $z = 1$.

§. 234. Sit $i = 1$ ideoque $b = 6$, eritque

$$A = \int\frac{(1-zz)\partial z}{(1+zz)^2} \text{ et } B = \int\frac{(1-zz)zz\partial z}{(1+zz)^2}.$$

Nunc igitur reperiemus per reductionem tertiam

$$\int\frac{\partial z}{(1+zz)^2} = \tfrac{1}{2}\int\frac{\partial z}{1+zz} + \tfrac{1}{2}\cdot\frac{z}{1+zz} = \tfrac{\pi}{8} + \tfrac{1}{4},$$

et per reductionem primam

$$\int\frac{zz\partial z}{(1+zz)^2}=\tfrac{1}{2}\int\frac{\partial z}{1+zz}-\tfrac{1}{2}\cdot\frac{z}{1+zz}=\frac{\pi}{8}-\frac{1}{4},$$

porro

$$\int\frac{z^4\partial z}{(1+zz)^2}=\tfrac{3}{2}\int\frac{zz\partial z}{1+zz}-\tfrac{1}{2}\cdot\frac{z^3}{1+zz}=\frac{5}{4}-\frac{3\pi}{8}.$$

Ex his jam valoribus colligitur $A=\frac{1}{2}$ et $B=\frac{\pi}{2}-\frac{3}{2}$, ideoque $\frac{B}{A}=\pi-3$, quocirca orietur ista summatio

$$3+\pi=6+\cfrac{1.1}{6+\cfrac{3.3}{6+\cfrac{5.5}{6+\cfrac{7.7}{6+\text{etc.}}}}}$$

§. 235. Sit nunc $i=2$ et $b=10$, eritque

$$A=\int\frac{(1-zz)^2\partial z}{(1+zz)^3}\quad\text{et}\quad B=\int\frac{zz(1-zz)^2\partial z}{(1+zz)^3}.$$

Quo harum integralium valores investigemus, sequentes evolvamus formulas

$$\int\frac{\partial z}{(1+zz)^3}=\tfrac{3}{4}\int\frac{\partial z}{(1+zz)^2}+\tfrac{1}{4}\cdot\frac{z}{(1+zz)^2}=\frac{3\pi}{32}+\frac{1}{4}$$

$$\int\frac{zz\partial z}{(1+zz)^3}=\tfrac{1}{4}\int\frac{\partial z}{(1+zz)^2}-\tfrac{1}{4}\cdot\frac{z}{(1+zz)^2}=\frac{\pi}{32}$$

$$\int\frac{z^4\partial z}{(1+zz)^3}=\tfrac{3}{4}\int\frac{zz\partial z}{(1+zz)^2}-\tfrac{1}{4}\cdot\frac{z^3}{(1+zz)^2}=\frac{3\pi}{32}-\frac{1}{4}$$

$$\int\frac{z^6\partial z}{(1+zz)^3}=\tfrac{5}{4}\int\frac{z^4\partial z}{(1+zz)^2}-\tfrac{1}{4}\cdot\frac{z^5}{(1+zz)^2}=\frac{3}{2}-\frac{15\pi}{32}.$$

Ex quibus jam valoribus deducitur $A=\frac{\pi}{8}$ et $B=2-\frac{5\pi}{8}$, ideoque $\frac{B}{A}=\frac{16-5\pi}{\pi}$, unde emergit sequens summatio

$$\frac{5\pi+16}{\pi}=10+\cfrac{1.1}{10+\cfrac{3.3}{10+\cfrac{5.5}{10+\text{etc.}}}}$$

§. 236. Si b esset numerus negativus, investigatio nulla prorsus laboraret difficultate. Si enim in genere fuerit

$$s = -a + \cfrac{\alpha}{-b + \cfrac{\beta}{-c + \cfrac{\gamma}{-d + \cfrac{\delta}{-e + \text{etc.}}}}}$$

semper erit

$$-s = a + \cfrac{\alpha}{b + \cfrac{\beta}{c + \cfrac{\gamma}{d + \cfrac{\delta}{\varepsilon + \text{etc.}}}}}$$

unde si habeatur valor istius expressionis, idem negative sumtus dabit valorem illius.

Exemplum 3.

§. 237. Proposita sit fractio continua, cujus valorem investigari oporteat, ista

$$1 + \cfrac{1 . 1}{3 + \cfrac{3 . 3}{5 + \cfrac{5 . 5}{7 + \cfrac{7 . 7}{9 + \text{etc.}}}}}$$

Quo fractiones supra allegatae, omisso membro supremo, sint

$$3 + \cfrac{3 \cdot 3}{5 + \cfrac{5 \cdot 5}{7 + \cfrac{7 \cdot 7}{9 + \text{etc.}}}}$$

eritque $\beta + b = 3$, $2\beta + b = 5$, ideoque $\beta = 2$ et $b = 1$; tum vero ut ante $\alpha = 2$, $a = -1$, $\gamma = 2$ et $c = +1$; invento autem s erit valor quaesitus $= 1 + \frac{1}{s}$. Nunc igitur habebimus

$$\frac{\partial Q}{Q} = -\frac{\partial x(1 + x + xx)}{2x(1 - x - xx)}.$$

Est vero

$$\frac{1 + x + xx}{x(1 - x - xx)} = \frac{1}{x} + \frac{2 + 2x}{1 - x - xx},$$

unde fit

$$lQ = -\tfrac{1}{2}lx - \int \frac{\partial x(1 + x)}{1 - x - xx}.$$

Porro vero pro formula $\int \frac{\partial x(1+x)}{1-x-xx}$ invenienda, statuamus denominatorem

$$1 - x - xx = (1 - fx)(1 - gx),$$

eritque $f + g = 1$ et $fg = -1$, unde fit

$$f = \frac{1 + \sqrt{5}}{2} \quad \text{et} \quad g = \frac{1 - \sqrt{5}}{2}.$$

Nunc statuatur

$$\frac{1 + x}{1 - x - xx} = \frac{\mathfrak{A}}{1 - fx} + \frac{\mathfrak{B}}{1 - gx},$$

unde reperietur

$$\mathfrak{A} = \frac{1 + f}{f - g} \quad \text{et} \quad \mathfrak{B} = -\frac{(1 + g)}{f - g},$$

sive substitutis pro f et g valoribus supra datis erit

$$\mathfrak{A} = \frac{\sqrt{5} + 3}{2\sqrt{5}} \quad \text{et} \quad \mathfrak{B} = \frac{\sqrt{5} - 3}{2\sqrt{5}},$$

quibus inventis erit

$$\int \frac{\partial x(1+x)}{1-x-xx} = -\frac{\mathfrak{A}}{f}\, l(1-fx) - \frac{\mathfrak{B}}{g}\, l(1-gx) =$$
$$-\frac{(1+\sqrt{5})}{2\sqrt{5}}\, l(1-fx) - \frac{(\sqrt{5}-1)}{2\sqrt{5}}\, l(1-gx);$$

quocirca fiet

$$l\,Q = -\tfrac{1}{2}\, lx + \frac{(\sqrt{5}+1)}{2\sqrt{5}}\, l(1-fx) + \frac{(\sqrt{5}-1)}{2\sqrt{5}}\, l(1-gx),$$

consequenter

$$Q = \frac{(1-fx)^{\frac{\sqrt{5}+1}{2\sqrt{5}}}(1-gx)^{\frac{\sqrt{5}-1}{2\sqrt{5}}}}{\sqrt{x}},$$

qui valor duobus casibus evanescit: altero quo

$$x = \frac{1}{f} = \frac{2}{1+\sqrt{5}} = \frac{\sqrt{5}-1}{2};$$

altero vero quo $x = \frac{1}{g} = -\frac{1-\sqrt{5}}{2}$; utrovis autem utamur, res eodem redibit.

§. 238. Ex hoc autem valore habebimus

$$A = \int \frac{Q\partial x}{1-x-xx} \quad \text{et} \quad B = \int \frac{Qx\partial x}{1-x-xx},$$

unde porro deducitur

$$s = (\alpha+a)\frac{A}{B} = \frac{A}{B},$$

et propositae fractionis summa erit $1+\frac{B}{A}$. Hinc autem nihil ulterius concludere licet, ob formulas differentiales non solum irrationales, sed etiam vere transcendentes ob exponentes surdos.

E x e m p l u m 4.

§. 239. Proposita sit haec fractio continua

$$b + \cfrac{1\,.\,1}{b + \cfrac{2\,.\,2}{b + \cfrac{3\,.\,3}{b + \cfrac{4\,.\,4}{b + \text{etc.}}}}}$$

ubi est $\beta = 0$, $b = b$. Nunc consideremus hanc formam

$$s = b + \cfrac{2.2}{b + \cfrac{3.3}{b + \text{etc.}}}$$

quippe quo valore invento quaesitus erit $= b + \frac{1}{s}$. Habebimus igitur $\gamma + c = 2$, $2\gamma + c = 3$, ideoque $\gamma = 1$ et $c = 1$, deinde erit $\alpha = \gamma = 1$, $a = 0$ et $c = 1$. Hinc igitur colligimus

$$\frac{\partial Q}{Q} = -\frac{\partial x(bx + xx)}{x(1 - xx)} = -\frac{\partial x(b + xx)}{1 - xx},$$

ideoque

$$lQ = -\tfrac{b}{2} l \frac{1+x}{1-x} + \tfrac{1}{2} l(1 - xx),$$

hincque

$$Q = \frac{(1-x)^{\frac{b}{2}} \sqrt{(1 - xx)}}{(1+x)^{\frac{b}{2}}} = \frac{(1-x)^{\frac{b+1}{2}}}{(1+x)^{\frac{b-1}{2}}},$$

quae quantitas manifesto evanescit posito $x = 1$. Hinc igitur fiet

$$A = \int \frac{Q\partial x}{1 - xx} = \int \frac{(1-x)^{\frac{b+1}{2}} \partial x}{(1+x)^{\frac{b-1}{2}}(1-xx)} = \int \frac{(1-x)^{\frac{b-1}{2}} \partial x}{(1+x)^{\frac{b+1}{2}}}, \text{ etc.}$$

$$B = \int \frac{x(1-x)^{\frac{b-1}{2}} \partial x}{(1+x)^{\frac{b+1}{2}}},$$

tum autem erit $s = (\alpha + a)\frac{A}{B} = \frac{A}{B}$, ideoque summa quaesita $= b + \frac{B}{A}$.

§. 240. Percurramus nunc casus praecipuos: ac primo sit $b = 1$, eritque

$$A = \int \frac{\partial x}{1+x} = l(1+x) = l2, \text{ et}$$

$$B = \int \frac{x\,\partial x}{1+x} = x - \int \frac{\partial x}{1+x} = 1 - l2,$$

ideoque $b + \frac{B}{A} = \frac{1}{l2}$; ergo hinc prodiit ista summatio

$$\frac{1}{l2} = 1 + \cfrac{1 \cdot 1}{1 + \cfrac{2 \cdot 2}{1 + \cfrac{3 \cdot 3}{1 + \text{etc.}}}}$$

§. 241. Sit nunc $b = 2$, eritque

$$A = \int \frac{\partial x \sqrt{(1-x)}}{(1+x)^{\frac{3}{2}}}, \text{ et } B = \int \frac{x\,\partial x \sqrt{(1-x)}}{(1+x)^{\frac{3}{2}}}.$$

Ad has formulas rationales reddendas statuamus

$$\frac{\sqrt{1-x}}{\sqrt{1+x}} = z, \text{ eritque } x = \frac{1-zz}{1+zz},$$

unde terminis integrationis $x = 0$ et $x = 1$ respondebunt $z = 1$ et $z = 0$; tum vero erit

$$1 + x = \frac{2}{1+zz}, \text{ et } \partial x = -\frac{4z\,\partial z}{(1+zz)^2},$$

hincque colligitur

$$A = -2\int \frac{zz\,\partial z}{1+zz} = -2z + 2 \text{ Arc. tang. } z + 2 - \tfrac{\pi}{2} = 2 - \tfrac{\pi}{2},$$

porro fit

$$B = -2\int \frac{zz\,\partial z}{(1+zz)^2} + 2\int \frac{z^4\,\partial z}{(1+zz)^2}.$$

Per reductiones igitur supra §. 234. monstratas, si hic scilicet terminos integrationis $z = 1$ et $z = 0$ permutemus, ut habeamus

$$B = +2\int \frac{zz\,\partial z}{(1+zz)^2} - 2\int \frac{z^4\,\partial z}{(1+zz)^2}, \text{ erit}$$

$$B = 2\left(\tfrac{\pi}{8} - \tfrac{1}{4}\right) - 2\left(\tfrac{3}{4} - \tfrac{\pi}{4}\right) = \pi - 3,$$

unde sequitur ista summatio

$$\frac{2}{4-\pi} = 2 + \cfrac{1.1}{2 + \cfrac{2.2}{2 + \cfrac{3.3}{2 + \text{etc.}}}}$$

quae *Brouncherianae* simplicitate nihil cedit.

§. 242. Si ponamus $b = 0$, fractio continua abit in sequens continuum productum

$$\frac{1\cdot 1}{2\cdot 2}\cdot\frac{3\cdot 3}{4\cdot 4}\cdot\frac{5\cdot 5}{6\cdot 6}\cdot\frac{7\cdot 7}{8\cdot 8}\cdot\text{etc.}$$

hoc autem casu fit

$$A = \int\frac{\partial x}{\sqrt{(1-xx)}} = \frac{\pi}{2} \text{ et } B = \int\frac{x\,\partial x}{\sqrt{(1-xx)}} = 1;$$

unde istius producti valor colligitur $\frac{2}{\pi}$, id quod egregie convenit cum jam dudum cognitis, quandoquidem hoc productum est ipsa progressio *Wallisiana*.

E x e m p l u m 5.

§. 243. Proposita sit haec fractio continua, ubi $\beta = 0$, $b = b$, et numeratores numeri trigonales.

$$b + \cfrac{1}{b + \cfrac{3}{b + \cfrac{6}{b + \cfrac{10}{b + \text{etc.}}}}}$$

Omisso supremo membro statuamus

$$s = b + \cfrac{3}{b + \cfrac{6}{b + \text{etc.}}}$$

et primo numeratores per producta repraesentemus hoc modo

$3 = 2 \cdot \frac{3}{2}; \; 6 = 3 \cdot \frac{4}{2}, \; 10 = 4 \cdot \frac{5}{2},$ etc.

quorum priores comparentur cum formulis

$\gamma + c, \; 2\gamma + c, \; 3\gamma + c,$ etc.

posteriores vero cum formulis $2\alpha + a, \; 3\alpha + a, \; 4\alpha + a,$ etc.
eritque $\gamma = 1, \; c = 1, \; \alpha = \frac{1}{2}, \; a = \frac{1}{2}$, unde erit

$$\frac{\partial Q}{Q} = \frac{\partial x\left(\frac{1}{2} - bx - xx\right)}{x\left(\frac{1}{2} - xx\right)} = \frac{\partial x\,(1 - 2bx - 2xx)}{x\,(1 - 2xx)}, \text{ sive}$$

$$\frac{\partial Q}{Q} = \frac{\partial x}{x} - \frac{2b\,\partial x}{1 - 2xx},$$

cujus integrale est

$$lQ = lx - \frac{b}{\sqrt{2}}\, l\, \frac{1 + x\sqrt{2}}{1 - x\sqrt{2}}, \text{ ergo}$$

$$Q = \frac{x\,(1 - x\sqrt{2})^{\frac{b}{\sqrt{2}}}}{(1 + x\sqrt{2})^{\frac{b}{\sqrt{2}}}},$$

quae formula evanescit casu $x = \frac{1}{\sqrt{2}}$. Hinc igitur erit

$$\partial v = \frac{2x\,(1 - x\sqrt{2})^{\frac{b}{\sqrt{2}}}\,\partial x}{(1 - 2xx)\,(1 + x\sqrt{2})^{\frac{b}{\sqrt{2}}}}.$$

Sit $\frac{b}{\sqrt{2}} = \lambda$, eritque

$$A = 2\int \frac{x\,(1 - x\sqrt{2})^{\lambda}\,\partial x}{(1 - 2xx)\,(1 + x\sqrt{2})^{\lambda}} = 2\int \frac{x\,(1 - x\sqrt{2})^{\lambda - 1}\,\partial x}{(1 + x\sqrt{2})^{\lambda + 1}}$$

et

$$B = 2\int \frac{xx\,(1 - x\sqrt{2})^{\lambda - 1}\,\partial x}{(1 + x\sqrt{2})^{\lambda + 1}},$$

ubi post integrationem statuitur $x = \frac{1}{\sqrt{2}}$; tum autem fit $s = \frac{A}{B}$,
hincque valor fractionis propositae $= b + \frac{B}{A}$.

§. 244. Nisi igitur fuerit $\lambda = \frac{b}{\sqrt{2}}$ numerus rationalis, hos valores commode assignare non licet. Sit igitur $b = \sqrt{2}$, sive $\lambda = 1$, eritque

$$A = 2\int \frac{x \partial x}{(1 + x\sqrt{2})^2}, \text{ et } B = 2\int \frac{xx \partial x}{(1 + x\sqrt{2})^2}.$$

Hinc integrando colligitur

$$A = l(1 + x\sqrt{2}) - \frac{x\sqrt{2}}{1 + x\sqrt{2}},$$

ideoque posito $x\sqrt{2} = 1$ fiet $A = l2 - \frac{1}{2}$; tum vero reperitur

$$B = \frac{3}{2\sqrt{2}} - \sqrt{2} . l2,$$

quare ob $b = \sqrt{2}$ erit

$$b + \frac{B}{A} = \frac{1}{\sqrt{2}(2l2 - 1)},$$

unde sequitur haec summatio

$$\frac{1}{\sqrt{2}(2l2 - 1)} = \sqrt{2} + \cfrac{1}{\sqrt{2} + \cfrac{3}{\sqrt{2} + \cfrac{6}{\sqrt{2} + \text{etc.}}}}$$

Scholion.

§. 245. Fractiones autem continuae, ad quas plerumque calculo numerico deducimur, hujusmodi formam habere solent

$$a + \cfrac{1}{b + \cfrac{1}{c + \cfrac{1}{d + \cfrac{1}{e + \text{etc.}}}}}$$

ubi omnes numeratoros sunt unitates, denominatores vero a, b, c, d, etc. numeri integri. Verum ope nostrae methodi difficulter talium formarum valores eruere licet, etiamsi numeri a, b, c, d, etc. progressionem arithmeticam constituant, id quod sequenti exemplo ostendamus.

Exemplum 6.

§. 246. Proposita sit ista fractio continua

$$\beta+b+\cfrac{1}{2\beta+b+\cfrac{1}{3\beta+b+\cfrac{1}{4\beta+b+\cfrac{1}{5\beta+b+\text{etc.}}}}}$$

ubi $\alpha=0$, $\gamma=0$, $a=1$, $c=1$.

Hinc fit

$$\frac{\partial Q}{Q}=-\frac{\partial x\,(1+bx-xx)}{\beta xx}, \text{ unde}$$

$$lQ=\frac{1}{\beta x}+\frac{b}{\beta}\,lx+\frac{x}{\beta} \text{ et}$$

$$Q=e^{\frac{1+xx}{\beta x}}\cdot x^{\frac{b}{\beta}},$$

quae autem expressio nullo casu evanescere potest, etiamsi per x^n multiplicetur, siquidem β fuerit numerus positivus. Verum si pro β sumamus numeros negativos, puta $\beta=-m$, tum valor $Q=x^{\frac{-b}{m}}\times e^{\frac{-(1+xx)}{mx}}$ manifesto evanescit, tam si $x=0$, quam si $x=\infty$. Hinc autem erit

$$\partial v=\frac{x^{\frac{-b}{m}}\cdot e^{\frac{-1-xx}{mx}}\,\partial x}{mxx},$$

quamobrem habebimus

$$A=\frac{1}{m}\int\frac{\partial x}{x^{2+\frac{b}{m}}\cdot e^{\frac{1+xx}{mx}}}, \text{ et}$$

$$B = \frac{1}{m}\int\frac{\partial x}{x^{1+\frac{b}{m}}\cdot e^{\frac{1+xx}{mx}}}.$$

His valoribus inventis formula $\frac{A}{B}$ exprimet summam hujus fractionis continuae

$$-m+b+\cfrac{1}{-2m+b+\cfrac{1}{-3m+b+\cfrac{1}{-4m+b+\cfrac{1}{-5m+b+\text{etc.}}}}}$$

quamobrem formula illa negative sumta $-\frac{A}{B}$ exprimet valorem hujus fractionis continuae

$$m-b+\cfrac{1}{2m-b+\cfrac{1}{3m-b+\cfrac{1}{4m-b+\text{etc.}}}}$$

quem igitur assignare liceret, si modo formulae integrales A et B expediri et a termino $x=0$ ad $x=\infty$ extendi possent. Verum istae formulae ita sunt comparatae, ut earum integratio nullo plane casu per quantitates cognitas exprimi queat, quod tamen non impedit, quo minus fractio $\frac{A}{B}$ valores satis cognitos involvere queat, etiamsi eos nullo adhuc modo assignare valeamus.

§. 247. Talium autem fractionum continuarum mihi quidem binae sequentes innotuere, quarum valores commode exhibere licet.

$$n+\cfrac{1}{3n+\cfrac{1}{5n+\cfrac{1}{7n+\cfrac{1}{9n+\text{etc.}}}}}=\frac{e^{\frac{2}{n}}}{e^{\frac{2}{n}}-1}, \text{ et}$$

$$n-\cfrac{1}{3n-\cfrac{1}{5n-\cfrac{1}{7n-\cfrac{1}{9n-\text{etc.}}}}}=\cot.\tfrac{1}{n}.$$

Harum fractionum prior cum formulis postremi exempli collata praebet $m-b=n$, $2m-b=3n$, ideoque $m=2n$ et $b=n$, unde fit

$$A=\tfrac{1}{2n}\int\frac{\partial x}{x^{\frac{5}{2}}\,e^{\frac{1+xx}{2nx}}}, \text{ et}$$

$$B=\tfrac{1}{2n}\int\frac{\partial x}{x^{\frac{3}{2}}\,e^{\frac{1+xx}{2n}}};$$

unde jam discimus si hae duae formulae integrentur a termino $x=0$ usque ad terminum $x=\infty$, tum fore

$$\frac{A}{B}=\frac{1+e^{\frac{2}{n}}}{1-e^{\frac{2}{n}}},$$

quanquam nulla adhuc via analytica patet, hanc convenientiam demonstrandi.