SUPPLEMENTUM IV.

AD TOM. I. CAP. V.

DE

INTEGRATIONE FORMULARUM ANGULOS SINUSVE ANGULORUM IMPLICANTIUM.

---

1) De formulis differentialibus angularibus maxime irrationalibus, quas tamen per logarithmos et arcus circulares integrare licet. *M. S. Academiae exhibit. die* 5. *Maii* 1777.

§. 1. Quae jam saepius sum commentatus de formulis differentialibus irrationalibus, quae nulla substitutione ad rationalitatem revocari possunt, nihilo vero minus integrationem per logarithmos et arcus circulares admittunt: etiam transferri possunt ad ejusmodi formulas angulares, quae sinus et cosinus cujuspiam anguli involvunt. Forma autem generalis hujusmodi differentialium, quae hoc modo tractari possunt, sequenti modo repraesentari potest: denotante ϕ angulum quemcunque, designet Φ functionem quamcunque rationalem ipsius tang. $n\phi$, atque inveni istam formulam

$$\frac{\Phi \partial \phi (f \sin. \lambda \phi + g \cos. \lambda \phi)}{\sqrt[n]{(a \sin. n \phi + b \cos. n \phi)^{\lambda}}}$$

semper per logarithmos et arcus circulares integrari posse, id quod a casibus simplicioribus inchoando in sequentibus problematibus ostendere constitui.

Problema 1.

§. 2. *Proposita formula differentiali* $\frac{\partial \Phi \cos. \Phi}{\sqrt[n]{\cos. n \Phi}}$, *ejus integrale per logarithmos et arcus circulares investigare.*

Solutio.

Quoniam mihi quidem alia adhuc via non patet istud praestandi, nisi per imaginaria procedendo, formulam $\sqrt{-1}$ littera i in posterum designabo, ita ut sit $ii = -1$, ideoque $\frac{1}{i} = -i$. Jam ante omnia in numeratore nostrae formulae loco $\cos. \Phi$ has duas partes substituamus

$$\tfrac{1}{2}(\cos. \Phi + i \sin. \Phi) + \tfrac{1}{2}(\cos. \Phi - i \sin. \Phi),$$

atque ipsam formulam propositam per duas hujusmodi partes repraesentemus, quae sint

$$\partial p = \frac{\partial \Phi (\cos. \Phi + i \sin. \Phi)}{\sqrt[n]{\cos. n \Phi}} \text{ et } \partial q = \frac{\partial \Phi (\cos. \Phi - i \sin. \Phi)}{\sqrt[n]{\cos. n \Phi}},$$

ita ut ipsa formula nostra proposita sit $\frac{1}{2}\partial p + \frac{1}{2}\partial q$, ideoque ejus integrale $\frac{p+q}{2}$.

§. 3. Nunc ambas istas partes seorsim sequenti modo tractemus. Pro formula scilicet priore

$$\partial p = \frac{\partial \Phi (\cos. \Phi + i \sin. \Phi)}{\sqrt[n]{\cos. n \Phi}} \text{ statuamus } \frac{\cos. \Phi + i \sin. \Phi}{\sqrt[n]{\cos. n \Phi}} = x,$$

ut sit $\partial p = x \partial \Phi$, ac sumtis potestatibus exponentis n habebimus

$$x^n = \frac{(\cos.\,\phi + i \sin.\,\phi)^n}{\cos.\,n\,\phi}.$$

Constat autem esse

$$(\cos.\,\phi + i \sin.\,\phi)^n = \cos.\,n\,\phi + i \sin.\,n\,\phi,$$

sicque erit $x^n = 1 + i \operatorname{tang.} n\,\phi$, unde colligitur

$$\operatorname{tang.} n\,\phi = \frac{x^n - 1}{i} = i\,(1 - x^n):$$

hinc cum posito in genere tang. $\omega = Z$, sit $\partial\,\omega = \frac{\partial Z}{1+ZZ}$, erit pro nostro casu

$$n\,\partial\,\phi = \frac{-n\,i\,x^{n-1}\,\partial\,x}{1 + i\,i - 2\,i\,i\,x^n + i\,i\,x^{2n}},$$

quae formula ob $i\,i = -1$ transmutatur in hanc

$$\partial\,\phi = \frac{-i\,x^{n-1}\,\partial\,x}{2\,x^n - x^{2n}},$$

hincque ipsa formula

$$\partial\,p = x\,\partial\,\phi = \frac{-i\,\partial\,x}{2 - x^n},$$

quae cum sit rationalis, ejus integratio nulli difficultati est subjecta.

§. 4. Quodsi jam simili modo pro altera formula

$$\partial\,q = \frac{\partial\,\phi\,(\cos.\,\phi - i \sin.\,\phi)}{\sqrt[n]{\cos.\,n\,\phi}}, \text{ statuatur } \frac{\cos.\,\phi - i \sin.\,\phi}{\sqrt[n]{\cos.\,n\,\phi}} = y,$$

ut sit $\partial\,q = y\,\partial\,\phi$, per similes operationes, quae a praecedentibus in hoc solo discrepabunt, quod littera i negative sit accipienda, resultabit ista transformatio $\partial\,q = \frac{i\,\partial\,y}{2 - y^n}$, quae cum priori prorsus sit similis, eadem integratione totum negotium conficietur, et pro ipso integrali quaesito habebimus

$$p+q=-i\int\frac{\partial x}{2-x^n}+i\int\frac{\partial y}{2-y^n}.$$

§. 5. Constat autem integralia talium formularum ex duplicis generis partibus, scilicet logarithmicis et arcubus circularibus constare, ita ut illarum forma generalis sit $f\,l\,(\alpha+\beta x+\gamma xx)$, harum vero g Arc. tang. $(\delta+\varepsilon x)$. Quare cum hic differentia inter binas formulas integrales similes occurrat, ex singulis partibus logarithmicis orietur talis forma $-\,i\,f\,l\frac{\alpha+\beta x+\gamma xx}{\alpha+\beta y+\gamma yy}$, ubi tam x quam y imaginaria involvit, hanc ob rem ponamus brevitatis gratia $x=r+is$ et $y=r-is$, ubi erit

$$r=\frac{\cos.\,\Phi}{\sqrt[n]{\cos.\,n\,\Phi}}\quad\text{et}\quad s=\frac{\sin.\,\Phi}{\sqrt[n]{\cos.\,n\,\Phi}};$$

his igitur valoribus substitutis, quaelibet pars logarithmica erit

$$-\,i\,f\,l\frac{\alpha+\beta r+\gamma rr-\gamma ss+i\,(\beta s+2\gamma rs)}{\alpha+\beta r+\gamma rr-\gamma ss-i\,(\beta s+2\gamma rs)}.$$

§. 6. Loco hujus expressionis prolixioris scribamus brevitatis gratia $-\,i\,f\,l\frac{t+iu}{t-iu}$, ita ut sit

$$t=\alpha+\beta r+\gamma rr-\gamma ss\quad\text{et}\quad u=\beta s+2\gamma rs,$$

sicque etiam hi valores per angulum Φ innotescunt. Quoniam igitur jam saepius est demonstratum, esse

$$l\frac{t+u\sqrt{-1}}{t-u\sqrt{-1}}=2\sqrt{-1}.\ \text{Arc. tang.}\ \frac{u}{t},$$

ista portio integralis erit $=+\,2\,f$ Arc. tang. $\frac{u}{t}$, quae ergo penitus est realis, dum imaginaria se mutuo sustulerunt, ita ut quaelibet portio logarithmica imaginaria producat arcum circularem realem.

§. 7. Simili modo conjungamus in genere binos arcus circulares per integrationem prodeuntes, qui ex forma assumta erunt

$$- i g \text{ Arc. tang. } (\delta + \varepsilon x) + i g \text{ Arc. tang. } (\delta + \varepsilon y),$$

quae forma ita in unum arcum contrahetur, qui erit

$$- i g \text{ Arc. tang. } \frac{\varepsilon(x-y)}{1+(\delta+\varepsilon x)(\delta+\varepsilon y)}:$$

quae introductis valoribus assumtis $x = r + is$ et $y = r - is$, induet hanc formam

$$- i g \text{ Arc. tang. } \frac{2 i \varepsilon s}{1+\delta\delta+2\varepsilon\delta r+\varepsilon\varepsilon(rr+ss)}.$$

Cum igitur in genere sit

$$\text{Arc. tang. } v\sqrt{-1} = \frac{\sqrt{-1}}{2} l \frac{1+v}{1-v},$$

ista pars circularis transformabitur in sequentem logarithmum realem

$$\frac{g}{2} l \frac{1+\delta\delta+2\delta\varepsilon r+\varepsilon\varepsilon(rr+ss)+2\varepsilon s}{1+\delta\delta+2\delta\varepsilon r+\varepsilon\varepsilon(rr+ss)-2\varepsilon s}:$$

hoc ergo modo sumendis omnium integralium partibus, tandem obtinebitur integrale quaesitum per meros logarithmos et arcus circulares realiter expressum.

Problema. 2.

§. 8. *Proposita formula differentiali* $\frac{\partial\Phi \sin.\Phi}{\sqrt[n]{\cos. n\Phi}}$, *ejus integrale per logarithmos et arcus circulares investigare.*

Solutio.

Hic loco sin. Φ scribatur haec forma duabus constans partibus

$$\frac{1}{2i}(\cos.\Phi + i\sin.\Phi) - \frac{1}{2i}(\cos.\Phi - i\sin.\Phi),$$

ac formula proposita resolvatur in has partes

$$\partial p = \frac{\partial \Phi (\cos.\Phi + i \sin.\Phi)}{\sqrt[n]{\cos.n\Phi}} \text{ et } \partial q = \frac{\partial \Phi (\cos.\Phi - i \sin.\Phi)}{\sqrt[n]{\cos.n\Phi}},$$

ita ut ipsa formula proposita jam fiat $\frac{\partial p - \partial q}{2i}$, ideoque ipsum integrale quaesitum $\frac{p-q}{2i}$.

§. 9. Quodsi jam rursus ut ante statuamus

$$\frac{\cos.\Phi + i \sin.\Phi}{\sqrt[n]{\cos.n\Phi}} = x \text{ et } \frac{\cos.\Phi - i \sin.\Phi}{\sqrt[n]{\cos.n\Phi}} = y,$$

reperietur ut supra

$$\partial p = -\frac{i\partial x}{2 - x^n} \text{ et } \partial q = \frac{i\partial y}{2 - y^n};$$

unde ergo fiet ipsum integrale quaesitum

$$\frac{p-q}{2i} = -\tfrac{1}{2}\int\frac{\partial x}{2 - x^n} - \tfrac{1}{2}\int\frac{\partial y}{2 - y^n},$$

ubi coefficientes evaserunt reales.

§. 10. Consideremus nunc ex forma integrali utriusque partis quamlibet portionem logarithmicam, quae sit $f\,l(\alpha + \beta x + \gamma xx)$, hincque pro integrali quaesito ex utraque parte orietur

$$-\tfrac{1}{2}f\,l(\alpha + \beta x + \gamma xx) - \tfrac{1}{2}f\,l(\alpha + \beta y + \gamma yy).$$

Quodsi jam ut supra ponamus brevitatis gratia $x = r + is$ et $y = r - is$, tum vero

$$t = \alpha + \beta r + \gamma rr - \gamma ss \text{ et } u = \beta s + 2\gamma rs,$$

hi ambo logarithmi evadunt

$$= -\tfrac{1}{2}f\,l(t + iu) - \tfrac{1}{2}f\,l(t - iu),$$

qui contrahuntur in $-\tfrac{1}{2}f\,l(tt + uu$, quae expressio jam est realis, neque ulla ulteriori reductione indiget.

§. 11. Eodem modo binae partes circulares ex integratione oriundae

$$-\tfrac{1}{2} g \text{ Arc. tang. } (\delta + \varepsilon x) - \tfrac{1}{2} g \text{ Arc. tang. } (\delta + \varepsilon y),$$

quae per r et s ita repraesentantur

$$-\tfrac{1}{2} g \, [\text{Arc. tang. } (\delta + \varepsilon r + i \varepsilon s) + \text{Arc. tang. } (\delta + \varepsilon r - i \varepsilon s)],$$

qui duo arcus ita in unum contrahuntur

$$-\tfrac{1}{2} g \text{ Arc. tang. } \frac{2\delta + 2\varepsilon r}{1 - (\delta + \varepsilon r)^2 - \varepsilon\varepsilon ss},$$

quae expressio jam ultro prodiit realis.

Problema 3.

§. 12. *Proposita formula differentiali* $\frac{\partial \phi \cos. \lambda \phi}{\sqrt[n]{\cos. n \phi^\lambda}}$, *ejus integrale per logarithmos et arcus circulares investigare.*

Solutio.

Cum sit

$$\cos. \lambda \phi = \tfrac{1}{2}(\cos. \phi + i \sin. \phi)^\lambda + \tfrac{1}{2}(\cos. \phi - i \sin. \phi)^\lambda,$$

formula proposita in has duas partes discerpatur

$$\partial p = \frac{\partial \phi (\cos. \phi + i \sin. \phi)^\lambda}{\sqrt[n]{\cos. n \phi^\lambda}} \text{ et } \partial q = \frac{\partial \phi (\cos. \phi - i \sin. \phi)^\lambda}{\sqrt[n]{\cos. n \phi^\lambda}},$$

ita ut integrale quaesitum fiat $\frac{p+q}{2}$.

§. 13. Jam statuamus, ut ante fecimus,

$$\frac{\cos. \phi + i \sin. \phi}{\sqrt[n]{\cos. n \phi}} = x \text{ et } \frac{\cos. \phi - i \sin. \phi}{\sqrt[n]{\cos. n \phi}} = y,$$

quo facto fiet $\partial p = x^\lambda \partial \phi$ et $\partial q = y^\lambda \partial \phi$. Calculo autem ut supra expedito obtinebimus

$$\partial \Phi = -\frac{i x^{n-1} \partial x}{2 x^n - x^{2n}}, \text{ hincque } \partial p = -\frac{i x^{\lambda-1} \partial x}{2 - x^n};$$

similique modo erit $\partial q = \frac{i y^{\lambda-1} \partial y}{2 - y^n}$, sicque totum integrale quaesitum erit

$$= -\tfrac{i}{2} \int \frac{x^{\lambda-1} \partial x}{2 - x^n} + \tfrac{i}{2} \int \frac{y^{\lambda-1} \partial y}{2 - y^n}.$$

§. 14. Quoniam haec duo integralia sibi sunt similia, ideoque similes partes tam logarithmicas quam circulares complectuntur, ex parte logarithmica, quae sit $f l (\alpha + \beta x + \gamma x x)$, ponendo ut supra $x = r + i s$ et $y = r - i s$, tum vero

$$t = a + \beta r + \gamma r r - \gamma s s \text{ et } u = \beta s + 2 \gamma r s,$$

hinc primo ista pars logarithmica colligitur $- i f l \frac{t + i u}{t - i u}$, quae cum sit imaginaria reducitur ad hunc arcum circularem realem $= 2 f$ Arc. tang. $\frac{u}{t}$; simili modo si forma arcus circularis ex integratione priorunda fuerit $- g$ Arc. tang. $(\delta + \varepsilon x)$, ex partibus circularibus primo oritur sequens arcus imaginarius

$$- i g \text{ Arc. tang. } \frac{2 i \varepsilon s}{1 + \delta\delta + 2 \varepsilon \delta r + \varepsilon \varepsilon (r r + s s)},$$

qui denique ad hunc logarithmum realem revocatur

$$\tfrac{g}{2} l \frac{1 + \delta\delta + 2 \delta \varepsilon r + \varepsilon\varepsilon (r r + s s) + 2 \varepsilon s}{1 + \delta\delta + 2 \delta \varepsilon r + \varepsilon\varepsilon (r r + s s) - 2 \varepsilon s}.$$

Problema 4.

§. 15. *Proposita formula differentiali* $\frac{\partial \Phi \sin. \lambda \Phi}{\sqrt[n]{\cos. n \Phi^{\lambda}}}$, *ejus integrale per logarithmos et arcus circulares investigare.*

Solutio.

Cum sit

$$\sin.\lambda\,\phi = \tfrac{1}{2i}(\cos.\phi + i\sin.\phi)^\lambda - \tfrac{1}{2i}(\cos.\phi - i\sin.\phi)^\lambda,$$

constituamus ut hactenus has duas partes

$$\partial p = \frac{\partial\phi(\cos.\phi + i\sin.\phi)^\lambda}{\sqrt[n]{\cos.n\phi^\lambda}} \text{ et } \partial q = \frac{\partial\phi(\cos.\phi - i\sin.\phi)^\lambda}{\sqrt[n]{\cos.n\phi^\lambda}},$$

ita ut integrale quaesitum sit $\frac{p-q}{2i}$. Statuamus nunc iterum

$$\frac{\cos.\phi + i\sin.\phi}{\sqrt[n]{\cos.n\phi}} = x \text{ et } \frac{\cos.\phi - i\sin.\phi}{\sqrt[n]{\cos.n\phi}} = y,$$

ut fiat $\partial p = x^\lambda \partial\phi$ et $\partial q = y^\lambda\partial\phi$, hincque calculo ut supra instituto, fiet

$$\partial p = -\frac{ix^{\lambda-1}\partial x}{2-x^n} \text{ et } \partial q = \frac{iy^{\lambda-1}\partial y}{2-y^n},$$

sicque integrale quaesitum erit

$$-\tfrac{1}{2}\int\frac{x^{\lambda-1}\partial x}{2-x^n} - \tfrac{1}{2}\int\frac{y^{\lambda-1}\partial y}{2-y^n}.$$

§. 16. Quodsi jam ut hactenus est factum, ponamus $x = r + is$ et $y = r - is$, et pro partibus logarithmicis, quarum forma sit $fl(a + \beta x + \gamma xx)$, ponamus

$$t = a + \beta r + \gamma rr - \gamma ss \text{ et } u = \beta\alpha + 2\gamma r s,$$

binae partes logarithmicae imaginariae uti in problemate secundo in unum logarithmum realem contrahentur, qui erit $-\tfrac{1}{2}fl(tt + uu)$. At si pro partibus circularibus, quarum forma sit g Arc. tang. $(\delta + \varepsilon x)$, bini tales arcus imaginarii jungantur, illi coalescent in unum arcum realem

$$-\tfrac{1}{2}g \text{ Arc. tang } \frac{2\delta + 2\varepsilon r}{1 - (\delta + \varepsilon r)^2 - \varepsilon\varepsilon ss}.$$

Problema generale.

§. 17. *Si Φ denotet functionem quamcunque rationalem ipsius tang. $n\,\phi$, ac proposita fuerit haec formula differentialis*

$$\frac{\Phi\,\partial\phi\,(F\sin.\lambda\phi + G\cos.\lambda\phi)}{\sqrt[n]{(a\cos.n\phi + b\sin.n\phi)^\lambda}},$$

ejus integrationem ad logarithmos et arcus circulares reducere.

Solutio.

Ex praecedentibus jam facile intelligitur, formulam numeratoris F sin. $\lambda\,\phi$ + G cos. $\lambda\,\phi$ semper ad talem formam revocari posse

$$F'(\cos.\phi + i\sin.\phi)^\lambda + G'(\cos.\phi - i\sin.\phi)^\lambda,$$

atque hinc ipsa forma proposita discerpatur in has duas partes

$$\partial p = \frac{\Phi\,\partial\phi\,(\cos.\phi + i\sin.\phi)^\lambda}{\sqrt[n]{(a\cos.n\phi + b\sin.n\phi)^\lambda}} \text{ et}$$

$$\partial q = \frac{\Phi\,\partial\phi\,(\cos.\phi + i\sin.\phi)^\lambda}{\sqrt[n]{(a\cos.n\phi + b\sin.n\phi)^\lambda}};$$

ita ut integrale quaesitum jam futurum sit F′ p + G′ q.

§. 18. Jam pro formula priori ∂p statuatur

$$\frac{\cos.\phi + i\sin.\phi}{\sqrt[n]{(a\cos.n\phi + b\sin.n\phi)}} = x, \text{ et pro posteriori}$$

$$\frac{\cos.\phi - i\sin.\phi}{\sqrt[n]{(a\cos.n\phi + b\sin.n\phi)}} = y.$$

ita ut hinc futurum sit

$$\partial p = \Phi\,x^\lambda\,\partial\phi \text{ et } \partial q = \Phi\,y^\lambda\,\partial\phi;$$

inde autem fiet

$$x^n = \frac{\cos. n\,\Phi + i \sin. n\,\Phi}{a \cos. n\,\Phi + i \sin. n\,\Phi},$$

unde colligitur

$$\text{tang. } n\,\Phi = \frac{1 - a x^n}{b x^n - i};$$

quare cum Φ denotet functionem rationalem ipsius tang. $n\,\Phi$, evadet quoque functio rationalis ipsius x, atque adeo ipsius x^n, quae designetur per X. Praeterea vero etiam differentiale $\partial\,\Phi$ rationaliter determinabitur; cum fiat

$$\partial\,\Phi = \frac{(a - b)\,x^{n-1}\,\partial x}{(a a + b b)\,x^{2n} - 2\,(a - i b)\,x^n},$$

hoc ergo modo habebimus

$$\partial\,p = \frac{(i a - b)\,\mathrm{X}\,x^{\lambda - 1}\,\partial x}{(a a + b b)\,x^n - 2\,(a + i b)},$$

quae cum sit penitus rationalis, certum est, ejus integrale, quantumcunque etiam laborem postulaverit, semper per logarithmos et arcus circulares expediri posse.

§. 19. Simili modo res se habet in altera formula $\partial\,q$, quae ab ista tantum ratione signi litterae i differet, et quoniam hic omnia rationaliter per y prodibunt expressa, quo pacto Φ abeat in Y, atque obtinebitur

$$\partial\,q = -\frac{(b + i a)\,\mathrm{Y}\,y^{\lambda - 1}\,\partial y}{(a a + b b) y^n - 2\,a + 2\,i b},$$

cujus integratio omnino similis erit praecedenti, et quasi eodem labore absolvetur.

§. 20. Manifestum autem est, in hujusmodi calculo imaginaria cum realibus multo arctius commisceri, quam in praecedentibus problematibus usu venit, quandoquidem jam statim ab initio coëfficientes derivati F′ et G′ jam imaginaria involvunt; deinde vero etiam utrinque tang. $n\,\phi$ imaginariis inquinatur, unde etiam in valores X et Y imaginaria ingredientur; quamobrem reductio ad realitatem plerumque maximum laborem exigere poterit, proque autem negotio praecepta necessaria jam satis sunt cognita.

---

2) Theorema maxime memorabile circa formulam integralem $\int \frac{\partial \phi \cos. \lambda \phi}{(1 + a a - 2 a \cos. \phi)^{n+1}}$. *M. S. Academiae exhib. die* 13. *Augusti* 1778.

§. 21. Haec formula aliam restrictionem non postulat nisi quod littera λ numeros tantum integros designat sive positivos sive negativos. Evidens autem est valores negativos non discrepare a positivis, cum semper sit $\cos. - \phi = \cos. + \phi$. Hoc notato si istius formulae integrale a termino $\phi = 0$ usque ad terminum $\phi = 180^\circ$ sive $\phi = \pi$ porrigatur, ejus valor semper sequenti formula exprimetur $\frac{\pi a}{(1 - a a)^{2n+1}} . V$, existente

$$V = \left(\frac{n-\lambda}{0}\right)\left(\frac{n+\lambda}{\lambda}\right) + \left(\frac{n-\lambda}{1}\right)\left(\frac{n+\lambda}{\lambda+1}\right) a a$$
$$+ \left(\frac{n-\lambda}{2}\right)\left(\frac{n+\lambda}{\lambda+2}\right) a^4 + \left(\frac{n-\lambda}{3}\right)\left(\frac{n+\lambda}{\lambda+3}\right) a^6$$
$$+ \left(\frac{n-\lambda}{4}\right)\left(\frac{n+\lambda}{\lambda+4}\right) a^8 + \left(\frac{n-\lambda}{5}\right)\left(\frac{n+\lambda}{\lambda+5}\right) a^{10} \text{ etc.}$$

Ubi formulae uncinulis inclusae non fractiones, sed eos characteres designant, quibus unciae potestatum Binomii designari solent, ita ut sit

$$\left(\frac{\alpha}{\beta}\right) = \frac{\alpha}{1} \cdot \frac{\alpha - 1}{2} \cdot \frac{\alpha - 2}{3} \cdot \cdot \cdot \cdot \cdot \cdot \cdot \frac{\alpha - \beta + 1}{\beta},$$

quae expressio quoniam nostro casu β ubique est numerus integer, determinatum valorem facile quovis casu exhibendam declarat, ubi notasse sufficiet, quoties fuerit $\beta = 0$ semper fore $(\frac{\alpha}{0}) = 1$; sin autem fuerit β numerus negativus, valorem hujus characteris in nihilum abire; tum vero etiam observari convenit, si fuerit $\beta = \alpha$ fore $(\frac{\alpha}{\alpha}) = 1$, et si $\beta > \alpha$ pariter valores evanescere. Cum semper sit $(\frac{\alpha}{\beta}) = (\frac{\alpha}{\alpha - \beta})$.

§. 22. His explicatis evolvamus praecipuos casus quibus exponenti n valores simpliciores 0, 1, 2, 3, 4 etc. tribuuntur.

Casus I.

quo $n = 0$, et formula integralis haec proponitur

$$\int \frac{\partial \phi \cos. \lambda \phi}{1 + aa - 2a \cos. \phi} \begin{bmatrix} b\, x = 0 \\ ad\, x = \pi \end{bmatrix}.$$

Quia hic $n = 0$, pro prioribus factoribus quantitatis V habebimus

$$\left(\frac{0-\lambda}{0}\right) = 1;\ \left(\frac{0-\lambda}{1}\right) = -\lambda;\ \left(\frac{0-\lambda}{2}\right) = \frac{\lambda}{1} \cdot \frac{\lambda+1}{2};$$
$$\left(\frac{0-\lambda}{3}\right) = -\frac{\lambda}{1} \cdot \frac{\lambda+1}{2} \cdot \frac{\lambda+2}{3};\ \left(\frac{0-\lambda}{4}\right) = \frac{\lambda}{1} \cdot \frac{\lambda+1}{2} \cdot \frac{\lambda+2}{3} \cdot \frac{\lambda+3}{4};\ \text{etc.}$$

Pro posterioribus vero factoribus babebimus

$$\left(\frac{0+\lambda}{\lambda}\right) = 1;\ \left(\frac{0+\lambda}{\lambda+1}\right) = 0;\ \left(\frac{0+\lambda}{\lambda+2}\right) = 0 \ \text{etc.}$$

hic scilicet omnes isti factores praeter primum evanescunt; unde colligitur valor quantitatis $V = 1$, ideoque integrale quaesitum hujus casus erit $= \frac{\pi a^{\lambda}}{1 - aa}$.

Hinc ergo si fuerit $n = 0$, erit $\int \frac{\partial \phi}{1 + aa - 2a \cos. \phi} = \frac{\pi}{1 - aa}$ quod egregie consentit cum integratione satis cognita

$$\int \frac{\partial \phi}{\alpha + \beta \cos. \phi} = \frac{1}{\sqrt{(\alpha\alpha - \beta\beta)}} \text{ Arc. cos. } \frac{\alpha \cos. \phi + \beta}{\alpha + \beta \cos. \phi},$$

quod integrale jam sponte evanescit sumto $\Phi = 0$. Statuatur igitur, ut hic perpetuo assumimus, $\Phi = 180^\circ = \pi$, atque ob cos. $\Phi = -1$, erit istud integrale

$$\frac{1}{\sqrt{(\alpha\alpha - \beta\beta)}} \text{Arc. cos.} -1 = \frac{\pi}{\sqrt{(\alpha\alpha - \beta\beta)}}.$$

Jam nostro casu est $\alpha = 1 + aa$ et $\beta = -2a$, unde fit $\sqrt{(\alpha\alpha - \beta\beta)} = 1 - aa$.

Casus II.

quo $n = 1$, et formula integralis haec proponitur

$$\int \frac{\partial\Phi \cos. \lambda\Phi}{(1 + aa - 2a\cos.\Phi)^2} \begin{bmatrix} a\ \Phi = 0 \\ ad\ \Phi = \pi \end{bmatrix}.$$

Quia hic est $n = 1$, erit pro prioribus factoribus quantitatis V

$$\left(\frac{1-\lambda}{0}\right) = 1;\ \left(\frac{1-\lambda}{1}\right) = -(\lambda - 1);$$
$$\left(\frac{1-\lambda}{2}\right) = \frac{\lambda(\lambda-1)}{1.\quad 2}.$$

Pro posterioribus vero factoribus habebimus

$$\left(\frac{1+\lambda}{\lambda}\right) = \lambda + 1;\ \left(\frac{1+\lambda}{\lambda+1}\right) = 1;$$

sequentes vero formulae evanescunt, sicque erit

$$V = \lambda + 1 - (\lambda - 1)\, aa;$$

quocirca valor integralis propositi erit

$$\frac{\pi a^\lambda}{(1-aa)^3}\,[(\lambda + 1) - (\lambda - 1)\,aa]:$$

hinc ergo sequentes casus speciales apposuisse juvabit, ubi brevitatis gratia loco formulae $1 + aa - 2a\cos.\Phi$ characterem Δ scribamus

$$\int \frac{\partial\Phi}{\Delta^2} = \frac{\pi(1+aa)}{(1-aa)^3},$$
$$\int \frac{\partial\Phi \cos.\Phi}{\Delta^2} = \frac{2\pi a}{(1-aa)^3},$$

$$\int \frac{\partial \Phi \cos. 2 \Phi}{\Delta^2} = \frac{\pi a^2 (3 - a a)}{(1 - a a)^3},$$
$$\int \frac{\partial \Phi \cos. 3 \Phi}{\Delta^2} = \frac{\pi a^3 (4 - 2 a a)}{(1 - a a)^3},$$
$$\int \frac{\partial \Phi \cos. 4 \Phi}{\Delta^2} = \frac{\pi a^4 (5 - 3 a a)}{(1 - a a)^3},$$
$$\int \frac{\partial \Phi \cos. 5 \Phi}{\Delta^2} = \frac{\pi a^5 (6 - 4 a a)}{(1 - a a)^3},$$
$$\int \frac{\partial \Phi \cos. 6 \Phi}{\Delta^2} = \frac{\pi a^6 (7 - 5 a a)}{(1 - a a)^3},$$
etc. etc.

Casus III.

quo $n = 2$, et formula integralis haec proponitur

$$\int \frac{\partial \Phi \cos. \lambda \Phi}{(1 + a a - 2 a \cos. \Phi)^3} \left[\begin{array}{l} a\ \Phi = 0 \\ ad\ \Phi = \pi \end{array} \right].$$

Hic factores priores, qui in valore quantitatis V occurrunt, erunt

$$\left(\frac{2-\lambda}{0}\right) = 1;\ \left(\frac{2-\lambda}{1}\right) = -(\lambda - 2);\ \left(\frac{2-\lambda}{2}\right) = \frac{(\lambda-2)(\lambda-1)}{1. \quad 2};$$
$$\left(\frac{2-\lambda}{3}\right) = \frac{\lambda - 2 . \lambda - 1 . \lambda}{1. \quad 2. \quad 3} \text{ etc.}$$

factores autem posteriores erunt

$$\left(\frac{2+\lambda}{\lambda}\right) = \frac{\lambda+2}{1} \cdot \frac{\lambda+1}{2};\quad \left(\frac{2+\lambda}{\lambda+1}\right) = \lambda + 2;\ \left(\frac{2+\lambda}{\lambda+2}\right) = 1;$$

et sequentes omnes evanescunt; hinc ergo colligimus

$$V = \frac{(\lambda+2)(\lambda+1)}{1. \quad 2} - (\lambda\lambda - 4)\, a a + \frac{(\lambda-2)(\lambda-1)}{1. \quad 2} a^4,$$

hocque valore invento erit integrale quaesitum $\frac{\pi a^\lambda}{(1 - a a)^5} . V$, unde sequentes casus speciales, statuendo ut ante $1 + a a - 2 a \cos. \Phi = \Delta$, evolvamus

$$\int \frac{\partial \Phi}{\Delta^3} = \frac{\pi}{(1 - a a)^5} (1 + 4 a a + a^4),$$
$$\int \frac{\partial \Phi \cos. \Phi}{\Delta^3} = \frac{3 \pi a}{(1 - a a)^5} (1 + a a),$$
$$\int \frac{\partial \Phi \cos. 2 \Phi}{\Delta^3} = \frac{6 \pi a^2}{(1 - a a)^5},$$

$$\int \frac{\partial \phi \cos. 3\phi}{\Delta^3} = \frac{\pi a^3}{(1-aa)^5}(10 - 5aa + a^4),$$
$$\int \frac{\partial \phi \cos. 4\phi}{\Delta^3} = \frac{3\pi a^4}{(1-aa)^5}(5 - 4aa + a^4),$$
$$\int \frac{\partial \phi \cos. 5\phi}{\Delta^3} = \frac{3\pi a^5}{(1-aa)^5}(7 - 7aa + 2a^4),$$
$$\int \frac{\partial \phi \cos. 6\phi}{\Delta^3} = \frac{2\pi a^6}{(1-aa)^5}(14 - 16aa + 5a^4),$$
etc. etc.

Casus IV.

quo $n = 3$, et formula integralis haec proponitur

$$\int \frac{\partial \phi \cos. \lambda \phi}{(1 + aa - 2a\cos.\phi)^4} \left[\begin{array}{l} a\,\phi = 0 \\ ad\,\phi = \pi \end{array}\right].$$

Hic pro prioribus factoribus quantitatis V habebimus

$$\left(\frac{3-\lambda}{0}\right) = 1;\ \left(\frac{3-\lambda}{1}\right) = -(\lambda - 3);\ \left(\frac{3-\lambda}{2}\right) = \frac{3-\lambda}{1}\cdot\frac{2-\lambda}{2};$$
$$\left(\frac{3-\lambda}{3}\right) = \frac{3-\lambda}{1}\cdot\frac{2-\lambda}{2}\cdot\frac{1-\lambda}{3};\ \ \left(\frac{3-\lambda}{4}\right) = \frac{3-\lambda}{1}\cdot\frac{2-\lambda}{2}\cdot\frac{1-\lambda}{3}\cdot\frac{-\lambda}{4};$$

factores autem posteriores erunt

$$\left(\frac{3+\lambda}{\lambda}\right) = \frac{3+\lambda}{1}\cdot\frac{2+\lambda}{2}\cdot\frac{1+\lambda}{3};\ \ \left(\frac{3+\lambda}{\lambda+1}\right) = \frac{3+\lambda}{1}\cdot\frac{2+\lambda}{2};$$
$$\left(\frac{3+\lambda}{\lambda+2}\right) = 3 + \lambda;\ \left(\frac{3+\lambda}{\lambda+3}\right) = 1;$$

et sequentes omnes evanescunt, hinc ergo colligimus

$$V = \frac{(\lambda+1)}{1.}\frac{(\lambda+2)}{2.}\frac{(\lambda+3)}{3} - \frac{(\lambda+2)}{1.}\frac{(\lambda\lambda-9)}{2}aa + \frac{(\lambda-2)}{1.}\frac{(\lambda\lambda-9)}{2}a^4$$
$$- \frac{(\lambda-1)}{1.}\frac{(\lambda-2)}{2.}\frac{(\lambda-3)}{3}a^6.$$

Quo valore invento colligimus integrale quaesitum $= \frac{\pi a^\lambda}{(1-aa)^7}\cdot V$, hincque sequentes casus speciales, ponendo ut hactenus $1 + aa - 2a\cos.\phi = \Delta$, evolvamus

$$\int \frac{\partial \phi}{\Delta^4} = \frac{\pi}{(1-aa)^7}(1 + 9aa + 9a^4 + a^6),$$
$$\int \frac{\partial \phi \cos.\phi}{\Delta^4} = \frac{4\pi a}{(1-aa)^7}(1 + 3aa + a^4),$$
$$\int \frac{\partial \phi \cos. 2\phi}{\Delta^4} = \frac{10\pi a^2}{(1-aa)^7}(1 + aa),$$

$$\int \frac{\partial \Phi \cos. 3 \Phi}{\Delta^4} = \frac{20 \pi a^3}{(1 - aa)^7},$$
$$\int \frac{\partial \Phi \cos. 4 \Phi}{\Delta^4} = \frac{\pi a^4}{(1 - aa)^7} (35 - 21 aa + 7 a^4 - a^6),$$
etc. etc.

§. 23. Hic longius progredi superfluum foret, cum forma generalis pro V inventa totum negotium facillime conficiat; verum haud inutile erit, litterae n etiam valores negativos tribuere, quibus casibus tota integratio per methodos consuetas haud difficulter expeditur, unde jucundum erit pulcherrimum consensum nostrae formae generalis perspicere.

Casus I.

quo $n = -1$, et formula integralis haec proponitur
$$\int \partial \Phi \cos. \lambda \Phi \begin{bmatrix} a \Phi = 0 \\ ad \Phi = \pi \end{bmatrix}.$$

Haec formula absolute est integrabilis, cum sit
$$\int \partial \Phi \cos. \lambda \Phi = \frac{1}{\lambda} \sin. \lambda \Phi,$$
quae formula cum jam evanescat posito $\Phi = 0$; sumendo $\Phi = \pi$, ob λ numerum integrum iste valor semper erit $= 0$, solo casu excepto $\lambda = 0$. Spectato enim λ tanquam infinite parvo, erit $\sin. \lambda \pi = \lambda \pi$, ideoque hoc casu valor erit $= \pi$. Nunc autem forma generalis pro quantitate V data erit

$$\begin{aligned} V = & \left(\frac{-1-\lambda}{0}\right)\left(\frac{-1+\lambda}{\lambda}\right) + \left(\frac{-1-\lambda}{1}\right)\left(\frac{-1+\lambda}{\lambda+1}\right) a^2 \\ & + \left(\frac{-1-\lambda}{2}\right)\left(\frac{-1+\lambda}{\lambda+2}\right) a^4 + \left(\frac{-1-\lambda}{3}\right)\left(\frac{-1+\lambda}{\lambda+3}\right) a^6 \\ & + \left(\frac{-1-\lambda}{4}\right)\left(\frac{-1+\lambda}{\lambda+4}\right) a^8 + \left(\frac{-1-\lambda}{5}\right)\left(\frac{-1+\lambda}{\lambda+5}\right) a^{10} \\ & \text{etc.} \qquad\qquad \text{etc.} \end{aligned}$$

Cujus expressionis factores posteriores omnes evanescunt, quoties fuerit vel $\lambda = 1$ vel $\lambda > 1$, propterea quod numeri inferiores majores, quam superiores, utrique vero positivi; quae conclusio autem non valet, quando superior numerus evadit negativus, uti evenit casu $\lambda = 0$, quem ergo solum perpendisse necesse est; hoc autem casu factores priores evadent

$$\left(\frac{-1}{0}\right) = 1;\ \left(\frac{-1}{1}\right) = -1;\ \left(\frac{-1}{2}\right) = +1;$$
$$\left(\frac{-1}{3}\right) = -1;\ \left(\frac{-1}{4}\right) = +1;\ \text{etc.}$$

at vero valores posteriores eosdem determinationes recipiunt; sicque habebimus

$$V = 1 + aa + a^4 + a^6 + a^8 + a^{10} + \text{etc.}$$

quae series cum sit geometrica, erit $V = \frac{1}{1-aa}$ quare cum, ob $n = -1$ et $\lambda = 0$, valor quaesitus per nostram formam generalem sit $\pi\ (1-aa)\ V$, iste valor nunc ob $V = \frac{1}{1-aa}$, abit in π, uti supra.

Casus II.

quo $n = -2$, et formula integralis haec proponitur

$$\int \partial\Phi \cos.\lambda\Phi\ (1 + aa - 2a\cos.\Phi) \left[\begin{smallmatrix} a\ \Phi = 0 \\ ad\ \Phi = \pi \end{smallmatrix}\right].$$

Per formam nostram generalem integrale quaesitum erit $\pi\ a^\lambda\ (1-aa)^3\ V$, existente

$$V = \left(\frac{-2-\lambda}{0}\right)\left(\frac{-2+\lambda}{\lambda}\right) + \left(\frac{-2-\lambda}{1}\right)\left(\frac{-2+\lambda}{\lambda+1}\right) aa + \left(\frac{-2-\lambda}{2}\right)\left(\frac{-2+\lambda}{\lambda+2}\right) a^4$$
$$+ \left(\frac{-2-\lambda}{3}\right)\left(\frac{-2+\lambda}{\lambda+3}\right) a^6 + \left(\frac{-2-\lambda}{4}\right)\left(\frac{-2+\lambda}{\lambda+4}\right) a^8 + \left(\frac{-2-\lambda}{5}\right)\left(\frac{-2+\lambda}{\lambda+5}\right) a^{10}$$
etc. etc.

Ubi iterum evidens est, si fuerit vel $\lambda = 2$ vel $\lambda > 2$, omnes factores posteriores evanescere, ideoque fieri $V = 0$, ita ut etiam valor integralis quaesitus semper evanescat, id quod ex ipsa natura formulae sponte sequitur, quippe cujus integrale, ob

$$\cos.\Phi \cos.\lambda\Phi = \tfrac{1}{2}\cos.(\lambda - 1)\Phi + \tfrac{1}{2}\cos.(\lambda + 1)\Phi,$$

in genere erit

$$\frac{1+aa}{\lambda}\sin.\lambda\Phi - \frac{a}{\lambda-1}\sin.(\lambda-1)\Phi - \frac{a}{\lambda+1}\sin.(\lambda+1)\Phi,$$

quod quia $\lambda > 1$ casu $\phi = \pi$ manifesto evanescit; unde duos casus perpendere superest, alterum quo $\lambda = 0$, et alterum quo $\lambda = 1$.

I°. Sit $\lambda = 0$, et integrale $\pi\,(1 - aa)^3\,V$, ubi pro V factores posteriores evadunt

$$(\tfrac{-2}{0}) = 1;\ (\tfrac{-2}{1}) = -2;\ (\tfrac{-2}{2}) = 3;\ (\tfrac{-2}{3}) = -4;$$
$$(\tfrac{-2}{4}) = +5;\ (\tfrac{-2}{5}) = -6;\ \text{etc.}$$

simili modo priores factores erunt

$$(\tfrac{-2}{0}) = 1;\ (\tfrac{-2}{1}) = -2;\ (\tfrac{-2}{2}) = 3;\ \text{etc.}$$

unde colligitur fore

$$V = 1 + 4\,aa + 9\,a^4 + 16\,a^6 + 25\,a^8 + 36\,a^{10} + \text{etc.}$$

Pro qua serie summanda, inde subtrahatur series $V\,a\,a$, et remanebit

$$V\,(1 - aa) = 1 + 3\,aa + 5\,a^4 + 7\,a^6 + 9\,a^8 + \text{etc.}$$

Multiplicitur denuo utrinque per $1 - aa$, ac prodibit

$$V\,(1 - aa)^2 = 1 + 2\,aa + 2\,a^4 + 2\,a^6 + 2\,a^8 + \text{etc.}$$

quae denuo ducta in $1 - aa$ praebet

$$V\,(1 - aa)^3 = 1 + aa, \text{ ideoque } V = \frac{1 + aa}{(1 - aa)^3}.$$

Consequenter integrale quaesitum erit $= \pi\,(1 + aa)$, id quod utique oritur ex integratione actuali, cum sit

$$\int \partial\phi\,(1 + aa - 2\,a\cos.\phi) = (1 + aa)\,\phi - 2\,a\sin.\phi,$$

quod facto $\phi = \pi$ abit in $(1 + aa)\,\pi$.

II°. Sit $\lambda = 1$, et integrale quaesitum $\pi\,a\,(1 - aa)^3\,V$; ubi pro factoribus posterioribus est

$$(\tfrac{-1}{1}) = -1;\ (\tfrac{-1}{2}) = +1;\ (\tfrac{-1}{3}) = -1;$$
$$(\tfrac{-1}{4}) = +1;\ (\tfrac{-1}{5}) = -1;\ \text{etc.}$$

Factores vero priores evadunt

$(\frac{-3}{0}) = 1$; $(\frac{-3}{1}) = -3$; $(\frac{-3}{2}) = 6$; $(\frac{-3}{3}) = -10$;

$(\frac{-3}{4}) = 15$; $(\frac{-3}{5}) = -21$; $(\frac{-3}{6}) = 28$;

$(\frac{-3}{7}) = -36$; etc.

hinc igitur habebimus

$$V = -1 - 3aa - 6a^4 - 10a^6 - 15a^8 - 21a^{10} - 28a^{12} - 36a^{14} - \text{etc.}$$

Pro cujus summatione multiplicetur utrinque per $1 - aa$, et prodibit

$$V(1 - aa) = -1 - 2aa - 3a^4 - 4a^6 - 5a^8 - 6a^{10} - 7a^{12} - 8a^{14} - \text{etc.}$$

multiplicando denuo per $1 - aa$, prodit

$$V(1 - aa)^2 = -1 - aa - a^4 - a^6 - a^8 - a^{10} - a^{12} - a^{14} - \text{etc.}$$

et multiplicando rursus per $1 - aa$, erit

$$V(1 - aa)^3 = -1, \text{ ita ut sit } V = -\frac{1}{(1 - aa)^3},$$

consequenter integrale quaesitum $= -\pi a$. Ipsa autem integratio ob $\cos.\Phi^2 = \frac{1}{2} + \frac{1}{2}\cos.2\Phi$ praebet

$$\int \partial\Phi \cos.\Phi(1 + aa - 2a\cos.\Phi) = (1 + aa)\sin.\Phi$$
$$- a\Phi - \tfrac{1}{2}a\sin.2\Phi,$$

unde statuendo $\Phi = \pi$, oritur integrale $= -a\pi$.

Casus III.

quo $n = -3$, et formula integralis haec proponitur

$$\int \partial\Phi \cos.\lambda\Phi(1 + aa - 2a\cos.\Phi)^2 \begin{bmatrix} a\,\Phi = 0 \\ ad\,\Phi = \pi \end{bmatrix}.$$

Hoc ergo casu ex forma generali erit integrale

$\pi a^\lambda (1 - aa)^5 V$, existente

$$V = (\frac{-3-\lambda}{0})(\frac{-3+\lambda}{\lambda}) + (\frac{-3-\lambda}{1})(\frac{-3+\lambda}{\lambda+1})a^2$$
$$+ (\frac{-3-\lambda}{2})(\frac{-3+\lambda}{\lambda+2})a^4 + (\frac{-3-\lambda}{3})(\frac{-3+\lambda}{\lambda+3})a^6$$

etc. etc.

ubi factores posteriores manifesto omnes evanescunt, quando fuerit vel $\lambda = 3$ vel $\lambda > 3$, quibus ergo casibus totum integrale evanescit, ut cuilibet calculum instituenti facile patebit: tres autem casus considerandi restant, quibus $\lambda < 3$.

I. Sit $\lambda = 0$, atque tam priores quam posteriores factores convenient, eruntque

$$(\tfrac{-3}{0}) = 1;\ (\tfrac{-3}{1}) = -3;\ (\tfrac{-3}{2}) = 6;\ (\tfrac{-3}{3}) = -10;$$
$$(\tfrac{-3}{4}) = 15;\ (\tfrac{-3}{5}) = -21;\ (\tfrac{-3}{6}) = 28;\ \text{etc.}$$

unde colligitur

$$V = 1 + 9aa + 36a^4 + 100a^6 + 225a^8 + 441a^{10} + \text{etc.}$$

quae series cum tandem perducat ad differentias constantes, simili modo ut hactenus summari poterit, prima enim multiplicatio per $1 - aa$ praebet

$$V(1-aa) = 1 + 8aa + 27a^4 + 64a^6 + 125a^8 + 216a^{10} + 343a^{12} + \text{etc.}$$

Secunda multiplicatio per $1 - aa$ praebet

$$V(1-aa)^2 = 1 + 7aa + 19a^4 + 37a^6 + 61a^8 + 91a^{10} + 127a^{12} + \text{etc.}$$

Tertia multiplicatio dat

$$V(1-aa)^3 = 1 + 6aa + 12a^4 + 18a^6 + 24a^8 + 30a^{10} + \text{etc.}$$

Quarta multiplicatio dat

$$V(1-aa)^4 = 1 + 5aa + 6a^4 + 6a^6 + 6a^8 + 6a^{10} + \text{etc.}$$

ac denique

$V(1-aa)^5 = 1 + 4aa + a^4$, ita ut sit $V = \frac{1+4aa+a^4}{(1-aa)^6}$;

consequenter valor integralis quaesitus hoc casu erit $\pi(1 + 4aa + a^4)$, quod egregie cum integrali more solito invento congruit.

II. Sit $\lambda = 1$, quo casu priores factores ipsius V erunt

$$(\tfrac{-4}{0}) = 1;\ (\tfrac{-4}{1}) = -4;\ (\tfrac{-4}{2}) = 10;\ (\tfrac{-4}{3}) = -20;$$
$$(\tfrac{-4}{4}) = 35;\ (\tfrac{-4}{5}) = -56;\ (\tfrac{-4}{6}) = 84;\ (\tfrac{-4}{7}) = -120 \text{ etc.}$$

posteriores vero ita se habent

$$(\tfrac{-2}{1}) = -2;\ (\tfrac{-2}{2}) = +3;\ (\tfrac{-2}{3}) = -4;\ (\tfrac{-2}{4}) = +5;$$
$$(\tfrac{-2}{5}) = -6;\ (\tfrac{-2}{6}) = +7;\ (\tfrac{-2}{7}) = -8:\ (\tfrac{-2}{8}) = +9; \text{ etc.}$$

ideoque

$$V = -2 - 12\,a^2 - 40\,a^4 - 100\,a^6 - 210\,a^8 - 392\,a^{10} - 672\,a^{12} - 1080\,a^{14} - \text{etc.}$$

quae series cum tandem perducat ad differentias constantes, simili modo ut ante summari poterit; prima enim multiplicatio per $1 - aa$ dat

$$V(1 - aa) = -2 - 10\,a^2 - 28\,a^4 - 60\,a^6 - 110\,a^8 - 182\,a^{10} - 280\,a^{12} - \text{etc.}$$

Secunda multiplicatio per $1 - aa$ praebet

$$V(1 - aa)^2 = -2 - 8\,a^2 - 18\,a^4 - 32\,a^6 - 50\,a^8 - 72\,a^{10} - 98\,a^{12} - \text{etc.}$$

Tertia multiplicatio dat

$$V(1 - aa)^3 = -2 - 6\,a^2 - 10\,a^4 - 14\,a^6 - 18\,a^8 - 22\,a^{10} - 26\,a^{12} - \text{etc.}$$

Quarta multiplicatio dat

$$V(1 - aa)^4 = -2 - 4a^2 - 4\,a^4 - 4\,a^6 - 4\,a^8 - 4\,a^{10} - 4\,a^{12} - \text{etc.}$$

ac denique quinta multiplicatio per $1 - aa$ praebet

$$V(1 - aa)^5 = -2 - 2\,aa = -2(1 + aa);$$

unde colligitur $V = -\frac{2(1+aa)}{(1-aa)^5}$, ideoque valor integralis quaesitus erit $= -2\pi a(1 + aa)$, qui egregie cum integrali more solito invento congruit.

III. Sit $\lambda = 2$, atque factores priores ipsius V erunt

$$(\tfrac{-5}{0}) = 1;\ (\tfrac{-5}{1}) = -5;\ (\tfrac{-5}{2}) = 15;\ (\tfrac{-5}{3}) = -35;$$
$$(\tfrac{-5}{4}) = 70;\ (\tfrac{-5}{5}) = -126;\ (\tfrac{-5}{6}) = 210;$$
$$(\tfrac{-5}{7}) = -330 \text{ etc.}$$

posteriores vero factores ita se habebunt

$$(\tfrac{-1}{2}) = 1;\ (\tfrac{-1}{3}) = -1;\ (\tfrac{-1}{4}) = 1;\ (\tfrac{-1}{5}) = -1;$$
$$(\tfrac{-1}{6}) = 1;\ (\tfrac{-1}{7}) = -1;\ (\tfrac{-1}{8}) = 1;\ (\tfrac{-1}{9}) = -1 \text{ etc.}$$

unde colligitur

$$V = 1 + 5a^2 + 15a^4 + 35a^6 + 70a^8 + 126a^{10} + 210a^{12} + 330a^{14} + \text{etc.}$$

quae series eodem modo ut ante summata praebet $V = + \frac{1}{(1-aa)^5}$, unde colligitur valor integralis quaesitus $= \pi aa$, qui cum integrali more solito invento utique egregie congruit.

§. 24. Quodsi haec integralia quibus n est numerus negativus cum iis comparemus, quibus n est numerus positivus, insignis analogia deprehenditur inter valores harum formularum

$$\int \Delta^n \partial \phi \cos. \lambda \phi \text{ et } \int \frac{\partial \phi \cos. \lambda \phi}{\Delta^{n+1}},$$

quae affinitas, si per plures casus exploretur, sequens nobis suppeditat theorema maxime notabile.

Theorema.

§. 25. Posito brevitatis gratia $\Delta = 1 + aa - 2a\cos.\phi$, atque integralia a termino $\phi = 0$ usque ad terminum $\phi = 180°$ extendantur, semper locum habebit sequens proportio

$$\int \Delta^n \partial \phi \cos. \lambda \phi : \int \frac{\partial \phi \cos. \lambda \phi}{\Delta^{n+1}} = (\tfrac{n}{\lambda})(1-aa)^n : (\tfrac{-n-1}{\lambda})(1-aa)^{-n-1},$$

vel si statuamus

$$\frac{\Delta}{1-aa} = \frac{1+aa-2a\cos.\Phi}{1-aa} = \Gamma,$$

simplicius erit

$$\int \Gamma^n \partial\Phi \cos.\lambda\Phi : \int \frac{\partial\Phi \cos.\lambda\Phi}{\Gamma^{n+1}} = (\tfrac{n}{\lambda}) : (\tfrac{-n-1}{\lambda}).$$

§. 26. Ita exempla gratia si ponamus $n = 2$, erit ex priore proportione

$$\int \Delta^2 \partial\Phi \cos.\lambda\Phi : \int \frac{\partial\Phi \cos.\lambda\Phi}{\Delta^3} = (\tfrac{2}{\lambda})(1-aa)^2 : (\tfrac{-3}{\lambda})(1-aa)^{-3}$$

unde si $\lambda = 0$, ob $(\tfrac{n}{0}) = 1$ et $(\tfrac{-3}{0}) = 1$, erit

$$\int \Delta^2 \partial\Phi : \int \frac{\partial\Phi}{\Delta^3} = (1-aa)^2 : \frac{1}{(1-aa)^3} = 1 : \frac{1}{(1-aa)^5},$$

ideoque erit

$$\int \frac{\partial\Phi}{\Delta^3} = \frac{1}{(1-aa)^5} \int \Delta^2 \partial\Phi.$$

Cum igitur sit

$$\int \Delta^2 \partial\Phi = \pi(1+4aa+a^4), \text{ erit}$$

$$\int \frac{\partial\Phi}{\Delta^3} = \frac{\pi}{(1-aa)^5}(1+4aa+a^4).$$

§. 27. Manente $n = 2$, sit $\lambda = 1$, ob $(\tfrac{2}{1}) = 1$ et $(\tfrac{-3}{1}) = -3$, erit

$$\int \Delta^2 \partial\Phi \cos.\Phi : \int \frac{\partial\Phi \cos.\Phi}{\Delta^3} = 2(1-aa)^2 : -3(1-aa)^{-3} = 1 : \frac{-3}{2(1-aa)^5}$$

unde fit

$$\int \frac{\partial\Phi \cos.\Phi}{\Delta^3} = \frac{-3}{2(1-aa)5} \int \Delta^2 \partial\Phi \cos.\Phi;$$

cum igitur sit

$$\int \Delta^2 \partial\Phi \cos.\Phi = -2\pi a(1+aa), \text{ erit}$$

$$\int \frac{\partial\Phi \cos.\Phi}{\Delta^3} = \frac{+3\pi a(1+aa)}{(1-aa)^5}.$$

§. 28. Simili modo sumatur $\lambda = 2$, et ob $(\tfrac{2}{2}) = 1$ et $(\tfrac{-3}{2}) = 6$, erit

$$\int \Delta^2 \partial \Phi \cos. 2\Phi : \int \frac{\partial \Phi \cos. 2\Phi}{\Delta^3} = (1-aa)^2 : 6(1-aa)^{-3} = 1 : \frac{6}{(1-aa)^5},$$

unde fit

$$\int \frac{\partial \Phi \cos. 2\Phi}{\Delta^3} = \frac{6}{(1-aa)^5} \int \Delta^2 \partial \Phi \cos. 2\Phi.$$

Erat autem

$$\int \Delta^2 \partial \Phi \cos. 2\Phi = \pi a a,$$

consequenter

$$\int \frac{\partial \Phi \cos. 2\Phi}{\Delta^3} = \frac{6\pi a a}{(1-aa)^5}.$$

§. 29. Cum character $(\frac{n}{\lambda})$ fiat $= 1$ casu $\lambda = n$, casibus vero quibus $\lambda > n$ semper sit $(\frac{n}{\lambda}) = 0$, siquidem λ fuerit numerus integer, uti hic perpetuo assumimus, evidens est istis casibus, quibus $\lambda > n$, semper valorem formulae $\int \Delta^n \partial \Phi \cos. \lambda \Phi$ in nihilum abire.

§. 30. Theorema, quod hic proposuimus, non solum ob simplicitatem rationis omni attentione est dignum, sed etiam quod id tantum per plures casus sola inductione conclusimus, neque adhuc ulla via patere videtur, qua ejus veritas directe demonstrari queat; hujusmodi autem theoremata summam Geometrarum attentionem merentur. Evolvamus autem adhuc alios quosdam casus memorabiles nostri theorematis initio propositi.

Evolutio casus quo $\lambda = n$, et formula integralis proposita

$$\int \frac{\partial \Phi \cos. n\Phi}{\Delta^{n+1}}.$$

Ex forma generali hoc casu integrale erit $\frac{\pi a^n}{(1-aa)^{2n+1}} V$, existente

$$V = (\tfrac{0}{0})(\tfrac{2n}{n}) + (\tfrac{0}{1})(\tfrac{2n}{n+1})aa + (\tfrac{0}{2})(\tfrac{2n}{n+2})a^4 + \text{etc.}$$

ubi manifesto omnes termini praeter primum evanescunt, ita ut sit $V = (\frac{2n}{n})$, ideoque nostrum integrale

$$\int \frac{\partial\Phi \cos. n\Phi}{\Delta^{n+1}} = \frac{\pi a^n}{(1-aa)^{2n+1}} \cdot (\tfrac{2n}{n});$$

ubi notetur, valores characteris $(\frac{2n}{n})$ pro variis valoribus numeri n sequenti modo se habere

n	0,	1,	2,	3,	4,	5,	6,	7	etc.
$(\frac{2n}{n})$	1,	2,	6,	20,	70,	252,	924,	3432	

quae series facillime per hos factores continuatur

$$\tfrac{2}{1} \cdot \tfrac{6}{2} \cdot \tfrac{10}{3} \cdot \tfrac{14}{4} \cdot \tfrac{18}{5} \cdot \tfrac{22}{6} \cdot \tfrac{26}{7} \text{ etc.}$$

Postremum vero theorema inventum ad hunc casum applicatum praebebit hanc proportionem

$$\int \Delta^n \partial\Phi \cos. n\Phi : \int \frac{\partial\Phi \cos. n\Phi}{\Delta^{n+1}} = (1-aa)^n : (\tfrac{-1-n}{n})(1-aa)^{-n-1},$$

unde fit

$$\int \Delta^n \partial\Phi \cos. n\Phi = \frac{\pi a^n}{(\frac{-n-1}{n})} \cdot (\tfrac{2n}{n}) = (\tfrac{2n}{n})\pi a^n : (\tfrac{-n-1}{n};$$

ubi notetur valores characteris $(\frac{-n-1}{n})$ pro variis valoribus ipsius n esse

n	0,	1,	2,	3,	4,	5,	6	etc.
$(\frac{-n-1}{n})$	−1,	−2,	6,	−20,	70,	−252,	924	

unde patet esse $(\frac{-n-n}{n}) = \pm(\frac{2n}{n})$, dum signum superius valet, quando n est numerus par, contra vero signum inferius, quando n est numerus impar; hinc ergo erit

$$\int \Delta^n \partial\Phi \cos. n\Phi = \pm \pi a^n.$$

His notatis evolvamus casus simpliciores pro utraque formula integrali

$n = 0$	$\int\frac{\partial\Phi}{\Delta} = \frac{\pi}{1-aa}$	$\int\partial\Phi = +\pi$
$n = 1$	$\int\frac{\partial\Phi\cos.\Phi}{\Delta^2} = \frac{2\pi a}{(1-aa)^3}$	$\int\Delta\partial\Phi\cos.\Phi = -\pi a$
$n = 2$	$\int\frac{\partial\Phi\cos.2\Phi}{\Delta^3} = \frac{2\pi a^2}{(1-aa)^5}$	$\int\Delta^2\partial\Phi\cos.2\Phi = +\pi a^2$
$u = 3$	$\int\frac{\partial\Phi\cos.3\Phi}{\Delta^4} = \frac{20\pi a^3}{(1-aa)^7}$	$\int\Delta^3\partial\Phi\cos.3\Phi = -\pi a^3$
$n = 4$	$\int\frac{\partial\Phi\cos.4\Phi}{\Delta^5} = \frac{70\pi a^4}{(1-aa)^9}$	$\int\Delta^4\partial\Phi\cos.4\Phi = +\pi a^4$
$n = 5$	$\int\frac{\partial\Phi\cos.5\Phi}{\Delta^6} = \frac{252\pi a^5}{(1-aa)^{11}}$	$\int\Delta^5\partial\Phi\cos.5\Phi = -\pi a^5$
$n = 6$	$\int\frac{\partial\Phi\cos.6\Phi}{\Delta^7} = \frac{924\pi a^6}{(1-aa)^{13}}$	$\int\Delta^6\partial\Phi\cos.6\Phi = +\pi a^6$
	etc.	etc.

Hic imprimis notatu dignum occurrit, quod his casibus $\lambda = n$ integralia tam succincte exprimuntur; nunc autem alios perpendamus casus, quibus litterae λ successive valores 0, 1, 2, 3 etc. tribuantur.

Evolutio casus
quo $\lambda = 0$, et formula integralis proposita

$$\int\frac{\partial\Phi}{\Delta^{n+1}}.$$

§. 31. Cum hic sit $\lambda = 0$, integrale quaesitum ex nostra formula erit $\frac{\pi}{(1-aa)^{2n+1}}$ V, existente

$$V = (\tfrac{n}{0})^2 + (\tfrac{n}{1})^2 aa + (\tfrac{n}{2})^2 a^4 + (\tfrac{n}{3})^2 a^6 + (\tfrac{n}{4})^2 a^8 + \text{etc.}$$

simul vero hinc etiam assignari poterit valor hujus formulae $\int\Delta^n\partial\Phi$, cum sit

$$\int \Delta^n \partial \Phi : \int \frac{\partial \Phi}{\Delta^{n+1}} = (1-aa)^n : (1-aa)^{-n-1} = (1-aa)^{2n+1} : 1,$$

ex qua proportione colligitur

$$\int \Delta^n \partial \Phi = \pi . V.$$

Percurramus igitur simpliciores casus pro exponente n, quos sequenti tabula subjungamus

$$n = 0 \begin{cases} \int \frac{\partial \Phi}{\Delta} = \frac{\pi}{1-aa} \\ \int \partial \Phi = \pi \end{cases}$$

$$n = 1 \begin{cases} \int \frac{\partial \Phi}{\Delta^2} = \frac{\pi}{(1-aa)^3} (1+aa) \\ \int \Delta \partial \Phi = \pi (1+aa) \end{cases}$$

$$n = 2 \begin{cases} \int \frac{\partial \Phi}{\Delta^3} = \frac{\pi}{(1-aa)^5} (1+2^2 aa + a^4) \\ \int \Delta^2 \partial \Phi = \pi (1+2^2 aa + a^4) \end{cases}$$

$$n = 3 \begin{cases} \int \frac{\partial \Phi}{\Delta^4} = \frac{\pi}{(1-aa)^7} (1+3^2 aa + 3^2 a^4 + a^6) \\ \int \Delta^3 \partial \Phi = \pi (1+3^2 aa + 3^2 a^4 + a^6) \end{cases}$$

$$n = 4 \begin{cases} \int \frac{\partial \Phi}{\Delta^5} = \frac{\pi}{(1-aa)^9} (1+4^2 aa + 6^2 a^4 + 4^2 a^6 + a^8) \\ \int \Delta^4 \partial \Phi = \pi (1+4^2 aa + 6^2 a^4 + 4^2 a^6 + a^8) \end{cases}$$

etc. etc.

Evolutio casuum

quibus $\lambda = 1$, et formula integralis proposita

$$\int \frac{\partial \Phi \cos. \Phi}{\Delta^{n+1}}$$

§. 32. Hoc igitur casu integrale quaesitum erit

$$\frac{\pi a}{(1-aa)^{2n+1}} . V$$

existente

$$V = \left(\tfrac{n-1}{0}\right)\left(\tfrac{n+1}{1}\right) + \left(\tfrac{n-1}{1}\right)\left(\tfrac{n+1}{2}\right) a a$$
$$+ \left(\tfrac{n-1}{2}\right)\left(\tfrac{n+1}{3}\right) a^4 + \left(\tfrac{n-1}{3}\right)\left(\tfrac{n+1}{4}\right) a^6$$
$$+ \left(\tfrac{n-1}{4}\right)\left(\tfrac{n+1}{5}\right) a^6 + \left(\tfrac{n-1}{5}\right)\left(\tfrac{n+1}{6}\right) a^8 + \text{etc.}$$

Tum vero cum ob $\lambda = 1$ fit

$$\int \Delta^n \partial \Phi \cos. \Phi : \int \frac{\partial \Phi \cos. \Phi}{\Delta^{n+1}} = n(1-aa)^n : -(n+1)(1-aa)^{-n-1}$$

unde fit

$$\int \Delta^n \partial \Phi \cos. \Phi = -\frac{n}{n+1} . \pi a V.$$

Pro casibus ergo simplicioribus ipsius n sequentem tabulam subjungamus

$$n=0 \begin{cases} \int \frac{\partial \Phi \cos. \Phi}{\Delta} = \frac{\pi a}{1-aa} \\ \int \partial \Phi \cos. \Phi = 0 \end{cases}$$

$$n=1 \begin{cases} \int \frac{\partial \Phi \cos. \Phi}{\Delta^2} = \frac{2 \pi a}{(1-aa)^3} \\ \int \Delta \partial \Phi \cos. \Phi = -\pi a \end{cases}$$

$$n=2 \begin{cases} \int \frac{\partial \Phi \cos. \Phi}{\Delta^3} = \frac{\pi a}{(1-aa)^5} (1.3 + 1.3\, aa) \\ \int \Delta^2 \partial \Phi \cos. \Phi = -\frac{2}{3} \pi a (1.3 + 1.3\, aa) \end{cases}$$

$$n=3 \begin{cases} \int \frac{\partial \Phi \cos. \Phi}{\Delta^4} = \frac{\pi a}{(1-aa)^7} (1.4 + 2.6\, aa + 1.4\, a^4) \\ \int \Delta^3 \partial \Phi \cos. \Phi = -\frac{3}{4} \pi a (1.4 + 2.6\, aa + 1.4\, a^4) \end{cases}$$

$$n=4 \begin{cases} \int \frac{\partial \Phi \cos. \Phi}{\Delta^5} = \frac{\pi a}{(1-aa)^9} (1.5 + 3.10\, aa + 3.10\, a^4 + 1.5 . a^6) \\ \int \Delta^4 \partial \Phi \cos. \Phi = -\frac{4}{5} \pi a (1.5 + 3.10\, aa + 3.10\, a^4 + 1.5 . a^6) \end{cases}$$

$$n=5 \begin{cases} \int \frac{\partial \Phi \cos. \Phi}{\Delta^6} = \frac{\pi a}{(1-aa)^{11}} (1.6 + 4.15\, aa + 6.20\, a^4 + 4.15\, a^6 + 1.6\, a^8) \\ \int \Delta^5 \partial \Phi \cos. \Phi = -\frac{5}{6} \pi (1.6 + \text{etc.}) \end{cases}$$

$$n=6 \begin{cases} \int \frac{\partial \Phi \cos. \Phi}{\Delta^7} = \frac{\pi a}{(1-aa)^{13}} (1.7 + 5.21\, aa + 10.35\, a^4 + 10.35\, a^6 + \text{etc.}) \\ \int \Delta^6 \partial \Phi \cos. \Phi = -\frac{6}{7} \pi a (1.7 + \text{etc.}) \end{cases}$$

Evolutio casuum

quibus $\lambda = 2$, et formula integralis proposita

$$\int \frac{\partial \phi \cos. 2\phi}{\Delta^{n+1}}.$$

§. 33. Hoc ergo casu integrale quaesitum erit

$$\frac{\pi a^2}{(1-aa)^{2n+1}} \cdot V$$

existente

$$V = \left(\frac{n-2}{0}\right)\left(\frac{n+2}{2}\right) + \left(\frac{n-2}{1}\right)\left(\frac{n+2}{3}\right) aa + \left(\frac{n-2}{2}\right)\left(\frac{n+2}{4}\right) a^4$$
$$+ \left(\frac{n-2}{3}\right)\left(\frac{n+2}{5}\right) a^6 + \left(\frac{n-2}{4}\right)\left(\frac{n+2}{6}\right) a^8 + \text{etc.}$$

tum vero erit altera forma

$$\int \Delta^n \partial \phi \cos. 2\phi = \frac{n(n-1)}{(n+1)(n+2)} \pi aa V.$$

Percurramus ergo ut hactenus casus simpliciores, et quia integratio formulae $\int \Delta^n \partial \phi \cos. 2\phi$ sponte patet ex ultima formula, superfluum foret haec integralia allegare

$n = 0 : \int \frac{\partial \phi \cos. 2\phi}{\Delta} = \frac{\pi aa}{1-aa}$

$n = 1 : \int \frac{\partial \phi \cos. 2\phi}{\Delta^2} = \frac{\pi aa}{(1-aa)^3}(1.3 - 1.1.aa)$

$n = 2 : \int \frac{\partial \phi \cos. 2\phi}{\Delta^3} = \frac{\pi aa}{(1-aa)^5}(1.6)$

$n = 3 : \int \frac{\partial \phi \cos. 2\phi}{\Delta^4} = \frac{\pi aa}{(1-aa)^7}(1.10 + 1.10\, aa)$

$n = 4 : \int \frac{\partial \phi \cos. 2\phi}{\Delta^5} = \frac{\pi aa}{(1-aa)^9}(1.15 + 2.20\, aa + 1.15\, a^4)$

$n = 5 : \int \frac{\partial \phi \cos. 2\phi}{\Delta^6} = \frac{\pi aa}{(1-aa)^{11}}(1.21 + 3.35\, a^2 + 3.35\, a^4 + 1.21\, a^6)$

$n = 6 : \int \frac{\partial \phi \cos. 2\phi}{\Delta^7} = \frac{\pi aa}{(1-aa)^{13}}(1.28 + 4.56\, aa + 6.70\, a^4 + 4.56\, a^6 - 1.28\, a^8)$

etc. etc.

Evolutio casuum quibus $\lambda = 3$ et formula integralis proposita

$$\int \frac{\partial \phi \cos. 3\,\phi}{\Delta^{n+1}}.$$

§. 34. Hoc ergo casu integrale erit

$$\frac{\pi a^3}{(1-aa)^{2n+1}} . V,$$

existente

$$V = (\tfrac{n-3}{0})(\tfrac{n+3}{3}) + (\tfrac{n-3}{1})(\tfrac{n+3}{4})\,aa + (\tfrac{n-3}{2})(\tfrac{n+3}{5})\,a^4$$
$$+ (\tfrac{n-3}{3})(\tfrac{n+3}{6})\,a^6 + \text{etc.}$$

pro altera autem formula habebimus

$$\int \Delta^n \partial \phi \cos. 3\,\phi = -\frac{n(n-1)(n-2)}{(n+1)(n+2)(n+3)}\,\pi a^3\,V.$$

Pro praecipuis igitur casibus habebimus sequentem tabellam

$n = 0 : \int \frac{\partial \phi \cos. 3 \phi}{\Delta} = \frac{\pi a^3}{1-aa}$

$n = 1 : \int \frac{\partial \phi \cos. 3 \phi}{\Delta^2} = \frac{\pi a^3}{(1-aa)^3}(1.4 - 2.1aa)$

$n = 2 : \int \frac{\partial \phi \cos. 3 \phi}{\Delta^3} = \frac{\pi a^3}{(1-aa)^5}(1.10 - 1.5aa)$

$n = 3 : \int \frac{\partial \phi \cos. 3 \phi}{\Delta^4} = \frac{\pi a^3}{(1-aa)^7}(1.20)$

$n = 4 : \int \frac{\partial \phi \cos. 3 \phi}{\Delta^5} = \frac{\pi a^3}{(1-aa)^9}(1.35 + 1.35aa)$

$n = 5 : \int \frac{\partial \phi \cos. 3 \phi}{\Delta^6} = \frac{\pi a^3}{(1-aa)^{11}}(1.56 + 2.70aa + 1.56a^4)$

$n = 6 : \int \frac{\partial \phi \cos. 3 \phi}{\Delta^7} = \frac{\pi a^3}{(1-aa)^{13}}(1.84 + 3.126aa + 3.126a^4 + 1.84a^6)$

etc. etc.

Observatio circa valores negativos ipsius λ.

§. 35. Jam initio monuimus, pro littera λ tantum numeros integros positivos sumi oportere, qua conditione generalitas nostrae quaestionis non restringitur cum semper sit cos. $-\lambda\phi =$ cos. $\lambda\phi$. Interim tamen hic ingens paradoxon se offert, quod solutiones supra inventae evadant falsae, quando ipsi λ valores negativi tribuuntur; quod quo clarius pateat consideremus casum $n = 0$; pro quo supra invenimus

$$\int \frac{\partial\phi \cos.\lambda\phi}{\Delta} = \frac{\pi a^\lambda}{1-aa},$$

unde videtur sequi debere, casu $\lambda = -i$ fore

$$\int \frac{\partial\phi \cos. i\phi}{\Delta} = \frac{\pi}{a^i(1-aa)},$$

quod autem manifesto est falsum, cum verum integrale utique sit $\frac{\pi a^i}{1-aa}$, perinde ac si esset $\lambda = + i$. At vero ista restrictio tantum est apparens, atque solutio nostra generalis nihilo minus veritati est consentanea, etiamsi litterae λ valores negativi tribuantur, dummodo fuerint integri; quandoquidem perpetuo assumimus, casu $\phi = \pi$ semper esse sin. $\lambda\phi = 0$; hoc igitur maxime operae erit pretium clarius ostendisse.

§. 36. Sufficiet autem, casum quo $n = 0$ perpendisse, pro quo nostra solutio generalis praebet

$$\int \frac{\partial\phi \cos.\lambda\phi}{\Delta} = \frac{\pi a^\lambda}{1-aa} V,$$

existente

$$V = \left(\frac{-\lambda}{0}\right)\left(\frac{\lambda}{\lambda}\right) + \left(\frac{-\lambda}{1}\right)\left(\frac{\lambda}{\lambda+1}\right) aa + \left(\frac{-\lambda}{2}\right)\left(\frac{\lambda}{\lambda+2}\right) a^4$$
$$+ \left(\frac{-\lambda}{3}\right)\left(\frac{\lambda}{\lambda+3}\right) a^6 + \text{etc.}$$

Cujus expressionis tantum prima pars remanet, quando λ est numerus positivus integer, propterea quod tum formulae $\left(\frac{\lambda}{\lambda+1}\right)$, $\left(\frac{\lambda}{\lambda+2}\right)$, $\left(\frac{\lambda}{\lambda+3}\right)$, etc. evanescunt; longe secus autem se res habet, quando pro λ assumitur numerus negativus, veluti si ponamus $\lambda = - i$ tum erit

$$V = (\tfrac{i}{0})(\tfrac{-i}{-i}) + (\tfrac{i}{1})(\tfrac{-i}{1-i})\,a\,a + (\tfrac{i}{2})(\tfrac{-i}{2-i})\,a^4$$
$$+ (\tfrac{i}{3})(\tfrac{-i}{3-i})\,a^6 + \text{etc.}$$

ubi notetur, omnium horum characterum, quamdiu denominator est negativus, valores evanescere; quoniam vero denominatores continuo crescunt, tandem evadent positivi, atque adeo valores determinatos exhibebunt. Ad hoc ostendendum ponamus primo $\lambda = - 1$ sive $i = + 1$, eritque $V = - a\,a$ ubi primum membrum sine dubio est $= 0$, secundum vero

$$(\tfrac{1}{1})(\tfrac{+1}{0})\,a\,a = a\,a,$$

Cum igitur sit $V = a\,a$ casu $\lambda = - 1$, nostra formula praebet hoc integrale

$$\int \frac{\partial\,\Phi\,\cos. - \Phi}{\Delta} = \frac{\pi\,a^{-1}}{1 - a\,a}\,.\,a\,a = \frac{\pi\,a}{1 - a\,a},$$

id quod prorsus convenit.

§. 37. Sumamus nunc $\lambda = - 2$ sive $i = 2$, manente $n = 0$, eritque

$$V = (\tfrac{2}{0})(\tfrac{-2}{-2}) + (\tfrac{2}{1})(\tfrac{-2}{-1})\,a\,a + (\tfrac{2}{2})(\tfrac{-2}{0})\,a^4,$$

ubi sequentes termini manifesto evanescunt; ob factores priores autem bini termini initiales etiam evanescunt ob denominatores negativos; tertius autem terminus ob $(\tfrac{-2}{0}) = 1$ praebet $V = a^4$, consequenter casu $\lambda = - 2$ habebimus

$$\int \frac{\partial\,\Phi\,\cos. - 2\,\Phi}{\Delta} = \frac{\pi\,a^{-2}}{1 - a\,a}\,.\,a^4 = \frac{\pi\,a\,a}{1 - aa},$$

prorsus atque invenimus pro $\int \frac{\partial\,\Phi\,\cos. 2\,\Phi}{\Delta}$.

§. 38. Simili modo facile intelligitur, casu $\lambda = -3$ prodíturum esse $V = a^6$, eodemque modo casu $\lambda = -4$ reperietur $V = a^8$, atque adeo in genere casu $\lambda = -i$ obtinebitur $V = a^{2i}$, sicque hujus formulae $\int \frac{\partial \phi \cos. - i \phi}{\Delta}$ integrale erit

$$\frac{\pi a^{-i}}{1 - aa} \cdot a^{2i} = \frac{\pi a^i}{1 - aa},$$

quod ipsum est integrale formulae $\int \frac{\partial \phi \cos. i \phi}{\Delta}$, uti natura rei postulat.

§. 39. Talis autem egregius consensus locum habebit pro omnibus valoribus ipsius n. Sit enim verbi gratia $n = 2$, et integratio nostra

$$\int \frac{\partial \phi \cos. \lambda \phi}{\Delta^3} = \frac{\pi a^\lambda}{(1 - aa)^5} \cdot V$$

existente

$$V = (\tfrac{2-\lambda}{0})(\tfrac{2+\lambda}{\lambda}) + (\tfrac{2-\lambda}{1})(\tfrac{2+\lambda}{\lambda+1}) aa + (\tfrac{2-\lambda}{2})(\tfrac{2+\lambda}{\lambda+2}) a^4 + \text{etc.}$$

quare sumto $\lambda = -3$, ut forma nostra sit

$$\int \frac{\partial \phi \cos. - 3 \phi}{\Delta^3} = \frac{\pi a^{-3}}{(1 - aa)^5} \cdot V,$$

existente

$$V = (\tfrac{5}{0})(\tfrac{-1}{-3}) + (\tfrac{5}{1})(\tfrac{-1}{-2}) aa + (\tfrac{5}{2})(\tfrac{-1}{-1}) a^4 + (\tfrac{5}{3})(\tfrac{-1}{0}) a^6$$
$$+ (\tfrac{5}{4})(\tfrac{-1}{1}) a^8 + (\tfrac{5}{5})(\tfrac{-1}{2}) a^{10},$$

ubi tria priora membra evanescunt, sequentia autem ob

$$(\tfrac{-1}{0}) = 1, \quad (\tfrac{-1}{1}) = -1, \quad (\tfrac{-1}{2}) = 1, \text{ erit}$$

$$V = 10 a^6 - 5 a^8 + a^{10} = a^6 (10 - 5 aa + a^4),$$

consequenter nostrum integrale fit

$$\int \frac{\partial \phi \cos. - 3 \phi}{\Delta^3} = \frac{\pi a^3}{(1 - aa)^5} (10 - 5 aa + a^4),$$

prorsus uti supra invenimus pro casu $\int \frac{\partial \phi \cos. 3 \phi}{\Delta^3}$; talis autem consensus perpetuo deprehendi debebit.

---

3) Disquisitio conjecturalis super formula integrali $$\int \frac{\partial \phi \cos. i \phi}{(\alpha + \beta \cos. \phi)^n}.$$

M. S. Academiae exhib. die 31. Augusti 1778.

§. 40. Incipiamus a casu simplicissimo quo $i = 0$ et $n = 1$, et formula integranda proponitur haec $\int \frac{\partial \phi}{\alpha + \beta \cos. \phi}$, ad quod praestandum commodissime in subsidium vocatur haec substitutio tang. $\frac{1}{2} \phi = t$, unde statim fit $\partial \phi = \frac{2 \partial t}{1 + tt}$: porro vero cum hinc sit

$$\sin. \tfrac{1}{2} \phi = \frac{t}{\sqrt{(1+tt)}} \text{ et } \cos. \tfrac{1}{2} \phi = \frac{1}{\sqrt{(1+tt)}},$$

erit $\cos. \phi = \frac{1 - tt}{1 + tt}$, ideoque denominator nostrae formulae

$$\alpha + \beta \cos. \phi = \frac{\alpha + \beta + (\alpha - \beta) tt}{1 + tt},$$

sicque nostra formula integranda erit

$$\int \frac{2 \partial t}{\alpha + \beta + (\alpha - \beta) tt}.$$

§. 41. Constat autem ex elementis esse

$$\int \frac{\partial t}{f + gtt} = \frac{1}{\sqrt{fg}} \text{ Arc. tang. } t \sqrt{\frac{g}{f}}.$$

Quare cum pro nostro casu sit $f = \alpha + \beta$ et $g = \alpha - \beta$, habebimus hanc integrationem

$$\int \frac{\partial \phi}{\alpha + \beta \cos. \phi} = \frac{2}{\sqrt{(\alpha\alpha - \beta\beta)}} \text{ Arc. tang. } t \sqrt{\frac{\alpha - \beta}{\alpha + \beta}},$$

existente $t =$ tang. $\frac{1}{2} \phi$; quod ergo integrale evanescit casu $t = 0$, ideoque casu $\phi = 0$. Quodsi ergo hoc integrale extendere velimus a termino $\phi = 0$ usque ad terminum $\phi = 180^\circ$, ubi fit $t = \infty$, istud integrale erit $\frac{2}{\sqrt{(\alpha\alpha - \beta\beta)}} \cdot \frac{\pi}{2}$, denotante π semiperipheriam circuli, cujus radius $= 1$.

§. 42. Quoniam igitur integrale nostrae formulae a termino $\phi = 0$ usque ad terminum $\phi = 180^\circ$ tam concinne et simpliciter exprimitur, etiam generatim in hac dissertatione in ea tantum integralia formulae generalis propositae

$$\int \frac{\partial \phi \cos. i \phi}{(\alpha + \beta \cos. \phi)^n},$$

sum inquisiturus, quae comprehenduntur inter terminos $\phi = 0$ et $\phi = 180^\circ$. Quia autem in casu tractato formula inest irrationalis $\sqrt{(\alpha\alpha - \beta\beta)}$, ad hoc incommodum tollendum, in sequentibus perpetuo assumemus $\alpha = 1 + aa$ et $\beta = -2a$, unde fit $\sqrt{(\alpha\alpha - \beta\beta)} = 1 - aa$, sicque nostrae disquisitiones versabuntur circa integrationem hujus formulae generalis

$$\int \frac{\partial \phi \cos. i \phi}{(1 + aa - 2a \cos. \phi)^n},$$

pro qua brevitatis gratia ubique statuamus

$$1 + aa - 2a \cos. \phi = \Delta,$$

ut nostra formula generalis jam sit

$$\int \frac{\partial \phi \cos. i \phi}{\Delta^n},$$

ubi ut jam notatum, eum tantum integralis valorem explorare nobis est propositum, qui intra terminos $\phi = 0$ et $\phi = 180^\circ$ contineatur, quem valorem ex casibus particularibus concludere conabimur. Praeterea vero hic in genere notetur, litteram i nobis perpetuo alios numeros non designare praeter integros, et quidem positivos, quandoquidem semper est

$$\cos. - i \phi = \cos. + i \phi.$$

I. De integratione formulae

$$\int \frac{\partial \Phi \cos. i \Phi}{\Delta} \begin{bmatrix} a \Phi = 0 \\ ad \Phi = 180^\circ \end{bmatrix}.$$

§. 43. Hic ergo casus in generali continetur, ponendo exponentem $n = 1$, quem casum ut simplicissimum spectamus, siquidem casus $n = 0$ nulla prorsus laborat difficultate, cum sit

$$\int \partial \Phi \cos. i \Phi = \frac{1}{i} \sin. i \Phi,$$

quod integrale jam evanescit casu $i = 0$, et quoniam i numeros tantum integros significat, sumto $\Phi = 180^\circ$ hoc integrale iterum evanescit, solo casu excepto quo $i = 0$, quippe quo casu integrale fiet $= \Phi$, ideoque sumto $\Phi = 180^\circ$ erit pro terminis integrationis constitutis $\int \partial \Phi = \pi$.

§. 44. Iste postremus casus fundamentum continet, unde integralia formae hic propositae haurire conveniet; cum enim sit

$$\partial \Phi = \frac{(1 + aa) \partial \Phi}{\Delta} - \frac{2 a \partial \Phi \cos. \Phi}{\Delta},$$

erit integrando pro terminis praescriptis

$$\pi = (1 + aa) \int \frac{\partial \Phi}{\Delta} - 2 a \int \frac{\partial \Phi \cos. \Phi}{\Delta};$$

supra autem invenimus esse $\int \frac{\partial \Phi}{\Delta} = \frac{\pi}{1 - aa}$, quo valore substituto adipiscimur integrationem casus $i = 1$, cum enim sit

$$\pi = \frac{(1 + aa) \pi}{1 - aa} - 2 a \int \frac{\partial \Phi \cos. \Phi}{\Delta}, \text{ erit } \int \frac{\partial \Phi \cos. \Phi}{\Delta} = \frac{\pi a}{1 - aa};$$

sicque jam duos casus sumus adepti, qui sunt

$$\int \frac{\partial \Phi}{\Delta} = \frac{\pi}{1 - aa} \text{ et } \int \frac{\partial \Phi \cos. \Phi}{\Delta} = \frac{\pi a}{1 - aa}.$$

§. 45. Ex his autem duobus casibus $i = 0$ et $i = 1$ sequentes omnes haud difficulter derivare licet ope hujus lemmatis; cum sit ut vidimus $\int \partial \Phi \cos. i \Phi = 0$, erit

$$0 = (1 + aa) \int \frac{\partial \Phi \cos. i \Phi}{\Delta} - 2 a \int \frac{\partial \Phi \cos. \Phi \cos. i \Phi}{\Delta}.$$

Constat autem esse

$$2\cos.\phi\cos.i\phi = \cos.(i-1)\phi + \cos.(i+1)\phi,$$

unde habebimus hanc aequationem

$$\frac{1+aa}{a}\int\frac{\partial\phi\cos.i\phi}{\Delta} = \int\frac{\partial\phi\cos.(i-1)\phi}{\Delta} + \int\frac{\partial\phi\cos.(i+1)\phi}{\Delta},$$

unde oritur istud lemma

$$\int\frac{\partial\phi\cos.(i+1)\phi}{\Delta} = \frac{1+aa}{a}\int\frac{\partial\phi\cos.i\phi}{\Delta} - \int\frac{\partial\phi\cos.(i-1)\phi}{\Delta}.$$

Sumto nunc $i=1$, istud lemma nobis suppeditat hunc casum

$$\int\frac{\partial\phi\cos.2\phi}{\Delta} = \frac{1+aa}{a}\int\frac{\partial\phi\cos.\phi}{\Delta} - \int\frac{\partial\phi}{\Delta},$$

qui ergo per binos praecedentes expeditur; fiet enim

$$\int\frac{\partial\phi\cos.2\phi}{\Delta} = \frac{\pi aa}{1-aa}.$$

Sumatur nunc $i=2$, et lemma nobis dabit

$$\int\frac{\partial\phi\cos.3\phi}{\Delta} = \frac{1+aa}{a}\int\frac{\partial\phi\cos.2\phi}{\Delta} - \int\frac{\partial\phi\cos.\phi}{\Delta}, \text{ sive}$$

$$\int\frac{\partial\phi\cos.3\phi}{\Delta} = \frac{\pi a^3}{1-aa}:$$

simili modo sumto $i=3$, lemma dabit

$$\int\frac{\partial\phi\cos.4\phi}{\Delta} = \frac{1+aa}{a}\int\frac{\partial\phi\cos.3\phi}{\Delta} - \int\frac{\partial\phi\cos.2\phi}{\Delta}, \text{ sive}$$

$$\int\frac{\partial\phi\cos.4\phi}{\Delta} = \frac{\pi a^4}{1-aa}.$$

Porro casus $i=4$ praebet

$$\int\frac{\partial\phi\cos.5\phi}{\Delta} = \frac{1+aa}{a}\int\frac{\partial\phi\cos.4\phi}{\Delta} - \int\frac{\partial\phi\cos.3\phi}{\Delta}, \text{ sive}$$

$$\int\frac{\partial\phi\cos.5\phi}{\Delta} = \frac{\pi a^5}{1-aa}, \text{ atque ita porro.}$$

§. 46. Hinc igitur patet, singulos istos casus ex binis praecedentibus determinari ope scalae relationis $\frac{1+aa}{a}$, -1, atque seriem recurrentem hinc oriundam abire in geometricam: quodsi enim bini termini postremi jam inventi fuerint

$$\frac{\pi a^\lambda}{1-aa} \text{ et } \frac{\pi a^{\lambda+1}}{1-aa},$$

sequens reperitur $= \frac{\pi a^{\lambda+2}}{1-aa}$, ex quo ergo sine ullo dubio sequitur, pro casu particulari hoc loco tractati in genere fore.

$$\int \frac{\partial \Phi \cos. i \Phi}{\Delta} = \frac{\pi a^i}{1-aa},$$

ubi autem probe est notandum, loco i non nisi numeros integros positivos assumi debere.

II. De integratione formulae.

$$\int \frac{\partial \Phi \cos. i \Phi}{\Delta^2} \begin{bmatrix} a \Phi = 0 \\ ad \Phi = 180^\circ \end{bmatrix}.$$

§. 47. Casus simplicissimus hic occurret $\int \frac{\partial \Phi}{\Delta^2}$, cujus ergo integrale ante omnia perscrutari oportet; hunc in finem consideremus hanc formulam finitam $\frac{\sin. \Phi}{\Delta} = V$, quae pro utroque termino $\Phi = 0$ et $\Phi = 180^\circ$ evanescit; hinc autem erit

$$\partial V = \frac{\partial \Phi \cos. \Phi}{\Delta} - \frac{2 a \partial \Phi \sin. \Phi^2}{\Delta^2}, \text{ sive}$$

$$\partial V = \frac{(1+aa) \partial \Phi \cos. \Phi - 2 a \partial \Phi}{\Delta^2};$$

unde integrando jam novimus esse

$$0 = (1+aa) \int \frac{\partial \Phi \cos. \Phi}{\Delta^2} - 2 a \int \frac{\partial \Phi}{\Delta^2}.$$

Porro vero quoniam ante habuimus $\int \frac{\partial \Phi}{\Delta} = \frac{\pi}{1-aa}$, hanc formulam integralem supra et infra per Δ multiplicando, erit quoque

$$\frac{\pi}{1-aa} = (1+aa) \int \frac{\partial \Phi}{\Delta^2} - 2 a \int \frac{\partial \Phi \cos \Phi}{\Delta^2}.$$

Ex praecedente autem colligitur

$$\int \frac{\partial \Phi \cos. \Phi}{\Delta^2} = \frac{2a}{1+aa} \cdot \int \frac{\partial \Phi}{\Delta^2},$$

quo valore substituto habebimus

$$\frac{\pi}{1-aa} = (1+aa) \int \frac{\partial \Phi}{\Delta^2} - \frac{4aa}{1+aa} \int \frac{\partial \Phi}{\Delta^2} = \frac{(1-aa)^2}{1+aa} \int \frac{\partial \Phi}{\Delta^2},$$

quamobrem hinc adipiscimur hanc integrationem principalem

$$\int \frac{\partial \Phi}{\Delta^2} = \frac{\pi (1+aa)}{(1-aa)^3},$$

ex quo immediate deducitur casus sequens

$$\int \frac{\partial \Phi \cos. \Phi}{\Delta^2} = \frac{2\pi a}{(1-aa)^3}.$$

§. 48. Pro sequentibus casibus consideremus integrationem in articulo praecedente inventam

$$\int \frac{\partial \Phi \cos. i \Phi}{\Delta} = \frac{\pi a^i}{1-aa},$$

quae formula integralis supra et infra per Δ multiplicando discerpitur in sequentes duas partes

$$\frac{\pi a^i}{1-aa} = (1+aa)\int \frac{\partial \Phi \cos. i \Phi}{\Delta^2} - 2a\int \frac{\partial \Phi \cos. \Phi \cos. i \Phi}{\Delta^2},$$

quae aequatio porro evolvitur in hanc formam

$$\frac{\pi a^i}{1-aa} = (1+aa)\int \frac{\partial \Phi \cos. i \Phi}{\Delta^2} - a\int \frac{\partial \Phi \cos. (i-1)\Phi}{\Delta^2}$$
$$- a\int \frac{\partial \Phi \cos. (i+1)\Phi}{\Delta^2};$$

unde deducitur hoc quasi lemma

$$\int \frac{\partial \Phi \cos. (i+1)\Phi}{\Delta^2} = \frac{1+aa}{a}\int \frac{\partial \Phi \cos. i \Phi}{\Delta^2}$$
$$- \int \frac{\partial \Phi \cos. (i-1)\Phi}{\Delta^2} - \frac{\pi a^{i-1}}{1-aa}.$$

§. 49. Sumamus nunc statim $i = 1$, atque istud lemma nobis praebet

$$\int \frac{\partial \Phi \cos. 2\Phi}{\Delta^2} = \frac{1+aa}{a}\int \frac{\partial \Phi \cos. \Phi}{\Delta^2} - \int \frac{\partial \Phi}{\Delta^2} - \frac{\pi}{1-aa};$$

hic jam bini valores jam inventi substituantur, atque reperietur

$$\int \frac{\partial \Phi \cos. 2\Phi}{\Delta^2} = \frac{\pi(1+aa) - \pi(1-aa)^2}{(1-aa)^3},$$

hinc ergo sequitur fore

$$\int \frac{\partial \Phi \cos. 2\Phi}{\Delta^2} = \frac{\pi(3aa - a^4)}{(1-aa)^3} = \frac{\pi aa(3-aa)}{(1-aa)^3}.$$

Sumatur nunc pro lemmate praemisso $i = 2$, eritque

$$\int \frac{\partial \Phi \cos. 3\Phi}{\Delta^2} = \frac{1+aa}{a} \int \frac{\partial \Phi \cos. 2\Phi}{\Delta^2} - \int \frac{\partial \Phi \cos. \Phi}{\Delta^2} - \frac{\pi a}{1-aa}, \text{ sive}$$

$$\int \frac{\partial \Phi \cos. 3\Phi}{\Delta^2} = \frac{(1+aa)\pi a(3-aa) - 2\pi a - \pi a(1-aa)^2}{(1-aa)^3},$$

quae expressio contrahitur in hanc

$$\int \frac{\partial \Phi \cos. 3\Phi}{\Delta^2} = \frac{\pi a^3(4-2aa)}{(1-aa)^3}.$$

Sit nunc in lemmate praemisso $i = 3$, eritque

$$\int \frac{\partial \Phi \cos. 4\Phi}{\Delta^2} = \frac{1+aa}{a} \int \frac{\partial \Phi \cos. 3\Phi}{\Delta^2} - \int \frac{\partial \Phi \cos. 2\Phi}{\Delta^2} - \frac{\pi aa}{1-aa}, \text{ sive}$$

$$\int \frac{\partial \Phi \cos. 4\Phi}{\Delta^2} = \frac{(1+aa)\pi aa(4-2aa) - \pi aa(3-aa) - \pi aa(1-aa)^2}{(1-aa)^3},$$

quae expressio contrahitur in hanc

$$\int \frac{\partial \Phi \cos. 4\Phi}{\Delta^2} = \frac{\pi a^4(5-3aa)}{(1-aa)^3}.$$

Sit nunc in lemmate nostro $i = 4$, eritque

$$\int \frac{\partial \Phi \cos. 5\Phi}{\Delta^2} = \frac{1+aa}{a} \int \frac{\partial \Phi \cos. 4\Phi}{\Delta^2} - \int \frac{\partial \Phi \cos. 3\Phi}{\Delta^2} = \frac{\pi a^3}{1-aa}, \text{ sive}$$

$$\int \frac{\partial \Phi \cos. 5\Phi}{\Delta^2} = \frac{(1+aa)\pi a^3(5-3aa) - \pi a^3(4-2aa) - \pi a^3(1-aa)^2}{(1-aa)^3},$$

quae expressio contrahitur in hanc

$$\int \frac{\partial \Phi \cos. 5\Phi}{\Delta^2} = \frac{\pi a^5(6-4aa)}{(1-aa)^3}.$$

Sit nunc in lemmate nostro $i = 5$, eritque

$$\int \frac{\partial \Phi \cos. 3\Phi}{\Delta^2} = \frac{1+aa}{a} \int \frac{\partial \Phi \cos. 5\Phi}{\Delta^2} - \int \frac{\partial \Phi \cos. 4\Phi}{\Delta^2} - \frac{\pi a^4}{1-aa}, \text{ sive}$$

$$\int \frac{\partial \Phi \cos. 6\Phi}{\Delta^2} = \frac{(1+aa)\pi a^4(6-4aa) - \pi a^4(5-3aa) - \pi a^4(1-aa)^2}{(1-aa)^3},$$

quae expressio contrahitur in hanc

$$\int \frac{\partial \Phi \cos. 6\Phi}{\Delta^2} = \frac{\pi a^6(7-5aa)}{(1-aa)^3}.$$

§. 50. Qui has formulas earumque generationem attentius perpendet, nullo certe modo dubitabit, inde hanc conclusionem deducere, quin in genere pro casu hic proposito futurum sit

$$\int \frac{\partial \Phi \cos. i \Phi}{\Delta^2} = \frac{\pi a^i [i + 1 - (i - 1) a a]}{(1 - a a)^3}$$

cujus lex cum non sit tam manifesta, quam in casu praecedente, omnes formulas inventas junctim ante oculos ponamus

$$\int \frac{\partial \Phi}{\Delta^2} = \frac{\pi (1 + a a)}{(1 - a a)^3}$$

$$\int \frac{\partial \Phi \cos. \Phi}{\Delta^2} = \frac{\pi a (2 - 0 a a)}{(1 - a a)^3}$$

$$\int \frac{\partial \Phi \cos. 2 \Phi}{\Delta^2} = \frac{\pi a a (3 - a a)}{(1 - a a)^3}$$

$$\int \frac{\partial \Phi \cos. 3 \Phi}{\Delta^2} = \frac{\pi a^3 (4 - 2 a a)}{(1 - a a)^3}$$

$$\int \frac{\partial \Phi \cos. 4 \Phi}{\Delta^2} = \frac{\pi a^4 (5 - 3 a a)}{(1 - a a)^3}$$

$$\int \frac{\partial \Phi \cos. 5 \Phi}{\Delta^2} = \frac{\pi a^5 (6 - 4 a a)}{(1 - a a)^3}$$

$$\int \frac{\partial \Phi \cos. 6 \Phi}{\Delta^2} = \frac{\pi a^6 (7 - 5 a a)}{(1 - a a)^3}$$

III. De integratione formulae.

$$\int \frac{\partial \Phi \cos. i \Phi}{\Delta^3} \begin{bmatrix} a \Phi = 0 \\ ad \Phi = 180 \end{bmatrix}.$$

§. 51. Pro casu simplicissimo $\int \frac{\partial \Phi}{\Delta^3}$ eruendo, utamur hac formula

$$V = \frac{\sin. \Phi}{\Delta^2}, \text{ eritque } \partial V = \frac{\partial \Phi \cos. \Phi}{\Delta^2} - \frac{2 \partial \Phi \sin. \Phi^2}{\Delta^3}, \text{ sive}$$

$$\partial V = \frac{(1 + a a) \partial \Phi \cos. \Phi - 2 a \partial \Phi \cos. \Phi^2 - 4 a \partial \Phi \sin. \Phi^2}{\Delta^3}.$$

Hic loco sin. Φ^2 scribatur $1 - \cos. \Phi^2$, atque integrando, ob $V = 0$ habebimus hanc aequationem

$$0 = (1 + a a) \int \frac{\partial \Phi \cos. \Phi}{\Delta^3} - 4 a \int \frac{\partial \Phi}{\Delta^3} + 2 a \int \frac{\partial \Phi \cos. \Phi^2}{\Delta^3}.$$

§. 52. Huc addamus hanc formam indefinitam

$$s = A\int\frac{\partial\phi}{\Delta} + B\int\frac{\partial\phi}{\Delta^2}$$

cujus differentiale ad denominationem Δ^3 perducatur, litterae vero A et B ita definiantur, ut membra $\partial\phi\cos.\phi$ et $\partial\phi\cos.\phi^2$ evanescant, eritque formulis differentialibus additis

$$\begin{aligned}\frac{\Delta^3(\partial V+\partial s)}{\partial\phi} = &-4a + (1+aa)\cos.\phi + 2a\cos.\phi^2\\ &+A(1+aa)^2 - 4Aa(1+aa)\cos.\phi + 4Aaa\cos.\phi^2\\ &+B(1+aa) - 2Ba\cos.\phi.\end{aligned}$$

Nunc igitur ut termini $\cos.\phi^2$ abigantur, statuatur

$$2a + 4Aaa = 0,\ \text{ideoque}\ A = \frac{-1}{2a}.$$

Nunc etiam termini $\cos.\phi$ e medio tollantur, eritque

$$1 + aa - 4Aa(1+aa) - 2Ba = 0,\ \text{unde fit}$$

$$B = \frac{3(1+aa)}{2a}.$$

Ex quibus valoribus nanciscimur

$$\frac{\Delta^3(\partial V+\partial s)}{\partial\phi} = \frac{(1-aa)^2}{a};$$

hinc ergo vicissim integrando habebimus

$$V + s = \frac{(1-aa)^2}{a}\int\frac{\partial\phi}{\Delta^3}.$$

§. 53. Cum igitur, ut jam notavimus, sit $V = 0$, atque ex casibus jam tractatis

$$s = \frac{-1}{2a}\cdot\frac{\pi}{1-aa} + \frac{3(1+aa)}{2a}\cdot\frac{\pi(1+aa)}{(1-aa)^3},$$

habebimus hanc aequationem

$$\frac{(1-aa)^2}{a}\int\frac{\partial\phi}{\Delta^3} = \frac{3\pi(1+aa)^2 - \pi(1-aa)^2}{2a(1-aa)^3},$$

unde colligitur

$$\int\frac{\partial\phi}{\Delta^3} = \frac{\pi(1+4aa+a^4)}{(1-aa)^5}.$$

§. 54. Cum sit $\int \frac{\partial \phi}{\Delta^2} = \frac{\pi(1+aa)}{(1-aa)^3}$, erit per reductionem hactenus usitatam

$$\frac{\pi(1+aa)}{(1-aa)^3} = (1+aa)\int \frac{\partial \phi}{\Delta^3} - 2a\int \frac{\partial \phi \cos.\phi}{\Delta^3},$$

unde concludimus

$$\int \frac{\partial \phi \cos.\phi}{\Delta^3} = \frac{1+aa}{2a}\int \frac{\partial \phi}{\Delta^3} - \frac{\pi(1+aa)}{2a(1-aa)^3}, \text{ ideoque}$$

$$\int \frac{\partial \phi \cos.\phi}{\Delta^3} = \frac{1+aa}{2a} \cdot \frac{\pi(1+4aa+a^4)}{(1-aa)^5} - \frac{\pi(1+aa)}{2a(1-aa)^3}$$

$$= \frac{3\pi a(1+aa)}{(1-aa)^5} = \frac{\pi a(3+3aa)}{(1-aa)^5}.$$

§. 55. Cum igitur in articulo praecedente invenimus

$$\int \frac{\partial \phi \cos.i\phi}{\Delta^2} = \frac{\pi a^i[i+1-(i-1)aa]}{(1-aa)^3},$$

hanc formulam integralem supra et infra per Δ multiplicando habebimus

$$\frac{\pi a^i[i+1-(i-1)aa]}{(1-aa)^3} = (1+aa)\int \frac{\partial \phi \cos.i\phi}{\Delta^3}$$
$$-2a\int \frac{\partial \phi \cos.i\phi \cos.\phi}{\Delta^3}, \text{ sive}$$

$$\frac{\pi a^i[i+1-(i-1)aa]}{(1-aa)^3} = (1+aa)\int \frac{\partial \phi \cos.i\phi}{\Delta^3}$$
$$-a\int \frac{\partial \phi \cos.(i-1)\phi}{\Delta^3} - a\int \frac{\partial \phi \cos.(i+1)\phi}{\Delta^3};$$

unde deducitur hoc quasi lemma

$$\int \frac{\partial \phi \cos.(i+1)\phi}{\Delta^3} = \frac{1+aa}{a}\int \frac{\partial \phi \cos.i\phi}{\Delta^3}$$
$$-\int \frac{\partial \phi \cos.(i-1)\phi}{\Delta^3} - \frac{\pi a^{i-1}[i+1-(i-1)aa]}{(1-aa)^3}$$

§. 56. Sumamus nunc statim $i = 1$, atque istud lemma nobis praebet

$$\int \frac{\partial \Phi \cos. 2\Phi}{\Delta^3} = \frac{1+aa}{2a} \int \frac{\partial \Phi \cos. \Phi}{\Delta^3} - \int \frac{\partial \Phi}{\Delta^3} - \frac{2\pi}{2(1-aa)^3};$$

hic jam bini valores jam inventi substituantur, reperieturque

$$\int \frac{\partial \Phi \cos. 2\Phi}{\Delta^3} = \frac{1+aa}{a} \cdot \frac{\pi a(3+3aa)}{(1-aa)^5} - \frac{\pi(1+4aa+a^4)}{(1-aa)^5}$$
$$- \frac{\pi(1-aa)^2}{(1-aa)^5} = \frac{\pi aa(6)}{(1-aa)^5};$$

sumto $i = 2$, erit

$$\int \frac{\partial \Phi \cos. 3\Phi}{\Delta^3} = \frac{\pi a^3(10-5aa+a^4)}{(1-aa)^5};$$

sumto $i = 3$, nanciscimur

$$\int \frac{\partial \Phi \cos. 4\Phi}{\Delta^3} = \frac{\pi a^4(15-12aa+3a^4)}{(1-aa)^5};$$

sumto $i = 4$, prodit

$$\int \frac{\partial \Phi \cos. 5\Phi}{\Delta^3} = \frac{\pi a^5(21-21aa+6a^4)}{(1-aa)^5};$$

posito $i = 5$, erit

$$\int \frac{\partial \Phi \cos. 6\Phi}{\Delta^3} = \frac{\pi a^6(28-32aa+10a^4)}{(1-aa)^5};$$

et in genere

$$\int \frac{\partial \Phi \cos. i\Phi}{\Delta^3} = \pi a^i \left[\frac{i(i+3)+2}{2} - 2(ii-4)aa + \left[\frac{i(i-3)+2}{2}\right]a^4\right],$$

quae forma facile transformatur in hanc

$$\int \frac{\partial \Phi \cos. i\Phi}{\Delta^3} = \frac{\pi a^i}{(1-aa)^5} \left[\frac{(i+1)(i+2)}{2} - (i+2)(i-2)aa + \frac{(i-1)(i-2)}{2}a^4\right].$$

§. 57. Hoc modo procedere liceret ad sequentes formulas, in quibus denominator est Δ^4, Δ^5, Δ^6, etc. verum integralium formae ita continuo magis fierent complicatae, ut vix ullus ordo in iis observari posset, quamobrem aliam viam inire conveniet, qua numerum i pro dato assumimus, et continuo a minoribus ad majores numeros n procedemus. Primo igitur sumamus $i = 0$, et investigemus valorem integralem formulae $\int \frac{\partial \Phi}{\varDelta^{n+1}}$.

Integratio formulae.

$$\int \frac{\partial \Phi}{\varDelta^{n+1}} \left[\begin{matrix} a\Phi = 0 \\ ad\Phi = 180 \end{matrix}\right]$$

existente $\varDelta = 1 + aa - 2a\cos.\Phi$.

§. 58. Ex praecedentibus colligere licet, quemlibet casum exponentis $n + 1$ a duobus praecedentibus pendere, ita ut sit sub terminis integrationis praescriptis

$$\int \frac{\partial \Phi}{\varDelta^{n+1}} = \alpha \int \frac{\partial \Phi}{\varDelta^{n}} + \beta \int \frac{\partial \Phi}{\varDelta^{n-1}};$$

ubi totum negotium eo redit, ut coefficientes α et β rite determinentur: hunc in finem statuamus in genere esse

$$\int \frac{\partial \Phi}{\varDelta^{n+1}} = \alpha \int \frac{\partial \Phi}{\varDelta^{n}} + \beta \int \frac{\partial \Phi}{\varDelta^{n-1}} + \gamma \frac{\sin.\Phi}{\varDelta^{n}};$$

quippe qui postremus terminus pro utroque integrationis termino evanescit.

§. 59. Differentietur nunc ista aequatio, et facta divisione per $\partial \Phi$, orietur sequens aequatio

$$\frac{1}{\varDelta^{n+1}} = \frac{\alpha}{\varDelta^{n}} + \frac{\beta}{\varDelta^{n-1}} + \frac{\gamma\cos.\Phi(1+aa-2a\cos.\Phi) - 2\gamma an\sin.\Phi^2}{\varDelta^{n+1}},$$

haecque aequatio multiplicata per $\varDelta^{n+1}$ abibit in hanc formam

$$1 = \alpha(1+aa-2a\cos.\Phi) + \beta(1+aa)^2 - 2\beta a\cos.\Phi(1+aa)$$
$$+ 4\beta aa\cos.\Phi^2 + \gamma\cos.\Phi(1+aa-2a\cos.\Phi) - 2\gamma an\sin.\Phi^2.$$

Cum nunc sit

$$2\cos.\Phi^2 = 1 + \cos.2\Phi \text{ et } 2\sin.\Phi^2 = 1 - \cos.2\Phi,$$

hac reductione adhibita pervenietur ad sequentem aequationem

$$\begin{aligned}1 = \alpha(1+aa) &- 2\,\alpha\,a\cos.\Phi &+ 2\,\beta\,a\,a\cos.2\Phi\\ +\beta(1+aa)^2 &- 4\,\beta\,a(1+aa)\cos.\Phi &- \gamma\,a\cos.2\Phi\\ +2\,\beta\,a\,a &+ \gamma(1+aa)\cos.\Phi &+ \gamma\,n\,a\cos.2\Phi\\ -\gamma\,a\\ -\gamma\,n\,a.\end{aligned}$$

§. 60. Ut nunc hanc aequationem resolvamus, necesse est, ut tam termini involventes cos. Φ, quam cos. $2\,\Phi$, seorsim ad nihilum redigantur; unde ex postremo termino deducimus

$$2\,\beta\,a\,a - \gamma\,a + \gamma\,n\,a = 0;$$

ideoque

$$\beta = \frac{\gamma(1-n)}{2\,a} = -\frac{\gamma(n-1)}{2\,a},$$

qui valor in terminis cos. Φ affectis substitutus perducit ad hanc aequationem

$$-2\,\alpha\,a + 2\,\gamma(n-1)(1+a\,a) + \gamma(1+a\,a) = 0,$$

unde fit

$$2\,\alpha\,a = 2\,\gamma\,n(1+a\,a) - \gamma(1+a\,a);$$

ideoque erit

$$\alpha = \frac{\gamma(1+a\,a)(2\,n-1)}{2\,a}.$$

Jam hic valores loco α et β inventi substituantur in prima parte, atque deducemur ad hanc aequationem

$$1 = \frac{\gamma\,n(1+a\,a)^2}{a} - \frac{\gamma(n-1)(1+a\,a)^2}{2\,a} - \gamma\,a(n-1) - \gamma a - \gamma n a, \text{ sive}$$

$$2a = 2\gamma n(1+aa)^2 - \gamma(n-1)(1+aa)^2 - 2\gamma aa(n-1) - 2\gamma aa - 2\gamma naa,$$

$$\text{vel } 2\,a = \gamma(n+1)(1+a\,a)^2 - 4\,\gamma\,n\,a\,a,$$

unde fit

$$\gamma = \frac{2\,a}{n(1-a\,a)^2}.$$

§. 61. Invento jam isto valore γ, hinc eliciemus

$$\alpha = \frac{(2n-1)(1+aa)}{n(1-aa)^2} \text{ et } \beta = \frac{-(n-1)}{n(1-aa)^2},$$

hincque per $n(1-aa)^2$ multiplicando, adipiscimur

$$n(1-aa)^2 \int \frac{a\,\Phi}{\Delta^{n+1}} = (2n-1)(1+aa)\int \frac{\partial\,\Phi}{\Delta^n} - (n-1)\int \frac{\partial\,\Phi}{\Delta^{n-1}},$$

cujus beneficio ex cognitis jam duobus casibus assignari poterit casus sequens.

§. 62. Jam ante autem invenimus esse $\int \frac{\partial\,\Phi}{\Delta} = \frac{\pi}{1-aa}$. Pro sequentibus vero ponamus

$$\int \frac{\partial\,\Phi}{\Delta^2} = \frac{\pi\,A}{(1-aa)^3};\ \int \frac{\partial\,\Phi}{\Delta^3} = \frac{\pi\,B}{(1-aa)^5};\ \int \frac{\partial\,\Phi}{\Delta^4} = \frac{\pi\,C}{(1-aa)^7};$$

$$\int \frac{\partial\,\Phi}{\Delta^5} = \frac{\pi\,D}{(1-aa)^9};\ \int \frac{\partial\,\Phi}{\Delta^6} = \frac{\pi\,E}{(1-aa)^{11}};\ \text{etc.}$$

Ubi jam ante invenimus $A = 1 + aa$ et $B = 1 + 4aa + a^4$, unde sequentes valores omnes C, D, E, etc. ope reductionis inventae definiri poterunt.

§. 63. Introducamus ergo istos valores, atque sequentes nanciscemur aequationes

I. $A = 1 + aa$,

II. $2B = 3(1+aa)A - (1-aa)^2$,

III. $3C = 5(1+aa)B - 2(1-aa)^2 A$,

IV. $4D = 7(1+aa)C - 3(1-aa)^2 B$,

V. $5E = 9(1+aa)D - 4(1-aa)^2 C$,

VI. $6F = 11(1+aa)E - 5(1-aa)^2 D$,

VII. $7G = 13(1+aa)F - 6(1-aa)^2 E$,

VIII. $8H = 15(1+aa)G - 7(1-aa)^2 F$,

etc.

§. 64 Harum aequationum prima statim dat valorem ante inventum $A = 1 + aa$; secunda vero praebet

$$2B = \left\{ \begin{array}{l} 3 + 6aa + 3a^4 \\ -1 + 2aa + a^4 \end{array} \right.$$

unde fit

$$B = 1 + 4aa + a^4.$$

Deinde vero tertia aequatio praebet

$$3C = \left\{ \begin{array}{l} 5 + 25aa + 25a^4 + 5a^6 \\ -2 + 2aa + 2a^4 - 2a^6 \end{array} \right.$$

unde elicitur

$$C = 1 + 9aa + 9a^4 + a^6.$$

Porro quarta aequatio

$$4D = \left\{ \begin{array}{l} 7 + 70aa + 126a^4 + 70a^6 + 7a^8 \\ -3 - 6aa + 18a^4 - 6a^6 - 3a^8 \end{array} \right.$$

unde colligitur

$$D = 1 + 16aa + 36a^4 + 16a^6 + a^8.$$

Simili modo ex aequatione quinta colligimus

$$5E = \left\{ \begin{array}{l} 9 + 153aa + 468a^4 + 468a^6 + 153a^8 + 9a^{10} \\ -4 - 28aa + 32a^4 + 32a^6 - 28a^8 - 4a^{10} \end{array} \right.$$

unde colligitur

$$E = 1 + 25aa + 100a^4 + 100a^6 + 25a^4 + a^{10}.$$

Evolvamus etiam sextam aequationem quae praebet

$$6F = \left\{ \begin{array}{l} 11 + 286aa + 1375a^4 + 2200a^6 + 1375a^8 + 286a^{10} + 11a^{12} \\ -5 - 70aa - 25a^4 + 200a^6 - 25a^8 - 70a^{10} - 5a^{12}, \end{array} \right.$$

hincque concluditur

$$F = 1 + 36aa + 225a^4 + 400a^6 + 225a^8 + 36a^{10} + a^{12}$$

§. 65. Hic non sine admiratione deprehendimus, omnes coefficientes harum formarum esse numeros quadratos, quorum radices occurrunt in potestatibus respondentibus binomii $1 + aa$, sicque pro littera sequente habebimus

$$G = 1 + 7^2 aa + 21^2 a^4 + 35^2 a^6 + 35^2 a^8 + 21^2 a^{10} + 7^2 a^{12} + a^{14},$$

quae littera respondet formulae integrali $\int \frac{\partial \Phi}{\Delta^{7+1}}$, ita ut hic sit $n = 7$. Quodsi ergo formae generalis $\int \frac{\partial \Phi}{\Delta^{n+1}}$ integrale statuamus $= \frac{\pi V}{(1 - aa)^{in+1}}$, erit valor litterae

$$V = 1 + (\tfrac{n}{1})^2 aa + (\tfrac{n}{2})^2 a^4 + (\tfrac{n}{3})^2 a^6 + (\tfrac{n}{4})^2 a^8 + (\tfrac{n}{5})^2 a^{10} + \text{etc.}$$

adhibitis scilicet characteribus, quibus coefficientes potestatum binomii designare consuevimus, dum scilicet est

$$(\tfrac{n}{1}) = n;\ (\tfrac{n}{2}) = \tfrac{n}{1} \cdot \tfrac{n-1}{2};\ (\tfrac{n}{3}) = \tfrac{n}{1} \cdot \tfrac{n-1}{2} \cdot \tfrac{n-2}{3} \text{ etc.}$$

§. 66. Haec quidem conclusio tantum per inductionem quasi conjectura est deducta; vix enim quisquam reperietur, cui ista conjectura suspecta videatur, quamquam rigorosa demonstratione nondum sit corroborata; casu enim fortuito neutiquam evenire certe potest, ut omnes istos coefficientes prodierint numeri quadrati, atque adeo ipsorum coefficientium qui in evolutione potestatis $(1 + aa)^n$ occurrunt, interim tamen deinceps vidi pro hac veritate solidam demonstrationem adornari posse.

§. 67. Hac igitur lege stabilita, valores litterarum A, B, C, D etc., quas in expressiones integralium induximus, sequenti modo se habebunt

$$A = 1^2 + 1^2 aa,$$
$$B = 1^2 + 2^2 aa + 1^2 a^4,$$
$$C = 1^2 + 3^2 aa + 3^2 a^4 + 1^2 a^6,$$

$$D = 1^2 + 4^2 aa + 6^2 a^4 + 4^2 a^6 + 1^2 a^8,$$
$$E = 1^2 + 5^2 aa + 10^2 a^4 + 10^2 a^6 + 5^2 a^8 + 1^2 a^8,$$
$$F = 1^2 + 6^2 aa + 15^2 a^4 + 20^2 a^6 + 15^2 a^8 + 6^2 a^{10} + 1^2 a^{12},$$
$$G = 1^2 + 7^2 aa + 21^2 a^4 + 35^2 a^6 + 35^2 a^8 + 21^2 a^{10} + 7^2 a^{12} + 1^2 a 14,$$

etc. etc.

Integratio formulae generalis

$$\int \frac{\partial \Phi \cos. i \Phi}{\Delta^{n+1}} \quad \left[\begin{array}{l} \text{a } \Phi = 0 \\ \text{ad } \Phi = 180 \end{array}\right]$$

existente $\Delta = 1 + aa - 2a \cos. \Phi$.

§. 68. Haec formula generalis perinde tractari potest ac praecedens, dum valor integralis cujusque casus etiam a duobus casibus praecedentibus pendet, ita ut ponere queamus

$$\int \frac{\partial \Phi \cos. i \Phi}{\Delta^{n+1}} = \alpha \int \frac{\partial \Phi \cos. i \Phi}{\Delta^n} + \beta \int \frac{\partial \Phi \cos. i \Phi}{\Delta^{n-1}},$$

quatenus scilicet integralia ad binos terminos integrationis stabilitos referuntur; quia autem necesse est, ut aequationem generalem ob ista conditione liberam constituamus, aliquot membra adjungi oportet, quae pro utroque termino evanescant, neque enim hic sufficit, ut ante unicum terminum adjunxisse, verum adeo ternos hujusmodi terminos adjungi debebunt, cujus ratio mox ex ipso calculo elucebit; hanc ob rem constituamus sequentem aequationem

$$\int \frac{\partial \Phi \cos. i \Phi}{\Delta^{n+1}} = \alpha \int \frac{\partial \Phi \cos. i \Phi}{\Delta^n} + \beta \int \frac{\partial \Phi \cos. i \Phi}{\Delta^{n-1}}$$
$$+ \gamma \frac{\sin i \Phi}{\Delta^n} + \delta \frac{\sin. (i-1) \Phi}{\Delta^n} + \varepsilon \frac{\sin. (i+1) \Phi}{\Delta^n},$$

quae postrema membra, quoniam i est numerus integer, pro utroque termino integrationis evanescunt.

§. 69. Differentietur igitur nunc ista aequatio, ac posito brevitatis gratia $1 + aa = b$, ut sit $\Delta = b - 2a \cos. \Phi$, negligantur denominatores, qui erunt Δ^{n+1} una cum elemento $\partial \Phi$. Primo notetur esse

$$\Delta \cos. i\Phi = b \cos. i\Phi - a \cos. (i-1)\Phi - a \cos. (i+1)\Phi,$$

tum vero ob

$$\Delta^2 = bb - 4ab\cos.\Phi + 4aa\cos.\Phi^2 = 2aa + bb - 4ab\cos.\Phi + 2aa\cos. 2\Phi, \text{ erit}$$

$$\Delta^2 \cos. i\Phi = (bb + 2aa)\cos. i\Phi - 2ab\cos.(i-1)\Phi - 2ab\cos.(i+1)\Phi + aa\cos.(i-2)\Phi + aa\cos.(i+2)\Phi.$$

Deinde vero habebitur

$$\partial . \frac{\sin. i\Phi}{\Delta^n} = i\Delta\cos. i\Phi - 2na\sin. i\Phi\sin.\Phi = ib\cos. i\Phi + ia\cos.(i-1)\Phi - ia\cos.(i+1)\Phi - na\cos.(i-1)\Phi + na\cos.(i+1)\Phi.$$

Simili modo erit

$$\partial . \frac{\sin. (i-1)\Phi}{\Delta^n} = (i-1)b\cos.(i-1)\Phi - (i-1)a\cos.(i-2)\Phi - (i-1)a\cos. i\Phi - na\cos.(i-2)\Phi + na\cos. i\Phi),$$

ac denique

$$\partial . \frac{\sin. (i+1)\Phi}{\Delta^n} = (i+1)b\cos.(i+1)\Phi - (i+1)a\cos. i\Phi - (i+1)a\cos. (i+2)\Phi - na\cos. i\Phi + na\cos.(i+2)\Phi.$$

§. 70. Hic igitur occurrunt quinque anguli scilicet

$$i\Phi,\ (i-1)\Phi,\ (i+1)\Phi,\ (i-2)\Phi \text{ et } (i+2)\Phi,$$

unde patet ratio, cur terni termini absoluti sint supra adjuncti; diffeerentiale ergo facta evolutione singulorum terminorum, per quinque columnas sequenti modo repraesentetur, ita ut membrum sinistrum, quod est cos. $i\,\Phi$, aequetur sequenti expressioni

$\cos. i\,\Phi$	$\cos. (i-1)\,\Phi$	$\cos. (i+1)\,\Phi$	$\cos. (i-2)\,\Phi$	$\cos. (i+2)\,\Phi$
$+\alpha b$	$-\alpha a$	$-\alpha a$		
$+\beta(bb+2aa)$	$-2\beta ab$	$-2\beta ab$	$+\beta aa$	$+\beta aa$
$+\gamma ib$	$-\gamma ia$	$-\gamma ia$		
	$-\gamma na$	$+\gamma na$		
$-\delta(i-1)a$	$+\delta(i-1)b$	$+\varepsilon(i+1)b$	$-\delta(i-1)a$	$-\varepsilon(i+1)a$
$+\delta na$			$-\delta na$	$+\varepsilon na$
$-\varepsilon(i+1)a$				
$-\varepsilon na$				

§. 71. Hic igitur omnes quatuor posteriores columnae ad nihilum redigi debent, propterea quod sola prima columna membro sinistro aequari potest; incipiamus igitur a binis columnis ultimis, unde deducimus

$$\delta = \frac{\beta a}{i+n-1} \text{ et } \varepsilon = \frac{\beta a}{i-n+1}.$$

His valoribus introductis, pro secunda columna erit

$$-2\beta ab + \delta(i-1)b = \frac{\beta ab(1-i-2n)}{i+n-1} = -\frac{\beta ab(i+2n-1)}{i+n-1}.$$

Pro tertia vero columna erit

$$-2\beta ab + \varepsilon(i+1)b = -\frac{\beta ab(i-2n+1)}{i-n+1};$$

unde haec binae columnae nobis praebent has duas aequationes

$$-\alpha a - \gamma(i+n)a - \frac{\beta ab(i+2n-1)}{i+n-1} = 0,$$

$$-\alpha a - \gamma(i-n)a - \frac{\beta ab(i-2n+1)}{i-n+1} = 0$$

§. 72. Harum duarum aequationum subtrahatur posterior a priore, ac prodibit

$$-2\gamma na - \frac{2\beta inab}{ii-(n-1)^2} = 0,$$

unde colligimus

$$\gamma = -\frac{\beta i b}{ii-(n-1)^2}.$$

Atque hinc porro ex secunda deduci potest valor ipsius α, cum sit

$$\alpha a = -\gamma(i+n)\, a - \frac{\beta a b (i+2n-1)}{i+n-1},$$

erit

$$\alpha = \frac{\beta i (i+n) b}{ii-(n-1)^2} - \frac{\beta(i+2n-1)b}{i+n-1} = \frac{\beta(2nn-3n+1)b}{ii-(n-1)^2}$$
$$= \frac{\beta(n-1)(2n-1)b}{ii-(n-1)^2}.$$

§. 73. Hi jam valores substituantur in prima columna, atque orietur sequens aequatio

$$\left.\begin{array}{ll} \frac{\beta(n-1)(2n-1)bb}{ii-(n-1)^2} & + 2\beta aa \\ + \beta bb & - \frac{\beta(i-n-1)aa}{i+n-1} \\ - \frac{\beta iibb}{ii-(n-1)^2} & - \frac{\beta(i+n+1)aa}{i-n+1} \end{array}\right\} = 1.$$

Multiplicando igitur per $ii-(n-1)^2$, prodibit haec aequatio

$$ii-(n-1)^2 = 2\beta aa[ii-(n-1)^2] + \beta bb(n-1)(2n-1)$$
$$-\beta aa(i-n-1)(i-n+1) + \beta bb[ii-(n-1)^2]$$
$$-\beta aa(i+n+1)(i+n-1) - \beta iibb.$$

Facta autem reductione, terminus βaa multiplicabitur per

$$2[ii-(n-1)^2] - (i-n)^2 + 1 - (i+n)^2 + 1,$$

sive per $-4n(n-1)$; at vero βbb multiplicabitur per

$$(n-1)(2n-1) + ii - (n-1)^2 - ii,$$

sive per $n(n-1)$, sicque erit

$$ii(n-1)^2 = -4\beta n(n-1)aa + \beta n(n-1)bb$$
$$= \beta n(n-1)(bb-4aa).$$

Cum igitur posuerimus $b = 1 + aa$, erit

$$bb - 4aa = (1 - aa)^2,$$

consequenter hinc elicimus

$$\beta = \frac{ii - (n-1)^2}{n(n-1)(1-aa)^2}:$$

§. 74. Invento jam valore litterae β, ex eo deducimus valorem $\alpha = \frac{(2n-1)b}{n(1-aa)^2}$: valores autem litterarum γ, δ, et ε non amplius in censum veniunt, et reductio quam quaerimus erit

$$\int \frac{\partial \Phi \cos. i\Phi}{\Delta^{n+1}} = \alpha \int \frac{\partial \Phi \cos. \Phi}{\Delta^n} + \beta \int \frac{\partial \Phi \cos. i\Phi}{\Delta^{n-1}},$$

sive sublatis fractionibus habebitur ista aequatio

$$n(n-1)(1-aa)^2 \int \frac{\partial \Phi \cos. i\Phi}{\Delta^{n+1}} = (n-1)(2n-1)(1+aa) \int \frac{\partial \Phi \cos. i\Phi}{\Delta^n}$$
$$+ [ii - (n-1)^2] \int \frac{\partial \Phi \cos. i\Phi}{\Delta^{n-1}},$$

quae aequatio casu $i = 0$ redit ad reductionem praecedentis sectionis.

§. 75. Inventa hac reductione generali, pro ejus applicatione cum sit

$$\int \frac{\partial \Phi \cos. i\Phi}{\Delta} = \frac{\pi a^i}{1-aa}, \text{ ubi } n = 0,$$

ponamus pro sequentibus

$$\int \frac{\partial \Phi \cos. i\Phi}{\Delta^2} = \frac{\pi a^i}{(1-aa)^3} \text{A, ubi } n = 1$$
$$\int \frac{\partial \Phi \cos. i\Phi}{\Delta^3} = \frac{\pi a^i}{(1-aa)^5} \text{B, ubi } n = 2$$
$$\int \frac{\partial \Phi \cos. i\Phi}{\Delta^4} = \frac{\pi a^i}{(1-aa)^7} \text{C, ubi } n = 3$$

$$\int \frac{\partial \phi \cos. i \phi}{\Delta^5} = \frac{\pi a^i}{(1-aa)^9} \text{ D, ubi } n = 4$$

$$\int \frac{\partial \phi \cos. i \phi}{\Delta^6} = \frac{\pi a^i}{(1-aa)^{11}} \text{ E, ubi } n = 5,$$

atque adeo in genere sit

$$\int \frac{\partial \phi \cos. i \phi}{\Delta^{n+1}} = \frac{\pi a^i}{(1-aa)^{2n+1}} \text{ V}:$$

supra autem jam i invenimus esse

$$A = i + 1 - (i - 1)\, a\, a,$$

sive terminos positive repraesentando

$$A = 1 + i + (1 - i)\, a\, a.$$

§. 76. Quodsi in reductione nostra inventa poneremus $n = 1$, ea daret $i\, i \int \partial \phi \cos. i \phi = 0$, quod primo verum est casu $i = 0$, tum vero ob $\int \partial \phi \cos. i \phi = \frac{1}{i} \sin. i \phi = 0$, quod quidem per se patet. Incipiamus igitur a casu $n = 2$, et procedendo per sequentes valores $n = 3$, $n = 4$, $n = 5$, etc. nanciscemur sequentes aequationes

I. Si $n = 2$, erit
$$2 \cdot 1\, B = 1 \cdot 3\, (1 + aa)\, A + (ii - 1)\, (1 - aa)^2.$$

II. Si $n = 3$, erit
$$3 \cdot 2\, C = 2 \cdot 5\, (1 + aa)\, B + (ii - 4)\, (1 - aa)^2\, A.$$

III. Si $n = 4$, erit
$$4 \cdot 3\, D = 3 \cdot 7\, (1 + aa)\, C + (ii - 9)\, (1 - aa)^2\, B.$$

IV. Si $n = 5$, erit
$$5 \cdot 4\, E = 4 \cdot 9\, (1 + aa)\, D + (ii - 16)\, (1 - aa)^2\, C.$$

V. Si $n = 6$, erit
$$6 \cdot 5\, F = 5 \cdot 11\, (1 + aa)\, E + (ii - 25)\, (1 - aa)^2\, D.$$

etc. etc.

§. 77. Cum igitur sit

$$A = 1 + i + (1 - i)\,aa,$$

pro prima aequatione erit

$$(1 + aa)\,A = 1 + i + 2\,aa + (1 - i)\,a^4,$$

hujus triplo addi oportet

$$(ii - 1)(1 - aa)^2 = ii - 1 - 2\,(ii - 1)\,aa + (ii - 1)\,a^4,$$

unde oritur primo terminus absolutus $= (2 + i)\,(1 + i)$, deinde coefficiens ipsius aa erit $8 - 2\,ii$, coefficiens vero ipsius a^4 erit $(2 - i)\,(1 - i)$, unde concludimus litteram

$$B = \frac{(2+i)\,(1+i)}{1.\quad 2} + (2 + i)\,(2 - i)\,aa + \frac{(2-i)\,(1-i)}{1.\quad 2}\,a^4.$$

§. 78. Ista forma nos manuducit ad coefficientes potestatum binomii, quos ut jam moninus per characteres peculiares repraesentamus, sicque per tales characteres erit

$$A = \left(\frac{1+i}{1}\right) + \left(\frac{1-i}{1}\right) aa, \text{ tum vero}$$

$$B = \left(\frac{2+i}{2}\right) + \left(\frac{2+i}{1}\right)\left(\frac{2-i}{1}\right) aa + \left(\frac{2-i}{2}\right) a^4$$

Videamus autem, quomodo haec lex in sequentibus valoribus se sit habitura.

§. 79. Evolvamus igitur aequationem secundam, pro qua sequentes duas multiplicationes institui oportet

$$10\left[\frac{2+3i+ii}{2} + (4 - ii)\,aa + \frac{2-3i+ii}{2}\,a^4\right] \text{ per } 1 + aa,$$

ultimum autem membrum postulat hanc multiplicationem

$$(ii - 4)\,(1 - 2\,aa + a^4) \text{ per } 1 + i + (1 - i)\,aa;$$

unde primo oritur iste terminus absolutus

$$10 + 15\,i + 5\,ii + (ii - 4)\,(1 + i),$$

quae reducitur ad hanc formam $(2 + i)\,(1 + i)\,(3 + i)$. Pro termino autem aa erit

$$40 - 10ii + 5(2+i)(1+i) + (ii-4)[-2(1+i) + 1 - i]$$
$$= (4-ii)(11+3i) + 5(2+i)(1+i),$$

quae expressio reducitur ad

$$(2+i)(27-3ii) = 3(2+i)(3+i)(3-i).$$

Porro coefficiens ipsius a^4 erit

$$(2-i)(27-3ii) = 3(2-i)(3+i)(3-i).$$

Denique coefficiens ipsius a^6 erit $(2-i)(1-i)(3-i)$.

§. 80. Calculo ergo hoc peracto habebimus

$$3.2\,C = (3+i)(2+i)(1+i) + 3(3+i)(2+i)(3-i)\,aa$$
$$+ 3(3+i)(2-i)(3-i)\,a^4 + (3-i)(2-i)(1-i)\,a^6,$$

quae forma commode redigitur ad istam per characteres coefficientium binomii

$$C = (\tfrac{3+i}{3}) + (\tfrac{3+i}{2})(\tfrac{3-i}{1})\,aa + (\tfrac{3+i}{1})(\tfrac{3-i}{2})\,a^4 + (\tfrac{3-i}{3})\,a^6.$$

Hic ordo maxime confirmat conjecturam ex casibus praecedentibus deductam, neque dubium ullum esse potest, quin sequentes litterae istos sortiantur valores

$$D = (\tfrac{4+i}{4}) + (\tfrac{4+i}{3})(\tfrac{4-i}{1})\,aa + (\tfrac{4+i}{2})(\tfrac{4-i}{2})\,a^4$$
$$+ (\tfrac{4+i}{1})(\tfrac{4-i}{3})\,a^6 + (\tfrac{4-i}{4})\,a^8.$$
$$E = (\tfrac{5+i}{5}) + (\tfrac{5+i}{4})(\tfrac{5-i}{1})\,aa + (\tfrac{5+i}{3})(\tfrac{5-i}{2})\,a^4$$
$$+ (\tfrac{5+i}{2})(\tfrac{5-i}{3})\,a^6 + (\tfrac{5+i}{1})(\tfrac{5-i}{4})\,a^8 + (\tfrac{5-i}{5})\,a^{10}.$$

etc. etc.

Interim tamen fatendum est, hunc ordinem egregium tantum per conjecturam se nobis obtulisse; cujus ergo demonstratio rigorosa adhuc desideratur.

§. 81. Cum igitur supra ingenere posuerimus

$$\int \frac{\partial \Phi \cos. i \Phi}{\Delta^{n+1}} \left[\begin{array}{l} \text{a } \Phi = 0 \\ \text{ad } \Phi = 180 \end{array}\right] = \frac{\pi a^i}{(1 - a a)^{2n+1}} \text{V},$$

erit nunc

$$\text{V} = \left(\tfrac{n+i}{n}\right) + \left(\tfrac{n+i}{n-1}\right)\left(\tfrac{n-i}{1}\right) a a + \left(\tfrac{n+i}{n-2}\right)\left(\tfrac{n-i}{2}\right) a^4$$
$$+ \left(\tfrac{n+i}{n-3}\right)\left(\tfrac{n-i}{3}\right) a^6 + \left(\tfrac{n+i}{n-4}\right)\left(\tfrac{n-i}{4}\right) a^8 + \text{etc.}$$

unde sponte deducitur forma in articulo praecedenti conclusa, ubi erat $i = 0$. Pro hoc enim casu erit

$$\text{V} = \left(\tfrac{n}{n}\right) + \left(\tfrac{n}{n-1}\right)\left(\tfrac{n}{1}\right) a a + \left(\tfrac{n}{n-2}\right)\left(\tfrac{n}{2}\right) a^4 + \left(\tfrac{n}{n-3}\right)\left(\tfrac{n}{3}\right) a^6 + \text{etc.}$$

Cum autem in hujusmodi characteribus perpetuo sit $\left(\frac{n}{p}\right) = \left(\frac{n}{n-p}\right)$, erit prorsus uti supra conjectavimus

$$\text{V} = \left(\tfrac{n}{0}\right) + \left(\tfrac{n}{1}\right)^2 a a + \left(\tfrac{n}{2}\right)^2 a^4 + \left(\tfrac{n}{3}\right)^2 a^6 + \left(\tfrac{n}{4}\right)^2 a^8 + \text{etc.}$$

Hinc igitur operae pretium erit sequens theorema constituere.

Theorema generale.

§. 82. Si formula integralis

$$\int \frac{\partial \Phi \cos. i \Phi}{(1 + a a - 2 a \cos. \Phi)^{n+1}},$$

a termino $\Phi = 0$ usque ad terminum $\Phi = 180°$ extendatur, valor integralis semper habebit talem formam

$$\frac{\pi a^i}{(1 - a a)^{2n+1}} \text{V}, \text{ existente}$$

$$\text{V} = \left(\tfrac{n+i}{i}\right) + \left(\tfrac{n+i}{i+1}\right)\left(\tfrac{n-i}{1}\right) a a + \left(\tfrac{n+i}{i+2}\right)\left(\tfrac{n-i}{2}\right) a^4$$
$$+ \left(\tfrac{n+i}{i+3}\right)\left(\tfrac{n-i}{3}\right) a^6 + \left(\tfrac{n+i}{i+4}\right)\left(\tfrac{n-i}{4}\right) a^8 + \text{etc.}$$

dummodo fuerit i numerus integer, atque adeo tam positivus quam negativus; quandoquidem etiam posteriori casu ista forma veritati consentanea deprehenditur, ita ut ista expressio latius pateat, quam omnes casus speciales junctim sumti, unde eam per conjecturam conclusimus; namque in omnibus casibus specialibus littera i necessario denotabat numeros integros tantum positivos.

---

4) Demonstratio Theorematis insignis per conjecturam eruti, circa integrationem formulae

$$\int \frac{\partial \phi \cos. i \phi}{(1 + aa - 2 a \cos. \phi)^{n+1}}$$

M. S. Academiae exhib. die 10 *Septembris* 1778.

§. 83. Cum nuper hanc formulam integralem tractassem, ac potissimum in ejus valorem inquisivissem, quem accipit, si integrale a termino $\phi = 0$ ad terminum $\phi = 180^\circ$ usque extendatur; ex pluribus casibus, quos evolvere licuit, conclusi ejus integrale in genere ita expressum iri

$$\frac{\pi a^i}{(1 - aa)^{2n+1}} \text{V},$$

ubi V denotat summam hujus seriei

$$\text{V} = \left(\tfrac{n-i}{0}\right)\left(\tfrac{n+i}{i}\right) + \left(\tfrac{n-i}{1}\right)\left(\tfrac{n+i}{i+1}\right) a^2 + \left(\tfrac{n-i}{2}\right)\left(\tfrac{n+i}{i+2}\right) a^4 + \text{etc.}$$

Hic scilicet isti characteres clausulis inclusi designant coefficientes potestatis binomialis, dum statuimus

$$(1 + x)^m = 1 + \left(\tfrac{m}{1}\right) x + \left(\tfrac{m}{2}\right) x^2 + \left(\tfrac{m}{3}\right) x^3 + \left(\tfrac{m}{4}\right) x^4 + \text{etc.}$$

§. 84. Circa hanc autem formulam integralem ante omnia tenendum est, litteram i perpetuo significare numeros integros, quandoquidem in analysi constanter assumitur, casu $\phi = 180^\circ$ semper esse $\sin. i\Phi = 0$; tum vero etiam ejus valores perpetuo ut positivi spectari possunt, propterea quod $\cos. (-i\Phi) = \cos. (+i\Phi)$. Interim tamen mox ostendemus nostram formam integralem etiam veritati esse consentaneam, quamvis litterae i valores negativi tribuantur. Ad hoc ostendendum circa characteres in subsidium vocatos sequentia sunt observanda.

1°). Si p et q designent numeros integros, ac primo quidem positivos, quoniam in evolutione potestatis binomialis omnes termini primum antecedentes sunt nulli, quoties fuerit q numerus negativus, semper erit $(\frac{p}{q}) = 0$.

2°). Quia coefficiens tam primi termini quam ultimi semper est unitas, erit tam $(\frac{p}{0}) = 1$ quam $(\frac{p}{p}) = 1$.

3°). Quia termini ultimum sequentes pariter sunt nulli, quoties fuerit $q > p$, valor characteris $(\frac{p}{q})$ semper pro nihilo haberi poterit.

4°). Quia in evolutione potestatis binomialis coefficientes ordinem tenent retrogradum, hinc sequitur semper fore $(\frac{p}{q}) = (\frac{p}{p-q})$. Sin autem superior numerus p fuerit negativus, ob rationem praecedentem semper etiam erit $(\frac{-p}{-q}) = 0$.

5°). At si q denotet numeros positivos, character $(\frac{-p}{q})$, perpetuo dabit valores alternatim positivos et negativos; cum sit

$$(\tfrac{-p}{0}) = 1;\ (\tfrac{-p}{1}) = -p;\ (\tfrac{-p}{2}) = +\tfrac{p(p+1)}{1.2};\ (\tfrac{-p}{3}) = -\tfrac{p(p+1)(p+2)}{1.2.3} \text{ etc.}$$

Atque hinc

6°). In genere tales characteres, ubi superior numerus est negativus, ad positivos reduci poterunt, cum sit $(\frac{-p}{q}) = \pm(\frac{p+q-1}{q})$, ubi signum $+$ valet si q fuerit numerus par, inferius $-$ vero, si impar.

§. 85. His proprietatibus circa characteres hic adhibitos notatis, in forma nostra integrali loco i scribamus $-i$, eritque

$$\int \frac{\partial \phi \cos. -i\phi}{(1+aa-2a\cos.\phi)^{n+1}} = \frac{\pi a^{-i}}{(1-aa)^{2n+1}} V,$$

existente

$$V = \left(\frac{n+i}{0}\right)\left(\frac{n-i}{-i}\right) + \left(\frac{n+i}{1}\right)\left(\frac{n-i}{-i+1}\right)a^2 + \left(\frac{n+i}{2}\right)\left(\frac{n-i}{-i+2}\right)a^4$$
$$+ \left(\frac{n+i}{3}\right)\left(\frac{n-i}{-i+3}\right)a^6 + \text{etc.}$$

ubi posteriores factores evanescunt, quamdiu denominatores sunt negativi: primum igitur membrum significatum habens erit $\left(\frac{n+i}{i}\right)$ $\left(\frac{n-i}{-i+i}\right)a^{2i}$, cujus valor erit $\left(\frac{n+i}{i}\right)a^{2i}$; sequentia autem membra erunt

$$\left(\frac{n+i}{i+1}\right)\left(\frac{n-i}{-i+i+1}\right)a^{2i+2} = \left(\frac{n+i}{i+1}\right)\left(\frac{n-i}{1}\right)a^{2i+2},$$

tum vero $\left(\frac{n+i}{i+2}\right)\left(\frac{n-i}{2}\right)a^{2i+4}$, etc. Hoc igitur modo erit

$$V = a^{2i}\left[\left(\frac{n+i}{i}\right)\left(\frac{n-i}{0}\right) + \left(\frac{n+i}{i+1}\right)\left(\frac{n-i}{1}\right)a^2 + \left(\frac{n+i}{i+2}\right)\left(\frac{n-i}{2}\right)a^4 + \text{etc.}\right]$$

qui valor ductus in $\frac{\pi a^{-i}}{(1-aa)^{2n+1}}$ praebet hanc formam

$$\frac{\pi a^i}{(1-aa)^{2n+1}}\left[\left(\frac{n+i}{i}\right)\left(\frac{n-i}{0}\right) + \left(\frac{n+i}{i+1}\right)\left(\frac{n-i}{1}\right)a^2 + \left(\frac{n+i}{i+2}\right)\left(\frac{n-i}{2}\right)a^4 + \text{etc.}\right]$$

quae prorsus congruit cum nostra formula valori positivo ipsius i respondente, qui egregius consensus haud contemnendum firmamentum pro veritate nostrae formae integralis continet.

§. 86. Praeterea vero circa formam nostram integralem imprimi notari debet, seriem pro V supra datam semper alicubi abrumpi quoties n fuerit numerus integer positivus, quippe quod eveniet, quando vel in priore factore, cujus forma est $\left(\frac{n-i}{\lambda}\right)$, pervenitur ad terminum quo $\lambda > n - i$, vel in posteriore factore, cujus forma est $\left(\frac{n+i}{i+\lambda}\right)$, evadet $\lambda > n$; quae proprietas eo magis est observanda, quod, si series V in infinitum porrigeretur, parum lucrati essemus censendi, id quod praecipue de iis casibus est notandum, quibus n foret numerus fractus, quos ergo casus penitus ab instituto nostro removemus, ita ut pro n tantum numeros integros simus assumturi.

§. 87. Consideremus ergo etiam casus, quibus n est numerus negativus, ac primo quidem jam per se clarum est, quamdiu is minor fuerit quam i, ideoque $n + i$ etiamnum numerus positivus, tum seriem pro V datam adeo citius abruptum iri; tum igitur demum in infinitum excurret, quando etiam $n + i$ fuerit numerus positivus. His autem casibus forma integralis supra data ita transformari potest, ut abruptio pariter locum inveniat.

§. 88. Ad hoc ostendendum statuamus $n = -m - 1$, ut formula nostra integralis evadat

$$\int \partial \phi \cos. i \phi \, (1 + aa - 2a \cos. \phi)^m,$$

ejusque igitur valor $= \pi a^i \, (1 - aa)^{2m+1}$ V, existente jam

$$V = \left(\frac{-m-1-i}{0}\right)\left(\frac{-m-1+i}{i}\right) + \left(\frac{-m-1-i}{i}\right)\left(\frac{-m-1+i}{i+1}\right) a^2$$
$$+ \left(\frac{-m-1-i}{2}\right)\left(\frac{-m-1+i}{i+2}\right) a^4 + \left(\frac{-m-1-i}{3}\right)\left(\frac{-m-1+i}{i+3}\right) a^6 + \text{etc.}$$

quae series manifesto in infinitum excurrit, quam autem ope sequentes lemmatis transformare poterimus.

Lemma.

§. 89. Ista series per characteres hic introductos procedens

$$\text{ђ} = \left(\frac{f}{0}\right)\left(\frac{h}{e}\right) + \left(\frac{f}{1}\right)\left(\frac{h}{e+1}\right) x + \left(\frac{f}{2}\right)\left(\frac{h}{e+2}\right) x^2 + \left(\frac{f}{3}\right)\left(\frac{h}{e+3}\right) x^3 + \text{etc.}$$

in hanc sui similem transmutari potest

$$♂ = \left(\frac{-h-1}{0}\right)\left(\frac{-f-1}{e}\right) + \left(\frac{-h-1}{1}\right)\left(\frac{-f-1}{e+1}\right)x + \left(\frac{-h-1}{2}\right)\left(\frac{-f-1}{e+2}\right)x^2 + \text{etc.}$$

quandoquidem inter earum valores ♄ et ♂ ista relatio semper locum habere, non ita pridem a me est demonstrata

$$\left(\frac{e+f}{1}\right) ♄ = \left(\frac{e-h-1}{e}\right)(1-x)^{f+h+1} ♂,$$

cujus demonstratio profundissimae est indaginis, dum adeo per aequationes differentiales secundi gradus procedit.

§. 90. Applicemus jam istud lemma ad casum nostrum propositum, atque ut series ♄ cum nostro V consentiens reddatur, ut fiat ♄ = V, sumi debet $f = -m-1-i$, $h = -m-1+i$, $e = i$ et $x = aa$, unde altera series ♂ hanc accipiet formam

$$♂ = \left(\frac{m-i}{0}\right)\left(\frac{m+i}{i}\right) + \left(\frac{m-i}{1}\right)\left(\frac{m+i}{i+1}\right)aa + \left(\frac{m-i}{2}\right)\left(\frac{m+i}{i+2}\right)a^4 + \text{etc.}$$

quae series jam certe abrumpitur alicubi, propterea quod hic m denotat numerum integrum positivum: at vero relatio inter superiorem V = ♄ et novam hanc seriem ♂ ita se habebit

$$\left(\frac{-m-1}{i}\right) V = \left(\frac{m}{i}\right)(1-aa)^{-2m-1} ♂.$$

§. 91. Hinc igitur formulae nostrae integralis hujus

$$\int \partial\phi \cos. i\phi (1 + aa - 2a\cos.\phi)^m = \frac{\left(\frac{m}{i}\right)\pi a^i ♂}{\left(\frac{-m-1}{i}\right)},$$

ubi ♂ denotat seriem modo ante §. 89. expositam, qui valor cum factorem habeat $\left(\frac{m}{i}\right)$ semper evanescet, quamdiu fuerit $i > m$, ita ut his casibus valor integralis semper nihilo sit aequalis. Ceterum hic notasse juvabit, facta evolutione esse

$$\left(\frac{m}{i}\right) : \left(\frac{-m-1}{i}\right) = \pm \frac{m(m-1) \;.\;.\;.\;.\;.\; (m-i+1)}{(m+1)(m+2) \;.\;.\;.\;.\; (m+i)},$$

ubi signum superius + valet si i fuerit numerus par, inferius —

vero si impar. His circa indolem nostri theorematis notatis, ipsam ejus demonstrationem aggrediamur, quam quo clarior evadat in varias partes distribuamus.

Demonstrationis pars prima.

§. 92. Quoniam valorem nostrum integralem ad duas formulas accommodavimus, eas distinctionis gratia signis ⊙ et ℭ designemus, sitque

$$\odot = \int \frac{\partial \phi \cos. i \phi}{(1 + aa - 2a \cos. \phi)^{n+1}} \begin{bmatrix} \text{a } \phi = 0 \\ \text{ad } \phi = 180^\circ \end{bmatrix},$$

$$\mathfrak{C} = \int \partial \phi \cos. i \phi (1 + aa - 2a \cos. \phi)^m \begin{bmatrix} a\, \phi = 0 \\ ad\, \phi = 180^\circ \end{bmatrix},$$

quarum posterior ℭ in priorem ⊙ convertitur si loco m scribamus $-n-1$; modo autem vidimus, has duas formulas a se invicem pendere, unde a posteriori tanquam simpliciori, siquidem denominatore $(1-aa)^{2n+1}$ caret, incipiamus, quam quo simpliciorem reddamus statuamus $\frac{a}{1+aa} = b$; sic enim habebimus

$$\mathfrak{C} = (1 + aa)^m \int \partial \phi \cos. i \phi (1 - 2b \cos. \phi)^m;$$

cujus ergo integrale nobis erit investigandum.

§. 93. Ante omnia igitur conveniet potestatem $(1 - 2b \cos. \phi)^m$ evolvi, unde fiet

$$(1 - 2b \cos. \phi)^m = 1 - \left(\tfrac{m}{1}\right) 2b \cos. \phi + \left(\tfrac{m}{2}\right) 4b^2 \cos. \phi^2$$
$$- \left(\tfrac{m}{3}\right) 8b^3 \cos. \phi^3 + \text{etc.}$$

cujus ergo terminus quicunque erit $\pm \left(\frac{m}{\lambda}\right) 2^\lambda b^\lambda \cos. \phi^\lambda$; ubi signum $+$ valet si λ fuerit numerus par, alterum vero $-$ si impar. Jam quia hic potestates ipsius cos. ϕ occurrunt, eas per praecepta satis cognita in cosinus simplices converti oportet, quibus fit

$$2^2\cos.\Phi^2 = 2\cos.2\Phi + 1\left(\tfrac{2}{1}\right),$$
$$2^3\cos.\Phi^3 = 2\cos.3\Phi + 2\left(\tfrac{3}{1}\right)\cos.\Phi,$$
$$2^4\cos.\Phi^4 = 2\cos.4\Phi + 2\left(\tfrac{4}{1}\right)\cos.2\Phi + 1\left(\tfrac{4}{2}\right).$$
$$2^5\cos.\Phi^5 = 2\cos.5\Phi + 2\left(\tfrac{5}{1}\right)\cos.3\Phi + 2\left(\tfrac{5}{2}\right)\cos.\Phi,$$
$$2^6\cos.\Phi^6 = 2\cos.6\Phi + 2\left(\tfrac{6}{1}\right)\cos.4\Phi + 2\left(\tfrac{6}{2}\right)\cos.2\Phi + 1\left(\tfrac{6}{3}\right),$$
etc. etc.

Ubi notandum, in potestatibus paribus postremum membrum cos. $0\,\Phi = 1$ dimidio tantum coefficiente esse affectum. Hinc igitur in genere erit

$$2^\lambda\cos.\Phi^\lambda = 2\cos.\lambda\Phi + 2\left(\tfrac{\lambda}{1}\right)\cos.(\lambda+2)\Phi + 2\left(\tfrac{\lambda}{2}\right)\cos.(\lambda-4)\Phi + 2\left(\tfrac{\lambda}{3}\right)\cos.(\lambda-6)\Phi + \text{etc.}$$

ubi notetur, quoties fuerit λ numerus par, puta $\lambda = 2\,i$, ultimum membrum fore tantum $1 \cdot \left(\frac{2\,i}{i}\right)$ cos. $0\,\Phi$.

§. 94 Postquam igitur omnes cosinuum potestates ad cosinus simplices fuerint reductae, integrationes nostrae semper ad talem formam redigentur $\int \partial\,\Phi \cos.\,i\,\Phi \cos.\,\lambda\,\Phi$, de qua forma hic imprimis est notandum, ejus integrale a $\Phi = 0$ ad $\Phi = 280^\circ$ extensum semper esse nullum, solo casu $\lambda = i$ excepto, Cum enim sit

$$\cos.\,i\,\Phi \cos.\,\lambda\,\Phi = \tfrac{1}{2}\cos.\,(i+\lambda)\,\Phi + \tfrac{1}{2}\cos.\,(i-\lambda)\,\Phi,$$

erit illud integrale indefinitum

$$= \frac{\sin.\,(i+\lambda)\,\Phi}{2\,(i+\lambda)} + \frac{\sin.\,(i-\lambda)\,\Phi}{2\,(i-\lambda)},$$

quod pro termino $\Phi = 0$ manifesto evanescit; pro altero vero termino $\Phi = 180^\circ = \pi$, ob i et λ numeros integros, manifestum est, hoc integrale denuo evanescere, solo casu excepto quo $\lambda = i$. Si enim $i - \lambda$ ut infinite parvum spectetur, puta $= \omega$, pars posterior hujus integralis erit $\frac{\sin.\,\omega\,\Phi}{2\,\omega} = \frac{\pi}{2}$, id quod etiam inde patet, quod sit

$$\cos. i\,\phi^2 = \tfrac{1}{2} + \tfrac{1}{2}\cos. 2\,i\,\phi,$$

ideoque

$$\int \partial\,\phi \cos. i\,\phi^2 = \tfrac{1}{2}\phi + \tfrac{1}{4}\sin. 2\,i\,\phi = \tfrac{1}{2}\pi.$$

§. 95. Ad integrale igitur quaesitum obtinendum, ex potestate $(1 - 2\,b\cos.\,\phi)^m$ evoluta, eos tantum terminos, qui $\cos.\,i\,\phi$ continent, excerpsisse sufficiet, cum reliqui omnes nihil plane producant, qui si junctim sumti praebeant $N \cos.\,i\,\phi$, totum nostrum integrale pro ☾ erit

$$☾ = (1 + a\,a)^m \cdot \tfrac{1}{2} N\,\pi;$$

quocirca nobis incumbet, in omnes superioris formae partes inquirere, quae formula $\cos.\,i\,\phi$ erunt affectae; unde evidens est, quamdiu in illo termino generali $\pm \left(\frac{m}{\lambda}\right) 2^\lambda b^\lambda \cos.\,\phi^\lambda$ exponens λ minor fuerit quam i, inde nihil plane in integrale inferri.

§. 96. Primus igitur terminus, qui hic in computum venit, erit $\pm \left(\frac{m}{i}\right) 2^i\,b^i \cos.\,\phi^i$, pro quo signum superius $+$ valebit si i fuerit numerus par, inferius $-$ vero si impar. Hinc autem par superiorem reductionem proveniet

$$2^i \cos.\,\phi^i = 2\cos.\,i\,\phi,$$

ita ut hinc pro N oriatur pars prima $\pm \left(\frac{m}{i}\right) 2\,b^i$. Tum vero ex termino immediate sequente, qui erit

$$\mp \left(\frac{m}{i+1}\right) 2^{i+1}\,b^{i+1} \cos.\,\phi^{i+1},$$

nullus angulus $i\,\phi$ oritur, cum sit

$$2^{i+1}\cos.\,\phi^{i+1} = 2\cos.(i+1)\phi + 2\left(\frac{i+1}{1}\right)\cos.\,(i-1)\,\phi + \text{etc.}$$

At vero terminus sequens

$$\pm \left(\frac{m}{i+2}\right) 2^{i+2}\,b^{i+2}\ \cos.\,\phi^{i+2},\ \text{ob}$$

$$2^{i+2}\cos.\,\phi^{i+2} = 2\cos.\,(i+2)\,\phi + 2\left(\frac{i+2}{1}\right)\cos.\,i\,\phi + \text{etc.}$$

partem hinc in litteram N resultantem dat

$$2\left(\frac{i+2}{1}\right)\left(\frac{m}{i+2}\right) b^{i+2}.$$

Simili modo ex casu $\lambda = i + 3$ nihil nascitur. At ex sequente

$$\pm\left(\frac{m}{i+4}\right) 2^{i+4} b^{i+4} \cos.\, \Phi^{i+4}, \text{ ob}$$

$$2^{i+4} \cos.\, \Phi^{i+4} = 2 \cos.\, (i+4)\Phi + 2\left(\frac{i+4}{1}\right) \cos.\, (i+2)\Phi$$
$$+ 2\left(\frac{i+4}{2}\right) \cos.\, i\Phi + \text{etc.}$$

pars ad litteram N accedens erit

$$2\left(\frac{i+4}{2}\right)\left(\frac{m}{i+4}\right) b^{i+4}.$$

Eodem modo ex casu $\lambda = i + 6$ pars ad litteram N accedens erit

$$2\left(\frac{i+6}{3}\right)\left(\frac{m}{i+6}\right) b^{i+6}, \text{ et ita porro.}$$

§. 97. His igitur omnibus partibus colligendis, nanciscemur valorem completum litterae N, qui erit

$$N = \pm 2\, b^i [\left(\frac{m}{i}\right) + \left(\frac{i+2}{1}\right)\left(\frac{m}{i+2}\right) b^2 + \left(\frac{i+4}{2}\right)\left(\frac{m}{i+4}\right) b^4 + \left(\frac{i+6}{3}\right)\left(\frac{m}{i+6}\right) b^6 + \text{etc.}]$$

ubi notasse juvabit esse, ut sequitur

$$\left(\frac{i+2}{1}\right)\left(\frac{m}{i+2}\right) = \left(\frac{m}{1}\right)\left(\frac{m-1}{i+1}\right),$$
$$\left(\frac{i+4}{2}\right)\left(\frac{m}{i+4}\right) = \left(\frac{m}{2}\right)\left(\frac{m-2}{i+2}\right),$$
$$\left(\frac{i+6}{3}\right)\left(\frac{m}{i+6}\right) = \left(\frac{m}{3}\right)\left(\frac{m-3}{i+3}\right),$$
$$\text{etc.}$$

Per hos igitur valores erit

$$N = \pm 2\, b^i [\left(\frac{m}{0}\right)\left(\frac{m}{i}\right) + \left(\frac{m}{1}\right)\left(\frac{m-1}{i+1}\right) b^2 + \left(\frac{m}{2}\right)\left(\frac{m-2}{i+2}\right) b^4 + \left(\frac{m}{3}\right)\left(\frac{m-3}{i+3}\right) b^6 \text{ etc.}]$$

quo valore invento, erit integrale nostrum quaesitum

$$\mathfrak{C} = \pm \pi (1 + aa)^m b^i [\left(\frac{m}{0}\right)\left(\frac{m}{i}\right) + \left(\frac{m}{1}\right)\left(\frac{m-1}{i+1}\right) b^2 + \text{etc.}]$$

quae series manifesto abrumpitur, quoties fuerit m numerus integer positivus. Statim enim atque in hoc charactere $(\frac{m-\lambda}{i+\lambda})$ denominator $i+\lambda$ superare incipit numeratorem $m-\lambda$, valor ejus in nihilum abit.

Demonstrationis pars secunda.

§. 98. Ut autem hanc integralis expressionem ad solam litteram a revocamus, prouti in nostro theoremate supra est repraesentata, hic loco b restituamus valorem assumtum $\frac{a}{1+aa}$, fietque

$$\mathbb{C} = \pm \pi a^i (1+aa)^{m-i} \left[(\tfrac{m}{0})(\tfrac{m}{i}) + (\tfrac{m}{1})(\tfrac{m-1}{i+1})\frac{a^2}{(i+aa)^2} + (\tfrac{m}{2})(\tfrac{m-2}{i+2})\frac{a^4}{(1+aa)^4} + \text{etc.}\right]$$

ubi, ut formam supra datam eliciamus, potestates ipsius $1+aa$ evolvi oportet. Hunc in finem statuamus $\mathbb{C} = \pm \pi a^i A$, ita ut jam sit

$$A = (\tfrac{m}{0})(\tfrac{m}{i})(1+aa)^{m-i} + (\tfrac{m}{1})(\tfrac{m-1}{i+1})a^2(1+aa)^{m-i-2} + (\tfrac{m}{2})(\tfrac{m-2}{i+2})a^4(1+aa)^{m-i-4} + (\tfrac{m}{3})(\tfrac{m-3}{i+3})a^6(1+aa)^{m-i-6} + \text{etc.}$$

Facta autem harum potestatum evolutione, fiat

$$A = \alpha + \beta a^2 + \gamma a^4 + \delta a^6 + \varepsilon a^8 + \zeta a^{10} + \eta a^{12} + \text{etc.}$$

quarum litterarum α, β, γ, δ, etc. valores investigemus.

§. 99. Primo igitur statim patet esse $\alpha = (\frac{m}{0})(\frac{m}{i})$; deinde vero reperietur

$$\beta = (\tfrac{m}{0})(\tfrac{m}{i})(\tfrac{m-i}{1}) + (\tfrac{m}{1})(\tfrac{m-1}{i+1}),$$

At vero pars posterior per priorem divisa, facta evolutione, praebet $\frac{m-i-1}{i+1}$, quo observato erit

$$\beta = \frac{m}{i+1}\,(\tfrac{m}{0})(\tfrac{m}{i})(\tfrac{m-i}{1}),$$

quod reducitur ad $\beta = (\frac{m}{1})(\frac{m}{i+1})$. Simili modo littera γ constabit ex tribus partibus: erit enim

$$\gamma = \left(\frac{m}{0}\right)\left(\frac{m}{i}\right)\left(\frac{m-i}{2}\right) + \left(\frac{m}{1}\right)\left(\frac{m-1}{i+1}\right)\left(\frac{m-i-2}{1}\right) + \left(\frac{m}{2}\right)\left(\frac{m-2}{i+2}\right),$$

ubi pars secunda per primam divisa dat $\frac{2(m-i-2)}{i+1}$. At tertius terminus per primum divisus praebet $\frac{(m-i-2)(m-i-3)}{(i+1)(i+2)}$, unde fit

$$\gamma = 1 + \frac{2(m-i-2)}{i+1} + \frac{(m-i-2)(m-i-3)}{(i+1)(i+2)}.$$

At vero est

$$1 + \frac{m-i-2}{i+1} = \frac{m-1}{i+1}; \text{ et}$$

$$\left(\frac{m-i-2}{i+1}\right)\left(1 + \frac{m-i-3}{i+2}\right) = \frac{m-1}{i+2} \cdot \frac{m-i-2}{i+1}.$$

unde colligitur

$$\gamma = \frac{m-1}{i+1} \cdot \frac{m}{i+2} \left(\frac{m}{0}\right)\left(\frac{m}{i}\right)\left(\frac{m-i}{2}\right),$$

quae expressio contrahitur in hanc $\left(\frac{m}{2}\right)\left(\frac{m}{i+2}\right)$.

§. 100. Cum igitur sit

$$\alpha = \left(\frac{m}{0}\right)\left(\frac{m}{i}\right), \quad \beta = \left(\frac{m}{1}\right)\left(\frac{m}{i+1}\right), \quad \gamma = \left(\frac{m}{2}\right)\left(\frac{m}{i+2}\right),$$

hinc jam satis tuto concludere liceret, fore

$$\delta = \left(\frac{m}{3}\right)\left(\frac{m}{i+3}\right), \quad \varepsilon = \left(\frac{m}{4}\right)\left(\frac{m}{i+4}\right), \text{ etc.}$$

Verum ne hic quicquam conjecturae vel inductioni tribuamus, in genere pro valore litterae A investigemus coëfficientem potestatis indefinitae $a^{2\lambda}$, quem vocemus $= \lambda$, eritque

$$\varLambda = \left(\frac{m-i}{\lambda}\right)\left(\frac{m}{0}\right)\left(\frac{m}{i}\right) + \left(\frac{m}{1}\right)\left(\frac{m-1}{i+1}\right)\left(\frac{m-i-2}{\lambda-1}\right) + \left(\frac{m}{2}\right)\left(\frac{m-2}{i+2}\right)\left(\frac{m-i-4}{\lambda-2}\right)$$
$$+ \left(\frac{m}{3}\right)\left(\frac{m-3}{i+3}\right)\left(\frac{m-i-6}{\lambda-3}\right) + \text{etc.}$$

§. 101. Hujus seriei pro $\varLambda$ inventae singulos terminos sub hac forma generali complecti licet $\left(\frac{m}{\theta}\right)\left(\frac{m-\theta}{i+\theta}\right)\left(\frac{m-i-2\theta}{\lambda-\theta}\right)$, quae secundum factores evoluta transmutatur in hanc formam

$$\frac{m(m-1) \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot (m-i-\lambda-\theta+1)}{1 \cdot \cdot \cdot \cdot \theta \times 1 \cdot \cdot \cdot \cdot \cdot (i+\theta) \times 1 \cdot \cdot (\lambda-\theta)},$$

ubi numeratoris factores ab m incipientes continuo unitate decrescent usque ad ultimum $(m-i-\lambda-\theta+1)$. Jam ista fractio supra et infra multiplicetur per hoc productum

$$\lambda(\lambda-1) \cdot \cdot \cdot \cdot \cdot \cdot \cdot (\lambda-\theta+1),$$

ac prodibit ista fractio

$$\frac{\lambda(\lambda-1) \cdot \cdot \cdot (\lambda-\theta+1) \times m(m-1) \cdot \cdot \cdot (m-i-\lambda-\theta+1)}{1.2.3 \cdot \cdot \cdot \theta \times 1.2.3 \cdot \cdot \cdot (i+\theta) \times 1.2.3 \cdot \cdot \cdot \lambda},$$

in qua primo continetur character $(\frac{\lambda}{\theta})$, deinde etiam ibi continetur character $(\frac{m}{\lambda})$; quod restat dabit characterem $(\frac{m-\lambda}{i+\theta})$, sicque habebitur forma A generalis $= (\frac{\lambda}{\theta})(\frac{m}{\lambda})(\frac{m-\lambda}{i+\theta})$. Unde si loco θ successive scribamus 0, 1, 2, 3, etc., quia in singulis terminis communis inest factor $(\frac{m}{\lambda})$, erit valor litterae

$$A = (\frac{m}{\lambda})[(\frac{\lambda}{0})(\frac{m-\lambda}{i}) + (\frac{\lambda}{1})(\frac{m-\lambda}{i+1}) + (\frac{\lambda}{2})(\frac{m-\lambda}{i+2}) + \text{etc.}]$$

Verum ante aliquod tempus demonstravi, hujus similis seriei

$$(\frac{p}{0})(\frac{q}{r}) + (\frac{p}{1})(\frac{q}{r+1}) + (\frac{p}{2})(\frac{q}{r+2}) + (\frac{p}{3})(\frac{q}{r+3}) + \text{etc.}$$

summam semper esse $= (\frac{p+q}{p+r}) = (\frac{p+q}{q-r})$. Facta ergo applicatione, erit $p=\lambda$, $q=m-\lambda$, $r=i$: sicque finito modo habebimus

$$A = (\frac{m}{\lambda})(\frac{m}{\lambda+i}) = (\frac{m}{\lambda})(\frac{m}{m-\lambda-i}),$$

quae est demonstratio conjecturae supra allatae et ex valoribus α, β, γ, conclusae.

§. 102. Quod si jam hic loco λ successive scribamus numeros 0, 1, 2, 3, etc., nanciscemur verum valorem seriei, quam sub littera A complexi; erit scilicet

$$A = (\frac{m}{0})(\frac{m}{i}) + (\frac{m}{1})(\frac{m}{i+1})\,a^2 + (\frac{m}{2})(\frac{m}{i+2})\,a^4$$
$$+ (\frac{m}{3})(\frac{m}{i+3})\,a^6 + \text{etc.}$$

atque hinc valor integralis sub signo $\mathfrak{C}$ indicatae formulae erit

$$\mathfrak{C} = \pm \pi a^i [(\tfrac{m}{0})(\tfrac{m}{i}) + (\tfrac{m}{1})(\tfrac{m}{i+1})a^2 + (\tfrac{m}{2})(\tfrac{m}{i+2})a^4 + \text{etc.}]$$

quae expressio manifesto semper abrumpitur, quoties m est numerus integer positivus. Hic autem meminisse oportet, signi ambigui $\pm$ superius locum habere quando i fuerit numerus par, inferius vero si impar.

Demonstrationis pars tertia.

§. 103. Ista forma, quam pro valore integrali $\mathfrak{C}$ hic sumus adepti multo adeo est simplicior ea, quam theorema nostrum nobis suppeditaverat, quippe quae, si loco $\mathfrak{S}$ seriem quam designat scribamus, erit

$$\mathfrak{C} = \frac{\pi a^i (\tfrac{m}{1})}{(\tfrac{-m-1}{i})}[(\tfrac{m-i}{0})(\tfrac{m+i}{i}) + (\tfrac{m-i}{1})(\tfrac{m+i}{i+1})a^2 + (\tfrac{m-i}{2})(\tfrac{m+i}{i+2})a^4 + \text{etc.}]$$

Superest igitur, ut perfectum consensum inter has duas expressiones specie multum a se invicem discrepantes ostendamus. Hic autem plurimum notasse juvabit, esse $(\tfrac{-m-1}{i}) = \pm (\tfrac{m+i}{i})$, propterea quod supra §. 88. jam observavimus, esse in genere $(\tfrac{-p}{q}) = \pm (\tfrac{p+q-1}{q})$, ubi signum superius valet si fuerit q numerus par, inferius vero si impar; quo notato posterior forma pro $\mathfrak{C}$ inventa erit

$$\mathfrak{C} = \pm \frac{\pi a^i (\tfrac{m}{i})}{(\tfrac{m+i}{i})}[(\tfrac{m-i}{0})(\tfrac{m+i}{i}) + (\tfrac{m-i}{1})(\tfrac{m+i}{i+1})a^2 + \text{etc.}].$$

§. 104. Quoniam nunc ambae formae affectae sunt signo ambiguo $\pm$, demonstrandum nobis incumbit, si utramque expressionem per $(\tfrac{m+i}{i})$ multiplicemus, duas sequentes series inter se prorsus esse aequales

$$\text{I.}\ (\tfrac{m}{0})(\tfrac{m}{i})(\tfrac{m+i}{i}) + (\tfrac{m}{1})(\tfrac{m}{i+1})(\tfrac{m+i}{i})\, a^2 + (\tfrac{m}{2})(\tfrac{m}{i+2})(\tfrac{m+i}{i})\, a^4 + \text{etc.}$$

$$\text{II.}\ (\tfrac{m-i}{0})(\tfrac{m+i}{i})(\tfrac{m}{i}) + (\tfrac{m-i}{1})(\tfrac{m+i}{i+1})(\tfrac{m}{i})\, a^2 + (\tfrac{m-i}{2})(\tfrac{m+i}{i+2})(\tfrac{m}{i})\, a^4 + \text{etc.}$$

ubi aequalitas primorum terminorum ob $(\frac{m}{0})$ et $(\frac{m-i}{0}) = 1$ sponte se prodit; deinde vero non difficulter aequalitas inter terminos secundos ipso $a\ a$ affectos ostendi poterit, similique modo etiam de sequentibus hoc idem est tenendum.

§. 105. Verum ne etiam hic inductione uti cogamur, convenientiam binorum terminorum eadem potestate $a^{2\lambda}$ demonstremus. In priore vero serie ista potestas $a^{2\lambda}$ hunc habet coëfficientem $(\frac{m}{\lambda})(\frac{m}{i+\lambda})(\frac{m+i}{i})$; in altera vero ejusdem coëfficiens est $(\frac{m-i}{\lambda})(\frac{m+i}{i+\lambda})(\frac{m}{i})$. Evolvatur igitur uterque in factores simplices, ac prior deducit ad hanc fractionem

$$\frac{m\ .\ .\ .\ (m-\lambda+1)\times m\ .\ .\ .\ (m-i-\lambda+1)\times(m+i)\ .\ .\ .\ (m+1)}{1\ .\ .\ .\ \lambda\times 1)\ .\ .\ .\ .\ (i+\lambda)\times 1\ .\ .\ .\ .\ .\ i};$$

posterior vero praebet istam

$$\frac{(m-i)\ .\ .\ .\ (m-i-\lambda+1)\times(m+i)\ .\ .\ .\ (m-\lambda+1)\times m\ .\ .\ .\ (m-i+1)}{1\ .\ .\ .\ .\ .\ \lambda\times 1\ .\ .\ .\ .\ .\ (i+\lambda)\times 1\ .\ .\ .\ .\ i};$$

ubi denominatores utrinque manifesto sunt iidem, ita ut tantum aequalitas inter numeratores sit demonstranda.

§. 106. Primo autem in priore numeratore tertius factor generalis cum primo conjunctus praebet hoc productum

$$(m+i)\ .\ .\ .\ .\ .\ (m-\lambda+1),$$

quod etiam in forma posteriori occurrit: his igitur sublatis aequalitatem monstrari oportet inter partes residuas quae sunt, in priori forma $m \quad . \quad . \quad . \quad . \quad . \quad (m - i - \lambda + 1)$

in altera $m \; . \; . \; . \; (m - i + 1) \times (m - i) \; . \; . \; . \; (m - i - l + 1)$

quae nunc iterum est manifesta. Sic igitur veritas nostri theorematis, quod demonstrandum suscepimus, jam rigide est ob oculos posita pro formula integrali

$$☾ = \int \partial \phi \cos. i \phi (1 + aa - 2 a \cos. \phi)^n \left[\begin{smallmatrix} a \phi = 0 \\ ad \phi = \pi \end{smallmatrix}\right].$$

Demonstrationis pars quarta.

§. 107. Invento valore formulae ☾, tota demonstratio jam confecta est censenda, quandoquidem jam initio ex valore formulae ⊙ ille rite est derivatus. Interim tamen hic quoque vicissim ex valore ☾ alterum valorem ⊙ derivari conveniet. Utamur autem forma simpliciori ipsius ☾, ad quem nos ipsa demonstratio immediate perduxit, qui erat

$$☾ = \pm \pi a^i [(\tfrac{m}{0})(\tfrac{m}{i}) + (\tfrac{m}{1})(\tfrac{m}{i+1}) a^2 + (\tfrac{m}{2})(\tfrac{m}{i+2}) a^4 + \text{etc.}]$$

ubi signum superius valet si i fuerit numerus par, inferius si impar.

§. 108. Ex hoc jam valore formulae ☾ alterius formulae ⊙ valor deducitur, si modo loco m scribamus $-n-1$, qui ergo valor hinc erit

$$⊙ = \pm \pi a^i [(\tfrac{-n-1}{0})(\tfrac{-n-1}{i}) + (\tfrac{-n-1}{1})(\tfrac{-n-1}{i+1}) a^2 + (\tfrac{-n-1}{2})(\tfrac{-n-1}{i+2}) a^4 + \text{etc.}]$$

quae autem series nunc in infinitum progreditur, siquidem n fuerit numerus integer positivus; quamobrem hanc seriem in aliam converti oportet, quae abrumpatur, quoties n fuerit numerus integer positivus, id quod ope lemmatis supra initio allati praestari poterit.

§. 109. Seriem igitur hic inventam cum serie ♄ in lemmate comparemus, id quod fit statuendo

$$f = -n - 1,\quad h = -n - 1 \text{ et } e = i,$$

ita ut jam sit $\odot = \pm \pi a^i$ ♄. Ex his autem valoribus altera series signo ♂ notata fiet, ob

$$-h - 1 = n,\ -f - 1 = n, \text{ et } x = a^2,$$

$$\text{♂} = \left(\frac{n}{0}\right)\left(\frac{n}{i}\right) + \left(\frac{n}{1}\right)\left(\frac{n}{i+1}\right) a^2 + \left(\frac{n}{2}\right)\left(\frac{n}{i+2}\right) a^4 + \text{etc.}$$

At vero relatio inter has duas series erit

$$\left(\frac{i-n-1}{i}\right) \text{♄} = \frac{\left(\frac{n+i}{i}\right) \text{♂}}{(1-aa)^{2n+1}};$$

ubi notetur, cum supra jam observaverimus esse

$$\left(\frac{-p}{q}\right) = \pm \left(\frac{p+q-1}{q}\right), \text{ hic fore } \left(\frac{-n-1+i}{i}\right) = \pm \left(\frac{n}{i}\right);$$

ubi iterum signum superius valet, si i fuerit numerus par. Hinc igitur erit

$$\text{♄} = \pm \frac{\left(\frac{n+i}{i}\right) \text{♂}}{\left(\frac{n}{i}\right)(1-aa)^{2n+1}}.$$

§. 110. Substituatur igitur iste valor loco ♄, quo ipso duplex signorum ambiguitas e medio tolletur, loco ♂ autem series modo data scribatur, atque pro $\odot$ sequentem nanciscemur expressionem

$$\odot = \frac{\pi a^i \left(\frac{n+i}{i}\right)}{\left(\frac{n}{i}\right)(1-aa)^{2n+1}} \left[\left(\frac{n}{0}\right)\left(\frac{n}{i}\right) + \left(\frac{n}{1}\right)\left(\frac{n}{i+1}\right) a^2 + \left(\frac{n}{2}\right)\left(\frac{n}{i+2}\right) a^4 + \text{etc.}\right]$$

quae series manifesto semper abrumpitur, quoties n fuerit numerus integer positivus. Verumtamen hoc laborat defectu, quod casibus quibus $n < i$, ob $\left(\frac{n}{i}\right) = 0$, infinita evadere videtur. Verum notandum est, his casibus etiam omnes terminos seriei ♂ in nihilum abire; ex quo necesse est, ut in ejus verum valorem totiusque expressionis inquiramus. At vero reliquis casibus, quibus $n > i$ haec expressio adeo illi quam in theoremate dedimus praeferenda videtur.

§. 111. Ostendi ergo hic debet, omnes terminos nostrae seriei ita transformari posse, ut per denominatorem $(\frac{n}{i})$ divisionem admittant. At vero quilibet nostrae seriei terminus sub hac forma continetur $(\frac{n}{\lambda})(\frac{n}{i+\lambda})$, quae per factorem comunem $(\frac{n+i}{i})$ multiplicata fit $(\frac{n+i}{i})(\frac{n}{\lambda})(\frac{n}{i+\lambda})$, quae in factores evoluta ad hanc fractionem reducitur

$$\frac{(n+i)\ldots\ldots(n+1)\times n\ldots\ldots(n-\lambda+1)\times n\ldots\ldots(n-i-\lambda+1)}{1\ldots\ldots i\times 1\ldots\ldots\lambda\times 1\ldots\ldots(i+\lambda)};$$

ubi tam numerator quam denominator tres habet factores principales; factores autem singulares in numeratore continuo unitate decrescunt, in denominatore unitate increscunt. Cum igitur sit $(\frac{n}{i}) = \frac{n\ldots\ldots(n-i+1)}{1\ldots\ldots i}$, superior fractio per hanc divisa, ob

$$\frac{n\ldots\ldots(n-i-\lambda+1)}{n\ldots\ldots(n-i+1)} = (n-i)\ldots\ldots(n-i-\lambda+1),$$

proveniet

$$\frac{(n+i)\ldots\ldots(n+i)\times n\ldots\ldots(n-\lambda+1)\times(n-i)\ldots\ldots(n-i-\lambda+1)}{1.2.3\ldots\ldots\lambda\times 1.2.3\ldots\ldots(i+\lambda)},$$

quae manifesto in hanc transit (ob duo priores factores cohaerentes)

$$\frac{n+i)\ldots\ldots\ldots(n-\lambda+1)\times(n-i)\ldots(n-i-\lambda+1)}{1.2\ldots\ldots\lambda\times 1.2\ldots\ldots(i+\lambda)},$$

ita ut omnibus ad characteres reductis, sit forma generalis cujusque termini $= (\frac{n+i}{i+\lambda})(\frac{n-i}{\lambda})$.

§. 112. Nunc igitur loco λ successive scribantur valores 0, 1, 2, 3, etc. atque valor integralis formulae ♁ prodibit, prorsus uti in theoremate est enunciatus, scilicet

$$\odot = \frac{\pi a^i}{(1-aa)^{2n+1}} [(\tfrac{n-i}{0})(\tfrac{n+i}{i}) + (\tfrac{n-i}{1})(\tfrac{n+i}{i+1})a^2 + (\tfrac{n-i}{2})(\tfrac{n+i}{i+2})a^4 + \text{etc.}]$$

quae expressio jam non solum semper abrumpitur, quoties n fuerit numerus integer positivus, nec ullo amplius laborat defectu, cum omnibus casibus valorem ipsius $\odot$ determinatum exhibeat, sicque adeo nostrum theorema, quod antea sola conjectura innitebatur, solidissima demonstratione est confirmatum.