SUPPLEMENTUM III.

AD TOM. I. CAP. IV.

DE

INTEGRATIONE FORMULARUM LOGARITHMICARUM ET EXPONENTIALIUM.

1) Evolutio formulae integralis $\int x^{f-1}\partial x(lx)^{\frac{m}{n}}$, integratione a valore $x=0$ ad $x=1$ extensa. *Nov. Commentarii Acad. Imp. Sc. Petropolitanae. Tom. XVI. Pag.* 91 — 139.

Theorema 1.

§. 1. Si n denotat numerum integrum positivum quemcunque, et formulae $\int x^{f-1}\partial x(1-x^g)^n$ integratio a valore $x=0$ usque ad $x=1$ extendatur, erit ejus valor:

$$=\frac{g^n}{f}\cdot\frac{1.\quad 2.\quad 3.\ \ldots\ldots\ldots\ n}{(f+g)(f+2g)(f+3g)\ldots\ldots(f+ng)}.$$

Demonstratio.

Notum est in genere, integrationem formulae

$$\int x^{f-1}\partial x(1-x^g)^m$$

reduci posse ad integrationem hujus $\int x^{f-1}\partial x(1-x^g)^{m-1}$, quoniam quantitates constantes A et B ita definire licet, ut fiat

$$\int x^{f-1}\partial x(1-x^g)^m = A\int x^{f-1}\partial x(1-x^g)^{m-1}+Bx^f(1-x^g)^m:$$

sumtis enim differentialibus prodit haec aequatio

$$x^{f-1}\partial x(1-x^g)^m = Ax^{f-1}\partial x(1-x^g)^{m-1} + Bfx^{f-1}\partial x(1-x^g)^m - Bmgx^{f+g-1}\partial x(1-x^g)^{m-1},$$

quae per $x^{f-1}\partial x(1-x^g)^{m-1}$ divisa dat

$$1-x^g = A + Bf(1-x^g) - Bmgx^g, \text{ seu}$$
$$1-x^g = A - Bmg + B(f+mg)(1-x^g),$$

quae aequatio ut consistere possit, necesse est sit

$$1 = B(f+mg) \text{ et } A = Bmg;$$

unde colligimus

$$B = \frac{1}{f+mg} \text{ et } A = \frac{mg}{f+mg}.$$

Quocirca habebimus sequentem reductionem generalem

$$\int x^{f-1}\partial x(1-x^g)^m = \frac{mg}{f+mg}\int x^{f-1}\partial x(1-x^g)^{m-1} + \frac{1}{f+mg} : x^f(1-x^g)^m$$

quae cum evanescat posito $x=0$, siquidem sit $f>0$, constantis additione haud est opus. Quare extenso utroque integrali usque ad $x=1$, pars integralis postrema sponte evanescit, eritque pro casu $x=1$

$$\int x^{f-1}\partial x(1-x^g)^m = \frac{mg}{f+mg}\int x^{f-1}\partial x(1-x^g)^{m-1}.$$

Cum igitur sumto $m=1$ sit $\int x^{f-1}\partial x(1-x^g)^0 = \frac{1}{f}x^f = \frac{1}{f}$, posito $x=1$, nanciscimur pro eodem casu $x=1$ sequentes valores

$$\int x^{f-1}\partial x(1-x^g)^1 = \frac{g}{f}\cdot\frac{1}{f+g}$$
$$\int x^{f-1}\partial x(1-x^g)^2 = \frac{g^2}{f}\cdot\frac{1}{f+g}\cdot\frac{2}{f+2g}$$
$$\int x^{f-1}\partial x(1-x^g)^3 = \frac{g^3}{f}\cdot\frac{1}{f+g}\cdot\frac{2}{f+2g}\cdot\frac{3}{f+3g}$$

etc.

hincque pro numero quocunque integro positivo n concludimus fore

$$\int x^{f-1}\partial x(1-x^g)^n = \frac{g^n}{f}\cdot\frac{1}{f+g}\cdot\frac{2}{f+2g}\cdot\frac{3}{f+3g}\cdots\cdots\frac{n}{f+ng}$$

si modo numeri f et g sint positivi.

Corollarium I.

§. 2. Hinc ergo vicissim valor hujusmodi producti ex quotcunque factoribus formati, per formulam integralem exprimi potest, ita ut sit

$$\frac{1. \quad 2. \quad 3. \quad \ldots\ldots \quad n}{(f+g)(f+2g)(f+3g)\ldots(f+ng)} = \frac{f}{g^n}\int x^{f-1}\partial x\,(1-x^g)^n,$$

integrali hoc a valore $x = 0$ usque ad $x = 1$ extenso.

Corollarium 2.

§. 3. Quodsi ergo hujusmodi habeatur progressio

$$\frac{1}{f+g};\ \frac{1. \quad 2}{(f+g)(f+2g)};\ \frac{1. \quad 2. \quad 3}{(f+g)(f+2g)(f+3g)};\ \frac{1. \quad 2. \quad 3. \quad 4.}{(f+g)(f+2g)(f+3g)(f+4g)};\ \text{etc.}$$

ejus terminus generalis qui indici indefinito n convenit, commode hac forma integrali $\frac{f}{g^n}\int x^{f-1}\partial x\,(1-x^g)^n$ repraesentatur, cujus ope ea progressio interpolari, ejusque termini indicibus fractis respondentes exhiberi poterunt.

Corollarium 3.

§. 4. Si loco n scribamus $n-1$, habebimus

$$\frac{1. \quad 2. \quad 3. \ldots\ldots (n-1)}{(f+g)(f+2g)(f+3g)\ldots[f+(n-1)g]} = \frac{f}{g^{n-1}}\int x^{f-1}\partial x\,(1-x^g)^{n-1},$$

quae per $\frac{n}{f+ng}$ multiplicata praebet

$$\frac{1. \quad 2. \quad 3. \ldots\ldots n}{(f+g)(f+2g)(f+3g)\ldots(f+ng)} = \frac{f \,.\, ng}{g^n(f+ng)}\int x^{f-1}\partial x\,(1-x^g)^{n-1}.$$

Scholion 1.

§. 5. Hanc posteriorem formam immediate ex praecedente derivare licuisset, cum modo demonstraverimus esse

$$\int x^{f-1}\partial x\,(1-x^g)^n = \frac{ng}{f+ng}\int x^{f-1}\partial x\,(1-x^g)^{n-1},$$

siquidem utrumque integrale a valore $x=0$ usque ad $x=1$ extendatur; quam integralium determinationem in sequentibus ubique subintelligi oportet. Deinde etiam perpetuo est tenendum, quantitates f et g esse positivas, quippe quam conditionem demonstratio allata absolute postulat. Quod autem ad numerum n attinet, quatenus eo index cujusque termini progressionis (§. 3.) designatur, nihil impedit, quominus eo numeri quicunque sive positivi sive negativi denotentur, quandoquidem ejus progressionis omnes termini etiam indicibus negativis respondentes per formulam integralem datam exhiberi censentur. Interim tamen probe tenendum est, hanc reductionem

$$\int x^{f-1}\partial x\,(1-x^g)^m = \frac{mg}{f+mg}\int x^{f-1}\partial x\,(1-x^g)^{m-1}$$

non esse veritati consentaneam, nisi sit $m>0$; quia alioquin pars algebraica $\frac{1}{f+mg}x^f(1-x^g)^m$ non evanesceret posito $x=1$.

Scholion 2.

§ 6. Hujusmodi series, quas transcendentes appellare licet, quia termini indicibus fractis respondentes sunt quantitates transcendentes, jam olim in Comment. Petrop. Tomo V. (*) fusius sum prosecutus; unde hoc loco non tam istas progressiones, quam eximias formularum integralium comparationes, quae inde derivantur, diligentius sum scrutaturus. Cum scilicet ostendissem, hujus producti indefiniti 1. 2. 3. . . . n valorem hac formula integrali $\int\partial x\,(l\frac{1}{x})^n$ ab $x=0$ ad $x=1$ extensa exprimi, quae res quoties n est numerus integer positivus per ipsam integrationem est manifesta, eos casus examini subjeci, quibus pro n numeri fracti accipiuntur; ubi quidem ex ipsa formula integrali neutiquam patet, ad quodnam genus quantitatum transcendentium hi termini referri debeant. Singulari autem

(*) Institut. Calc. integralis Tom. I. Sect. I. Cap. IV.

artificio eosdem terminos ad quadraturas magis cognitas reduxi, quod propterea maxime dignum videtur, ut majori studio perpendatur.

Problema 1.

§. 7. *Cum demonstratum sit esse*

$$\frac{1.\quad 2.\quad 3\ldots\ldots n}{(f+g)(f+2g)(f+3g)\ldots(f+ng)}=\frac{f}{g^n}\int x^{f-1}\partial x\,(1-x^g)^n$$ *integrali ab* $x=0$ *ad* $x=1$ *extenso, ejusdem producti casu quo* $g=0$ *valorem per formulam integralem assignare.*

Solutio.

Posito $g=0$ in formula integrali membrum $(1-x^g)^n$ evanescit, simul vero etiam denominator g^n, unde quaestio huc redit, ut fractionis $\frac{(1-x^g)^n}{g^n}$ valor definiatur casu $g=0$, quo tam numerator quam denominator evanescit. Hunc in finem spectetur g ut quantitas infinite parva, et cum sit $x^g=e^{g\,l\,x}$, fiet $x^g=1+g\,l\,x$, ideoque

$$(1-x^g)^n=g^n(-l\,x)^n=g^n\,(l\tfrac{1}{x})^n;$$

ex quo pro hoc casu formula nostra integralis abit in

$$f\int x^{f-1}\partial x\,(l\tfrac{1}{x})^n;$$

ita ut jam hebeatur

$$\frac{1.\quad 2.\quad 3\ldots\ldots n}{f^n}=f\int x^{f-1}\partial x\,(l\tfrac{1}{x})^n \text{ seu}$$

$$1.\quad 2.\quad 3\ldots\ldots n=f^{n+1}\int x^{f-1}\partial x\,(l\tfrac{1}{x})^n.$$

Collarium 1.

§. 8. Quoties n est numerus integer positivus, integratio formulae $\int x^{f-1}\partial x\,(l\frac{1}{x})^n$ succedit, eaque ab $x=0$ ad $x=1$ extensa revera prodit id productum, cui istam formulam aequalem invenimus. Sin autem pro n capiantur numeri fracti, eadem formula integralis inserviet huic progressioni hypergeometricae interpolandae

1; 1. 2; 1. 2. 3; 1. 2. 3. 4; 1. 2. 3. 4. 5; 1. 2. 3. 4. 5. 6; etc. seu
1; 2; 24; 120; 720; etc.

Corollarium 2.

§. 9. Si expressio modo inventa per principalem dividatur, orietur productum, cujus factores in progressione arithmetica quacunque progrediuntur

$$(f+g)(f+2g)(f+3g)\ldots(f+ng)=f^n g^n \cdot \frac{\int x^{f-1}\,\partial x\,(l\frac{1}{x})^n}{\int x^{f-1}\,\partial x\,(1-x^g)^n},$$

cujus ergo etiam valores, si n sit numerus fractus, hinc assignare licebit.

Corollarium 3.

§. 10. Cum sit

$$\int x^{f-1}\,\partial x\,(1-x^g)^n=\frac{ng}{f+ng}\int x^{f-1}\,\partial x\,(1-x^g)^{n-1},$$

erit etiam simili modo pro casu $g=0$

$$\int x^{f-1}\,\partial x\,(l\tfrac{1}{x})^n=\frac{n}{f}\int x^{f-1}\,\partial x\,(l\tfrac{1}{x})^{n-1},$$

hincque per istas alteras formulas integrales

$$1.\ 2.\ 3\ldots\ldots n=nf^n\int x^{f-1}\,\partial x\,(l\tfrac{1}{x})^{n-1} \text{ et}$$

$$(f+g)(f+2g)\ldots(f+ng)=f^{n-1}g^{n-1}(f+ng)\cdot\frac{\int x^{f-1}\,\partial x\,(l\frac{1}{x})^{n-1}}{\int x^{f-1}\,\partial x\,(1-x^g)^{n-1}}.$$

Scholion.

§. 11. Cum invenerimus esse

$$1.\ 2.\ 3\ldots\ldots n=f^{n+1}\int x^{f-1}\,\partial x\,(l\tfrac{1}{x})^n,$$

patet hanc formulam integralem non a valore quantitatis f pendere, quod etiam facile perspicitur ponendo $x^f = y$, unde fit $f x^{f-1} \partial x = \partial y$, et $l\frac{1}{x} = -lx = -\frac{1}{f} ly = \frac{1}{f} l\frac{1}{y}$, ideoque $f^n (l\frac{1}{x})^n = (l\frac{1}{y})^n$, ita ut sit

$$1.\ 2.\ 3 \ldots\ldots n = \int \partial y \,(l\tfrac{1}{y})^n,$$

quae formula ex priori nascitur ponendo $f = 1$. Pro interpolatione ergo hujusmodi formarum totum negotium huc reducitur, ut istius formulae integralis $\int \partial x \,(l\frac{1}{x})^n$ valores difiniantur, quando exponens n est numerus fractus. Veluti si n sit $= \frac{1}{2}$, assignari oportet valorem hujus formulae $\int \partial x \sqrt{l\frac{1}{x}}$; quem olim jam ostendi esse $= \frac{1}{2}\sqrt{\pi}$, denotante π circuli peripheriam cujus diameter $= 1$: pro aliis autem numeris fractis cujus valorem ad quadraturas curvarum algebraicarum altioris ordinis revocare docui. Quae reductio cum minime sit obvia, atque tum solum locum habeat, quando formulae $\int \partial x \,(l\frac{1}{x})^n$ integratio a valore $x = 0$ ad $x = 1$ extenditur, singulari attentione digna videtur. Etsi autem jam olim hoc argumentum tractavi, tamen quia per plures ambages eo sum perductus, idem hic resumere et concinnius evolvere constitui.

Theorema 2.

§. 12. Si formulae integrales a valore $x = 0$ usque ad $x = 1$ extendantur, et n denotet numerum integrum positivum, erit

$$\frac{1.\ 2.\ 3 \ldots\ldots n}{(n+1)(n+2)(n+3)\ldots 2n} = \tfrac{1}{2} ng \int x^{f+ng-1} \partial x (1-x^g)^{n-1} \times \frac{\int x^{f-1} \partial x \,(1-x^g)^{n-1}}{\int x^{f-1} \partial x (1-x^g)^{2n-1}}$$

quicunque numeri positivi loco f et g accipiantur.

Demonstratio.

Cum supra (§. 4.) ostenderimus esse

$$\frac{1.\quad 2.\quad 3\ldots\ldots n}{(f+g)(f+2g)\ldots(f+ng)}=\frac{f.\,ng}{g^n(f+ng)}\int x^{f-1}\partial x(1-x^g)^{n-1},$$

habebimus, si loco n scribamus $2n$,

$$\frac{1.\quad 2.\quad 3\ldots\ldots 2n}{(f+g)(f+2g)\ldots(f+2ng)}=\frac{f.\,2ng}{g^{2n}(f+2ng)}\int x^{f-1}\partial x(1-x^g)^{2n-1}.$$

Dividatur nunc prima aequatio per secundam, ac prodibit ista tertia

$$\frac{[f+(n+1)g][f+(n+2)g]\ldots(f+2ng)}{(n+1)(n+2)\ldots\ldots 2n}=\frac{g^n(f+2ng)}{2(f+ng)}\cdot\frac{\int x^{f-1}\partial x(1-x^g)^{n-1}}{\int x^{f-1}\partial x(1-x^g)^{2n-1}}.$$

At si in prima aequatione loco f scribatur $f+ng$, orietur haec aequatio quarta

$$\frac{1.\quad 2.\quad 3\ldots\ldots\ldots\ldots n}{[f+(n+1)g][f+(n+2)g]\ldots(f+2ng)}=\frac{(f+ng)ng}{g^n(f+2ng)}\int x^{f+ng-1}\partial x(1-x^g)^{n-1}.$$

Multiplicetur haec quarta aequatio per illam tertiam, ac reperietur ipsa aequatio demonstranda:

$$\frac{1.\quad 2.\quad 3\ldots\ldots n}{(n+1)(n+2)(n+3)\ldots 2n}=\tfrac{1}{2}ng\int x^{f+ng-1}\partial x(1-x^g)^{n-1}\times\frac{\int x^{f-1}\partial x(1-x^g)^{n-1}}{\int x^{f-1}\partial x(1-x^g)^{2n-1}}.$$

Corollarium 1.

§. 13. Si in prima aequatione statuatur $f=n$ et $g=1$, orietur idem productum

$$\frac{1.\quad 2.\quad\ldots\ldots n}{(n+1)(n+2)\ldots\ldots 2n}=\tfrac{1}{2}n\int x^{n-1}\partial x(1-x)^{n-1},$$

qua aequatione cum illa collata adipiscimur

$$\frac{\int x^{n-1}\partial x(1-x)^{n-1}}{g\int x^{f+ng-1}\partial x(1-x^g)^{n-1}}=\frac{\int x^{f-1}\partial x(1-x^g)^{n-1}}{\int x^{f-1}\partial x(1-x^g)^{2n-1}}.$$

Corollarium 2.

§. 14. Si in illa aequatione loco x scribamus x^g, fiet

$$\frac{1.\ 2.\ 3 \ldots\ldots n}{(n+1)(n+2)\ldots 2n} = \tfrac{1}{2} n g \int x^{ng-1} \partial x (1-x^g)^{n-1};$$

ita ut jam consequamur istam comparationem inter sequentes formulas integrales

$$\int x^{ng-1} \partial x (1-x^g)^{n-1} = \int x^{f+ng-1} \partial x (1-x^g)^{n-1} \times \frac{\int x^{f-1} \partial x (1-x^g)^{n-1}}{\int x^{f-1} \partial x (1-x^g)^{2n-1}}.$$

Corollarium 3.

§. 15. Si in aequatione theorematis ponamus $g = 0$, ob $(1-x^g)^m = g^m (l\frac{1}{x})^m$, potestates ipsius g se destruent, orieturque haec aequatio

$$\frac{1.\ 2.\ 3 \ldots\ldots n}{(n+1)(n+2)\ldots 2n} = \tfrac{1}{2} n \int x^{f-1} \partial x (l\tfrac{1}{x})^{n-1} \times \frac{\int x^{f-1} \partial x (l\frac{1}{x})^{n-1}}{\int x^{f-1} \partial x (l\frac{1}{x})^{2n-1}};$$

unde colligimus

$$\frac{[\int x^{f-1} \partial x (l\frac{1}{x})^{n-1}]^2}{\int x^{f-1} \partial x (l\frac{1}{x})^{2n-1}} = g \int x^{ng-1} \partial x (1-x^g)^{n-1},$$

seu ob

$$\int x^{f-1} \partial x (l\tfrac{1}{x})^{n-1} = \frac{f}{n} \int x^{f-1} \partial x (l\tfrac{1}{x})^n, \text{ hanc}$$

$$\frac{2f}{n} \cdot \frac{[\int x^{f-1} \partial x (l\frac{1}{x})^n]^2}{\int x^{f-1} \partial x (l\frac{1}{x})^{2n}} = g \int x^{ng-1} \partial x (1-x^g)^{n-1}.$$

Corollarium 4.

§. 16. Ponamus hic $f = 1$, $g = 2$ et $n = \frac{m}{2}$, ut m sit numerus integer positivus, et ob $\int \partial x (l\frac{1}{x})^m = 1.\ 2.\ 3 \ldots\ldots m$, erit

$$\frac{4}{m} \cdot \frac{[\int \partial x\,(l\tfrac{1}{x})^{\frac{m}{2}}]^2}{1.\ 2.\ 3 \ldots m} = 2\int x^{m-1}\,\partial x\,(1-x^2)^{\frac{m}{2}-1},$$

hincque

$$\int \partial x\,(l\tfrac{1}{x})^{\frac{m}{2}} = \sqrt{1.\ 2.\ 3 \ldots m \,.\, \tfrac{m}{2}\int x^{m-1}\,\partial x\,(1-x^2)^{\frac{m}{2}-1}},$$

et sumendo $m = 1$, ob $\int \frac{\partial x}{\sqrt{(1-xx)}} = \frac{\pi}{2}$ habebitur

$$\int \partial x \sqrt{l\tfrac{1}{x}} = \sqrt{\tfrac{1}{2}\int \frac{\partial x}{\sqrt{(1-xx)}}} = \tfrac{1}{2}\sqrt{\pi}.$$

Scholion.

§. 17. En ergo succinctam demonstrationem theorematis olim a me prolati, quod sit $\int \partial x \sqrt{l\frac{1}{x}} = \frac{1}{2}\sqrt{\pi}$, eamque ab interpolationis ratione, qua tum usus fueram, libera. Deducta scilicet hic ea ex hoc theoremate, quo inveni esse

$$\frac{[\int x^{f-1}\,\partial x\,(l\frac{1}{x})^{n-1}]^2}{\int x^{f-1}\,\partial x\,(l\frac{1}{x})^{2n-1}} = g\int x^{ng-1}\,\partial x\,(1-x^g)^{n-1}.$$

Principale autem theorema, unde hoc est deductum ita se habet

$$g \cdot \frac{\int x^{f-1}\partial x\,(1-x^g)^{n-1} \times \int x^{f+ng-1}\,\partial x\,(1-x^g)^{n-1}}{\int x^{f-1}\,\partial x\,(1-x^g)^{2n-1}} = \int x^{n-1}\partial x\,(1-x)^{n-1};$$

utrumque enim membrum per intergrationem ab $x = 0$ ad $x = 1$ extensam evolvitur in hoc productum numericum

$$\frac{1.\quad 2.\quad 3 \ldots\ldots (n-1)}{(n+1)\,(n+2) \ldots\ldots (2n-1)}.$$

Ac si alteri membro speciem latius patentem tribuere velimus, theorema ita proponi poterit ut sit

$$g \cdot \frac{\int x^{f-1}\partial x\,(1-x^g)^{n-1} \times \int x^{f+ng-1}\partial x\,(1-x^g)^{n-1}}{\int x^{f-1}\partial x\,(1-x^g)^{2n-1}} = k\int x^{nk-1}\partial x\,(1-x^k)^{n-1},$$

hicque si capiatur $g = 0$, fit

$$\frac{[\int x^{f-1}\partial x\,(l\tfrac{1}{x})^{n-1}]^2}{\int x^{f-1}\partial x\,(l\tfrac{1}{x})^{2n-1}} = k\int x^{nk-1}\partial x\,(1-x^k)^{n-1}.$$

Imprimis igitur notandum est, quod illa aequalitas subsistat, quicunque numeri loco f et g accipiantur: casu quidem $f = g$, ea est manifesta, cum sit

$$\int x^{g-1}\partial x\,(1-x^g)^{n-1} = \frac{1-(1-x^g)^n}{ng} = \frac{1}{ng},$$

fiet enim

$$2g\int x^{ng+g-1}\partial x\,(1-x^g)^{n-1} = k\int x^{nk-1}\partial x\,(1-x^k)^{n-1},$$

et quia

$$\int x^{ng+g-1}\partial x\,(1-x^g)^{n-1} = \tfrac{1}{2}\int x^{ng-1}\partial x\,(1-x^g)^{n-1},$$

aequalitas est perspicua, quia k pro lubitu accipere licet. Eodem autem modo, quo ad hoc theorema perveni, ad alia similia pertingere licet.

Theorema 3.

§. 18. Si sequentes formulae integrales a valore $x = 0$ ad $x = 1$ extendantur, et n denotet numerum integrum positivum quemcunque, erit

$$\frac{1.\ 2.\ 3\ \ldots\ldots n}{(2n+1)\,(2n+2)\ldots 3n} = \tfrac{2}{3}ng\int x^{f+2ng-1}\partial x\,(1-x^g)^{n-1}\times \frac{\int x^{f-1}\partial x\,(1-x^g)^{2n-1}}{\int x^{f-1}\partial x\,(1-x^g)^{3n-1}},$$

quicunque numeri positivi pro f et g accipiantur.

Demonstratio.

In praecedente theoremate jam vidimus esse

$$\frac{1.\ 2.\ 3\ldots\ldots 2n}{(f+g)(f+2g)\ldots(f+2ng)g}=\frac{f.2ng}{g^{2n}(f+2ng)}\int x^{f-1}\partial x(1-x^g)^{2n-1}:$$

simili autem modo, si in forma principali loco n scribamus $3n$ habebimus

$$\frac{1.\ 2.\ 3\ldots\ldots 3n}{(f+g)(f+2g)\ldots(f+3ng)}=\frac{f.3ng}{g^{3n}(f+3ng)}\int x^{f-1}\partial x(1-x^g)^{3n-1},$$

ex quo illa aequatio per hanc divisa producit

$$\frac{[f+(2n+1)g][f+(2n+2)g]\ldots\ldots(f+3ng)}{(2n+1)(2n+2)\ldots\ldots\ldots 3n}=\frac{2g^n(f+3ng)}{3(f+2ng)}\times\frac{\int x^{f-1}\partial x(1-x^g)^{2n-1}}{\int x^{f-1}\partial x(1-x^g)^{3n-1}}.$$

Verum si in aequatione principali (§. 4.) loco f scribamus $f+2ng$, adipiscimur hanc aequationem

$$\frac{1.\ 2.\ 3\ldots\ldots\ldots\ldots n}{[f+(2n+1)g][f+(2n+2)g]\ldots\ldots(f+3ng)}=\frac{(f+2ng).ng}{g^n(f+3ng)}\times\int x^{f+2ng-1}\partial x(1-x^g)^{n-1}.$$

Multiplicetur nunc haec aequatio per praecedentem, et orietur ipsa aequatio, quam demonstrari oportet

$$\frac{1.\ 2.\ 3.\ldots\ldots n}{(2n+1)(2n+2)\ldots\ldots 3n}=\tfrac{2}{3}ng.\int x^{f+2ng-1}\partial x(1-x^g)^{n-1}\times\frac{\int x^{f-1}\partial x(1-x^g)^{2n-1}}{\int x^{f-1}\partial x(1-x^g)^{3n-1}}.$$

Corollarium 1.

§. 19. Eundem valorem ex aequatione principali nanciscimur, ponendo $f=2n$ et $g=1$, ita ut sit

$$\frac{1.\ 2\ 3.\ldots\ldots n}{(2n+1)(2n+2)\ldots 3n}=\tfrac{2}{3}n\int x^{2n-1}\partial x(1-x)^{n-1};$$

quae formula integralis, loco x scribendo x^k, transformatur in hanc

$$\tfrac{2}{3} n k \int x^{2nk-1} \partial x \, (1 - x^k)^{n-1},$$

ita ut sit

$$g \int x^{f+2ng-1} \partial x \, (1 - x^g)^{n-1} \times \frac{\int x^{f-1} \partial x \, (1 - x^g)^{2n-1}}{\int x^{f-1} \partial x \, (1 - x^g)^{3n-1}}$$
$$= k \int x^{2nk-1} \partial x \, (1 - x^k)^{n-1}.$$

Corollarium 2.

§. 20. Si hic statuamus $g = 0$, ob $1 - x^g = g\, l\frac{1}{x}$ habebimus hanc aequationem

$$\int x^{f-1} \partial x \, (l\tfrac{1}{x})^{n-1} \times \frac{\int x^{f-1} \partial x \, (l\frac{1}{x})^{2n-1}}{\int x^{f-1} \partial x \, (l\frac{1}{x})^{3n-1}} = k \int x^{2nk-1} \partial x \, (1 - x^k)^{n-1}:$$

cum igitur ante invenissemus

$$\frac{[\int x^{f-1} \partial x \, (l\frac{1}{x})^{n-1}]^2}{\int x^{f-1} \partial x \, (l\frac{1}{x})^{2n-1}} = k \int x^{nk-1} \partial x \, (1 - x^k)^{n-1},$$

habebimus has aequationes in se multiplicando

$$\frac{[\int x^{f-1} \partial x \, (l\frac{1}{x})^{n-1}]^3}{\int x^{f-1} \partial x \, (l\frac{1}{x})^{3n-1}} = k^2 \int x^{nk-1} \partial x \, (1 - x^k)^{n-1} \times \int x^{2nk-1} \partial x \, (1 - x^k)^{n-1}.$$

Corollarium 3.

§. 21. Sine ulla restrictione hic ponere licet $f = 1$; tum ergo sumto $n = \frac{1}{3}$ et $k = 3$, erit

$$\frac{[\int \partial x \, (l\frac{1}{x})^{-\frac{2}{3}}]^3}{\int \partial x \, (l\frac{1}{x})^0} = 9 \int \partial x \, (1 - x^3)^{-\frac{2}{3}} \times \int x \partial x \, (1 - x^3)^{-\frac{2}{3}},$$

et ob

$$\int \partial x \, (l\tfrac{1}{x})^{-\frac{2}{3}} = 3 \int \partial x \, (l\tfrac{1}{x})^{\frac{1}{3}} \text{ et } \int \partial x \, (l\tfrac{1}{x})^0 = 1, \text{ obtinebimus}$$

$$[\int \partial x\,(l\tfrac{1}{x})^{\frac{1}{3}}]^3 = \int \partial x\,(1-x^3)^{-\frac{2}{3}} \times \int x\,\partial x\,(1-x^3)^{-\frac{2}{3}}$$

tum vero sumto $n = \frac{2}{3}$ et $k = 3$, erit

$$\frac{[\int \partial x\,(l\frac{1}{x})^{-\frac{1}{3}}]^3}{\int \partial x\,(l\frac{1}{x})} = 9 \int x\,\partial x\,(1-x^3)^{-\frac{1}{3}} \times \int x^3\,\partial x\,(1-x^3)^{-\frac{1}{3}}$$

seu

$$[\int \partial x\,(l\tfrac{1}{x})^{\frac{2}{3}}]^3 = 4 \int x\,\partial x\,(1-x^3)^{-\frac{1}{3}} \times \int x^3\,\partial x\,(1-x^3)^{-\frac{1}{3}}.$$

Theorema generale.

§. 22. Si sequentes formulae integrales a valore $x = 0$ usque ad $x = 1$ extendantur, et n denotet numerum integrum positivum quemcunque, erit

$$\frac{1.\ 2.\ 3.\ \ldots\ n}{(\lambda n+1)(\lambda n+2)\ldots(\lambda+1)n} = \frac{n}{\lambda+1}\, n g \int x^{f+\lambda n g-1}\,\partial x\,(1-x^g)^{n-1} \times \frac{\int x^{f-1}\,\partial x\,(1-x^g)^{\lambda n-1}}{\int x^{f-1}\,\partial x\,(1-x^g)^{(\lambda+1)n-1}},$$

quicunque numeri positivi pro litteris f et g accipiantur.

Demonstratio.

Cum sit uti supra ostendimus

$$\frac{1.\ 2.\ \ldots\ldots\ n}{(f+g)(f+2g)\ldots(f+ng)} = \frac{f.\ ng}{g^n\,(f+ng)} \int x^{f-1}\,\partial x\,(1-x^g)^{n-1},$$

si hic loco n scribamus primo λn, tum vero $(\lambda+1)n$, nanciscemur has duas aequationes

$$\frac{1.\ 2.\ \ldots\ldots\ \lambda n}{(f+g)(f+2g)\ldots(f+\lambda n g)} = \frac{f.\ \lambda n g}{g^{\lambda n}\,(f+\lambda n g)} \int x^{f-1}\,\partial x\,(1-x^g)^{\lambda n-1} \text{ et}$$

$$\frac{1. \quad 2. \quad \ldots \ldots \quad (\lambda+1)n}{(f+g)(f+2g)\ldots[f+(\lambda+1)ng]} = \frac{f.\,(\lambda+1)ng}{g^{(\lambda+1)n}[f+(\lambda+1)ng]} \times \int x^{f-1}\partial x(1-x^g)^{(\lambda+1)n-1},$$

quarum illa per hanc divisa praebet

$$\frac{(f+\lambda ng+g)(f+\lambda ng+2g)\ldots\ldots(f+\lambda ng+ng)}{(\lambda n+1)(\lambda n+2)\ldots\ldots\ldots(\lambda n+n)}$$

$$= g^n\frac{\lambda(f+\lambda ng+ng)}{(\lambda+1)(f+\lambda ng)}\cdot\frac{\int x^{f-1}\partial x(1-x^g)^{\lambda n-1}}{\int x^{f-1}\partial x(1-x^g)^{(\lambda+1)n-1}}.$$

At si in aequatione prima loco f scribamus $f+\lambda ng$, obtinebus

$$\frac{1. \quad 2. \quad \ldots\ldots\ldots \quad n}{(f+\lambda ng+g)(f+\lambda ng+2g)\ldots(f+\lambda ng+ng)}$$

$$= \frac{(f+\lambda ng)ng}{g^n(f+\lambda ng+ng)}\cdot\int x^{f+\lambda ng-1}\partial x(1-x^g)^{n-1},$$

quae duae aequationes in se ductae producunt ipsam aequalitatem demonstrandam

$$\frac{1. \quad 2. \quad \ldots\ldots \quad n}{(\lambda n+1)(\lambda n+2)\ldots(\lambda n+n)} = \frac{\lambda ng}{\lambda+1}\int x^{f+\lambda ng-1}\partial x(1-x^g)^{n-1}\times \frac{\int x^{f-1}\partial x(1-x^g)^{\lambda n-1}}{\int x^{f-1}\partial x(1-x^g)^{(\lambda+1)n-1}}.$$

Corollarium 1.

§. 23. Si in aequatione principali statuamus $f=\lambda n$ et $g=1$, reperiemus etiam

$$\frac{1. \quad 2. \quad \ldots\ldots \quad n}{(\lambda n+1)(\lambda n+2)\ldots(\lambda n+n)} = \frac{\lambda n}{\lambda+1}\int x^{\lambda n-1}\partial x(1-x)^{n-1},$$

quae forma loco x scribendo x^k abit in hanc

$$\frac{\lambda nk}{\lambda+1}\int x^{\lambda nk-1}\partial x(1-x^k)^{n-1};$$

ita ut habeamus hoc theorema latissime patens

$$g\int x^{f+\lambda n g-1}\partial x\,(1-x^g)^{n-1}\times\frac{\int x^{f-1}\partial x\,(1-x^g)^{\lambda n-1}}{\int x^{f-1}\partial x\,(1-x^g)^{\lambda n+n-1}}$$
$$=k\int x^{\lambda n k-1}\partial x\,(1-x^k)^{n-1}$$

Corollarium 2.

§. 24. Hoc jam theorema locum habet, etiamsi n non fit numerus integer; quin etiam cum numerum λ pro lubitu accipere liceat, loco λn scribamus m, et perveniemus ad hoc theorema

$$\frac{\int x^{f-1}\partial x\,(1-x^g)^{m-1}}{\int x^{f-1}\partial x\,(1-x^g)^{m+n-1}}=\frac{k\int x^{mk-1}\partial x\,(1-x^k)^{n-1}}{g\int x^{f+mg-1}\partial x\,(1-x^g)^{n-1}}.$$

Corollarium 3.

§. 25. Si ponamus $g=0$, ob $1-x^g=g\,l\frac{1}{x}$, hoc theorema istam induet formam

$$\frac{\int x^{f-1}\partial x\,(l\frac{1}{x})^{m-1}}{\int x^{f-1}\partial x\,(l\frac{1}{x})^{m+n-1}}=\frac{k\int x^{mk-1}\partial x\,(1-x^k)^{n-1}}{\int x^{f-1}\partial x\,(l\frac{1}{x})^{n-1}},$$

quae commodius ita repraesentatur

$$\frac{\int x^{f-1}\partial x\,(l\frac{1}{x})^{n-1}\times\int x^{f-1}\partial x\,(l\frac{1}{x})^{m-1}}{\int x^{f-1}\partial x\,(l\frac{1}{x})^{m+n-1}}=k\int x^{mk-1}\partial x\,(1-x^k)^{n-1};$$

ubi evidens est numeros m et n inter se permutari posse.

Scholion.

§. 26. Duplicem ergo deteximus fontem, unde innumerabiles formularum integralium comparationes haurire licet; alter fons §. 24. patefactus complectitur hujusmodi formulas integrales

$$\int x^{p-1}\partial x\,(1-x^g)^{q-1},$$

quas jam ante aliquod tempus pertractavi in observationibus circa integralia formularum (*)

$$\int x^{p-1}\partial x\,(1-x^n)^{\frac{q}{n}-1}$$

a valore $x=0$ usque ad $x=1$ extensa, ubi ostendi primo litteras p et q inter se permutari posse, ut sit

$$\int x^{p-1}\partial x\,(1-x^n)^{\frac{q}{n}-1}=\int x^{q-1}\partial x\,(1-x^n)^{\frac{p}{n}-1},$$

tum vero etiam esse

$$\int\frac{x^{p-1}\partial x}{(1-x^n)^{\frac{p}{m}}}=\frac{\pi}{n\sin.\frac{p\pi}{n}}:$$

imprimis autem demonstravi esse

$$\int\frac{x^{p-1}\partial x}{\sqrt[n]{(1-x^n)^{n-q}}}\times\int\frac{x^{p+q-1}\partial x}{\sqrt[n]{(1-x^n)^{n-r}}}=\int\frac{x^{p-1}\partial x}{\sqrt[n]{(1-x^n)^{n-r}}}\times\int\frac{x^{p+r-1}\partial x}{\sqrt[n]{(1-x^n)^{n-q}}},$$

in qua aequatione comparatio in §. 24. inventa jam continetur; ita ut hinc nihil novi, quod non jam evolvi, deduci queat. Alterum igitur fontem §. 25. indicatum hic potissimum investigandum suscipio, ubi cum sive ulla restrictione sumi queat $f=1$, aequasio nostra primaria erit

$$\frac{\int\partial x\,(l\frac{1}{x})^{n-1}\times\int\partial x\,(l\frac{1}{x})^{m-1}}{\int\partial x\,(l\frac{1}{x})^{m+n-1}}=k\int x^{mk-1}\partial x\,(1-x^k)^{n-1},$$

cujus beneficio valores formulae integralis $\int\partial x\,(l\frac{1}{x})^\lambda$, quando λ non est numerus integer, ad quadraturas curvarum algebraicarum revocare licebit; quandoquidem quoties λ est numerus integer, integratio habetur absoluta quoniam est

$$\int\partial x\,(l\frac{1}{x})^\lambda=1.\ 2.\ 3.\ \ldots\ldots\ \lambda.$$

Maximi autem momenti quaestio versatur circa eos casus, quibus λ est numerus fractus, quos ergo pro ratione denominationis hic successive sum definiturus.

(*) Miscellanea Taurinensia. Tom. III.

Problema 2.

§. 27. *Denotante i numerum integrum positivum, definire valorem formulae integralis* $\int \partial x \left(l\frac{1}{x}\right)^{\frac{i}{2}}$, *integratione ab* $x = 0$ *usque ad* $x = 1$ *extensa.*

Solutio.

In aequatione nostra generali faciamus $m = n$, eritque

$$\frac{\left[\int \partial x \left(l\frac{1}{x}\right)^{n-1}\right]^2}{\int \partial x \left(l\frac{1}{x}\right)^{2n-1}} = k \int x^{nk-1} \partial x (1 - x^k)^{n-1}.$$

Sit jam $n - 1 = \frac{i}{2}$, et ob $2n - 1 = i + 1$, erit

$$\int \partial x \left(l\frac{1}{x}\right)^{2n-1} = 1.\ 2.\ 3.\ \ldots\ldots\ (i+1):$$

sumatur porro $k = 2$, ut sit $nk - 1 = i + 1$, fietque

$$\frac{\left[\int \partial x \sqrt{\left(l\frac{1}{x}\right)^i}\right]^2}{1.\ 2.\ 3.\ldots(i+1)} = 2 \int x^{i+1} \partial x (1 - x^2)^{\frac{i}{2}},$$

ideoque

$$\frac{\int \partial x \sqrt{\left(l\frac{1}{x}\right)^i}}{\sqrt{[1.\ 2.\ 3.\ldots(i+1)]}} = \sqrt{2} \int x^{i+1} \partial x \sqrt{(1 - x^2)^i},$$

ubi evidens est, pro i numeros tantum impares sumi convenire; quoniam pro paribus evolutio per se est manifesta.

Corollarium 1.

§. 28. Omnes autem casus facile reducuntur ad $i = 1$, vel adeo ad $i = -1$; dummodo enim $i + 1$ non sit numerus negativus, reductio inventa locum habet. Pro hoc ergo casu erit

$$\int \frac{\partial x}{\sqrt{l\frac{1}{x}}} = \sqrt{2} \int \frac{\partial x}{\sqrt{(1 - xx)}} = \sqrt{\pi}, \text{ ob } \int \frac{\partial x}{\sqrt{(1 - xx)}} = \frac{\pi}{2}.$$

Corollarium 2.

§. 29. Hoc autem casu principali expedito, ob

$$\int \partial x (\tfrac{1}{x})^n = n \int \partial x (l\tfrac{1}{x})^{n-1}$$

habebimus

$$\int \partial x \sqrt{l\tfrac{1}{x}} = \tfrac{1}{2}\sqrt{\pi}; \quad \int \partial x (l\tfrac{1}{x})^{\frac{3}{2}} = \tfrac{1 \cdot 3}{2 \cdot 2}\sqrt{\pi}$$

atque in genere

$$\int \partial x (l\tfrac{1}{x})^{\frac{2n+1}{2}} = \tfrac{1}{2}\cdot\tfrac{3}{2}\cdot\tfrac{5}{2}\cdot\tfrac{7}{2}\cdot \ldots \ldots \tfrac{(2n+1)}{2}\sqrt{\pi}$$

Problema 3.

§. 30. *Denotante i numerum integrum positivum, definire valorem formulae integralis $\int \partial x (l\tfrac{1}{x})^{\frac{i}{3}-1}$, integratione ab $x=0$ ad $x=1$ extensa.*

Solutio.

Inchoemus ab aequatione praecedentis problematis

$$\frac{[\int \partial x (l\tfrac{1}{x})^{n-1}]^2}{\int \partial x (l\tfrac{1}{x})^{2n-1}} = k \int x^{nk-1} \partial x (1-x^k)^{n-1},$$

atque in forma generali statuamus $m = 2n$, ut habeatur

$$\frac{\int \partial x (l\tfrac{1}{x})^{n-1} \times \int \partial x (l\tfrac{1}{x})^{3n-1}}{\int \partial x (l\tfrac{1}{x})^{3n-1}} = k \int x^{2nk-1} \partial x (1-x^k)^{n-1},$$

ac multiplicando has duas aequalitates adipiscimur

$$\frac{[\int \partial x (l\tfrac{1}{x})^{n-1}]^3}{\int \partial x (l\tfrac{1}{x})^{3n-1}} = kk \int x^{nk-1} \partial x (1-x^k)^{n-1} \times \int x^{2nk-1} \partial x (1-x^k)^{n-1}.$$

Hic jam ponatur $n = \frac{i}{3}$ ut sit

$$\int \partial x (l\tfrac{1}{x})^{i-1} = 1.\ 2.\ 3.\ \ldots \ldots (i-1),$$

sumaturque $k=3$, ac prodibit

$$\frac{[\int \partial x \sqrt[3]{(l\tfrac{1}{x})^{i-3}}]^3}{1.\,2.\,3\ldots(i-1)} = 9\int x^{i-1}\partial x \sqrt[3]{(1-x^3)^{i-3}} \times \int x^{2i-1}\partial x \sqrt[3]{(1-x^3)^{i-3}};$$

unde concludimus

$$\frac{\int \partial x \sqrt[3]{(l\tfrac{1}{x})^{i-3}}}{\sqrt[3]{1.\,2.\,3\ldots(i-1)}} = \sqrt[3]{9}\int\frac{x^{i-1}\partial x}{\sqrt[3]{(1-x^3)^{3-i}}} \times \int\frac{x^{2i-1}\partial x}{\sqrt[3]{(1-x^3)^{3-i}}}.$$

Corollarium 1.

§. 31. Bini hic occurrunt casus principales, a quibus reliqui omnes pendent, ponendo scilicet vel $i=1$ vel $i=2$, qui sunt

$$\text{I. } \int\frac{\partial x}{\sqrt[3]{(l\tfrac{1}{x})^2}} = \sqrt[3]{9}\int\frac{\partial x}{\sqrt[3]{(1-x^3)^2}} \times \int\frac{x\,\partial x}{\sqrt[3]{(1-x^3)^2}}$$

$$\text{II. } \int\frac{\partial x}{\sqrt[3]{l\tfrac{1}{x}}} = \sqrt[3]{9}\int\frac{x\,\partial x}{\sqrt[3]{(1-x^3)}} \times \int\frac{x\,\partial x}{\sqrt[3]{(1-x^3)}},$$

quae posterior forma ob

$$\int\frac{x^3\,\partial x}{\sqrt[3]{(1-x^3)}} = \frac{1}{3}\int\frac{\partial x}{\sqrt[3]{(1-x^3)}}$$

abit in

$$\int\frac{\partial x}{\sqrt[3]{l\tfrac{1}{x}}} = \sqrt[3]{3}\int\frac{\partial x}{\sqrt[3]{(1-x^3)}} \times \int\frac{x\,\partial x}{\sqrt[3]{(1-x^3)}}.$$

Corollarium 2.

§. 32. Si uti in observationibus meis ante allegatis brevitatis gratia ponamus

$$\int \frac{x^{p-1}\,\partial x}{\sqrt[3]{(1-x^3)^{3-q}}} = \left(\frac{p}{q}\right),$$

atque ut ibi pro hac classe

$$\left(\frac{2}{1}\right) = \frac{\pi}{3\sin.\frac{\pi}{3}} = \alpha,$$

tum vero

$$\left(\frac{1}{1}\right) = \frac{\partial x}{\sqrt[3]{(1-x^3)^2}} = A, \text{ erit}$$

$$\text{I. } \int \frac{\partial x}{\sqrt[3]{(l\frac{1}{x})^2}} = \sqrt[3]{9\left(\frac{1}{1}\right)\left(\frac{2}{1}\right)} = \sqrt[3]{9\,\alpha\,A},$$

$$\text{II. } \int \frac{\partial x}{\sqrt[3]{(l\frac{1}{x})^1}} = \sqrt[3]{3\left(\frac{1}{2}\right)\left(\frac{2}{2}\right)} = \sqrt[3]{\frac{3\,\alpha\,\alpha}{A}}.$$

Corollarium 3.

§. 33. Pro casu ergo priori habebimus

$$\int \partial x \sqrt[3]{(l\tfrac{1}{x})^{-2}} = \sqrt[3]{9\,\alpha\,A}, \quad \int \partial x \sqrt[3]{l\tfrac{1}{x}} = \tfrac{1}{3}\sqrt[3]{9\,\alpha\,A}, \text{ et}$$

$$\int \partial x \sqrt[3]{(l\tfrac{1}{x})^{3n+1}} = \tfrac{1}{3}\cdot\tfrac{4}{3}\cdot\tfrac{7}{3}\cdots\cdots\tfrac{3n+1}{3}\sqrt[3]{9\,\alpha\,A}:$$

pro altero vero casu

$$\int \partial x \sqrt[3]{(l\tfrac{1}{x})^{-1}} = \sqrt[3]{\tfrac{3\alpha\alpha}{A}}, \quad \int \partial x \sqrt[3]{(l\tfrac{1}{x})^{2}} = \tfrac{2}{3}\sqrt[3]{\tfrac{3\alpha\alpha}{A}}, \text{ et}$$

$$\int \partial x \sqrt[3]{(l\tfrac{1}{x})^{3n-1}} = \tfrac{2}{3}\cdot\tfrac{5}{3}\cdot\tfrac{8}{3}\cdots\cdots\cdots\tfrac{3n-1}{3}\sqrt[3]{\tfrac{3\alpha\alpha}{A}}.$$

Problema 4.

§. 34. *Denotante i numerum integrum positivum, definire valorem formulae integralis $\int \partial x\, (l\frac{1}{x})^{\frac{i}{4}-1}$, integratione ab $x=0$ ad $x=1$ extensa.*

Solutio.

In solutione problematis praecedentis perducti sumus ad hanc aequationem

$$\frac{[\int \partial x\,(l\frac{1}{x})^{n-1}]^3}{\int \partial x\,(l\frac{1}{x})^{3n-1}} = k\,k\int \frac{x^{nk-1}\,\partial x}{(1-x^k)^{1-n}} \times \int \frac{x^{2nk-1}\,\partial x}{(1-x^k)^{1-n}};$$

forma generalis autem sumendo $m = 3\,n$ praebet

$$\frac{\int \partial x\,(l\frac{1}{x})^{n-1} \times \int \partial x\,(l\frac{1}{x})^{3n-1}}{\int \partial x\,(l\frac{1}{x})^{4n-1}} = k\int \frac{x^{3nk-1}\,\partial x}{(1-x^k)^{1-n}},$$

quibus conjungendis adipiscimur

$$\frac{[\int \partial x\,(l\frac{1}{x})^{n-1}]^4}{\int \partial x\,(l\frac{1}{x})^{4n-1}} = k^3\int \frac{x^{nk-1}\,\partial x}{(1-x^k)^{1-n}} \times \int \frac{x^{2nk-1}\,\partial x}{(1-x^k)^{1-n}} \times \int \frac{x^{3nk-1}\,\partial x}{(1-x^k)^{1-n}}.$$

Sit nunc $n = \frac{i}{4}$, et sumatur $k = 4$, fietque

$$\frac{\int \partial x\,(l\frac{1}{x})^{\frac{i}{4}-1}}{\sqrt[4]{1.2.3\ldots\ldots(i-1)}} = \sqrt[4]{4^3}\int \frac{x^{i-1}\,\partial x}{\sqrt[4]{(1-x^4)^{4-i}}} \times \int \frac{x^{2i-1}\,\partial x}{\sqrt[4]{(1-x^4)^{4-i}}} \times \int \frac{x^{3i-1}\,\partial x}{\sqrt[4]{(1-x^4)^{4-i}}}$$

Corollarium 1.

§. 35. Si igitur sit $i = 1$, habebimus

$$\int \partial x\,\sqrt[4]{(l\tfrac{1}{x})^{-3}} = \sqrt[4]{4^3}\int \frac{\partial x}{\sqrt[4]{(1-x^4)^3}} \times \int \frac{x\,\partial x}{\sqrt[4]{(1-x^4)^3}} \times \int \frac{x\,x\,\partial x}{\sqrt[4]{(1-x^4)^3}}$$

quae expressio si littera P designetur, erit in genere

$$\int \partial x \sqrt[4]{(l\tfrac{1}{x})^{4n-3}} = \tfrac{1}{4}\cdot\tfrac{5}{4}\cdot\tfrac{9}{4}\ldots\ldots\ldots\ldots \tfrac{4n-3}{4}\cdot P.$$

Corollarium 2.

§. 36. Pro altero casu principali sumamus $i = 3$, eritque

$$\int \partial x \sqrt[4]{(l\tfrac{1}{x})^{-1}} = \sqrt[4]{2.4^3} \int \frac{x^2 \partial x}{\sqrt[4]{(1-x^4)}} \times \int \frac{x^5 \partial x}{\sqrt[4]{(1-x^4)}} \times \int \frac{x^8 \partial x}{\sqrt[4]{(1-x^4)}},$$

seu facta reductione ad simpliciores formas

$$\int \partial x \sqrt[4]{(l\tfrac{1}{x})^{-1}} = \sqrt[4]{8} \int \frac{x x \partial x}{\sqrt[4]{(1-x^4)}} \times \int \frac{x \partial x}{\sqrt[4]{(1-x^4)}} \times \int \frac{\partial x}{\sqrt[4]{(1-x^4)}},$$

quae expressio si littera Q designetur, erit generatim

$$\int \partial x \sqrt[4]{(l\tfrac{1}{x})^{4n-1}} = \tfrac{3}{4} \cdot \tfrac{7}{4} \cdot \tfrac{11}{4} \ldots\ldots\ldots \tfrac{4n-1}{4} \cdot Q.$$

Scholion.

§. 37. Si formulam integralem

$$\int \frac{x^{p-1} \partial x}{\sqrt[4]{(1-x^4)^{4-q}}}$$

hoc signo $\left(\frac{p}{q}\right)$ indicemus, solutio problematis ita se habebit

$$\int \partial x \sqrt[4]{(l\tfrac{1}{x})^{i-4}} = \sqrt[4]{1.2.3\ldots\ldots(i-1).4^3 \left(\tfrac{i}{i}\right)\left(\tfrac{2i}{i}\right)\left(\tfrac{3i}{i}\right)},$$

et pro binis casibus evolutis fit

$$P = \sqrt[4]{4^3 \left(\tfrac{1}{1}\right)\left(\tfrac{2}{1}\right)\left(\tfrac{3}{1}\right)} \text{ et } Q = \sqrt[4]{8 \left(\tfrac{3}{3}\right)\left(\tfrac{2}{3}\right)\left(\tfrac{1}{3}\right)}.$$

Statuamus nunc pro iis formulis quae a circulo pendent

$$\left(\tfrac{3}{1}\right) = \frac{\pi}{4 \sin. \frac{\pi}{4}} = \alpha \text{ et } \left(\tfrac{2}{2}\right) = \frac{\pi}{4 \sin. \frac{2\pi}{4}} = \beta,$$

pro transcendentibus autem altioris ordinis

$$\left(\tfrac{2}{1}\right) = \int \frac{x \partial x}{\sqrt[4]{(1-x^4)^3}} = \int \frac{\partial x}{\sqrt[4]{(1-x^4)}} = A,$$

quippe a qua omnes reliquae pendent ac reperiemus

$$P = \sqrt[4]{4^5 \cdot \frac{\alpha\alpha}{\beta} \cdot AA} \text{ et } Q = \sqrt[4]{4 \cdot \alpha\alpha\beta \cdot \frac{1}{AA}};$$

unde patet esse

$$PQ = 4\alpha = \frac{\pi}{\sin.\frac{\pi}{4}}.$$

Cum autem sit

$$\alpha = \frac{\pi}{2\sqrt{2}} \text{ et } \beta = \frac{\pi}{4}, \text{ erit}$$

$$P = \sqrt[4]{32\pi AA}, \quad Q = \sqrt[4]{\frac{\pi^3}{8AA}} \text{ et } \frac{P}{Q} = \frac{4A}{\sqrt{\pi}}.$$

Problema 5.

§. 38. *Denotante i numerum integrum positivum, definire valorem formulae integralis $\int \partial x \sqrt[5]{(l\frac{1}{x})^{i-5}}$, integratione ab $x = 0$ ad $x = 1$ extensa.*

Solutio.

Ex praecedentibus solutionibus jam satis est perspicuum pro hoc casu perventum iri ad hanc formam

$$\frac{\int \partial x \sqrt[5]{(l\frac{1}{x})^{i-5}}}{\sqrt[5]{1.\,2.\,3 \ldots (i-1)}} = \sqrt[5]{5^4 \int \frac{x^{i-1}\partial x}{\sqrt[5]{(1-x^5)^{5-i}}} \times \int \frac{x^{2i-1}\partial x}{\sqrt[5]{(1-x^5)^{5-i}}} \times \int \frac{x^{3i-1}\partial x}{\sqrt[5]{(1-x^5)^{5-i}}} \times \int \frac{x^{4i-1}\partial x}{\sqrt[5]{(1-x^5)^{5-i}}}},$$

quae formulae integrales ad classem quintam dissertationis meae supra allegatae sunt referendae. Quare si modo ibi recepto signum $\left(\frac{p}{q}\right)$ denotet hanc formulam

$$\int \frac{x^{p-1}\partial x}{\sqrt[5]{(1-x^5)^{5-q}}},$$

valorem quaesitum ita commodius exprimere licebit, ut sit

$$\int \partial x \sqrt[5]{(l\tfrac{1}{x})^{i-5}} = \sqrt[5]{1.2.3\ldots(i-1)5^4\left(\tfrac{i}{i}\right)\left(\tfrac{2i}{i}\right)\left(\tfrac{3i}{i}\right)\left(\tfrac{4i}{i}\right)},$$

ubi quidem sufficit ipsi i valores quinario minores tribuisse, quando autem numeratores quinarium superant, tenendum est esse

$$\left(\frac{5+m}{i}\right) = \frac{m}{m+i}\left(\frac{m}{i}\right);$$

tum vero porro

$$\left(\frac{10+m}{i}\right) = \frac{m}{m+i}\cdot\frac{m+5}{m+i+5}\left(\frac{m}{i}\right)$$

$$\left(\frac{15+m}{i}\right) = \frac{m}{m+i}\cdot\frac{m+5}{m+i+5}\cdot\frac{m+10}{m+i+10}\left(\frac{m}{i}\right).$$

Deinde vero pro hac classe binae formulae quadraturam circuli involvunt, quae sint

$$\left(\tfrac{4}{1}\right) = \frac{\pi}{5\sin.\frac{\pi}{5}} = \alpha \text{ et } \left(\tfrac{3}{2}\right) = \frac{\pi}{5\sin.\frac{2\pi}{5}} = \beta,$$

duae autem quadraturas altiores continent, quae ponantur

$$\left(\tfrac{3}{1}\right) = \int \frac{xx\,\partial x}{\sqrt[5]{(1-x^5)^4}} = \int \frac{\partial x}{\sqrt[5]{(1-x^5)^2}} = \mathrm{A} \text{ et}$$

$$\left(\tfrac{2}{2}\right) = \int \frac{x\,\partial x}{\sqrt[5]{(1-x^5)^3}} = \mathrm{B};$$

atque ex his valores omnium reliquarum formularum hujus classis assignavi, scilicet

$$\left(\tfrac{5}{1}\right) = 1;\ \left(\tfrac{5}{2}\right) = \tfrac{1}{2};\ \left(\tfrac{5}{3}\right) = \tfrac{1}{3};\ \left(\tfrac{5}{4}\right) = \tfrac{1}{4};\ \left(\tfrac{5}{5}\right) = \tfrac{1}{5}$$

$$\left(\tfrac{4}{1}\right) = \alpha;\ \left(\tfrac{4}{2}\right) = \frac{\beta}{\mathrm{A}};\ \left(\tfrac{4}{3}\right) = \frac{\beta}{2\mathrm{B}};\ \left(\tfrac{4}{4}\right) = \frac{\alpha}{3\mathrm{A}}$$

$$\left(\tfrac{3}{1}\right) = \mathrm{A};\ \left(\tfrac{3}{2}\right) = \beta;\ \left(\tfrac{3}{3}\right) = \frac{\beta\beta}{\alpha\mathrm{B}}$$

$$\left(\tfrac{2}{1}\right) = \frac{\alpha\mathrm{B}}{\beta};\ \left(\tfrac{2}{2}\right) = \mathrm{B}$$

$$\left(\tfrac{1}{1}\right) = \frac{\alpha\mathrm{A}}{\beta}.$$

Corollarium 1.

§. 39. Sumto exponente $i = 1$, erit

$$\int \partial x \sqrt[5]{(l\tfrac{1}{x})^{-4}} = \sqrt[5]{5^4 \left(\tfrac{1}{1}\right)\left(\tfrac{2}{1}\right)\left(\tfrac{3}{1}\right)\left(\tfrac{4}{1}\right)} = \sqrt[5]{5^4 \cdot \frac{\alpha^5}{\beta^2} A^2 B};$$

unde in genere concludimus fore, denotante n numerum integrum quemcunque

$$\int \partial x \sqrt[5]{(l\tfrac{1}{x})^{5n-4}} = \tfrac{1}{5} \cdot \tfrac{6}{5} \cdot \tfrac{11}{5} \ldots \tfrac{5n-4}{5} \cdot \sqrt[5]{5^4 \cdot \frac{\alpha^5}{\beta^2} A^2 B}$$

Corollarium 2.

§. 40. Sit nunc $i = 2$, et cum prodeat

$$\int \partial x \sqrt[5]{(l\tfrac{1}{x})^{-3}} = \sqrt[5]{1.\, 5^4 \left(\tfrac{2}{2}\right)\left(\tfrac{4}{2}\right)\left(\tfrac{6}{2}\right)\left(\tfrac{8}{2}\right)}, \text{ ob}$$

$$\left(\tfrac{6}{2}\right) = \tfrac{1}{3}\left(\tfrac{1}{2}\right) = \tfrac{1}{3}\left(\tfrac{2}{1}\right) \text{ et } \left(\tfrac{8}{2}\right) = \tfrac{3}{5}\left(\tfrac{3}{2}\right),$$

erit haec expressio

$$\sqrt[5]{5^3 \left(\tfrac{2}{2}\right)\left(\tfrac{4}{2}\right)\left(\tfrac{2}{1}\right)\left(\tfrac{3}{2}\right)} \sqrt[5]{5^3 \cdot \alpha\beta \cdot \frac{BB}{A}}$$

et in genere

$$\int \partial x \sqrt[5]{(l\tfrac{1}{x})^{5n-3}} = \tfrac{2}{5} \cdot \tfrac{7}{5} \cdot \tfrac{12}{5} \ldots \tfrac{5n-3}{5} \sqrt[5]{5^3 . \alpha\beta . \frac{BB}{A}}.$$

Corollarium 3.

§. 41. Sit $i = 3$, et forma inventa

$$\int \partial x \sqrt[5]{(l\tfrac{1}{x})^{-2}} = \sqrt[5]{2.\, 5^4 \left(\tfrac{3}{3}\right)\left(\tfrac{6}{3}\right)\left(\tfrac{9}{3}\right)\left(\tfrac{12}{3}\right)}, \text{ ob}$$

$$\left(\tfrac{6}{3}\right) = \tfrac{1}{4}\left(\tfrac{3}{1}\right);\ \left(\tfrac{9}{3}\right) = \tfrac{4}{7}\left(\tfrac{4}{3}\right);\ \left(\tfrac{12}{3}\right) = \tfrac{2}{5} \cdot \tfrac{7}{10}\left(\tfrac{3}{2}\right), \text{ abit in}$$

$$\sqrt[5]{2.\, 5^2 \left(\tfrac{3}{3}\right)\left(\tfrac{3}{1}\right)\left(\tfrac{4}{3}\right)\left(\tfrac{3}{2}\right)} = \sqrt[5]{5^2 \cdot \frac{\beta^4}{\alpha} \cdot \frac{A}{BB}};$$

unde in genere colligitur

$$\int \partial x \sqrt[5]{(l\tfrac{1}{x})^{5n-2}} = \tfrac{3}{5} \cdot \tfrac{8}{5} \cdot \tfrac{13}{5} \ldots \tfrac{5n-2}{5} \sqrt[5]{5^2 \cdot \frac{\beta^4}{\alpha} \cdot \frac{A}{BB}}.$$

Corollarium 4.

§. 42. Posito denique $i = 4$, forma nostra

$$\int \partial x \sqrt[5]{(l\tfrac{1}{x})^{-1}} = \sqrt[5]{6 \,.\, 5^4 \left(\tfrac{4}{4}\right)\left(\tfrac{8}{4}\right)\left(\tfrac{12}{4}\right)\left(\tfrac{16}{4}\right)}, \text{ ob}$$

$$\left(\tfrac{8}{4}\right) = \tfrac{3}{7}\left(\tfrac{4}{3}\right);\ \left(\tfrac{12}{4}\right) = \tfrac{2}{6}\cdot\tfrac{7}{11}\left(\tfrac{4}{2}\right);\ \left(\tfrac{16}{4}\right) = \tfrac{1}{5}\cdot\tfrac{6}{10}\cdot\tfrac{11}{15}\left(\tfrac{4}{1}\right),$$

transformabitur in hanc

$$\sqrt[5]{6 \,.\, 5 \left(\tfrac{4}{4}\right)\left(\tfrac{4}{3}\right)\left(\tfrac{4}{2}\right)\left(\tfrac{4}{1}\right)} = \sqrt[5]{5 \,.\, \tfrac{\alpha\alpha\beta\beta}{AAB}};$$

ita ut sit in genere

$$\int \partial x \sqrt[5]{(l\tfrac{1}{x})^{5n-1}} = \tfrac{4}{5}\cdot\tfrac{9}{5}\cdot\tfrac{14}{5}\ldots\tfrac{5n-1}{5}\sqrt[5]{5 \,.\, \alpha\alpha\beta\beta \,.\, \tfrac{1}{AAB}}.$$

Scholion.

§. 43. Si valorem formulae integralis $\int \partial x (l\frac{1}{x})^\lambda$ hoc signo $[\lambda]$ repraesentemus, casus hactenus evoluti praebent

$$[-\tfrac{4}{5}] = \sqrt[5]{5^4 \,.\, \tfrac{\alpha^3}{\beta^2} \,.\, A^2 B};\ [+\tfrac{1}{5}] = \tfrac{1}{5}\sqrt[5]{5^4 \,.\, \tfrac{\alpha^3}{\beta^2} \,.\, A^2 B}$$

$$[-\tfrac{3}{5}] = \sqrt[5]{5^3 \,.\, \alpha\beta \,.\, \tfrac{BB}{A}};\ [+\tfrac{2}{5}] = \tfrac{2}{5}\sqrt[5]{5^3 \,.\, \alpha\beta \,.\, \tfrac{BB}{A}}$$

$$[-\tfrac{2}{5}] = \sqrt[5]{5^2 \,.\, \tfrac{\beta^4}{\alpha} \,.\, \tfrac{A}{BB}};\ [+\tfrac{3}{5}] = \tfrac{3}{5}\sqrt[5]{5^2 \,.\, \tfrac{\beta^4}{\alpha} \,.\, \tfrac{A}{BB}}$$

$$[-\tfrac{1}{5}] = \sqrt[5]{5 \,.\, \alpha^2\beta^2 \,.\, \tfrac{1}{AAB}};\ [+\tfrac{4}{5}] = \tfrac{4}{5}\sqrt[5]{5 \cdot \alpha^2\beta^2 \cdot \tfrac{1}{BBA}};$$

unde binis, quarum indices simul sumti fiunt $= 0$, conjungendis colligimus

$$[+\tfrac{1}{5}] \cdot [-\tfrac{1}{5}] = \alpha = \frac{\pi}{5 \sin. \frac{\pi}{5}}$$

$$[+\tfrac{2}{5}] \cdot [-\tfrac{2}{5}] = 2\beta = \frac{2\pi}{5 \sin. \frac{2\pi}{5}}$$

$$[+\tfrac{3}{5}]\cdot[-\tfrac{3}{5}] = 3\beta = \frac{3\pi}{5\sin.\frac{3\pi}{5}}$$

$$[+\tfrac{4}{5}]\cdot[-\tfrac{4}{5}] = 4\alpha = \frac{2\pi}{5\sin.\frac{4\pi}{5}}$$

Ex antecedente autem problemate simili modo deducimus

$$[-\tfrac{3}{4}] = \mathrm{P} = \sqrt[4]{4^3.\tfrac{\alpha\alpha}{\beta}.\mathrm{AA}};\quad [+\tfrac{1}{4}] = \tfrac{1}{4}\sqrt[4]{4^3.\tfrac{\alpha\alpha}{\beta}.\mathrm{AA}}$$

$$[-\tfrac{1}{4}] = \mathrm{Q} = \sqrt[4]{4.\alpha\alpha\beta.\tfrac{1}{\mathrm{AA}}};\quad [+\tfrac{3}{4}] = \tfrac{3}{4}\sqrt[4]{4.\alpha\alpha\beta.\tfrac{1}{\mathrm{AA}}}$$

hincque

$$[+\tfrac{1}{4}]\cdot[-\tfrac{1}{4}] = \alpha = \frac{\pi}{4\sin.\frac{\pi}{4}}$$

$$[+\tfrac{3}{4}]\cdot[-\tfrac{3}{4}] = 3\alpha = \frac{3\pi}{4\sin.\frac{3\pi}{4}};$$

unde in genere hoc Theorema adipiscimur, quod sit

$$[\lambda]\cdot[-\lambda] = \frac{\lambda\pi}{\sin.\lambda\pi},$$

cujus ratio ex methodo interpolandi olim exposita ita reddi potest

$$\text{cum sit } [\lambda] = \frac{1^{1-\lambda}.2^{\lambda}}{1+\lambda}\cdot\frac{2^{1-\lambda}.3^{\lambda}}{2+\lambda}\cdot\frac{3^{1-\lambda}.4^{\lambda}}{3+\lambda} \text{ etc.}$$

$$\text{erit } [-\lambda] = \frac{1^{1+\lambda}.2^{-\lambda}}{1-\lambda}\cdot\frac{2^{1+\lambda}.3^{-\lambda}}{2-\lambda}\cdot\frac{3^{1+\lambda}.4^{-\lambda}}{3-\lambda} \text{ etc.}$$

hincque

$$[\lambda]\cdot[-\lambda] = \frac{1.1}{1-\lambda\lambda}\cdot\frac{2.2}{4-\lambda\lambda}\cdot\frac{3.3}{9-\lambda\lambda} \text{ etc.} = \frac{\lambda\pi}{\sin.\lambda\pi};$$

uti alibi demonstravi.

Problema 6 generale.

§. 44. *Si litterae i et n denotent numeros integros positivos, definire valorem formulae integralis*

$$\int \partial x \left(l\tfrac{1}{x}\right)^{\frac{i-n}{n}}, \text{ seu } \int \partial x \sqrt[n]{\left(l\tfrac{1}{x}\right)^{i-n}},$$

integratione ab $x=0$ ad $x=1$ extensa.

Solutio.

Methodus hactenus usitata quaesitum valorem sequenti modo per quadraturas curvarum algebraicarum expressum exhibebit

$$\frac{\int \partial x \sqrt[n]{\left(l\frac{1}{x}\right)^{i-n}}}{\sqrt[n]{1.2.3\ldots(i-1)}} = \sqrt[n]{n^{n-1}} \int \frac{x^{i-1}\partial x}{\sqrt[n]{(1-x^n)^{n-i}}} \times \int \frac{x^{2i-1}\partial x}{\sqrt[n]{(1-x^n)^{n-i}}} \times \ldots \int \frac{x^{(n-1)i-1}\partial x}{\sqrt[n]{(1-x^n)^{n-i}}}.$$

Quod si jam brevitatis gratia formulam integralem

$$\int \frac{x^{p-1}\partial x}{\sqrt[n]{(1-x^n)^{n-q}}} \text{ hoc charactere } \left(\frac{p}{q}\right),$$

formulam vero $\int \partial x \sqrt[n]{\left(l\frac{1}{x}\right)^m}$ isthoc $\left[\frac{m}{n}\right]$ designemus, ita ut $\left[\frac{m}{n}\right]$ valorem hujus producti indefiniti $1.2.3\ldots\ldots z$ denotet, existente $z=\frac{m}{n}$, succinctius valor quaesitus hoc modo expressus prodibit

$$\left[\frac{i-n}{n}\right] = \sqrt[n]{1.2.3\ldots(i-1)n^{n-1}\cdot\left(\frac{i}{i}\right)\left(\frac{2i}{i}\right)\left(\frac{3i}{i}\right)\ldots\ldots\left(\frac{ni-i}{i}\right)};$$

unde etiam colligitur

$$\left[\frac{i}{n}\right] = \frac{i}{n}\sqrt[n]{1.2.3\ldots(i-1)n^{n-1}\cdot\left(\frac{i}{i}\right)\left(\frac{2i}{i}\right)\left(\frac{3i}{i}\right)\ldots\ldots\left(\frac{ni-i}{i}\right)}.$$

Hic semper numerum i ipso n minorem accepisse sufficiet, quoniam pro majoribus notum est esse:

$$\left[\frac{i+n}{n}\right]=\frac{i+n}{n}\left[\frac{i}{n}\right],\text{ item }\left[\frac{i+2n}{n}\right]=\frac{i+n}{n}\cdot\frac{i+2n}{n}\left[\frac{i}{n}\right]\text{ etc.}$$

hocque modo tota investigatio ad eos tantum casus reducitur, quibus fractionis $\frac{i}{n}$ numerator i denominatore n est minor. Praeterea vero de formulis integralibus

$$\int\frac{x^{p-1}\partial x}{\sqrt[n]{(1-x^n)^{n-q}}}=\left(\frac{q}{p}\right),$$

sequentia notasse juvabit

I. Litteras p et q inter se esse permutabiles ut sit

$$\left(\frac{p}{p}\right)=\left(\frac{q}{p}\right).$$

II. Si alteruter numerorum p vel q ipsi exponenti n aequetur, valorem formulae integralis fore algebraicum, scilicet

$$\left(\frac{n}{p}\right)=\left(\frac{p}{n}\right)=\frac{1}{p},\text{ seu }\left(\frac{n}{q}\right)=\left(\frac{q}{n}\right)=\frac{1}{q}.$$

III. Si summa numerorum $p+q$ ipsi exponenti n aequatur, formulae integralis $\left(\frac{p}{q}\right)$ valorem per circulum exhiberi posse, cum sit

$$\left(\frac{p}{n-p}\right)=\left(\frac{n-p}{p}\right)=\frac{\pi}{n\sin.\frac{p\pi}{n}};\text{ et }\left(\frac{q}{n-q}\right)=\left(\frac{n-q}{q}\right)=\frac{\pi}{n\sin.\frac{q\pi}{n}}.$$

IV. Si alteruter numerorum p vel q major sit exponente n, formulam integralem $\left(\frac{p}{q}\right)$ ad aliam revocari posse, cujus termini sint ipso n minores, quod fit ope hujus reductionis

$$\left(\frac{p+n}{q}\right)=\frac{p}{p+q}\left(\frac{p}{q}\right).$$

V. Inter plures hujusmodi formulas integrales talem relationem intercedere, ut sit

*

$$\left(\frac{p}{q}\right)\left(\frac{p+q}{r}\right)=\left(\frac{p}{r}\right)\left(\frac{p+r}{q}\right)=\left(\frac{q}{r}\right)\left(\frac{q+r}{p}\right),$$

cujus ope omnes reductiones reperiuntur, quas in observationibus circa has formulas exposui.

Corollarium 1.

§. 45. Si hoc modo ope reductionis N°. IV. indicatae formam inventam ad singulos casus accommodemus, eos sequenti ratione simplicissime exhibere poterimus. Ac primo quidem pro casu $n=2$, quo nulla opus est reductione habebimus

$$\left[\tfrac{1}{2}\right]=\tfrac{1}{2}\sqrt[2]{2\left(\tfrac{1}{1}\right)}=\tfrac{1}{2}\sqrt[2]{\frac{\pi}{\sin.\frac{\pi}{2}}}=\tfrac{1}{2}\sqrt{\pi}.$$

Corollarium 2.

§. 46. Pro casu $n=3$ habebimus has reductiones

$$\left[\tfrac{1}{3}\right]=\tfrac{1}{3}\sqrt[3]{3^2.\left(\tfrac{1}{1}\right)\left(\tfrac{2}{1}\right)}$$

$$\left[\tfrac{2}{3}\right]=\tfrac{2}{3}\sqrt[3]{3.\,1.\left(\tfrac{2}{2}\right)\left(\tfrac{1}{2}\right)}.$$

Corollarium 3.

§. 47. Pro casu $n=4$ hae tres reductiones obtinentur

$$\left[\tfrac{1}{4}\right]=\tfrac{1}{4}\sqrt[4]{4^3\left(\tfrac{1}{1}\right)\left(\tfrac{2}{1}\right)\left(\tfrac{3}{1}\right)}$$

$$\left[\tfrac{2}{4}\right]=\tfrac{2}{4}\sqrt[4]{4^2.\,2.\left(\tfrac{2}{2}\right)^2\left(\tfrac{1}{2}\right)}=\tfrac{1}{2}\sqrt[2]{4\left(\tfrac{2}{2}\right)},\ \text{ob}\ \left(\tfrac{1}{2}\right)=\tfrac{1}{2}$$

$$\left[\tfrac{3}{4}\right]=\tfrac{3}{4}\sqrt[4]{4.\,1.\,2\left(\tfrac{3}{3}\right)\left(\tfrac{2}{3}\right)\left(\tfrac{1}{3}\right)};$$

cum in media sit $\left(\frac{2}{2}\right)=\left(\frac{2}{4-2}\right)=\frac{\pi}{4}$, erit utique ut ante

$$\left[\tfrac{2}{4}\right]=\left[\tfrac{1}{2}\right]=\tfrac{1}{2}\sqrt{\pi}.$$

Corollarium 4.

§. 48. Sit nunc $n = 5$, et prodeunt hae quatuor reductiones

$$[\tfrac{1}{5}] = \tfrac{1}{5}\sqrt[5]{5^4.(\tfrac{1}{1})(\tfrac{2}{1})(\tfrac{3}{1})(\tfrac{4}{1})}$$

$$[\tfrac{2}{5}] = \tfrac{2}{5}\sqrt[5]{5^3.1(\tfrac{2}{2})(\tfrac{4}{2})(\tfrac{1}{2})(\tfrac{3}{2})}$$

$$[\tfrac{3}{5}] = \tfrac{3}{5}\sqrt[5]{5^2.1.2(\tfrac{3}{3})(\tfrac{1}{3})(\tfrac{4}{3})(\tfrac{2}{3})}$$

$$[\tfrac{4}{5}] = \tfrac{4}{5}\sqrt[5]{5.1.2.3(\tfrac{4}{4})(\tfrac{3}{4})(\tfrac{2}{4})(\tfrac{1}{4})}.$$

Corollarium 5.

§. 49. Sit $n = 6$, et habebimus has reductiones

$$[\tfrac{1}{6}] = \tfrac{1}{6}\sqrt[6]{6^5.(\tfrac{1}{1})(\tfrac{2}{1})\tfrac{3}{1})(\tfrac{4}{1})(\tfrac{5}{1})}$$

$$[\tfrac{2}{6}] = \tfrac{2}{6}\sqrt[6]{6^4.2(\tfrac{2}{2})^2(\tfrac{4}{2})^2(\tfrac{6}{2})} = \tfrac{2}{6}\sqrt[3]{6^2(\tfrac{2}{2})(\tfrac{4}{2})}$$

$$[\tfrac{3}{6}] = \tfrac{3}{6}\sqrt[6]{6^3.3,3(\tfrac{3}{3})^3(\tfrac{6}{3})^2} = \tfrac{1}{2}\sqrt[2]{6(\tfrac{3}{3})}$$

$$[\tfrac{4}{6}] = \tfrac{4}{6}\sqrt[6]{6^2.2.4.2(\tfrac{4}{4})^2(\tfrac{2}{4})^2(\tfrac{6}{4})} = \tfrac{2}{3}\sqrt[3]{6.2(\tfrac{4}{4})(\tfrac{2}{4})}$$

$$[\tfrac{5}{6}] = \tfrac{5}{6}\sqrt[6]{6.1.2,3.4(\tfrac{5}{5})(\tfrac{4}{5})(\tfrac{3}{5})(\tfrac{2}{5})(\tfrac{1}{5})}.$$

Corollarium 6.

§. 50. Posito $n = 7$, sequentes sex prodeunt aequationes

$$[\tfrac{1}{7}] = \tfrac{1}{7}\sqrt[7]{7^6(\tfrac{1}{1})(\tfrac{2}{1})(\tfrac{3}{1})(\tfrac{4}{1})(\tfrac{5}{1})(\tfrac{6}{1})}$$

$$[\tfrac{2}{7}] = \tfrac{2}{7}\sqrt[7]{7^5.1(\tfrac{2}{2})(\tfrac{4}{2})(\tfrac{6}{2})(\tfrac{1}{2})(\tfrac{3}{2})(\tfrac{5}{2})}$$

$$[\tfrac{3}{7}] = \tfrac{3}{7}\sqrt[7]{7^4.1.2(\tfrac{3}{3})(\tfrac{6}{3})(\tfrac{2}{3})(\tfrac{5}{3})(\tfrac{1}{3})(\tfrac{4}{3})}$$

$$[\tfrac{4}{7}] = \tfrac{4}{7}\sqrt[7]{7^3.1.2.3(\tfrac{4}{4})(\tfrac{1}{4})(\tfrac{5}{4})(\tfrac{2}{4})(\tfrac{6}{4})(\tfrac{3}{4})}$$

$$\left[\tfrac{5}{7}\right] = \tfrac{5}{7} \sqrt[7]{7^2.\, 1.\, 2.\, 3.\, 4 \left(\tfrac{5}{5}\right) \left(\tfrac{3}{5}\right) \left(\tfrac{1}{5}\right) \left(\tfrac{6}{5}\right) \left(\tfrac{4}{5}\right) \left(\tfrac{2}{5}\right)}$$

$$\left[\tfrac{6}{7}\right] = \tfrac{6}{7} \sqrt[7]{7.\, 1.\, 2.\, 3.\, 4.\, 5 \left(\tfrac{6}{6}\right) \left(\tfrac{5}{6}\right) \left(\tfrac{4}{6}\right) \left(\tfrac{3}{6}\right) \left(\tfrac{2}{6}\right) \left(\tfrac{1}{6}\right)}.$$

Corollarium 7.

§. 51. Sit $n = 8$, et septem hae reductiones impetrabuntur

$$\left[\tfrac{1}{8}\right] = \tfrac{1}{8} \sqrt[8]{8^7 \left(\tfrac{1}{1}\right) \left(\tfrac{2}{1}\right) \left(\tfrac{3}{1}\right) \left(\tfrac{4}{1}\right) \left(\tfrac{5}{1}\right) \left(\tfrac{6}{1}\right) \left(\tfrac{7}{1}\right)}$$

$$\left[\tfrac{2}{8}\right] = \tfrac{2}{8} \sqrt[8]{8^6.\, 2 \left(\tfrac{2}{2}\right)^2 \left(\tfrac{4}{2}\right)^2 \left(\tfrac{6}{2}\right)^2 \left(\tfrac{8}{2}\right)} = \tfrac{1}{4} \sqrt[4]{8^3 \left(\tfrac{2}{2}\right) \left(\tfrac{4}{2}\right) \left(\tfrac{6}{2}\right)}$$

$$\left[\tfrac{3}{8}\right] = \tfrac{3}{8} \sqrt[8]{8^5.\, 1.\, 2 \left(\tfrac{3}{3}\right) \left(\tfrac{6}{3}\right) \left(\tfrac{1}{3}\right) \left(\tfrac{4}{3}\right) \left(\tfrac{7}{3}\right) \left(\tfrac{2}{3}\right) \left(\tfrac{5}{3}\right)}$$

$$\left[\tfrac{4}{8}\right] = \tfrac{4}{8} \sqrt[8]{8^4.\, 4.\, 4.\, 4 \left(\tfrac{4}{4}\right)^4 \left(\tfrac{8}{4}\right)^3} = \tfrac{1}{2} \sqrt[2]{8 \left(\tfrac{4}{4}\right)}$$

$$\left[\tfrac{5}{8}\right] = \tfrac{5}{8} \sqrt[8]{8^3.\, 1.\, 2.\, 3.\, 4 \left(\tfrac{5}{5}\right) \left(\tfrac{2}{5}\right) \left(\tfrac{7}{5}\right) \left(\tfrac{4}{5}\right) \left(\tfrac{1}{5}\right) \left(\tfrac{6}{5}\right) \left(\tfrac{3}{5}\right)}$$

$$\left[\tfrac{6}{8}\right] = \tfrac{6}{8} \sqrt[8]{8^2.\, 4.\, 2.\, 6.\, 4.\, 2 \left(\tfrac{6}{6}\right)^2 \left(\tfrac{4}{6}\right)^2 \left(\tfrac{2}{6}\right)^2 \left(\tfrac{8}{6}\right)} = \tfrac{3}{4} \sqrt[4]{8.\, 2.\, 4 \left(\tfrac{6}{6}\right) \left(\tfrac{4}{6}\right) \left(\tfrac{2}{6}\right)}$$

$$\left[\tfrac{7}{8}\right] = \tfrac{7}{8} \sqrt[8]{8.\, 1.\, 2.\, 3.\, 4.\, 5.\, 6 \left(\tfrac{7}{7}\right) \left(\tfrac{6}{7}\right) \left(\tfrac{5}{7}\right) \left(\tfrac{4}{7}\right) \left(\tfrac{3}{7}\right) \left(\tfrac{2}{7}\right) \left(\tfrac{1}{7}\right)}.$$

Scholion.

§. 52. Superfluum foret hos casus ulterius evolvere, cum ex allatis ordo istarum formularum satis perspiciatur. Si enim in formula proposita $\left[\frac{m}{n}\right]$ numeri m et n sint inter se primi lex est manifesta, cum fiat

$$\left[\tfrac{m}{n}\right] = \tfrac{m}{n} \sqrt[n]{n^{n-m}\,.\, 1.\, 2 \ldots (m-1) . \left(\tfrac{1}{m}\right) \left(\tfrac{2}{m}\right) \left(\tfrac{3}{m}\right) \cdots \left(\tfrac{n-1}{m}\right)},$$

sin autem hi numeri m et n communem habeant divisorem, expediet quidem fractionem $\frac{m}{n}$ ad minimam formam reduci, et ex casibus praecedentibus quaesitum valorem peti; interim tamen etiam operatio hoc modo instituí poterit. Cum expressio quaesita certe hanc habeat formam

$$\left[\tfrac{m}{n}\right] = \tfrac{m}{n} \sqrt[n]{n^{n-m}\, \mathrm{P}\,.\,\mathrm{Q}},$$

ubi Q est productum ex $n-1$ formulis integralibus, P vero productum ex aliquot numeris absolutis, primum pro illo producto Q inveniendo, continuetur haec formularum series $\left(\frac{m}{m}\right)\left(\frac{2m}{m}\right)\left(\frac{3m}{m}\right)$, donec numerator superet exponentem n, ejusque loco excessus supra n scribatur, qui si ponatur $=\alpha$, ut jam formula nostra sit $\left(\frac{\alpha}{m}\right)$, hic ipse numerator α dabit factorem producti P, tum hinc fomularum series porro statuatur $\left(\frac{\alpha}{m}\right)\left(\frac{\alpha+m}{m}\right)\left(\frac{\alpha+2m}{m}\right)$ etc. donec iterum ad numeratorem exponente n majorem perveniatur, formulaque prodeat $\left(\frac{n+\beta}{m}\right)$, cujus loco scribi oportet $\left(\frac{\beta}{m}\right)$, simulque hinc factor β in productum P inferatur, sicque progredi conveniet, donec pro Q prodierint $n-1$ formulae. Quae operationes quo facilius intelligantur, casum formulae $\left[\frac{9}{12}\right] = \frac{9}{12} \sqrt[12]{12^3\, \mathrm{P}\,.\,\mathrm{Q}}$ hoc modo evolvamus, ubi investigatio litterarum Q et P ita instituetur,

pro Q . . . . . $\left(\frac{9}{9}\right)\left(\frac{6}{9}\right)\left(\frac{3}{9}\right)\left(\frac{12}{9}\right)\left(\frac{9}{9}\right)\left(\frac{6}{9}\right)\left(\frac{3}{9}\right)\left(\frac{12}{9}\right)\left(\frac{9}{9}\right)\left(\frac{6}{9}\right)\left(\frac{3}{9}\right)$,
pro P 6. 3   9. 6. 3   9. 6. 3,

sicque reperitur

$$\mathrm{Q} = \left(\tfrac{9}{9}\right)^3 \left(\tfrac{6}{9}\right)^3 \left(\tfrac{3}{9}\right)^3 \left(\tfrac{12}{9}\right)^2 \text{ et}$$
$$\mathrm{P} = 6^3.\, 3^3.\, 9^2.$$

Cum igitur sit $\left(\frac{12}{9}\right) = \frac{1}{9}$, fit $\mathrm{P\,Q} = 6^3.\, 3^3 \left(\frac{9}{9}\right)^3 \left(\frac{6}{9}\right)^3 \left(\frac{3}{9}\right)^3$, ideoque

$$\left[\tfrac{9}{12}\right] = \tfrac{3}{4} \sqrt[4]{12.\, 6.\, 3.\, \left(\tfrac{9}{9}\right) \left(\tfrac{6}{9}\right) \left(\tfrac{3}{9}\right)}.$$

Theorema.

§. 33. Quicunque numeri integri positivi litteris m et n indicentur, erit semper signandi modo ante exposito nunc vero per theorema

$$\left[\tfrac{7}{6}\right] = \tfrac{2}{6}\sqrt[6]{6^4}\,.\,1\left(\tfrac{1}{2}\right)\left(\tfrac{2}{2}\right)\left(\tfrac{3}{2}\right)\left(\tfrac{4}{2}\right)\left(\tfrac{5}{2}\right),$$

ideoque necesse est sit

$$\left(\tfrac{2}{2}\right)\left(\tfrac{4}{2}\right) = \left(\tfrac{1}{2}\right)\left(\tfrac{3}{2}\right)\left(\tfrac{5}{2}\right),$$

cujus veritas indidem patet.

Corollarium 3.

§. 56. Si $m = 3$ et $n = 6$, pervenitur ad hanc aequationem

$$\left(\tfrac{3}{3}\right)^2 = 1\,.\,2\,.\left(\tfrac{1}{3}\right)\left(\tfrac{2}{3}\right)\left(\tfrac{4}{3}\right)\left(\tfrac{5}{3}\right),$$

at si $m = 4$ et $n = 6$, fit simili modo

$$2^2\left(\tfrac{4}{4}\right)\left(\tfrac{2}{4}\right) = 1.2.3.\left(\tfrac{1}{4}\right)\left(\tfrac{3}{4}\right)\left(\tfrac{5}{4}\right),\text{ seu}$$

$$\left(\tfrac{4}{4}\right)\left(\tfrac{2}{4}\right) = \tfrac{3}{2}\left(\tfrac{1}{4}\right)\left(\tfrac{3}{4}\right)\left(\tfrac{5}{4}\right),$$

quod etiam verum deprehenditur.

Corollarium 4.

§. 57. Casus $m = 2$ et $n = 8$ praebet hanc aequalitatem

$$\left(\tfrac{2}{2}\right)\left(\tfrac{4}{2}\right)\left(\tfrac{6}{2}\right) = \left(\tfrac{1}{2}\right)\left(\tfrac{3}{2}\right)\left(\tfrac{5}{2}\right)\left(\tfrac{7}{2}\right);$$

at casus $m = 4$ et $n = 8$ hanc

$$\left(\tfrac{4}{4}\right)^3 = 1.2.3\left(\tfrac{1}{4}\right)\left(\tfrac{2}{4}\right)\left(\tfrac{3}{4}\right)\left(\tfrac{5}{4}\right)\left(\tfrac{6}{4}\right)\left(\tfrac{7}{4}\right);$$

casus denique $m = 6$ et $n = 8$ istam

$$2.4\left(\tfrac{6}{6}\right)\left(\tfrac{4}{6}\right)\left(\tfrac{2}{6}\right) = 1\,.\,3\,.\,5\left(\tfrac{1}{6}\right)\left(\tfrac{3}{6}\right)\left(\tfrac{5}{6}\right)\left(\tfrac{7}{6}\right),$$

quae etiam veritati sunt consentaneae.

Scholion.

§. 58. In genere autem si numeri m et n communem habeant factorem 2, et formula proposita sit $\left[\frac{2m}{2n}\right] = \left[\frac{m}{n}\right]$ quia est

$$[\tfrac{m}{n}] = \tfrac{m}{n} \sqrt[n]{n^{n-m} . 1.2.3 \ldots (m-1) (\tfrac{1}{m}) (\tfrac{2}{m}) (\tfrac{3}{m}) \ldots (\tfrac{n-1}{m})}.$$

Demonstratio.

Pro casu, quo m et n sunt numeri inter se primi, veritas theorematis in antecedentibus est evicta; quod autem etiam locum habeat, si illi numeri m et n commune divisore gaudeant, inde quidem non liquet: verum ex hoc ipso, quod pro casibus, quibus m et n sunt numeri primi, veritas constat, tuto concludere licet, theorema in genere esse verum. Minime quidem diffiteor hoc concludendi genus prorsus esse singulare, ac plerisque suspectum videri debere. Quare quo nullum dubium relinquatur, quoniam pro casibus, quibus numeri m et n inter se sunt compositi, geminam expressionem sumus nacti, utriusquo consensum pro casibus ante evolutis ostendisse juvabit. Insigne autem jam suppeditat firmamentum casus $m = n$, quo forma nostra manifestò unitatem producit.

Corollarium 1.

§. 54. Primus casus consensus demonstrationem postulans est quo $m = 2$ et $n = 4$, pro quo supra §. 47 invenimus

$$[\tfrac{2}{4}] = \tfrac{2}{4} \sqrt[4]{4^2 \cdot (\tfrac{2}{2})^2},$$

nunc autem vi theorematis est

$$[\tfrac{2}{4}] = \tfrac{2}{4} \sqrt[4]{4^2 . 1 (\tfrac{1}{2}) (\tfrac{2}{2}) (\tfrac{3}{2})},$$

unde comparatione instituta fit $(\tfrac{2}{2}) = (\tfrac{1}{2})(\tfrac{3}{2})$, cujus veritas in observationibus supra allegatis est confirmata.

Corollarium 2.

§. 55. Si $m = 2$ et $n = 6$, ex superioribus §. 49 est

$$[\tfrac{2}{6}] = \tfrac{2}{6} \sqrt[6]{6^4 . (\tfrac{2}{2})^2 (\tfrac{4}{2})^2}$$

$$\left[\tfrac{m}{n}\right] = \tfrac{m}{n}\sqrt[n]{n^{n-m}.1.2.3\ldots(m-1)\left(\tfrac{1}{m}\right)\left(\tfrac{2}{m}\right)\left(\tfrac{3}{m}\right)\cdots\left(\tfrac{n-1}{m}\right)},$$

erit eadem ad exponentem $2n$ reducta

$$\frac{m}{n}\sqrt[2n]{2n^{2n-2m}.2^2.4^2.6^2\ldots(2m-2)^2\left(\frac{2}{2m}\right)^2\left(\frac{4}{2m}\right)^2\left(\frac{6}{2m}\right)^2\cdots\left(\frac{2n-2}{2m}\right)^2}.$$

Per theorema vero eadem expressio fit

$$\frac{m}{n}\sqrt[2n]{2n^{2n-2m}.1.2.3\ldots(2m-1)\left(\frac{1}{2m}\right)\left(\frac{2}{2m}\right)\left(\frac{3}{2m}\right)\cdots\left(\frac{2n-1}{2m}\right)},$$

unde pro exponente $2n$ erit

$$2.4.6\ldots(2m-2)\left(\frac{2}{2m}\right)\left(\frac{4}{2m}\right)\left(\frac{6}{2m}\right)\cdots\left(\frac{2n-2}{2m}\right) =$$
$$1.3.5\ldots(2m-1)\left(\frac{1}{2m}\right)\left(\frac{3}{2m}\right)\left(\frac{5}{2m}\right)\cdots\left(\frac{2n-1}{2m}\right).$$

Simili modo si communis divisor sit 3, pro exponente $3n$ reperietur

$$3^2.6^2.9^2\ldots(3m-3)^2\left(\frac{3}{3m}\right)^2\left(\frac{6}{3m}\right)^2\left(\frac{9}{3m}\right)^2\cdots\left(\frac{3n-3}{3m}\right)^2 =$$
$$1.2.4.5..(3m-2)(3m-1)\left(\frac{1}{3m}\right)\left(\frac{2}{3m}\right)\left(\frac{4}{3m}\right)\left(\frac{5}{3m}\right)\cdots\left(\frac{3n-1}{3m}\right).$$

quae aequatio concinnius ita exhiberi potest

$$\frac{1.2.4.5.7.8.10\ldots(3m-2)(3m-1)}{3^2.6^2.9^2\ldots(3m-3)^2} = \frac{\left(\frac{3}{3m}\right)^2\cdot\left(\frac{6}{3m}\right)^2\cdots\cdots\left(\frac{3n-3}{3m}\right)^2}{\left(\frac{1}{3m}\right)\left(\frac{2}{3m}\right)\left(\frac{4}{3m}\right)\left(\frac{5}{3m}\right)\left(\frac{7}{3m}\right)\cdots\left(\frac{3n-2}{3m}\right)\left(\frac{3n-1}{3m}\right)}.$$

In genere autem si communis divisor sit d et exponens dn, habebitur

$$\left[d.2d.3d\ldots(dm-d)\left(\frac{d}{dm}\right)\left(\frac{2d}{dm}\right)\left(\frac{3d}{dm}\right)\cdots\left(\frac{dn-d}{dm}\right)\right]^d =$$
$$1.2.3.4\ldots(dm-1)\left(\frac{1}{dm}\right)\left(\frac{2}{dm}\right)\left(\frac{3}{dm}\right)\cdots\left(\frac{dn-1}{dm}\right),$$

quae aequatio facile ad quosvis casus accommodari potest, unde sequens Theorema notari meretur.

Theorema.

§. 59. Si α fuerit divisor communis numerorum m et n, haecque formula $\left(\frac{p}{q}\right)$ denotet valorem integralis

$$\int \frac{x^{p-1}\partial x}{\sqrt[n]{(1-x^n)^{n-q}}}$$

ab $x=0$ usque ad $x=1$ extensi, erit

$$\left[\alpha.2\alpha.3\alpha.\ldots.(m-\alpha)\left(\frac{\alpha}{m}\right)\left(\frac{2\alpha}{m}\right)\left(\frac{3\alpha}{m}\right)\ldots\ldots\left(\frac{n-\alpha}{m}\right)\right]^{\alpha} =$$
$$1.2.3.\ldots.(m-1)\left(\frac{1}{m}\right)\left(\frac{2}{m}\right)\left(\frac{3}{m}\right)\ldots\ldots\left(\frac{n-1}{m}\right).$$

Demonstratio.

Ex praecedente scholio veritas hujus theorematis perspicitur, cum enim ibi divisor communis esset $=d$, binique numeri propositi dm et dn, horum loco hic scripsi m et n, loco divisoris eorum autem d litteram α, quam divisoris rationem aequalitas enunciata ita complectitur, ut in progressione arithmetica α, 2α, 3α, etc. continuata occurrere assumantur ipsi numeri m et n ideoque etiam $m-\alpha$ et $n-\alpha$. Caeterum fateri cogor, hanc demonstrationem utpote inductioni potissimum innixam, neutiquam pro rigorosa haberi posse: cum autem nihilominus de ejus veritate simus convicti, hoc theorema eo majori attentione dignum videtur, interim tamen nullum est dubium, quin uberior hujusmodi formularum integralium evolutio tandem perfectam demonstrationem sit largitura, quod autem jam ante hanc veritatem nobis perspicere licuerit, insigne hinc specimen analyticae investigationis elucet.

Corollarium 1.

§. 60. Si loco signorum adhibitorum ipsas formulas integrales substituamus, theorema nostrum ita se habebit ut sit

*

$$\alpha\, 2\alpha . 3\alpha \ldots (m-\alpha)\int\frac{x^{\alpha-1}\partial x}{\sqrt[n]{(1-x^n)^{n-m}}}\cdot\int\frac{x^{2\alpha-1}\partial x}{\sqrt[n]{(1-x^n)^{n-m}}}\cdots\int\frac{x^{n-\alpha-1}\partial x}{\sqrt[n]{(1-x^n)^{n-m}}}=$$

$$\sqrt{1.2.3\ldots(m-1)}\int\frac{\partial x}{\sqrt[n]{(1-x^n)^{n-m}}}\cdot\int\frac{x\,\partial x}{\sqrt[n]{(1-x^n)^{n-m}}}\cdots\int\frac{x^{n-2}\partial x}{\sqrt[n]{(1-x^n)^{n-m}}}.$$

Corollarium 2.

§. 60. Vel si ad abbreviandum statuamus

$\sqrt[n]{(1-x^n)^{n-m}}=X$, erit

$$\alpha . 2\alpha . 3\alpha \ldots (m-\alpha)\int\frac{x^{\alpha-1}dx}{X}\cdot\int\frac{x^{2\alpha-1}dx}{X}\cdots\cdots\int\frac{x^{n-\alpha-1}dx}{X}=$$

$$\sqrt[\alpha]{1.2.3\ldots(m-1)}\int\frac{dx}{X}\cdot\int\frac{x\,dx}{X}\cdot\int\frac{x^2\,dx}{X}\cdots\cdots\int\frac{x^{n-2}\,dx}{X}.$$

Theorema generale.

§. 62. Si binorum numerorum m et n divisores communes sint α, β, γ etc. formulaque $\left(\frac{p}{q}\right)$ denotet valorem integralis

$$\int\frac{x^{p-1}\partial x}{\sqrt[n]{(1-x^n)^{n-q}}}$$

ab $x=0$ ad $x=1$ extensi; sequentes expressiones ex hujusmodi formulis integralibus formatae inter se erunt aequales

$$\left[\alpha . 2\alpha . 3\alpha \ldots\ldots (m-\alpha)\left(\frac{\alpha}{m}\right)\left(\frac{2\alpha}{m}\right)\left(\frac{3\alpha}{m}\right)\cdots\cdots\left(\frac{n-\alpha}{m}\right)\right]^{\alpha}=$$

$$\left[\beta . 2\beta . 3\beta \ldots\ldots (m-\beta)\left(\frac{\beta}{m}\right)\left(\frac{2\beta}{m}\right)\left(\frac{3\beta}{m}\right)\cdots\cdots\left(\frac{n-\beta}{m}\right)\right]^{\beta}=$$

$$\left[2\gamma . \gamma . 3\gamma \ldots\ldots (m-\gamma)\left(\frac{\gamma}{m}\right)\left(\frac{2\gamma}{m}\right)\left(\frac{3\gamma}{m}\right)\cdots\cdots\left(\frac{n-\gamma}{m}\right)\right]^{\gamma}=\text{etc.}$$

Demonstratio.

Ex precedente Theoremate hujus veritas manifesto sequitur, cum quaelibet harum expressionum seorsim aequetur huic

$$1.2.3\ldots\ldots(m-1)\left(\frac{1}{m}\right)\left(\frac{2}{m}\right)\left(\frac{3}{m}\right)\ldots\ldots\left(\frac{n-1}{m}\right),$$

quae unitati utpote minimo communi divisori numerorum m et n convenit. Tot igitur hujusmodi expressiones inter se aequales exhiberi possunt, quot fuerint divisores communes binorum numerorum m et n.

Corollarium 1.

§. 63. Cum sit haec formula $\left(\frac{n}{m}\right)=\frac{1}{m}$, ideoque $m\left(\frac{n}{m}\right)=1$, expressiones nostrae aequales succinctius hoc modo repraesentari possunt

$$\left[\alpha.2\alpha.3\alpha\ldots\ldots m\left(\frac{\alpha}{m}\right)\left(\frac{2\alpha}{m}\right)\left(\frac{3\alpha}{m}\right)\ldots\ldots\left(\frac{n}{m}\right)\right]^{\alpha}=$$

$$\left[\beta.2\beta.3\beta\ldots\ldots m\left(\frac{\beta}{m}\right)\left(\frac{2\beta}{m}\right)\left(\frac{3\beta}{m}\right)\ldots\ldots\left(\frac{n}{m}\right)\right]^{\beta}=$$

$$\left[\gamma.2\gamma.3\gamma\ldots\ldots m\left(\frac{\gamma}{m}\right)\left(\frac{2\gamma}{m}\right)\left(\frac{3\gamma}{m}\right)\ldots\ldots\left(\frac{n}{m}\right)\right]^{\gamma}=\text{etc.}$$

Etsi enim hic factorum numerus est auctus, tamen ratio compositionis facilius in oculos incurrit.

Corollarium 2.

§. 64. Si ergo sit $m=6$ et $n=12$, ob horum numerorum divisores communes 6, 3, 2, 1, quatuor sequentes formae inter se aequales habebuntur

$$[6(\tfrac{6}{6})(\tfrac{12}{6})]^6=[3.6(\tfrac{3}{6})(\tfrac{6}{6})(\tfrac{9}{6})(\tfrac{12}{6})]^3=$$

$$[2.4.6(\tfrac{2}{6})(\tfrac{4}{6})(\tfrac{6}{6})(\tfrac{8}{6})(\tfrac{10}{6})(\tfrac{12}{6})]^2=$$

$$1.2.3.4.5.6(\tfrac{1}{6})(\tfrac{2}{6})(\tfrac{3}{6})\ldots\ldots(\tfrac{12}{6}).$$

Corollarium 3.

§. 65. Si ultima cum penultima combinetur, nascetur haec aequatio

$$\frac{1.3.5}{2.4.6} = \frac{\left(\frac{2}{6}\right)\left(\frac{4}{6}\right)\left(\frac{6}{6}\right)\left(\frac{8}{6}\right)\left(\frac{10}{6}\right)\left(\frac{12}{6}\right)}{\left(\frac{1}{6}\right)\left(\frac{3}{6}\right)\left(\frac{5}{6}\right)\left(\frac{7}{6}\right)\left(\frac{9}{6}\right)\left(\frac{11}{6}\right)},$$

ultima autem cum antepenultima comparata praebet

$$\frac{1.2.4.5}{3.3.5.5} = \frac{\left(\frac{3}{6}\right)\left(\frac{3}{6}\right)\left(\frac{6}{6}\right)\left(\frac{6}{6}\right)\left(\frac{9}{6}\right)\left(\frac{9}{6}\right)\left(\frac{12}{6}\right)\left(\frac{12}{6}\right)}{\left(\frac{1}{6}\right)\left(\frac{2}{6}\right)\left(\frac{4}{6}\right)\left(\frac{5}{6}\right)\left(\frac{7}{6}\right)\left(\frac{8}{6}\right)\left(\frac{10}{6}\right)\left(\frac{11}{6}\right)}.$$

Scholion.

§. 66. Infinitae igitur hinc consequuntur relationes inter formulas integrales formae

$$\int \frac{x^{p-1}\,\partial x}{\sqrt[n]{(1-x^n)^{n-q}}} = \left(\frac{p}{q}\right),$$

quae eo magis sunt notatu dignae, quod singulari prorsus methodo ad eas hic sumus perducti. Ac si quis de earum veritate adhuc dubitet, observationes meas circa has formulas integrales consulat, indeque pro quovis casu oblato de veritate facile convincetur. Etsi autem illa tractatio huic confirmandae inservit, tamen relationes hic erutae eo majoris sunt momenti, quod in iis certus ordo cernitur, eaeque per omnes classes, quantumvis exponentem n accipere lubeat, facili negotio continuentur; in priori vero tractatione calculus pro classibus altioribus continuo fiat operosior et intricatior.

Supplementum continens demonstrationem Theorematis §. 53. propositi.

§. 67. Demonstrationem hanc altius peti convenit; sumatur scilicet aequatio §. 25. data, quae posito $f = 1$ et mutatis litteris est

$$\frac{\int dx\,(l\tfrac{1}{x})^{\nu-1}\times\int dx\,(l\tfrac{1}{x})^{\mu-1}}{\int dx\,(l\tfrac{1}{x})^{\mu+\nu-1}}=\varkappa\int\frac{x^{\varkappa\mu-1}\,dx}{(1-x^{\varkappa})^{1-\nu}},$$

eaque per reductiones notas hac forma repraesentetur

$$\frac{\int dx\,(l\tfrac{1}{x})^{\nu}\times\int dx\,(l\tfrac{1}{x})^{\mu}}{\int dx\,(l\tfrac{1}{x})^{\mu+\nu}}=\frac{\varkappa\,\mu\,\nu}{\mu+\nu}\int\frac{x^{\varkappa\mu-1}\,dx}{(1-x^{\varkappa})^{1-\nu}}.$$

Statuatur nunc $\nu=\frac{m}{n}$ et $\mu=\frac{\lambda}{n}$, tum vero $\varkappa=n$, ut habeamus

$$\frac{\int dx\,(l\tfrac{1}{x})^{\frac{m}{n}}\times\int dx\,(l\tfrac{1}{x})^{\frac{\lambda}{n}}}{\int dx\,(l\tfrac{1}{x})^{\frac{\lambda+m}{n}}}=\frac{\lambda m}{\lambda+m}\int\frac{x^{\lambda-1}\,dx}{\sqrt[n]{(1-x^{n})^{n-m}}}$$

quae brevitatis gratia, more supra usitato, ita concinne referatur

$$\frac{\left[\frac{m}{n}\right]\cdot\left[\frac{\lambda}{n}\right]}{\left[\frac{\lambda+m}{m}\right]}=\frac{\lambda m}{\lambda+m}\cdot\left(\frac{\lambda}{m}\right).$$

Jam loco λ successive scribantur numeri 1, 2, 3, 4 . . . . . n omnesque hae aequationes, quarum numerus est $=n$, in se invicem ducantur, et aequatio resultans erit

$$\left[\frac{m}{n}\right]^{n}\cdot\frac{\left[\frac{1}{n}\right]\left[\frac{2}{n}\right]\left[\frac{3}{n}\right]\cdots\cdots\left[\frac{n}{n}\right]}{\left[\frac{m+1}{n}\right]\left[\frac{m+2}{n}\right]\left[\frac{m+3}{n}\right]\cdots\left[\frac{m+n}{n}\right]}=$$

$$m^{n}\cdot\frac{1}{m+1}\cdot\frac{2}{m+2}\cdot\frac{3}{m+3}\cdots\cdots\frac{n}{m+n}\left(\frac{1}{m}\right)\left(\frac{2}{m}\right)\left(\frac{3}{m}\right)\cdots\left(\frac{n}{m}\right)=$$

$$m^{n}\cdot\frac{1\;.\;2\;.\;3\cdots\cdots m}{(n+1)(n+2)(n+3)\cdots\cdots(n+m)}\left(\frac{1}{m}\right)\left(\frac{2}{m}\right)\left(\frac{3}{m}\right)\cdots\cdots\left(\frac{n}{m}\right).$$

Simili autem modo pars prior transformetur ut sit

$$\left[\frac{m}{n}\right]^{n}\cdot\frac{\left[\frac{1}{n}\right]\left[\frac{2}{n}\right]\left[\frac{3}{n}\right]\cdots\cdots\left[\frac{m}{n}\right]}{\left[\frac{n+1}{n}\right]\left[\frac{n+2}{n}\right]\left[\frac{n+3}{n}\right]\cdots\cdots\left[\frac{n+m}{n}\right]},$$

cujus convenientia cum forma praecedente multiplicando per crucem, ut ajunt, sponte se prodit. Cum vero ex natura harum formularum sit

$$\left[\frac{n+1}{n}\right]=\frac{n+1}{n}\left[\frac{1}{n}\right],\ \left[\frac{n+2}{n}\right]=\frac{n+2}{n}\left[\frac{2}{n}\right],\ \left[\frac{n+3}{n}\right]=\frac{n+3}{n}\left[\frac{3}{n}\right], \text{ etc.}$$

ob harum formularum numerum $= m$, evadet haec prior pars

$$\left[\frac{m}{n}\right]^n \cdot \frac{n^m}{(n+1)(n+2)(n+3)\ldots\ldots(n+m)},$$

quae cum aequalis sit parti alteri ante exhibitae

$$m^n \cdot \frac{1.\ \ 2.\ \ 3.\ldots\ldots\ldots m}{(n+1)(n+2)(n+3)\ldots\ldots(n+m)}\left(\frac{1}{m}\right)\left(\frac{2}{m}\right)\left(\frac{3}{m}\right)\ldots\ldots\left(\frac{n}{m}\right),$$

adpiscimur hanc aequationem

$$\left[\frac{m}{n}\right]^n=\frac{m^n}{n^m}\cdot 1.2.3\ldots\ldots m\left(\frac{1}{m}\right)\left(\frac{2}{m}\right)\left(\frac{3}{m}\right)\ldots\ldots\left(\frac{n}{m}\right),$$

ita ut sit

$$\left[\frac{m}{n}\right]=m\sqrt[n]{\frac{1.2.3\ldots\ldots m}{n^m}\left(\frac{1}{m}\right)\left(\frac{2}{m}\right)\left(\frac{3}{m}\right)\ldots\ldots\left(\frac{n}{m}\right)},$$

quae cum proposita in (§. 53.) ob $\left(\frac{n}{m}\right)=\frac{1}{m}$ omnino congruit, ex quo ejus veritas nuuc quidem ex principiis certissimis est evicta.

Demonstratio Theorematis §. 59. propositi.

§. 68. Etiam hoc Theorema firmiori demonstratione indiget, quam ex aequalitate ante stabilita

$$\frac{\left[\frac{m}{n}\right]\cdot\left[\frac{\lambda}{n}\right]}{\left[\frac{\lambda+n}{n}\right]}=\frac{\lambda m}{\lambda+m}\left(\frac{\lambda}{m}\right)$$

ita adorno. Existente α communi divisore numerorum m et n, loco λ successive scribantur numeri α, 2α, 3α, etc. usque ad n, quorum multitudo est $=\frac{n}{\alpha}$, atque omnes aequalitates hoc modo resultantes in se invicem ducantur, ut prodeat haec aequatio

$$\left[\frac{m}{n}\right]^{\frac{n}{\alpha}} \cdot \frac{\left[\frac{\alpha}{n}\right]\left[\frac{2\alpha}{n}\right]\left[\frac{3\alpha}{n}\right]\cdots\cdots\left[\frac{n}{n}\right]}{\left[\frac{m+\alpha}{n}\right]\left[\frac{m+2\alpha}{n}\right]\left[\frac{m+3\alpha}{n}\right]\cdots\left[\frac{m+n}{n}\right]} =$$

$$m^{\frac{n}{\alpha}} \cdot \frac{1\alpha}{m+\alpha} \cdot \frac{2\alpha}{m+2\alpha} \cdot \frac{3\alpha}{m+3\alpha} \cdots\cdots \frac{n}{m+n}\left(\frac{\alpha}{m}\right)\left(\frac{2\alpha}{m}\right)\left(\frac{3\alpha}{m}\right)\cdots\left(\frac{n}{m}\right).$$

Jam prior pars in hanc formam ipsi aequalem transmutetur

$$\left[\frac{m}{n}\right]^{\frac{n}{\alpha}} \cdot \frac{\left[\frac{\alpha}{n}\right]\cdot\left[\frac{2\alpha}{n}\right]\left[\frac{3\alpha}{n}\right]\cdots\cdots\left[\frac{m}{n}\right]}{\left[\frac{n+\alpha}{n}\right]\left[\frac{n+2\alpha}{n}\right]\left[\frac{n+3\alpha}{n}\right]\cdots\left[\frac{n+m}{n}\right]},$$

quae ob $\left[\frac{n+\alpha}{n}\right] = \frac{n+\alpha}{n}\left[\frac{\alpha}{n}\right]$, sicque de caeteris, reducitur ad hanc

$$\left[\frac{m}{n}\right]^{\frac{n}{\alpha}} \frac{n}{n+\alpha} \cdot \frac{n}{n+2\alpha} \cdot \frac{n}{n+3\alpha} \cdots\cdots \frac{n}{n+m}.$$

Posterior vero aequationis pars simili modo transformatur in

$$m^{\frac{n}{\alpha}} \frac{\alpha}{n+\alpha} \cdot \frac{2\alpha}{n+2\alpha} \cdot \frac{3\alpha}{n+3\alpha} \cdots\cdots \frac{m}{n+m}\left(\frac{\alpha}{m}\right)\left(\frac{2\alpha}{m}\right)\left(\frac{3\alpha}{m}\right)\cdots\cdots\left(\frac{n}{m}\right),$$

unde enascitur haec aequatio

$$\left[\frac{m}{n}\right]^{\frac{n}{\alpha}} n^{\frac{m}{\alpha}} = m^{\frac{n}{\alpha}} . \alpha . 2\alpha . 3\alpha \ldots\ldots m\left(\frac{\alpha}{m}\right)\left(\frac{2\alpha}{m}\right)\left(\frac{3\alpha}{m}\right)\cdots\cdots\left(\frac{n}{m}\right),$$

hincque

$$\left[\tfrac{m}{n}\right] = m\sqrt[n]{\frac{1}{m^n}\left[\alpha . 2\alpha\, 3\alpha \ldots\ldots m\left(\frac{\alpha}{m}\right)\left(\frac{2\alpha}{m}\right)\left(\frac{3\alpha}{m}\right)\cdots\cdots\left(\frac{n}{m}\right)\right]^{\alpha}},$$

quae expressio cum praecedente comparata praebet hanc aequationem

$$\left[\alpha . 2\alpha . 3\alpha \ldots\ldots m\left(\frac{\alpha}{m}\right)\left(\frac{2\alpha}{m}\right)\left(\frac{3\alpha}{m}\right)\cdots\cdots\left(\frac{n}{m}\right)\right]^{\alpha} =$$

$$1 . 2 . 3 \ldots\ldots m\left(\frac{1}{m}\right)\left(\frac{2}{m}\right)\left(\frac{3}{m}\right)\cdots\cdots\left(\frac{n}{m}\right),$$

quod de omnibus divisoribus communibus binorum numerorum m et n est intelligendum.

2) De valore formulae integralis $\int \frac{z^{\lambda-\omega} \pm z^{\lambda+\omega}}{1 \pm z^{2\lambda}} \cdot \frac{dz}{z} (l\, z)^{\mu}$, casu quo post integrationem ponitur $z = 1$. *Nov. Commentarii Acad. Imp. Sc. Petropolitanae. Tom. XIX. Pag.* 30—64.

§. 69. Ex consideratione innumerabilium arcuum circularium, qui communem habent vel sinum vel tangentem, jam olim summationem duarum serierum infinitarum deduxi, quae ob summam generalitatem maxime memoratu dignae videbantur. Si enim litterae m et n numeros quoscunque denotant, posita diametri ratione ad peripheriam ut 1 ad π, illae duae summationes hoc modo se habebant

$$\frac{1}{m} + \frac{1}{n-m} - \frac{1}{n+m} - \frac{1}{2n-m} + \frac{1}{2n+m} + \frac{1}{3n-m} - \text{etc.} = \frac{\pi}{n \sin. \frac{m\pi}{n}} \text{ et}$$

$$\frac{1}{m} - \frac{1}{n-m} + \frac{1}{n+m} - \frac{1}{2n-m} + \frac{1}{2n+m} - \frac{1}{3n-m} + \text{etc.} = \frac{\pi}{n \,\text{tang.} \frac{m\pi}{n}},$$

atque ex his duabus seriebus jam tum temporis elicueram summationes omnium serierum illarum, quarum denominatores secundum potestates numerorum naturalium progrediuntur, quemadmodum in introductione in analysin infinitorum et alibi fusius exposui. Nunc autem eaedem series me perduxerunt ad integrationem formulae in titulo expressae, quae eo magis attentione digna videtur, quod hujusmodi integrationes aliis methodis neutiquam exsequi liceat.

§. 70 Statim autem patet, has duas series infinitas oriri ex evolutione quarundam formularum integralium, si post integrationem quantitati variabili certus valor, veluti unitas tribuatur; ita prior series deducitur ex evolutione hujus formulae integralis

$$\int \frac{z^{m-1} + z^{n-m-1}}{1 + z^n} dz,$$

posterior vero ex evolutione istius

$$\int \frac{z^{m-1} - z^{n-m-1}}{1 - z^n} dz,$$

si quidem post integrationem statuatur $z = 1$. Deinceps autem ex ipsis principiis calculi integralis demonstravi, valorem integralis prioris harum duarum formularum, si quidem ponatur $z = 1$, reduci ad hanc formulam simplicem

$$\frac{\pi}{n \sin. \frac{m\pi}{n}},$$

integrale autem posterius, eodem casu $z = 1$, ad istam

$$\frac{\pi}{n \operatorname{tang.} \frac{m\pi}{n}},$$

ita, ut ex ipsis calculi integralis principiis certum sit esse

$$\int \frac{z^{m-1} + z^{n-m-1}}{1 + z^n} dz = \frac{\pi}{n \sin. \frac{m\pi}{n}} \quad \text{et}$$

$$\int \frac{z^{m-1} - z^{n-m-1}}{1 - z^n} dz = \frac{\pi}{n \operatorname{tang.} \frac{m\pi}{n}},$$

si quidem post integrationem ita institutam, ut integrale evanescat posito $z = 0$, statuatur $z = 1$.

§. 71. Quo jam hanc duplicem integrationem ad formam propositam reducamus, faciamus $n = 2\lambda$ et $m = \lambda - \omega$, unde binae illae series infinitae hanc induent formam

$$\frac{1}{\lambda - \omega} + \frac{1}{\lambda + \omega} - \frac{1}{3\lambda - \omega} - \frac{1}{3\lambda + \omega} + \frac{1}{5\lambda - \omega} + \frac{1}{5\lambda + \omega} - \text{etc. et}$$

$$\frac{1}{\lambda - \omega} - \frac{1}{\lambda + \omega} + \frac{1}{3\lambda - \omega} - \frac{1}{3\lambda + \omega} + \frac{1}{5\lambda - \omega} - \frac{1}{5\lambda + \omega} + \text{etc.}$$

harum igitur serierum prioris summa erit

$$\frac{\pi}{2\lambda\sin.\frac{\pi(\lambda-\omega)}{2\lambda}}=\frac{\pi}{2\lambda\cos.\frac{\pi\omega}{2\lambda}},$$

posterioris vero summa erit

$$\frac{\pi}{2\lambda\,\mathrm{tang}.\frac{\pi(\lambda-\omega)}{2\lambda}}=\frac{\pi}{2\lambda\,\mathrm{cotang}.\frac{\pi\omega}{2\lambda}}=\frac{\pi\,\mathrm{tang}.\frac{\pi\omega}{2\lambda}}{2\lambda}.$$

Quod si ergo brevitatis gratia ponamus

$$\frac{\pi}{2\lambda\cos.\frac{\pi\omega}{2\lambda}}=S,\ \text{et}\ \frac{\pi}{2\lambda}\mathrm{tang}.\frac{\pi\omega}{2\lambda}=T,$$

habebimus sequentes duas integrationes

$$\int\frac{z^{\lambda-\omega}+z^{\lambda+\omega}}{1+z^{2\lambda}}\cdot\frac{dz}{z}=S,\ \text{et}$$

$$\int\frac{z^{\lambda-\omega}-z^{\lambda+\omega}}{1-z^{2\lambda}}\cdot\frac{dz}{z}=T.$$

§. 72. Circa has binas integrationes ante omnia observo, eas perinde locum habere, sive pro litteris λ et ω accipiantur numeri integri, sive fracti. Sint enim λ et ω numeri fracti quicunque, qui evadant integri, si multiplicentur per α, quo posito fiat $z=x^\alpha$, eritque $\frac{dz}{z}=\frac{\alpha dx}{x}$, et potestas quaecunque $z^\delta=x^{\alpha\delta}$; prior igitur formula erit

$$\int\frac{x^{\alpha(\lambda-\omega)}+x^{\alpha(\lambda+\omega)}}{1+x^{2\alpha\lambda}}\cdot\frac{\alpha dx}{x},$$

ubi, cum jam omnes exponentes sint numeri integri, valor hujus formulae posito post integrationem $x=1$, quandoquidem tunc etiam fit $z=1$, a praecedente eo tantum differt, quod hic habemus $\alpha\lambda$ et $\alpha\omega$ loco λ et ω, ac praeterea hic adsit factor α; quocirca valor istius formulae erit

$$\alpha . \frac{\pi}{2\alpha\lambda \cos. \frac{\pi\omega}{2\lambda}} = \frac{\pi}{2\lambda \cos. \frac{\pi\omega}{2\lambda}},$$

qui ergo valor est $=$ S prorsus ut ante; quae identitas etiam manifesto est in altera formula, unde patet, etiamsi pro λ et ω fractiones quaecunque accipiantur, integrationem hic exhibitam nihilo minus locum esse habituram; quae circumstantia probe notari meretur, quoniam in sequentibus litteram ω tanquam variabilem sumus tractaturi.

§. 73. Postquam igitur binae istae formulae integrales litteris S et T indicatae fuerint integratae, ita ut evanescant posito $z = 0$, integralia spectari poterunt non solum ut functiones quantitatis z, sed etiam ut functiones binarum variabilium z et ω, quandoquidem numerum ω tanquam quantitatem variabilem tractare licet, quin etiam exponentem λ pro quantitate variabili habere liceret: sed quia hinc formulae integrales alius generis essent prodituræ, atque hic contemplari constitui, solam quantitatem ω, praeter ipsam variabilem z, hic ut quantitatem variabilem sum tractaturus.

§. 74. Cum igitur sit

$$S = \int \frac{z^{\lambda-\omega} + z^{\lambda+\omega}}{1 + z^{2\lambda}} \cdot \frac{dz}{z}$$

in qua integratione sola z ut variabilis spectatur, erit utique secundum signandi morem jam satis usu receptum

$$\left(\frac{dS}{dz}\right) = \frac{z^{\lambda-\omega} + z^{\lambda+\omega}}{1 + z^{2\lambda}} \cdot \frac{1}{z};$$

haec jam formula denuo differentietur, posita sola littera ω variabili, eritque

$$\left(\frac{ddS}{dzd\omega}\right) = \frac{-z^{\lambda-\omega} + z^{\lambda+\omega}}{1+z^{2\lambda}} \cdot \frac{1}{z} lz,$$

quae formula ducta in dz, ac denuo integrata sola z habita pro variabili, dabit

$$\int dz \left(\frac{ddS}{dzd\omega}\right) = \int \frac{-z^{\lambda-\omega} + z^{\lambda+\omega}}{1+z^{2\lambda}} \cdot \frac{dz}{z} lz,$$

ubi notetur esse

$$S = \frac{\pi}{2\lambda \cos. \frac{\pi\omega}{2\lambda}};$$

ita ut hinc deducamus

$$\left(\frac{dS}{d\omega}\right) = \frac{\pi\pi \sin. \frac{\pi\omega}{2\lambda}}{4\lambda\lambda \left(\cos. \frac{\pi\omega}{2\lambda}\right)^2},$$

hoc igitur valore substituto, nanciscimur hanc integrationem

$$\int \frac{-z^{\lambda-\omega} + z^{\lambda+\omega}}{1+z^{2\lambda}} \cdot \frac{dz}{z} lz = \frac{\pi\pi \sin. \frac{\pi\omega}{2\lambda}}{4\lambda\lambda \left(\cos. \frac{\pi\omega}{2\lambda}\right)^2}.$$

§. 75. Quod si jam altera formula simili modo tractetur, cum sit $T = \frac{\pi}{2\lambda} \operatorname{tang.} \frac{\pi\omega}{2\lambda}$, erit

$$\left(\frac{dT}{d\omega}\right) = \frac{\pi\pi}{4\lambda\lambda \left(\cos. \frac{\pi\omega}{2\lambda}\right)^2};$$

ex formula autem integrali erit

$$\left(\frac{dT}{d\omega}\right) = \int \frac{-z^{\lambda-\omega} - z^{\lambda+\omega}}{1-z^{2\lambda}} \cdot \frac{dz}{z} lz,$$

unde colligimus sequentem integrationem

$$\int \frac{z^{\lambda-\omega} + z^{\lambda+\omega}}{1-z^{2\lambda}} \cdot \frac{dz}{z} lz = \frac{-\pi\pi}{4\lambda\lambda \left(\cos. \frac{\pi\omega}{2\lambda}\right)^2}.$$

§. 76. Quoniam litteras S et T etiam per series expressas dedimus, erit etiam per similes series

$$\left(\frac{dS}{d\omega}\right) = \frac{1}{(\lambda-\omega)^2} - \frac{1}{(\lambda+\omega)^2} - \frac{1}{(3\lambda-\omega)^2} + \frac{1}{(3\lambda+\omega)^2} + \frac{1}{(5\lambda-\omega)^2} - \text{etc.}$$

$$= \frac{\pi\pi \sin.\frac{\pi\omega}{2\lambda}}{4\lambda\lambda\left(\cos.\frac{\pi\omega}{2\lambda}\right)^2}.$$

Similique modo etiam pro altera serie

$$\left(\frac{dT}{d\omega}\right) = \frac{1}{(\lambda-\omega)^2} + \frac{1}{(\lambda+\omega)^2} + \frac{1}{(3\lambda-\omega)^2} + \frac{1}{(3\lambda+\omega)^2} + \frac{1}{(5\lambda-\omega)^2} + \text{etc.}$$

$$= \frac{\pi\pi}{4\lambda\lambda\left(\cos.\frac{\pi\omega}{2\lambda}\right)^2};$$

sicque summas harum serierum quoque duplici modo repraesentavimus, scilicet per formulam evolutam quantitatem π involventem, tum vero etiam per formulam integralem, quae ita est comparata, ut ejus integrale nulla methodo adhuc consueta assignari possit.

§. 77. Applicemus has integrationes ad aliquot casus particulares: ac primo quidem sumamus $\omega = 0$, quo quidem casu prior integratio sponte in oculos incurrit, at posterior praebet

$$\int \frac{2z^\lambda}{1-z^{2\lambda}} \cdot \frac{dz}{z} lz = -\frac{\pi\pi}{4\lambda\lambda}, \text{ sive}$$

$$\int \frac{z^{\lambda-1} dz lz}{1-z^{2\lambda}} = -\frac{\pi\pi}{8\lambda\lambda};$$

hincque simul istam summationem adipiscimur

$$\frac{1}{\lambda\lambda} + \frac{1}{\lambda\lambda} + \frac{1}{9\lambda\lambda} + \frac{1}{9\lambda\lambda} + \frac{1}{25\lambda\lambda} + \frac{1}{25\lambda\lambda} + \text{etc.} = \frac{\pi\pi}{4\lambda\lambda}, \text{ sive}$$

$$1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \frac{1}{81} + \text{etc.} = \frac{\pi\pi}{8},$$

id quod jam dudum a me est demonstratum.

§. 78. Hic statim patet, perinde esse, quiam numerus pro λ accipiatur; sit igitur $\lambda = 1$, et habebitur ista integratio

$$\int \frac{dz\,lz}{1-z^2} = -\frac{\pi\pi}{8};$$

ex qua sequentia integralia simpliciora

$$\int \frac{dz\,lz}{1-z} \text{ et } \int \frac{dz\,lz}{1+z}$$

derivare licet ope hujus ratiocinii; statuatur

$$\int \frac{z\,dz\,lz}{1-zz} = \mathrm{P},$$

et posito $zz = v$, ut sit $z\,dz = \frac{dv}{2}$ et $lz = \frac{1}{2}lv$, prodibit

$$\frac{1}{4}\int \frac{dv\,lv}{1-v} = \mathrm{P},$$

si scilicet post integrationem fiat $v = 1$, quippe quo casu etiam sit $z = 1$; sic igitur erit

$$\int \frac{dv\,lv}{1-v} = 4\mathrm{P}:$$

nunc prior illa formula addatur ad inventam, eritque

$$\int \frac{dz\,lz + z\,dz\,lz}{1-zz} = \mathrm{P} - \frac{\pi\pi}{8},$$

haec autem formula sponte reducitur ad hanc

$$\int \frac{dz\,lz}{1-z} = \mathrm{P} - \frac{\pi\pi}{8},$$

modo autem vidimus esse

$$\int \frac{dv\,lv}{1-v} \text{ sive } \int \frac{dz\,lz}{1-z} = 4\mathrm{P}, \text{ ita ut sit } 4\mathrm{P} = \mathrm{P} - \frac{\pi\pi}{8},$$

unde manifesto sit $\mathrm{P} = -\frac{\pi\pi}{24}$, ex quo sequitur fore

$$\int \frac{dz\,lz}{1-z} = -\frac{\pi\pi}{6};$$

simili modo erit

$$\int \frac{dz\,lz - z\,dz\,lz}{1-zz} = -\mathrm{P} - \frac{\pi\pi}{8} = -\frac{\pi\pi}{12},$$

quae, supra et infra per $1-z$ dividendo, praebet

$$\int \frac{dz\,lz}{1+z} = -\frac{\pi\pi}{12},$$

quare jam adepti sumus tres integrationes memoratu maxime dignas

$$\text{I. } \int \frac{dz\, lz}{1+z} = -\frac{\pi\pi}{12},$$

$$\text{II. } \int \frac{dz\, lz}{1-z} = -\frac{\pi\pi}{6},$$

$$\text{III. } \int \frac{dz\, lz}{1-zz} = -\frac{\pi\pi}{8},$$

quibus adiungi potest

$$\text{IV. } \int \frac{z\, dz\, lz}{1-zz} = -\frac{\pi\pi}{24}.$$

§. 79. Quemadmodum igitur hae formulae ex ipsis calculi integralis principiis sunt deductae, ita etiam earum veritas per resolutionem in series facile comprobatur; cum enim sit

$$\frac{1}{1+z} = 1 - z + zz - z^3 + z^4 - z^5 + \text{etc.},$$

et in genere

$$\int z^n\, dz\, lz = \frac{z^{n+1}}{n+1}\, lz - \frac{z^{n+1}}{(n+1)^2},$$

qui valor posito $z = 1$ reducitur ad $\frac{1}{(n+1)^2}$, patet fore

$$\int \frac{dz\, lz}{1+z} = -1 + \frac{1}{4} - \frac{1}{9} + \frac{1}{16} - \frac{1}{25} + \text{etc.} = -\frac{\pi\pi}{12}, \text{ sive}$$

$$1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} - \text{etc.} = \frac{\pi\pi}{12},$$

simili modo ob

$$\frac{1}{1-z} = 1 + z + zz + z^3 + z^4 + \text{etc. erit}$$

$$\int \frac{dz\, lz}{1-z} = -1 - \frac{1}{4} - \frac{1}{9} - \frac{1}{16} - \frac{1}{25} - \text{etc.} = -\frac{\pi\pi}{6}, \text{ seu}$$

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \text{etc.} = \frac{\pi\pi}{6},$$

tum vero ob

$$\frac{1}{1-zz} = 1 + zz + z^4 + z^6 + z^8 + \text{etc. erit}$$

$$\int \frac{dz\, lz}{1-zz} = -1 - \frac{1}{9} - \frac{1}{25} - \frac{1}{49} - \frac{1}{81} - \text{etc.} = -\frac{\pi\pi}{8}, \text{ sive}$$

$$1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \frac{1}{81} + \text{etc.} = \frac{\pi\pi}{8}.$$

Eodem modo etiam

$$\int\frac{z\,dz\,lz}{1-zz}=-\frac{1}{4}-\frac{1}{16}-\frac{1}{36}-\frac{1}{64}-\text{etc.}=-\frac{\pi\pi}{24},\ \text{sive}$$

$$\frac{1}{4}+\frac{1}{16}+\frac{1}{36}+\frac{1}{64}+\text{etc.}=\frac{\pi\pi}{24},$$

quae quidem summationes jam sunt notissimae. Neque tamen quisquam adhuc methodo directa ostendit esse

$$\int\frac{dz\,lz}{1+z}=-\frac{\pi\pi}{12}.$$

§. 80. Ponamus nunc $\omega=1$, et nostrae integrationes has induent formas

$$1^0.\ \int\frac{-z^{\lambda-2}(1-zz)\,dz\,lz}{1+z^{2\lambda}}=\frac{\pi\pi\sin.\frac{\pi}{2\lambda}}{4\lambda\lambda(\cos.\frac{\pi}{2\lambda})^2}\ \text{et}$$

$$2^0.\ \int\frac{-z^{\lambda-2}(1+zz)\,dz\,lz}{1-z^{2\lambda}}=+\frac{\pi\pi}{4\lambda\lambda\left(\cos.\frac{\pi}{2\lambda}\right)^2},$$

unde pro diversis valoribus ipsius λ, quos quidem binario non minores accipere licet, sequentes obtinentur integrationes

I^0. si $\lambda=2$, erit

$$1^0.\ \int\frac{-(1-zz)\,dz\,lz}{1+z^4}=\frac{\pi\pi}{8\sqrt{2}},$$

$$2^0.\ \int\frac{-(1+zz)\,dz\,lz}{1-z^4}=+\frac{\pi\pi}{8},\ \text{sive}\int\frac{-dz\,lz}{1-zz}=+\frac{\pi\pi}{8}.$$

II^0. si $\lambda=3$, habebimus

$$1^0.\ \int\frac{-z(1-zz)\,dz\,lz}{1+z^6}=\frac{\pi\pi}{54},\ \text{et}$$

$$2^0.\ \int\frac{-z(1+zz)\,dz\,lz}{1-z^6}=\int\frac{-z\,dz\,lz}{1-zz+z^4}=\frac{\pi\pi}{27}.$$

Hae autem duae formulae ponendo $zz=v$, abibunt in sequentes

$$1^0.\ \int\frac{-dv(1-v)\,lv}{1+v^3}=\frac{2\pi\pi}{27},\ \text{et}$$

$$2^0.\ \int\frac{dv\,lv}{1-v+vv}=\frac{4\pi\pi}{27}.$$

III°. Sit $\lambda = 4$ et consequemur

$$1^0.\ \int \frac{-zz(1-zz)\,dz\,lz}{1+z^4} = \frac{\pi\pi\sqrt{\frac{\sqrt{2}-1}{2\sqrt{2}}}}{16(2+\sqrt{2})} = \frac{\pi\pi\sqrt{(2-\sqrt{2})}}{32(2+\sqrt{2})} \text{ et}$$

$$2^0.\ \int \frac{-zz(1+zz)\,dz\,lz}{1-z^8} = \int \frac{-zz\,dz\,lz}{(1-zz)(1+z^4)} = \frac{\pi\pi}{16(2+\sqrt{2})},$$

quae postrema forma reducitur ad hanc

$$\int -\frac{dz\,lz}{1-zz} + \int \frac{(1-zz)\,dz\,lz}{1+z^4} = \frac{\pi\pi}{8(2+\sqrt{2})},$$

est vero $\int \frac{-dz\,lz}{1-zz} = \frac{\pi\pi}{8}$, unde reperitur

$$\int \frac{(1-zz)\,dz\,lz}{1+z^4} = -\frac{\pi\pi(1+\sqrt{2})}{8(2+\sqrt{2})} = -\frac{\pi\pi}{8\sqrt{2}},$$

qui valor jam in superiori casu $\lambda = 2$ est inventus.

§. 81. Nihil autem impedit, quo minus etiam faciamus $\lambda = 1$, dummodo integralia ita capiantur ut evanescant, posito $z = 0$, tum autem reperiemus

$$1^0.\ \int \frac{-(1-zz)\,dz\,lz}{z(1+zz)} = \infty \text{ et}$$

$$2^0.\ \int \frac{-(1+zz)\,dz\,lz}{z(1-zz)} = \infty,$$

unde hinc nihil concludere licet. Caeterum etiam nostrae series supra inventae manifesto declarant, earum summas esse infinitas, quandoquidem primus terminus utriusque $\frac{1}{(\lambda-\omega)^2}$ fit infinitus, sumto uti fecimus $\lambda = 1$ et $\omega = 1$.

§. 82. His casibus evolutis, ulterius progrediamur ac ponamus formulas integrales inventas

$$\int \frac{-z^{\lambda-\omega}+z^{\lambda+\omega}}{1+z^{2\lambda}} \cdot \frac{dz}{z}\,lz = S' \text{ et}$$

$$\int \frac{-z^{\lambda-\omega}-z^{\lambda+\omega}}{1-z^{2\lambda}} \cdot \frac{dz}{z}\,lz = T'$$

ita ut sit

$$S' = \frac{\pi\pi \sin.\frac{\pi\omega}{2\lambda}}{4\lambda\lambda\left(\cos.\frac{\pi\omega}{2\lambda}\right)^2}, \text{ et } T' = \frac{\pi\pi}{4\lambda\lambda\left(\cos.\frac{\pi\omega}{2\lambda}\right)^2},$$

atque ut ante jam differentiemus solo numero ω pro variabili habito; quo facto sequentes nanciscimur integrationes

$$\int \frac{z^{\lambda-\omega} + z^{\lambda+\omega}}{1 + z^{2\lambda}} \cdot \frac{dz}{z} (lz)^2 = \left(\frac{dS'}{d\omega}\right), \text{ et}$$

$$\int \frac{z^{\lambda-\omega} - z^{\lambda+\omega}}{1 - z^{2\lambda}} \cdot \frac{dz}{z} (lz)^2 = \left(\frac{dT'}{d\omega}\right).$$

Hunc in finem ponamus brevitatis ergo angulum $\frac{\pi\omega}{2\lambda} = \varphi$, ut sit

$$S' = \frac{\pi\pi \sin.\varphi}{4\lambda\lambda \cos.^2\varphi} = \frac{\pi\pi}{4\lambda\lambda} \cdot \frac{\sin.\varphi}{\cos.^2\varphi}, \text{ et}$$

$$T' = \frac{\pi\pi}{4\lambda\lambda} \cdot \frac{1}{\cos.^2\varphi},$$

ac reperiemus

$$d \cdot \frac{\sin.\varphi}{\cos.^2\varphi} = \frac{\cos.^2\varphi + 2\sin.^2\varphi}{\cos.^3\varphi} d\varphi = \frac{1 + \sin.^2\varphi}{\cos.^3\varphi} d\varphi,$$

ubi est $d\varphi = \frac{\pi d\omega}{2\lambda}$; unde colligimus

$$\left(\frac{dS'}{d\omega}\right) = \frac{\pi^3}{8\lambda^3}\left(\frac{1 + \left(\sin.\frac{\pi\omega}{2\lambda}\right)^2}{\left(\cos.\frac{\pi\omega}{2\lambda}\right)^3}\right) = \frac{\pi^3}{8\lambda^3}\left(\frac{2}{\left(\cos.\frac{\pi\omega}{2\lambda}\right)^3} - \frac{1}{\cos.\frac{\pi\omega}{2\lambda}}\right);$$

simili modo ob $T' = \frac{\pi\pi}{4\lambda\lambda} \cdot \frac{1}{\cos.^2\varphi}$, erit

$$d \cdot \frac{1}{\cos.^2\varphi} = \frac{2\, d\varphi \sin.\varphi}{\cos.^3\varphi},$$

hincque

$$\left(\frac{dT'}{d\omega}\right) = \frac{\pi^3}{8\lambda^3} \cdot \frac{2\sin.\frac{\pi\omega}{2\lambda}}{\left(\cos.\frac{\pi\omega}{2\lambda}\right)^3}.$$

Consequenter integrationes hinc natae erunt

$$\int \frac{z^{\lambda-\omega}+z^{\lambda+\omega}}{1+z^{2\lambda}} \cdot \frac{dz}{z}(lz)^2 = \frac{\pi^3}{8\lambda^3}\left(\frac{2}{\left(\cos.\frac{\pi\omega}{2\lambda}\right)^3} - \frac{1}{\cos.\frac{\pi\omega}{2\lambda}}\right),$$

$$\int \frac{z^{\lambda-\omega}-z^{\lambda+\omega}}{1-z^{2\lambda}} \cdot \frac{dz}{z}(lz)^2 = \frac{\pi^3}{8\lambda^3}\,\frac{2\sin.\frac{\pi\omega}{2\lambda}}{\left(\cos.\frac{\pi\omega}{2\lambda}\right)^3}.$$

§. 83. Si jam eodem modo series §. 76. inventas denuo differentiemus, sumta sola ω variabili, perveniamus ad sequentes summationes

$$\frac{\pi^3}{8\lambda^3}\left\{\frac{2}{\left(\cos.\frac{\pi\omega}{2\lambda}\right)^3} - \frac{1}{\cos.\frac{\pi\omega}{2\lambda}}\right\} = +\frac{2}{(\lambda-\omega)^3} + \frac{2}{(\lambda+\omega)^3} - \frac{2}{(3\lambda-\omega)^3} - \frac{2}{(3\lambda+\omega)^3} + \frac{2}{(5\lambda-\omega)^3} + \frac{2}{(5\lambda+\omega^3)} - \text{etc.}$$

$$\frac{\pi^3}{8\lambda^3}\cdot\frac{2\sin.\frac{\pi\omega}{2\lambda}}{\left(\cos.\frac{\pi\omega}{2\lambda}\right)^3} = \frac{2}{(\lambda-\omega)^3} - \frac{2}{(\lambda+\omega)^3} + \frac{2}{(3\lambda-\omega)^3} - \frac{2}{(3\lambda+\omega)^3} + \frac{2}{(5\lambda-\omega)^3} - \text{etc.}$$

§. 84. Si jam hic sumamus $\omega = 0$ et $\lambda = 1$, prior integratio hanc induit formam

$$\int \frac{2\,dz(lz)^2}{1+zz} = \frac{\pi^3}{8} = \frac{2}{1^3} + \frac{2}{1^3} - \frac{2}{3^3} - \frac{2}{3^3} + \frac{2}{5^3} + \frac{2}{5^3} - \frac{2}{7^3} - \frac{2}{7^3} + \text{etc.}$$

ita ut sit

$$\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \frac{1}{9^3} - \frac{1}{11^3} + \text{etc.} = \frac{\pi^3}{32},$$

quemadmodum jam dudum demonstravi. Altera autem integratio hoc casu in nihilum abit. Ex priori vero integrali

$$\int \frac{dz\,lz^2}{1+zz} = \frac{\pi^3}{16},$$

alia derivare non licet, uti supra fecimus ex formula

$$\int \frac{dz\,lz}{1-zz} = -\frac{\pi\pi}{8},$$

propterea quod hic denominator $1+zz$ non habet factores reales.

§. 85. Sumamus igitur $\lambda = 2$ et $\omega = 1$, ac prior integratio dabit

$$\int \frac{(1+zz)\,dz\,(lz)^2}{1+z^4} = \frac{5\pi^3}{32\sqrt{2}};$$

series autem hinc nata erit

$$\frac{2}{1^3} + \frac{2}{3^3} - \frac{2}{5^3} - \frac{2}{7^3} + \frac{2}{9^3} + \frac{2}{11^3} - \text{etc.},$$

ita ut sit

$$\frac{1}{1^3} + \frac{1}{3^3} - \frac{1}{5^3} - \frac{1}{7^3} + \frac{1}{9^3} + \frac{1}{11^3} - \text{etc.} = \frac{5\pi^3}{64\sqrt{2}},$$

quae superiori addita praebet

$$\frac{1}{1^3} - \frac{1}{7^3} + \frac{1}{9^3} - \frac{1}{15^3} + \frac{1}{17^3} - \frac{1}{23^3} + \text{etc.} = \frac{\pi^3(5+\sqrt{2})}{128\sqrt{2}}.$$

Altera vero integratio hoc casu dat

$$\int \frac{dz\,(lz)^2}{1+zz} = \frac{\pi^3}{16},$$

quae cum paragrapho praecedenti perfecte congruit, quemadmodum etiam series hinc nata est

$$\frac{2}{1^3} - \frac{2}{3^3} + \frac{2}{5^3} - \frac{2}{7^3} + \frac{2}{9^3} - \frac{2}{11^3} + \frac{2}{13^3} - \text{etc.}$$

§. 86. Quo autem facilius sequentes integrationes per continuam differentiationem elicere valeamus, eas in genere repraesentemus; et cum pro priore sit

$$S = \frac{\pi}{2\lambda \cos.\frac{\pi\omega}{2\lambda}},$$

integrationes hinc ortae ita ordine procedent

$$\text{I.} \quad \int \frac{z^{\lambda-\omega} + z^{\lambda+\omega}}{1+z^{2\lambda}} \cdot \frac{dz}{z} = S,$$

$$\text{II.} \quad \int \frac{-z^{\lambda-\omega} + z^{\lambda+\omega}}{1+z^{2\lambda}} \cdot \frac{dz}{z}\, lz = \left(\frac{dS}{d\omega}\right),$$

$$\text{III.} \quad \int \frac{z^{\lambda-\omega} + z^{\lambda+\omega}}{1+z^{2\lambda}} \cdot \frac{dz}{z}\,(lz)^2 = \left(\frac{ddS}{d\omega^2}\right),$$

$$\text{IV.} \quad \int \frac{-z^{\lambda-\omega}+z^{\lambda+\omega}}{1+z^{2\lambda}} \cdot \frac{dz}{z}(lz)^3 = \left(\frac{d^3 S}{d\omega^3}\right),$$

$$\text{V.} \quad \int \frac{z^{\lambda-\omega}+z^{\lambda+\omega}}{1+z^{2\lambda}} \cdot \frac{dz}{z}(lz)^4 = \left(\frac{d^4 S}{d\omega^4}\right),$$

$$\text{VI.} \quad \int \frac{-z^{\lambda-\omega}+z^{\lambda+\omega}}{1+z^{2\lambda}} \cdot \frac{dz}{z}(lz)^5 = \left(\frac{d^5 S}{d\omega^5}\right),$$

$$\text{VII.} \quad \int \frac{z^{\lambda-\omega}+z^{\lambda+\omega}}{1+z^{2\lambda}} \cdot \frac{dz}{z}(lz)^6 = \left(\frac{d^6 S}{d\omega^6}\right).$$

etc. etc. etc.

§. 87. Pro his differentiationibus continuis facilius absoluendis, ponamus brevitatis ergo $\frac{\pi}{2\lambda} = \alpha$, ut sit

$$S = \frac{\alpha}{\cos. \alpha\omega};$$

tum vero sit

$$\sin \alpha\omega = p \text{ et } \cos. \alpha\omega = q,$$

eritque

$$dp = \alpha q\, d\omega \text{ et } dq = -\alpha p\, d\omega.$$

Praeterea vero notetur esse

$$d \cdot \frac{p^n}{q^{n+1}} = \alpha\, d\omega \left\{\frac{n p^{n-1}}{q^n} + \frac{(n+1) p^{n+1}}{q^{n+2}}\right\}.$$

His praemissis ob $S = \alpha \cdot \frac{1}{q}$ erit

$$\left(\frac{dS}{d\omega}\right) = \alpha^2 \cdot \frac{p}{qq}, \text{ deinde}$$

$$\left(\frac{ddS}{d\omega^2}\right) = \alpha^3 \left(\frac{1}{q} + \frac{2pp}{q^3}\right), \text{ porro}$$

$$\left(\frac{d^3 S}{d\omega^3}\right) = \alpha^4 \left(\frac{5p}{qq} + \frac{6p^3}{q^4}\right),$$

$$\left(\frac{d^4 S}{d\omega^4}\right) = \alpha^5 \left(\frac{5}{q} + \frac{28pp}{q^3} + \frac{24p^4}{q^5}\right),$$

$$\left(\frac{d^5 S}{d\omega^5}\right) = \alpha^6 \left(\frac{61p}{qq} + \frac{180p^3}{q^4} + \frac{120p^5}{q^6}\right),$$

$$\left(\frac{d^6 S}{d\omega^6}\right) = \alpha^7 \left(\frac{61}{q} + \frac{662\, pp}{q^3} + \frac{1320\, p^4}{q^5} + \frac{720\, p^6}{q^7}\right),$$

$$\left(\frac{d^7 S}{d\omega^7}\right) = \alpha^8 \left(\frac{1385\, p}{qq} + \frac{7266\, p^3}{q^4} + \frac{10920\, p^5}{q^6} + \frac{5040\, p^7}{q^8}\right), \text{ etc.}$$

hi autem valores ob $pp = 1 - qq$ ad sequentes reducuntur

$$S = \alpha \cdot \frac{1}{q},$$

$$\left(\frac{dS}{d\omega}\right) = \alpha^2 p \cdot \frac{1}{qq},$$

$$\left(\frac{ddS}{d\omega^2}\right) = \alpha^3 \left(\frac{1.2}{q^3} - \frac{1}{q}\right),$$

$$\left(\frac{d^3 S}{d\omega^3}\right) = \alpha^4 p \left(\frac{1.2.3}{q^4} - \frac{1}{qq}\right),$$

$$\left(\frac{d^4 S}{d\omega^4}\right) = \alpha^5 \left(\frac{1.2.3.4}{q^5} - \frac{20}{q^3} + \frac{1}{q}\right),$$

$$\left(\frac{d^5 S}{d\omega^5}\right) = \alpha^6 p \left(\frac{1.2.3.4.5}{q^6} - \frac{60}{q^4} + \frac{1}{qq}\right),$$

$$\left(\frac{d^6 S}{d\omega^6}\right) = \alpha^7 \left(\frac{1 \ldots\ldots 6}{q^7} - \frac{840}{q^5} + \frac{182}{q^3} - \frac{1}{q}\right), \text{ etc.}$$

§. 88. Has posteriores formas reperire licet ope horum duorum lemmatum

$$\text{I.} \quad \partial \cdot \frac{1}{q^{n+1}} = \alpha\, d\omega \frac{(n+1)p}{q^{n+2}}, \text{ et}$$

$$\text{II.} \quad \partial \cdot \frac{p}{q^{n+1}} = \alpha\, d\omega \left\{\frac{n+1}{q^{n+2}} - \frac{n}{q^n}\right\}.$$

hinc enim reperiemus

$$S = \alpha \frac{1}{q},$$

$$\left(\frac{dS}{d\omega}\right) = \alpha^2 \cdot \frac{p}{qq},$$

$$\left(\frac{ddS}{d\omega^2}\right) = \alpha^3 \left(\frac{2}{q^3} - \frac{1}{q}\right),$$

$$\left(\frac{d^3 S}{d^3 \omega}\right) = \alpha^4 \left(\frac{2.3\, p}{q^4} - \frac{p}{qq}\right),$$

$$\left(\frac{d^4 S}{d\omega^4}\right) = \alpha^5 \left(\frac{2.3.4}{q^5} - \frac{20}{q^3} + \frac{1}{q}\right),$$

$$\left(\frac{d^5 S}{d\omega^5}\right) = \alpha^6 \left(\frac{2.3.4.5\, p}{q^6} - \frac{3.20\, p}{q^4} + \frac{p}{qq}\right),$$

$$\left(\frac{d^6 S}{d\omega^6}\right) = \alpha^7 \left(\frac{2.3.4.5.6}{q^7} - \frac{840}{q^5} + \frac{182}{q^3} - \frac{1}{q}\right),$$

$$\left(\frac{d^7 S}{d\omega^7}\right) = \alpha^8 \left(\frac{2\ldots 7 p}{q^8} - \frac{5.840 p}{q^6} + \frac{3.182 p}{q^4} - \frac{p}{qq}\right) \text{ etc.}$$

§. 89. Ipsae autem series his formulis respondentes erunt

$$S = \frac{1}{\lambda - \omega} + \frac{1}{\lambda + \omega} - \frac{1}{3\lambda - \omega} - \frac{1}{3\lambda + \omega} + \frac{1}{5\lambda - \omega} + \frac{1}{5\lambda + \omega} - \text{etc,}$$

$$\left(\frac{dS}{d\omega}\right) = \frac{1}{(\lambda - \omega)^2} - \frac{1}{(\lambda + \omega)^2} - \frac{1}{(3\lambda - \omega)^2} + \frac{1}{(3\lambda + \omega)^2} + \frac{1}{(5\lambda - \omega)^2} - \frac{1}{(5\lambda + \omega)^2} - \text{etc.}$$

$$\left(\frac{ddS}{d\omega^2}\right) = \frac{1.2}{(\lambda - \omega)^3} + \frac{1.2}{(\lambda + \omega)^3} - \frac{1.2}{(3\lambda - \omega)^3} - \frac{1.2}{(3\lambda + \omega)^3} + \frac{1.2}{(5\lambda - \omega)^3} + \text{etc.}$$

$$\left(\frac{d^3 S}{d\omega^3}\right) = \frac{1.2.3}{(\lambda - \omega)^4} - \frac{1.2.3}{(\lambda + \omega)^4} - \frac{1.2.3}{(3\lambda - \omega)^4} + \frac{1.2.3}{(3\lambda + \omega)^4} + \frac{1.2.3}{(5\lambda - \omega)^4} - \text{etc.}$$

$$\left(\frac{d^4 S}{d\omega^4}\right) = \frac{1.2.3.4}{(\lambda - \omega)^5} + \frac{1.2.3.4}{(\lambda + \omega)^5} - \frac{1.2.3.4}{(3\lambda - \omega)^5} - \frac{1.2.3.4}{(3\lambda + \omega)^5} + \frac{1.2.3.4}{(5\lambda - \omega)^5} + \text{etc.}$$

$$\left(\frac{d^5 S}{d\omega^5}\right) = \frac{1.2.3.4.5}{(\lambda - \omega)^6} - \frac{1.2.3.4.5}{(\lambda + \omega)^6} - \frac{1.2.3.4.5}{(3\lambda - \omega)^6} + \frac{1.2.3.4.5}{(3\lambda + \omega)^6} + \frac{1.2.3.4.5}{(5\lambda - \omega)^6} - \text{etc.}$$

$$\left(\frac{d^6 S}{d\omega^6}\right) = \frac{1\ldots 6}{(\lambda - \omega)^7} + \frac{1\ldots 6}{(\lambda + \omega)^7} - \frac{1\ldots 6}{(3\lambda - \omega)^7} - \frac{1\ldots 6}{(3\lambda + \omega)^7} + \frac{1\ldots 6}{(5\lambda - \omega)^7} + \text{etc.}$$

$$\left(\frac{d^7 S}{d\omega^7}\right) = \frac{1\ldots 7}{(\lambda - \omega)^8} - \frac{1\ldots 7}{(\lambda + \omega)^8} - \frac{1\ldots 7}{(3\lambda - \omega)^8} + \frac{1\ldots 7}{(3\lambda + \omega)^8} + \frac{1\ldots 7}{(5\lambda - \omega)^8} - \text{etc.}$$

etc. etc. etc.

Circa hos autem valores probe meminisse opportet, esse

$$\alpha = \frac{\pi}{2\lambda},\ p = \sin.\alpha\omega = \sin.\frac{\pi\omega}{2\lambda}, \text{ et } q = \cos.\alpha\omega = \cos.\frac{\pi\omega}{2\lambda}.$$

§. 90. Eodem modo expediamus valores seu formulas integrales alterius generis, pro quibus est

$$T = \frac{\pi}{2\lambda} \text{ tang. } \frac{\pi\omega}{2\lambda},$$

unde continuo differentiando oriuntur sequentes integrationes

$$\text{I. } \int \frac{z^{\lambda - \omega} - z^{\lambda + \omega}}{1 - z^{2\lambda}} \cdot \frac{dz}{z} = T,$$

$$\text{II. } \int \frac{-z^{\lambda - \omega} - z^{\lambda + \omega}}{1 - z^{2\lambda}} \cdot \frac{dz}{z}\, lz = \left(\frac{dT}{d\omega}\right),$$

$$\text{III. } \int \frac{z^{\lambda - \omega} - z^{\lambda + \omega}}{1 - z^{2\lambda}} \cdot \frac{dz}{z} (lz)^2 = \left(\frac{ddT}{d\omega^2}\right),$$

$$\text{IV.}\quad \int \frac{-z^{\lambda-\omega}-z^{\lambda+\omega}}{1-z^{2\lambda}}\cdot\frac{dz}{z}(lz)^3=\left(\frac{d^3T}{d\omega^3}\right),$$

$$\text{V.}\quad \int \frac{z^{\lambda-\omega}-z^{\lambda+\omega}}{1-z^{2\lambda}}\cdot\frac{dz}{z}(lz)^4=\left(\frac{d^4T}{d\omega^4}\right),$$

$$\text{VI.}\quad \int \frac{-z^{\lambda-\omega}-z^{\lambda+\omega}}{1-z^{2\lambda}}\cdot\frac{dz}{z}(lz)^5=\left(\frac{d^5T}{d\omega^5}\right),$$

$$\text{VII.}\quad \int \frac{z^{\lambda-\omega}-z^{\lambda+\omega}}{1-z^{2\lambda}}\cdot\frac{dz}{z}(lz)^6=\left(\frac{d^6T}{d\omega^6}\right).$$

etc.

§. 91. Ponatur iterum $\frac{\pi}{2\lambda}=\alpha$, sin. $\alpha\omega=p$, et cos. $\alpha\omega=q$, ut sit

$$T=\frac{\alpha p}{q},$$

quae formula secundum lemmata §. 88. continuo differentiata dabit

$$T=\alpha\cdot\frac{p}{q},$$

$$\left(\frac{dT}{d\omega}\right)=\alpha^2\cdot\frac{1}{qq},$$

$$\left(\frac{ddT}{d\omega^2}\right)=\alpha^3\frac{2p}{q^3},$$

$$\left(\frac{d^3T}{d\omega^3}\right)=\alpha^4\left(\frac{6}{q^4}-\frac{4}{qq}\right),$$

$$\left(\frac{d^4T}{d\omega^4}\right)=\alpha^5\left(\frac{24p}{q^5}-\frac{8p}{q^3}\right),$$

$$\left(\frac{d^5T}{d\omega^5}\right)=\alpha^6\left(\frac{120}{q^6}-\frac{120}{q^4}+\frac{16}{qq}\right),$$

$$\left(\frac{d^6T}{d\omega^6}\right)=\alpha^7\left(\frac{720p}{q^7}-\frac{480p}{q^5}+\frac{32p}{q^3}\right),$$

$$\left(\frac{d^7T}{d\omega^7}\right)=\alpha^8\left(\frac{5040}{q^8}-\frac{6720}{q^6}+\frac{2016}{q^4}-\frac{64}{qq}\right).$$

etc.

§. 92. Series autem infinitae, quae hinc nascuntur, erunt

$$T=\frac{1}{\lambda-\omega}-\frac{1}{\lambda+\omega}+\frac{1}{3\lambda-\omega}-\frac{1}{3\lambda+\omega}+\frac{1}{5\lambda-\omega}-\frac{1}{5\lambda+\omega}+\text{etc.}$$

$$\left(\frac{dT}{d\omega}\right)=\frac{1}{(\lambda-\omega)^2}+\frac{1}{(\lambda+\omega)^2}+\frac{1}{(3\lambda-\omega)^2}+\frac{1}{(3\lambda+\omega)^2}+\frac{1}{(5\lambda-\omega)^2}+\text{etc.}$$

$$\left(\frac{ddT}{d\omega^2}\right)=\frac{1.2}{(\lambda-\omega)^3}-\frac{1.2}{(\lambda+\omega)^3}+\frac{1.2}{(3\lambda-\omega)^3}-\frac{1.2}{(3\lambda+\omega)^3}+\frac{1.2}{(5\lambda-\omega)^3}-\text{etc.}$$

$$\left(\frac{d^3T}{d\omega^3}\right)=\frac{1.2.3}{(\lambda-\omega)^4}+\frac{1.2.3}{(\lambda+\omega)^4}+\frac{1.2.3}{(3\lambda-\omega)^4}+\frac{1.2.3}{(3\lambda+\omega)^4}+\frac{1.2.3}{(5\lambda-\omega)^4}+\text{etc.}$$

$$\left(\frac{d^4T}{d\omega^4}\right)=\frac{1.2.3.4}{(\lambda-\omega)^5}-\frac{1.2.3.4}{(\lambda+\omega)^5}+\frac{1.2.3.4}{(3\lambda-\omega)^5}-\frac{1.2.3.4}{(3\lambda+\omega)^5}+\frac{1.2.3.4}{(5\lambda-\omega)^5}-\text{etc.}$$

$$\left(\frac{d^5T}{d\omega^5}\right)=\frac{1.2.3.4.5}{(\lambda-\omega)^6}+\frac{1.2.3.4.5}{(\lambda+\omega)^6}+\frac{1.2.3.4.5}{(3\lambda-\omega)^6}+\frac{1.2.3.4.5}{(3\lambda+\omega)^6}+\frac{1.2.3.4.5}{(5\lambda-\omega)^6}+\text{etc.}$$

$$\left(\frac{d^6T}{d\omega^6}\right)=\frac{1\ldots\ldots6}{(\lambda-\omega)^7}-\frac{1\ldots\ldots6}{(\lambda+\omega)^7}+\frac{1\ldots\ldots6}{(3\lambda-\omega)^7}-\frac{1\ldots\ldots6}{(3\lambda+\omega)^7}+\frac{1\ldots\ldots6}{(5\lambda-\omega)^7}-\text{etc.}$$

etc. etc. etc.

§. 93. Operae pretium erit, hinc casus simplicissimos evolvere, qui oriuntur ponendo $\lambda=1$ et $\omega=0$, ita ut sit $\alpha=\frac{\pi}{2}$, $p=0$ et $q=1$, unde habebimus

Pro ordine priore	Pro ordine posteriore
$S=\frac{\pi}{2}$	$T=0$
$\left(\frac{dS}{d\omega}\right)=0$	$\left(\frac{dT}{d\omega}\right)=\frac{\pi\pi}{4}$
$\left(\frac{ddS}{d\omega^2}\right)=\frac{\pi^3}{8}$	$\left(\frac{ddT}{d\omega^2}\right)=0$
$\left(\frac{d^3S}{d\omega^3}\right)=0$	$\left(\frac{d^3T}{d\omega^3}\right)=\frac{\pi^4}{2}$
$\left(\frac{d^4S}{d\omega^4}\right)=\frac{5\pi^5}{32}$	$\left(\frac{d^4T}{d\omega^4}\right)=0$
$\left(\frac{d^5S}{d\omega^5}\right)=0$	$\left(\frac{d^5T}{d\omega^5}\right)=\frac{\pi^6}{4}$
$\left(\frac{d^6S}{d\omega^6}\right)=\frac{61\pi^7}{128}$	$\left(\frac{d^6T}{d\omega^6}\right)=0$
$\left(\frac{d^7S}{d\omega^7}\right)=0$	$\left(\frac{d^7T}{d\omega^7}\right)=\frac{79\pi^8}{32}$
etc.	etc.

§. 94. Hinc ergo, omissis valoribus evanescentibus, ex priore ordine habebimus sequentes formulas integrales cum seriebus inde natis

$$\int\frac{dz}{1+zz}=\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\frac{1}{13}-\text{etc.}$$

$$\int\frac{dz(lz)^2}{1+zz}=\frac{\pi^3}{16}=\frac{2}{1^3}-\frac{2}{3^3}+\frac{2}{5^3}-\frac{2}{7^3}+\frac{2}{9^3}-\frac{2}{11^3}+\text{etc.}$$

$$\int\frac{dz\,(lz)^4}{1+zz}=\frac{5\pi^5}{64}=\frac{24}{1^5}-\frac{24}{3^5}+\frac{24}{5^5}-\frac{24}{7^5}+\frac{24}{9^5}-\frac{24}{11^5}+\text{etc.}$$

$$\int\frac{dz\,(lz)^6}{1+zz}=\frac{61\,\pi^7}{256}=\frac{720}{1^7}-\frac{720}{3^7}+\frac{720}{5^7}-\frac{720}{7^7}+\frac{720}{9^7}-\text{etc.}$$

etc. etc. etc.

§. 95. Ex altero autem ordine pro eodem casu oriuntur

$$\int\frac{-dz\,lz}{1-zz}=\frac{\pi\pi}{8}=\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\frac{1}{9^2}+\frac{1}{11^2}+\frac{1}{13^2}\text{ etc.}$$

$$\int\frac{-dz\,(lz)^3}{1-zz}=\frac{\pi^4}{16}=\frac{6}{1^4}+\frac{6}{3^4}+\frac{6}{5^4}+\frac{6}{7^4}+\frac{6}{9^4}+\frac{6}{11^4}+\frac{6}{13^4}+\text{etc.}$$

$$\int\frac{-dz\,(lz)^5}{1-zz}=\frac{\pi^6}{8}=\frac{120}{1^6}+\frac{120}{3^6}+\frac{120}{5^6}+\frac{120}{7^6}+\frac{120}{9^6}+\frac{120}{11^6}+\frac{120}{13^6}+\text{etc.}$$

etc. etc. etc.

§. 96. Quemadmodum ex primo integrali ordinis posterioris deduximus has formulas

$$\int\frac{dz\,lz}{1-z}=-\frac{\pi\pi}{6},\text{ et }\int\frac{dz\,lz}{1+z}=-\frac{\pi\pi}{12},$$

similes quoque formulae integrales ex sequentibus deduci possunt; cum enim sit

$$\int\frac{dz\,(lz)^3}{1-zz}=-\frac{\pi^4}{16},$$

ponamus esse

$$\int\frac{z\,dz\,(lz)^3}{1-zz}=\mathrm{P},\text{ eritque}$$

$$\int\frac{dz\,(lz)^3}{1-z}=\mathrm{P}-\frac{\pi^4}{16},\text{ et}$$

$$\int\frac{dz\,(lz)^3}{1+z}=-\mathrm{P}-\frac{\pi^4}{16},$$

nunc vero statuatur $zz=v$, ut sit $z\,dz=\frac{1}{2}dv$, et $lz=\frac{1}{2}lv$, ideoque $(lz)^3=\frac{1}{8}(lv)^3$, quibus substitutis erit

$$\mathrm{P}=\frac{1}{16}\int\frac{dv\,(lv)^3}{1-v}=\frac{1}{16}\left(\mathrm{P}-\frac{\pi^4}{16}\right),$$

unde fit

$$16\,\mathrm{P}=\mathrm{P}-\frac{\pi^4}{16},\text{ ideoque }\mathrm{P}=-\frac{\pi^4}{240},$$

sicque has duas habebimus integrationes novas

$$\int \frac{dz\,(lz)^3}{1-z} = -\frac{\pi^4}{15}, \text{ et}$$

$$\int \frac{dz\,(lz)^3}{1+z} = -\frac{7\pi^4}{120}:$$

hinc autem per series erit

$$\int \frac{-dz\,(lz)^3}{1-z} = +\frac{\pi^4}{15} = 6\left(1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \frac{1}{6^4} + \frac{1}{7^4} + \frac{1}{8^4} + \text{etc.}\right) \text{ et}$$

$$\int \frac{-dz\,(lz)^3}{1+z} = +\frac{7\pi^4}{120} = 6\left(1 - \frac{1}{2^4} + \frac{1}{3^4} - \frac{1}{4^4} + \frac{1}{5^4} - \frac{1}{6^4} + \frac{1}{7^4} - \frac{1}{8^4} + \text{etc.}\right)$$

§. 97. Porro cum $\int \frac{dz\,(lz)^5}{1-zz} = -\frac{\pi^6}{8}$, ponamus esse $\int \frac{z\,dz\,(lz)^5}{1-zz} = P$, ut hinc obtineamus

$$\int \frac{dz\,(lz)^5}{1-z} = P - \frac{\pi^6}{8}, \text{ et } \int \frac{dz\,(lz)^5}{1+z} = -P - \frac{\pi^6}{8},$$

nunc igitur statuamus $zz = v$, eritque

$$P = \frac{1}{64} \int \frac{dv\,(lv)^5}{1-v} = \frac{1}{64}\left(P - \frac{\pi^6}{8}\right),$$

unde sit

$$P = -\frac{\pi^6}{504},$$

novaeque integrationes hinc deductae sunt

$$\int \frac{dz\,(lz)^5}{1-z} = -\frac{8\pi^6}{63}, \text{ et}$$

$$\int \frac{dz\,(lz)^5}{1+z} = -\frac{31\pi^6}{252}:$$

et vero per series reperitur

$$\int \frac{dz\,(lz)^5}{1-z} = -\frac{8\pi^6}{63} = -120\left(1 + \frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{4^6} + \frac{1}{5^6} + \frac{1}{6^6} + \frac{1}{7^6} + \text{etc.}\right), \text{ et}$$

$$\int \frac{dz\,(lz)^5}{1+z} = -\frac{31\pi^6}{252} = -120\left(1 - \frac{1}{2^6} + \frac{1}{3^6} - \frac{1}{4^6} + \frac{1}{5^6} - \frac{1}{6^6} + \frac{1}{7^6} - \text{etc.}\right)$$

ita ut sit

$$1 + \frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{4^6} + \frac{1}{5^6} + \frac{1}{6^6} + \frac{1}{7^6} + \text{etc.} = \frac{\pi^6}{945}, \text{ et}$$

$$1 - \frac{1}{2^6} + \frac{1}{3^6} - \frac{1}{4^6} + \frac{1}{5^6} - \frac{1}{6^6} + \frac{1}{7^6} - \frac{1}{8^6} + \text{etc.} = \frac{31\pi^6}{30240} = \frac{31\pi^6}{32.945}.$$

§. 98. Consideremus etiam casus, quibus $\lambda = 2$ et $\omega = 1$, ita ut sit $\alpha = \frac{\pi}{4}$, et $\alpha\omega = \frac{\pi}{4}$, hinc $p = q = \frac{1}{\sqrt{2}}$, unde pro utroque ordine sequentes habebimus valores

Pro ordine priore	Pro ordine posteriore
$S = \frac{\pi}{2\sqrt{2}}$	$T = \frac{\pi}{4}$
$\left(\frac{dS}{d\omega}\right) = \frac{\pi\pi}{8\sqrt{2}}$	$\left(\frac{dT}{d\omega}\right) = \frac{\pi\pi}{8}$
$\left(\frac{ddS}{d\omega^2}\right) = \frac{3\pi^3}{32\sqrt{2}}$	$\left(\frac{ddT}{d\omega^2}\right) = \frac{\pi^3}{16}$
$\left(\frac{d^3S}{d\omega^3}\right) = \frac{11\pi^4}{128\sqrt{2}}$	$\left(\frac{d^3T}{d\omega^3}\right) = \frac{\pi^4}{16}$
$\left(\frac{d^4S}{d\omega^4}\right) = \frac{57\pi^5}{512\sqrt{2}}$	$\left(\frac{d^4T}{d\omega^4}\right) = \frac{5\pi^5}{64}$
$\left(\frac{d^5S}{d\omega^5}\right) = \frac{361\pi^6}{2048\sqrt{2}}$	$\left(\frac{d^5T}{d\omega^5}\right) = \frac{\pi^6}{8}$
$\left(\frac{d^6S}{d\omega^6}\right) = \frac{2763\pi^7}{8192\sqrt{2}}$	$\left(\frac{d^6T}{d\omega^6}\right) = \frac{61\pi^7}{256}$
$\left(\frac{d^7S}{d\omega^7}\right) = \frac{24611\pi^8}{32768\sqrt{2}}$	$\left(\frac{d^7T}{d\omega^7}\right) = \frac{79\pi^8}{32}$
etc.	etc.

§. 99. Hinc igitur sequentes integrationes, cum seriebus respondentibus resultant; ac primo quidem ex ordine primo

$$\int \frac{(1+zz)\,dz}{1+z^4} = \frac{\pi}{2\sqrt{2}} = 1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \frac{1}{13} - \text{etc.}$$

$$\int \frac{-(1-zz)\,dz\,lz}{1+z^4} = \frac{\pi\pi}{8\sqrt{2}} = 1 - \frac{1}{3^2} - \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} - \frac{-1}{11^2} - \frac{1}{13^2} + \text{etc}$$

$$\int \frac{(1+zz)\,dz\,(lz)^2}{1+z^4} = \frac{3\pi^3}{32\sqrt{2}} = \frac{2}{1^3} + \frac{2}{3^3} - \frac{2}{5^3} - \frac{2}{7^3} + \frac{2}{9^3} + \frac{2}{11^3} - \frac{2}{13^3} - \text{etc.}$$

$$\int \frac{-(1-zz)\,dz\,(lz)^3}{1+z^4} = \frac{11\pi^4}{128\sqrt{2}} = \frac{6}{1^4} - \frac{6}{3^4} - \frac{6}{5^4} + \frac{6}{7^4} + \frac{6}{9^4} - \frac{6}{11^4} - \frac{6}{13^4} + \text{etc.}$$

$$\int \frac{(1+zz)\,dz\,(lz)^4}{1+z^4} = \frac{57\pi^5}{512\sqrt{2}} = \frac{24}{1^5} + \frac{24}{3^5} - \frac{24}{5^5} - \frac{24}{7^5} + \frac{24}{9^5} + \frac{24}{11^5} - \frac{24}{13^5} - \text{etc}$$

$$\int \frac{-(1-zz)\,dz\,(lz)^5}{1+z^4} = \frac{361\pi^6}{2048\sqrt{2}} = \frac{120}{1^6} - \frac{120}{3^6} - \frac{120}{5^6} + \frac{120}{7^6} + \frac{120}{9^6} - \frac{120}{11^6} - \frac{120}{13^6} + \text{etc.}$$

$$\int \frac{(1+zz)\,dz\,(lz)^6}{1+z^4} = \frac{2763\pi^7}{8192\sqrt{2}} = \frac{720}{1^7} + \frac{720}{3^7} - \frac{720}{5^7} - \frac{720}{7^7} + \frac{720}{9^7} + \frac{720}{11^7} - \frac{720}{13^7} - \text{etc.}$$

$$\int \frac{-(1-zz)\,dz\,(lz)^7}{1+z^4} = \frac{24611\pi^8}{32768\sqrt{2}} = \frac{5040}{1^8} - \frac{5040}{3^8} - \frac{5040}{5^8} + \frac{5040}{7^8} + \frac{5040}{9^8} - \frac{5040}{11^8} - \frac{5040}{13^8} + \text{etc.}$$

etc. etc. etc.

§. 100. Eodem modo integrationes alterius ordinis cum seriebus erunt

$$\int \frac{dz}{1+zz} = \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \text{etc.}$$

$$\int \frac{-dz\,lz}{1-zz} = \frac{\pi\pi}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \frac{1}{11^2} + \frac{1}{13^2} + \text{etc.}$$

$$\int \frac{dz\,(lz)^2}{1+zz} = \frac{\pi^3}{16} = \frac{2}{1^3} - \frac{2}{3^3} + \frac{2}{5^3} - \frac{2}{7^3} + \frac{2}{9^3} - \frac{2}{11^3} + \frac{2}{13^3} - \text{etc.}$$

$$\int \frac{-dz\,(lz)^3}{1-zz} = \frac{\pi^4}{16} = \frac{6}{1^4} + \frac{6}{3^4} + \frac{6}{5^4} + \frac{6}{7^4} + \frac{6}{9^4} + \frac{6}{11^4} + \frac{6}{13^4} + \text{etc.}$$

$$\int \frac{dz\,(lz)^4}{1+zz} = \frac{5\pi^5}{64} = \frac{24}{1^5} - \frac{24}{3^5} + \frac{24}{5^5} - \frac{24}{7^5} + \frac{24}{9^5} - \frac{24}{11^5} + \frac{24}{13^5} - \text{etc.}$$

$$\int \frac{-dz\,(lz)^5}{1-zz} = \frac{\pi^6}{8} = \frac{120}{1^6} + \frac{120}{3^6} + \frac{120}{5^6} + \frac{120}{7^6} + \frac{120}{9^6} + \frac{120}{11^6} + \text{etc.}$$

$$\int \frac{dz\,(lz)^6}{1+zz} = \frac{61\pi^7}{256} = \frac{720}{1^7} - \frac{720}{3^7} + \frac{720}{5^7} - \frac{720}{7^7} + \frac{720}{9^7} - \frac{720}{11^7} + \text{etc.}$$

$$\int \frac{-dz\,(lz)^7}{1-zz} = \frac{79\pi^8}{32} = \frac{5040}{1^8} + \frac{5040}{3^8} + \frac{5040}{5^8} + \frac{5040}{7^8} + \frac{5040}{9^8} + \frac{5040}{11^8} + \text{etc.}$$

etc. etc. etc.

Hae autem series sunt eae ipsae, quas jam supra §§. 94. et 95. sumus consecuti.

§. 101. Praeterea autem ii casus imprimis notari merentur, quibus formulae integrales in formas simpliciores resolvi possunt. Haec autem resolutio tantum spectat ad fractionem

$$\pm \frac{z^{\lambda-\omega} + z^{\lambda+\omega}}{1 \pm z^{2\lambda}},$$

omisso factore $\frac{dz}{z}(lz)^{\mu}$; ad quod ostendendum sumamus primo $\lambda = 3$ et $\omega = 1$, unde fit $\alpha = \frac{\pi}{6}$, $p = \sin.\frac{\pi}{6}$, et $q = \cos.\frac{\pi}{6}$, tum autem, in priori ordine occurrunt alternatim sequentes fractiones

$$\text{I. } \frac{zz(1+zz)}{1+z^6} = \frac{zz}{1-zz+z^4},$$

quae posito $zz = v$ abit in $\frac{v}{1-v+vv}$: ergo cum sit

$$\frac{dz}{z} = \frac{1}{2}\frac{dv}{v}, \text{ et } lz = \frac{1}{2}lv,$$

hinc talis forma

$$\frac{1}{2^{2i+1}}\int\frac{dv\,(lv)^{2i}}{1-v+vv}$$

integrari poterit, casu scilicet $v=1$.

$$\text{II. } -\frac{zz(1-zz)}{1+z^6}=+\frac{2}{3(1+zz)}-\frac{(2-zz)}{3(1-zz+z^4)},$$

quae posito $zz=v$, abit in $\frac{2}{3(1+v)}+\frac{2-v}{3(1-v+vv)}$, quae ergo forma ducta in $\frac{dz}{z}(lz)^{2i+1}$ vel in

$$\frac{1}{2^{2i+2}}\cdot\frac{dv}{v}(lv)^{2i+1},$$

semper integrari potest posito $v=1$.

§. 102. Eodem casu ordo posterior sequentes suppeditat resolutiones

$$\text{I. } \frac{zz(1-zz)}{1-z^6}=\frac{zz}{1+zz+z^4}=\frac{v}{1+v+vv},$$

quae in $\frac{dz}{z}(lz)^{2i}$, vel in $\frac{1}{2^{2i+1}}\cdot\frac{dv}{v}(lz)^{2i}$ ducta semper est integrabilis

$$\text{II. } \frac{-zz(1+zz)}{1-z^6}=\frac{-2}{3(1-zz)}+\frac{2+zz}{3(1+zz+z^4)},$$

quae facto $zz=v$ fit

$$\frac{-2}{3(1-v)}+\frac{2+v}{3(1+v+vv)},$$

quae ergo formulae in $\frac{dv}{v}(lv)^{2i+1}$ ductae fiunt integrabiles; quia autem in hac resolutione numeratores per z vel v dividere non licet, alia resolutione est opus, quae reperitur

$$\frac{-zz(1+zz)}{1-z^6}=\frac{-2zz}{3(1-zz)}-\frac{zz(1+2zz)}{3(1+zz+z^4)}, \text{ sive}$$

$$\frac{-2v}{3(1-v)}-\frac{v(1+2v)}{3(1+v+vv)},$$

quae formulae ductae in $\frac{dz}{z}(l\,z)^{2i+1}$, vel in $\frac{1}{2^{2i+2}}\cdot\frac{dv}{v}(l\,v)^{2i+1}$, integrationem quoque admittunt.

§. 103. Porro manente $\lambda = 3$ sumatur $\omega = 2$, ut sit $\alpha = \frac{\pi}{6}$, $p = \sin.\frac{\pi}{3}$, et $q = \cos.\frac{\pi}{3}$, et ex ordine priore oriuntur sequentes reductiones.

$$\text{I.}\quad \frac{z(1+z^4)}{1+z^6} = \frac{2z}{3(1+zz)} + \frac{z(1+zz)}{3(1-zz+z^4)},$$

unde multiplicando per $\frac{dz}{z}\,l\,z^{2i}$ oriuntur formulae integrationem admittentes cusu $z = 1$.

$$\text{II.}\quad \frac{-z(1-z^4)}{1+z^6} = -\frac{z(1-zz)}{1-zz+z^4},$$

quae per $\frac{dz}{z}(l\,z)^{2i+1}$ multiplicata integrari poterit casu $z = 1$. Ex ordine vero posteriori sequentes prodibunt reductiones.

$$\text{I.}\quad \frac{z(1-z^4)}{1-z^6} = \frac{z(1+zz)}{1+zz+z^4},$$

quae ducta in $\frac{dz}{z}(l\,z)^{2i}$ fit integrabilis.

$$\text{II.}\quad \frac{-z(1+z^4)}{1-z^6} = \frac{-2z}{3(1-zz)} - \frac{z(1-zz)}{3(1+zz+z^4)},$$

quae formulae in $\frac{dz}{z}(l\,z)^{2i+1}$ ductae fiunt integrabiles.

§. 104. Operae jam erit pretium haec integralia actu evolvere, quare ex §. 101. ubi $\omega = 1$, ejusque numero I nanciscimur sequentes integrationes

$$1^0.\quad \frac{1}{2}\int\frac{dv}{1-v+vv} = \alpha\frac{1}{q} = \frac{\pi}{3\sqrt{3}}$$

$$2^0.\quad \frac{1}{8}\int\frac{dv\,(l\,v)^2}{1-v+vv} = \alpha^3\left(\frac{2}{q^3} - \frac{l}{q}\right) = \frac{5\pi^3}{324\sqrt{3}},$$

deinde vero ex ejusdem §. numero II. ubi etiam haec reductio locum habet

$$-\frac{zz(1-zz)}{1+z^6} = -\frac{2zz}{3(1+zz)} - \frac{zz(1-2zz)}{3(1-zz+z^4)} = -\frac{2v}{3(1+v)} - \frac{v(1-2v)}{3(1-v+vv)},$$

quae ducta in $\frac{1}{4}\cdot\frac{dv}{v}\,l\,v$ dabit

$$-\frac{1}{6}\int\frac{dv\,lv}{1+v}-\frac{1}{12}\int\frac{dv(1-2v)lv}{1-v+vv}=\alpha\alpha\frac{p}{qq}=\frac{\pi\pi}{54},$$

quarum formularum prior integrationem admittit, est enim

$$\int\frac{dv\,lv}{1+v}=-\frac{\pi\pi}{12},$$

unde invenitur posterior

$$\int\frac{dv\,(1-2v)\,lv}{1-v+vv}=-\frac{\pi\pi}{18}.$$

§. 105. Ex §. 102. ejusque numero I sequitur

$$1^0.\ \frac{1}{2}\int\frac{dv}{1+v+vv}=\frac{\alpha p}{q}=\frac{\pi}{6\sqrt{3}}$$

$$2^0.\ \frac{1}{8}\int\frac{dv\,(lv)^2}{1+v+vv}=\alpha^3\frac{2p}{q^3}=\frac{\pi^3}{81\sqrt{3}};$$

deinde vero ex numero II fit

$$-\frac{1}{6}\int\frac{dv\,lv}{1-v}-\frac{1}{12}\int\frac{dv\,(1+2v)\,lv}{1+v+vv}=\alpha\alpha\cdot\frac{1}{qq}=\frac{\pi\pi}{27};$$

supra autem invenimus esse

$$\int\frac{dv\,lv}{1-v}=-\frac{\pi\pi}{6},$$

quo valore substituto fit

$$\int\frac{dv\,(1+2v)\,lv}{1+v+vv}=-\frac{\pi\pi}{9}:$$

maxime igitur operae pretium est visum, has postremas integrationes evolvisse.

§. 106. Quod si ambae formulae integrales

$$\int\frac{dv\,(1-2v)\,lv}{1-v+vv}\ \text{et}\ \int\frac{dv\,(1+2v)\,lv}{1+v+vv}$$

in series convertantur, reperitur

$$\int\frac{dv\,(1-2v)\,lv}{1-v+vv}=-1+\frac{1}{4}+\frac{2}{9}+\frac{1}{16}-\frac{1}{25}-\frac{2}{36}-\frac{1}{49}+\text{etc. et}$$

$$\int\frac{dv\,(1+2v)\,lv}{1+v+vv}=-1-\frac{1}{4}+\frac{2}{9}-\frac{1}{16}-\frac{1}{25}+\frac{2}{36}-\frac{1}{49}-\text{etc.}$$

unde has duas summationes attentione nostra non indignas assequimur

$$\text{I. } 1-\frac{1}{4}-\frac{2}{9}-\frac{1}{16}+\frac{1}{25}+\frac{2}{36}+\frac{1}{49}-\frac{1}{64}-\frac{2}{81}-\frac{1}{100}+\text{etc.}=\frac{\pi\pi}{18},$$

$$\text{II. } 1+\frac{1}{4}-\frac{2}{9}+\frac{1}{16}+\frac{1}{25}-\frac{2}{36}+\frac{1}{49}+\frac{1}{64}-\frac{2}{81}+\frac{1}{100}+\text{etc.}=\frac{\pi\pi}{9},$$

quarum prior a posteriore ablata praebet

$$\frac{2}{4}+\frac{2}{16}-\frac{4}{36}+\frac{2}{64}+\frac{2}{100}\text{ etc.}=\frac{\pi\pi}{18},$$

cujus duplum perducit ad hanc

$$1+\frac{1}{4}-\frac{2}{9}+\frac{1}{16}+\frac{1}{25}-\frac{2}{36}+\text{etc.}=\frac{\pi\pi}{9},$$

quae quoniam cum secunda congruit, veritas utriusque summationis satis confirmatur. Quod si vero secunda a duplo primae subtrahatur, remanebit ista series memorabilis

$$1-\frac{3}{4}-\frac{2}{9}-\frac{3}{16}+\frac{1}{25}+\frac{6}{36}+\frac{1}{49}-\frac{3}{64}-\frac{2}{81}-\frac{3}{100}+\text{etc.}=0$$

quae in periodos sex terminos complectentes distributa, manifestum ordinem in numerationibus declarat, quippe qui sunt

$$1-3-2-3+1+6.$$

Additamentum.

§. 107. Quemadmodum superiores integrationes per continuam differentiationem formularum S et T deduximus, ita etiam per integrationem alias et prorsus singulares integrationes impetrabimus; si enim ut supra fuerit $S=\int\frac{T\,dz}{z}$, existente T formula illa

$$+\frac{z^{\lambda-\omega}\pm z^{\lambda+\omega}}{1\pm z^{2\lambda}},$$

quae praeter z etiam exponentem variabilem ω involvere concipitur, erit per naturam integralium duas variabiles involventium

$$\int S\,d\omega=\int\frac{dz}{z}\int T\,d\omega,$$

ubi in priore formula integrali $\int S\,d\omega$, ubi z pro constanti habetur, statim scribi potest $z=1$; hoc igitur lemmate praemisso, quia est

$$\int T\, d\omega = \frac{-z^{\lambda-\omega} \pm z^{\lambda+\omega}}{(1 \pm z^{2\lambda})\, lz},$$

ambas formulas supra tractatas nempe S et T hoc modo evolvamus, et quia utramque triplici modo expressam dedimus; primo scilicet per seriem infinitam, secundo, per formulam finitam, ac tertio per formulam integralem, etiam quantitates, quae pro integralibus $\int S\, d\omega$ et $\int T\, d\omega$ resultabunt, erunt inter se aequales.

§. 108 Incipiamus a formula S, et cum per seriem fuerit

$$S = \frac{1}{\lambda - \omega} + \frac{1}{\lambda + \omega} - \frac{1}{3\lambda - \omega} - \frac{1}{3\lambda + \omega} + \frac{1}{5\lambda - \omega} + \frac{1}{5\lambda + \omega} - \text{etc.}$$

erit

$$\int S\, d\omega = -l(\lambda - \omega) + l(\lambda + \omega) + l(3\lambda - \omega) - l(3\lambda + \omega) - \text{etc.} + C,$$

quam constantem ita definire decet, ut integrale evanescat posito $\omega = 0$, quo facto erit

$$\int S\, d\omega = l\frac{\lambda + \omega}{\lambda - \omega} + l\frac{3\lambda - \omega}{3\lambda + \omega} + l\frac{5\lambda + \omega}{5\lambda - \omega} + l\frac{7\lambda - \omega}{7\lambda + \omega} + \text{etc.}$$

quae expressio reducitur ad sequentem

$$\int S\, d\omega = l\frac{(\lambda + \omega)(3\lambda - \omega)(5\lambda + \omega)(7\lambda - \omega)(9\lambda + \omega)\ \text{etc.}}{(\lambda - \omega)(3\lambda + \omega)(5\lambda - \omega)(7\lambda + \omega)(9\lambda - \omega)\ \text{etc.}}.$$

Deinde quia per formulam finitam erat

$$S = \frac{\pi}{2\lambda \cos. \frac{\pi\omega}{2\lambda}}, \quad \text{erit } \int S\, d\omega = \int \frac{\pi\, d\omega}{2\lambda \cos. \frac{\pi\omega}{2\lambda}};$$

ubi si brevitatis gratia ponatur $\frac{\pi\omega}{2\lambda} = \varphi$, ut sit

$$d\omega = \frac{2\lambda\, d\varphi}{\pi}, \quad \text{erit } \int S\, d\omega = \int \frac{d\varphi}{\cos. \varphi};$$

quia igitur novimus esse

$$\int \frac{d\theta}{\sin. \theta} = l \text{ tang. } \tfrac{1}{2}\theta,$$

sumamus $\sin. \theta = \cos. \varphi$, sive $\theta = 90^0 - \varphi = \frac{\pi}{2} - \varphi$, eritque $d\theta = -d\varphi$, unde fit

$$\int \frac{-d\varphi}{\cos.\varphi} = l \text{ tang. } \left(\frac{\pi}{4} - \frac{1}{2}\varphi\right);$$

quoniam autem est

$$\varphi = \frac{\pi\omega}{2\lambda}, \text{ erit } \frac{\pi}{4} - \frac{1}{2}\varphi = \frac{\pi(\lambda-\omega)}{4\lambda},$$

unde nostrum integrale erit

$$\int S\, d\omega = -l \text{ tang. } \frac{\pi(\lambda-\omega)}{4\lambda} = +l \text{ tang. } \frac{\pi(\lambda+\omega)}{4\lambda}.$$

Ex tertia autem formula integrali

$$S = \int \frac{z^{\lambda-\omega} + z^{\lambda+\omega}}{1+z^{2\lambda}} \cdot \frac{dz}{z} \text{ colligitur fore}$$

$$\int S\, d\omega = \int \frac{-z^{\lambda-\omega} + z^{\lambda+\omega}}{1+z^{2\lambda}} \cdot \frac{dz}{z\, l z},$$

quod integrale a termino $z = 0$ usque ad terminum $z = 1$ extendi assumitur; sicque tres isti valores inventi inter se erunt aequales. Ac ne ob constantes forte addendas ullum dubium supersit, singulae istae expressiones sponte evanescunt casu $\omega = 0$.

§. 109. Consideremus hinc primo aequalitatem inter formulam primam et secundam: et quia utraque est logarithmus, erit

$$\text{tang. } \frac{\pi(\lambda+\omega)}{4\lambda} = \frac{(\lambda+\omega)(3\lambda-\omega)(5\lambda+\omega)(7\lambda-\omega) \text{ etc.}}{(\lambda-\omega)(3\lambda+\omega)(5\lambda-\omega)(7\lambda+\omega) \text{ etc.}}$$

cum igitur hujus fractionis numerator evanescat casibus, vel $\omega = -\lambda$, vel $\omega = +3\lambda$, vel $\omega = -5\lambda$, vel $\omega = +7\lambda$ etc. evidens est iisdem casibus quoque tangentem fieri $= 0$; denominator vero evanescit casibus vel $\omega = \lambda$, vel $\omega = -3\lambda$, vel $\omega = +5\lambda$, vel $\omega = -7\lambda$ etc. quibus ergo casibus tangens in infinitum excrescere debet, id quod etiam pulcherrime evenit. Caeterum haec expressio congruit cum ea, quam jam dudum inveni et in introductione exposui.

§. 110. Productum autem istud infinitum per principia alibi stabilita ad formulas integrales reduci potest ope hujus lemmatis latissime patentis

$$\frac{a(c+b)(a+k)(c+b+k)(a+2k)(c+b+2k) \text{ etc.}}{b(c+a)(b+k)(c+a+k)(b+2k)(c+a+2k) \text{ etc.}} = \frac{\int z^{c-1}\,dz\,(1-z^k)^{\frac{b-k}{k}}}{\int z^{c-1}\,dz\,(1-z^k)^{\frac{a-k}{k}}},$$

si quidem post utramque integrationem fiat $z = 1$. Nostro igitur casu erit $a = \lambda + \omega$, $b = \lambda - \omega$, $c = 2\lambda$, et $k = 4\lambda$; unde valor nostri producti erit

$$\frac{\int z^{2\lambda-1}\,dz\,(1-z^{4\lambda})^{\frac{-3\lambda-\omega}{4\lambda}}}{\int z^{2\lambda-1}\,dz\,(1-z^{4\lambda})^{\frac{-3\lambda+\omega}{4\lambda}}} = \text{tang.}\,\frac{\pi(\lambda+\omega)}{4\lambda};$$

formulae autem istae integrales concinniores evadunt, statuendo $z^{2\lambda} = y$, tum enim erit

$$\text{tang.}\,\frac{\pi(\lambda+\omega)}{4\lambda} = \frac{\int dy\,(1-yy)^{\frac{-3\lambda-\omega}{4\lambda}}}{\int dy\,(1-yy)^{\frac{-3\lambda+\omega}{4\lambda}}},$$

quae expressio utique omni attentione digna videtur. Denique ex formula integrali inventa erit quoque

$$\int \frac{-z^{\lambda-\omega}+z^{\lambda+\omega}}{1+z^{2\lambda}} \cdot \frac{dz}{z\,lz} = l\,\text{tang.}\,\frac{\pi(\lambda+\omega)}{4\lambda}.$$

§. 111. Operae erit pretium, etiam aliquot casus particulares evolvere: sit igitur primo $\lambda = 2$ et $\omega = 1$, ac per expressionem infinitam erit

$$\int S\,d\omega = l\frac{3.5}{1.7} \cdot \frac{11.13}{9.15} \cdot \frac{19.21}{17.23} \cdot \frac{27.29}{25.31} \cdot \frac{35.37}{33.39} \cdot \text{etc.}$$

deinde per expressionem finitam habebimus

$$\int S\,d\omega = l\,\text{tang.}\,\frac{3\pi}{8},$$

at per formulam integralem

$$\int S\,d\omega = \int \frac{-(1-zz)}{1+z^4}\cdot\frac{dz}{lz}.$$

Tum vero ex aequalitate duarum priorum expressionum

$$\text{tang.}\,\frac{3\pi}{8} = \frac{3.5}{1.7}\cdot\frac{11.13}{9.15}\cdot\frac{19.21}{17.23}\text{ etc.}$$

hincque per binas formulas integrales

$$\text{tang.}\,\frac{3\pi}{8} = \frac{\int dy\,(1-yy)^{-\frac{7}{8}}}{\int dy\,(1-yy)^{-\frac{5}{8}}}.$$

§. 112. Ponamus nunc esse $\lambda = 3$ et $\omega = 1$, ac per expressionem infinitam erit

$$\int S\,d\omega = l\,\frac{2}{1}\cdot\frac{4}{5}\cdot\frac{8}{7}\cdot\frac{10}{11}\cdot\frac{14}{13}\cdot\frac{16}{17}\cdot\frac{20}{19}\cdot\frac{22}{23}\text{ etc.}$$

secundo, per expressionem finitam

$$\int S\,d\omega = l\,\text{tang.}\,\frac{\pi}{3} = l\sqrt{3} = \tfrac{1}{2}\,l\,3,$$

ita, ut futurum sit

$$\sqrt{3} = \frac{2.4}{1.5}\cdot\frac{8.10}{7.11}\cdot\frac{14.16}{13.17}\text{ etc.}$$

hujusque producti valor per formulas integrales erit

$$\frac{\int dy\,(1-yy)^{-\frac{5}{6}}}{\int dy\,(1-yy)^{-\frac{2}{3}}}.$$

Denique formula integralis praebebit

$$\int S\,d\omega = \int \frac{-z(1-zz)}{1+z^6}\cdot\frac{dz}{lz}.$$

§. 113. Eodem modo etiam evolvamus alteram formulam T, cujus valor per seriem erat

$$T = \frac{1}{\lambda-\omega} - \frac{1}{\lambda+\omega} + \frac{1}{3\lambda-\omega} - \frac{1}{3\lambda+\omega} + \frac{1}{5\lambda-\omega} - \frac{1}{5\lambda+\omega} + \text{etc.}$$

unde sit

$$\int T\,d\omega = -l(\lambda-\omega) - l(\lambda+\omega) - l(3\lambda-\omega) - l(3\lambda+\omega) - \text{etc.}$$

quae expressio, ut evanescat posito $\omega = 0$, erit

$$\int T\,d\omega = l\frac{\lambda\lambda}{\lambda\lambda-\omega\omega}\cdot\frac{9\lambda\lambda}{9\lambda\lambda-\omega\omega}\cdot\frac{25\lambda\lambda}{25\lambda\lambda-\omega\omega}\ \text{etc.}$$

deinde vero cum per formulam finitam fuerit

$$T = \frac{\pi}{2\lambda}\,\text{tang.}\,\frac{\pi\omega}{2\lambda},\ \text{erit}$$

$$\int T\,d\omega = \int\frac{\pi\,d\omega}{2\lambda}\,\text{tang.}\,\frac{\pi\omega}{2\lambda},\ \text{ubi posito}\ \frac{\pi\omega}{2\lambda} = \varphi,\ \text{erit}$$

$$\int T\,d\omega = \int d\varphi\,\text{tang.}\,\varphi = -l\cos.\varphi,\ \text{ita ut sit}$$

$$\int T\,d\omega = -l\cos.\frac{\pi\omega}{2\lambda};$$

cujus valor casu $\omega = 0$ fit sponte $= 0$; denique per formulam integralem habebimus

$$\int T\,d\omega = -\frac{z^{\lambda-\omega} - z^{\lambda+\omega}}{1-z^{2\lambda}}\cdot\frac{dz}{z\,lz},$$

ubi integrale itidem a termino $z = 0$ usque ad terminum $z = 1$ extendi debet.

§. 114. Jam comparatio duorum priorum valorum hanc praebet aequationem.

$$\frac{1}{\cos.\frac{\pi\omega}{2\lambda}} = \frac{\lambda\lambda}{\lambda\lambda-\omega\omega}\cdot\frac{9\lambda\lambda}{9\lambda\lambda-\omega\omega}\cdot\frac{25\lambda\lambda}{25\lambda\lambda-\omega\omega}\cdot\frac{49\lambda\lambda}{49\lambda\lambda-\omega\omega}\ \text{etc. vel}$$

$$\cos.\frac{\pi\omega}{2\lambda} = \left(1-\frac{\omega\omega}{\lambda\lambda}\right)\left(1-\frac{\omega\omega}{9\lambda\lambda}\right)\left(1-\frac{\omega\omega}{25\lambda\lambda}\right)\left(1-\frac{\omega\omega}{49\lambda\lambda}\right)\ \text{etc.}$$

vel si factores singuli iterum in simplices evolvantur,

$$\cos.\frac{\pi\omega}{2\lambda} = \frac{\lambda+\omega}{\lambda}\cdot\frac{\lambda-\omega}{\lambda}\cdot\frac{3\lambda+\omega}{3\lambda}\cdot\frac{3\lambda-\omega}{3\lambda}\cdot\frac{5\lambda+\omega}{5\lambda}\cdot\frac{5\lambda-\omega}{5\lambda}\ \text{etc.}$$

quae formula cum reductione generali supra allata comparata dat, $a = \lambda+\omega$, $b = \lambda$, $c = -\omega$, et $k = 2\lambda$, unde colligimus

$$\cos.\frac{\pi\omega}{2\lambda} = \frac{\int z^{-\omega-1}\,dz\,(1-z^{2\lambda})^{-\frac{1}{2}}}{\int z^{-\omega-1}\,dz\,(1-z^{2\lambda})^{\frac{\omega-\lambda}{2\lambda}}}.$$

Ut autem exponentes negativos $z^{-\omega-1}$ evitemus, superius productum ita repraesentemus

$$\cos.\frac{\pi\omega}{2\lambda}=\frac{\lambda-\omega}{\lambda}\cdot\frac{\lambda+\omega}{\lambda}\cdot\frac{3\lambda-\omega}{3\lambda}\cdot\frac{3\lambda+\omega}{3\lambda}\text{ etc.}$$

eritque facta comparatione $a=\lambda-\omega$, $b=\lambda$, $c=+\omega$, et $k=2\lambda$, sicque per formulas integrales erit

$$\cos.\frac{\pi\omega}{2\lambda}=\frac{\int z^{\omega-1}dz(1-z^{2\lambda})^{-\frac{1}{2}}}{\int z^{\omega-1}dz(1-z^{2\lambda})^{\frac{-\lambda-\omega}{2\lambda}}},$$

quae expressio ad simpliciorem formam reduci nequit.

§. 115. Sit nunc etiam $\lambda=2$, et $\omega=1$, eruntque ternae nostrae expressiones

$$\text{I. }\int \mathrm{T}\,d\omega=l\,\frac{4}{3}\cdot\frac{36}{35}\cdot\frac{100}{99}\cdot\frac{196}{195}\text{ etc. sive}$$

$$\int \mathrm{T}\,d\omega=l\,\frac{2.2}{1.3}\cdot\frac{6.6}{5.7}\cdot\frac{10.10}{9.11}\cdot\frac{14.14}{13.15}\text{ etc.}$$

$$\text{II. }\int \mathrm{T}\,d\omega=-l\cos.\frac{\pi}{4}=+\frac{1}{2}l\,2,\text{ ita ut sit}$$

$$\sqrt{2}=\frac{2.2}{1.3}\cdot\frac{6.6}{5.7}\cdot\frac{10.10}{9.11}\cdot\frac{14.14}{13.15}\text{ etc.}$$

quod productum per formulas integrales ita exprimitur

$$\frac{\int dz(1-z^4)^{-\frac{1}{2}}}{\int dz(1-z^4)^{-\frac{3}{4}}}=\tfrac{1}{2}\sqrt{2}:$$

$$\text{III. }\int \mathrm{T}\,d\omega=\int\frac{-(1+zz)}{1-z^4}\cdot\frac{dz}{lz}=\int\frac{-dz}{(1-zz)lz};$$

quod ergo integrale a termino $z=0$ usque ad $z=1$ extensum praebet eundem valorem $+\frac{1}{2}l\,2$, cujus aequalitatis ratio utique difficillime patet.

§. 116. Sit denique ut supra $\lambda=3$ et $\omega=1$, ac ternae formulae ita se habebunt

$$\text{I. }\int \mathrm{T}\,d\omega=l\,\frac{9}{8}\cdot\frac{81}{80}\cdot\frac{225}{224}\text{ etc.}=l\,\frac{3.3}{2.4}\cdot\frac{9.9}{8.10}\cdot\frac{15.15}{14.16}\cdot\frac{21.21}{20.22}\text{ etc.}$$

II. $\int T\,d\omega = -l\cos.\frac{\pi}{6} = -l\frac{\sqrt{3}}{2} = +l\frac{2}{\sqrt{3}}$, ita ut sit

$$\frac{2}{\sqrt{3}} = \frac{3.3}{2.4}\cdot\frac{9.9}{8.10}\cdot\frac{15.15}{14.16}\cdot\frac{21.21}{20.22};$$

ideoque per binas formulas integrales

$$\frac{3}{4}\cdot\frac{2}{\sqrt{3}} = \frac{\int dz\,(1-z^6)^{-\frac{1}{2}}}{\int dz\,(1-z^6)^{-\frac{1}{3}}}.$$

III. $\int T\,d\omega = \int\frac{-(1+zz)}{1-z^6}\cdot\frac{dz}{lz}$,

quae posito $zz = v$ abit in hanc

$$\int T\,d\omega = \int\frac{-dv\,(1+v)}{(1-v^3)\,lv}.$$

Hinc igitur patet, hac methodo plane nova perveniri ad formulas integrales, quas per methodos adhuc cognitas nullo modo evolvere, vel saltem inter se comparare, licuit.

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3). De integratione formulae $\int\frac{dx\,lx}{\sqrt{(1-xx)}}$, ab $x=0$ ad $x=1$ extensa. *Acta Acad. Imp. Sc. Tom. I. P. II. Pag.* 3—28.

§. 117. Methodus maxime naturalis hujusmodi formulas $\int p\,dx\,lx$ tractandi in hoc consistit, ut eae ad alias hujusmodi formas $\int q\,dx$ reducantur, in quibus littera q sit functio algebraica ipsius x; quandoquidem regulae integrandi potissimum ad tales formulas sunt accommodatae. Hujusmodi autem reductio nulla prorsus laborat difficultate, quando functio p ita est comparata, ut integrale $\int p\,dx$ algebraice exhiberi queat. Si enim fuerit $\int p\,dx = P$, ita ut formula proposita sit $\int dP\,lx$, ea sponte reducitur ad hanc expressionem $P\,lx - \int\frac{P\,dx}{x}$, sicque jam totum negotium ad integrationem hujus formulae $\int \frac{P\,dx}{x}$ est perductum. Quando vero formula $\int p\,dx$ integrationem algebraicam non admittit, quemadmodum evenit in nostra formula proposita $\int \frac{dx\,lx}{\sqrt{(1-xx)}}$, talis reductio successu penitus caret. Cum enim sit $\int \frac{dx}{\sqrt{(1-xx)}} = \text{A}.\sin.x$, ista reductio daret

$$\int \frac{dx\,lx}{\sqrt{(1-xx)}} = \text{A}.\sin.x \times lx - \int \frac{dx}{x} \cdot \text{A}\sin.x,$$

sicque post signum integrationis nova quantitas transcendens A sin. x occurreret, cujus integratio aeque est abscondita ac ipsius propositae. Quare cum nuper singulari methodo invenissem esse

$$\int \frac{dx\,lx}{\sqrt{(1-xx)}} \begin{bmatrix} ab\;x=0 \\ ad\;x=1 \end{bmatrix} = -\frac{1}{2}\pi\,l\,2,$$

expressio integralis eo majori attentione digna est censenda, quod ejus investigatio neutiquam est obvia; unde operae pretium esse duxi ejus veritatem etiam ex aliis fontibus ostendisse, ante quam ipsam methodum, quae me eo perduxit, exponerem.

Prima demonstratio integrationis propositae.

§. 118. Quoniam hic potissimum ad series infinitas est recurrendum, formula autem lx talem resolutionem simplicem respuit, adhibeamus substitutionem $\sqrt{(1-xx)} = y$, unde fit $x = \sqrt{(1-yy)}$, hincque porro

$$lx = -\frac{yy}{2} - \frac{y^4}{4} - \frac{y^6}{6} - \frac{y^8}{8} - \text{etc.}$$

hoc igitur modo formula integralis proposita $\int \frac{dx\,lx}{\sqrt{(1-xx)}}$ transformatur in sequentem formam

$$\int \frac{dy}{\sqrt{(1-yy)}} \left(\frac{yy}{2} + \frac{y^4}{4} + \frac{y^6}{6} + \frac{y^8}{8} + \text{etc.}\right.$$

ubi, cum sit $y = \sqrt{(1-xx)}$, notetur integrationem extendi debere ab $y=1$ usque ad $y=0$; quare si hos terminos integrationis permutare velimus, signum totius formae mutari oportet.

§. 119. Quo autem minus tali signorum mutatione confundamur, designemus valorem quesitum littera S, ut sit

$$S = \int \frac{dx\, lx}{\sqrt{(1-xx)}} \begin{bmatrix} a\, b\, x = 0 \\ a\, d\, x = 1 \end{bmatrix}$$

atque facta substitutione $y = \sqrt{(1-xx)}$, habebimus, uti modo monuimus

$$S = -\int \frac{dy}{\sqrt{(1-yy)}} \left(\frac{yy}{2} + \frac{y^4}{4} + \frac{y^6}{6} + \text{etc.}\right) \begin{bmatrix} a\, b\, y = 0 \\ a\, d\, y = 1 \end{bmatrix}.$$

Sub his autem integrationis terminis, scilicet ab $y = 0$ ad $y = 1$, jam satis notum est, singulas partes, quae hic occurrunt, ad sequentes valores reduci

$$\int \frac{yy\, dy}{\sqrt{(1-yy)}} = \frac{1}{2} \cdot \frac{\pi}{2}$$

$$\int \frac{y^4\, dy}{\sqrt{(1-yy)}} = \frac{1.3}{2.4} \cdot \frac{\pi}{2}$$

$$\int \frac{y^6\, dy}{\sqrt{(1-yy)}} = \frac{1.3.5}{2.4.6} \cdot \frac{\pi}{2}$$

$$\int \frac{y^8\, dy}{\sqrt{(1-yy)}} = \frac{1.3.5.7}{2.4.6.8} \cdot \frac{\pi}{2}$$

$$\int \frac{y^{10}\, dy}{\sqrt{(1-yy)}} = \frac{1.3.5.7.9}{2.4.6.8.10} \cdot \frac{\pi}{2} \text{ etc.}$$

ubi nimirum est $\frac{\pi}{2} = \int \frac{dy}{\sqrt{(1-yy)}}$, ita ut $1 : \pi$ exprimat rationem diametri ad peripheriam circuli.

§. 120. Quodsi ergo singulos istos valores introducamus, pro valore quaesito S impetrabimus sequentem seriem infinitam

$$S = -\frac{\pi}{2}\left(\frac{1}{2^2} + \frac{1.3}{2.4^2} + \frac{1.3.5}{2.4.6^2} + \frac{1.3.5.7}{2.4.6.8^2} + \text{etc.}\right.$$

sicque nunc totum negotium eo est reductum, ut istius seriei infinitae summa investigetur; qui labor fortasse haud minus operosus videri potest, quam id ipsum, quod nobis exsequi est propositum. Interim tamen ad cognitionem summae hujus seriei haud difficulter sequenti modo nobis pertingere licebit.

§. 121. Cum sit

$$\frac{1}{\sqrt{(1-zz)}}=1+\frac{1}{2}zz+\frac{1.3}{2.4}z^4+\frac{1.3.5}{2.4.6}z^6+\text{etc.}$$

si utrinque per $\frac{dz}{z}$ multiplicemus et integremus, obtinebimus

$$\int\frac{dz}{z\sqrt{(1-zz)}}=lz+\frac{1}{2^2}zz+\frac{1.3}{2.4^2}z^4+\frac{1.3.5}{2.4.6^2}z^6+\text{etc.}$$

sicque ad ipsam seriem nostram sumus perducti, cujus ergo valor quaeri debet ex hac expressione $\int\frac{dz}{z\sqrt{(1-zz)}}-lz$, integrali scilicet ita sumto, ut evanescat posito $z=0$, quo facto statuatur $z=1$, ac prodibit ipsa series

$$\frac{1}{2^2}+\frac{1.3}{2.4^2}+\frac{1.3.5}{2.4.6^2}+\frac{1.3.5.7}{2.4.6.8^2}+\text{etc.}$$

Hoc igitur modo totum negotium perductum est ad istam formulam integralem $\int\frac{dz}{z\sqrt{(1-zz)}}$, quae posito $\sqrt{(1-zz)}=v$ transit in hanc formam $\frac{-dv}{1-vv}$, cujus integrale constat esse $-\frac{1}{2}l\frac{1+v}{1-v}=-l\frac{1+v}{\sqrt{(1-vv)}}$. Quodsi loco v restituatur valor $\sqrt{(1-zz)}$, tota expressio, qua indigemus, ita se habebit

$$\int\frac{dz}{z\sqrt{(1-zz)}}-lz=-l\frac{[1+\sqrt{(1-zz)}]}{z}-lz+C$$
$$=C-l[1+\sqrt{(1-zz)}],$$

ubi constans ita accipi debet, ut valor evanescat, posito $z=0$, ideoque erit $C=l2$. Quamobrem, posito $z=1$, summa seriei quaesita erit $l2$, hincque valor ipsius formulae integralis propositae erit

$$\int\frac{dx\,lx}{\sqrt{(1-xx)}}=S=-\frac{\pi}{2}l2:$$

prorsus uti longe alia methodo inveneram, ex quo jam satis intelligitur, istam veritatem utique altioris esse indaginis, ideoque attentione Geometrarum maxime dignam.

Alia demonstratio integrationis propositae.

§. 122. Cum sit $\frac{dx}{\sqrt{(1-xx)}}$ elementum arcus circuli cujus sinus $=x$, ponamus istum angulum $=\varphi$, ita ut sit

$$x = \sin.\varphi \text{ et } \frac{dx}{\sqrt{(1-xx)}} = d\varphi,$$

atque facta hac substitutione valor quantitatis S, in quem inquirimus, ita repraesentabitur

$$S = \int d\varphi\, l \sin.\varphi \begin{bmatrix} a\ \varphi = 0 \\ ad\ \varphi = 90^\circ \end{bmatrix}.$$

Cum enim ante termini fuissent $x = 0$ et $x = 1$, iis nunc respondent $\varphi = 0$ et $\varphi = 90^0$, sive $\varphi = \frac{\pi}{2}$. Hic igitur totum negotium eo redit, ut formula $l \sin.\varphi$ commode in seriem infinitam convertatur. Hunc in finem ponamus $l \sin.\varphi = s$ eritque $ds = \frac{d\varphi \cos.\varphi}{\sin.\varphi}$. Novimus autem esse

$$\frac{\cos.\varphi}{\sin.\varphi} = 2\sin.2\varphi + 2\sin.4\varphi + 2\sin.6\varphi + 2\sin.8\varphi + \text{etc.}$$

Si enim utrinque per $\sin.\varphi$ multiplicemus, ob

$$2\sin.n\varphi \sin.\varphi = \cos.(n-1)\varphi - \cos.(n+1)\varphi,$$

utique prodit

$$\begin{aligned} \cos.\varphi = \cos.\varphi &+ \cos.3\varphi + \cos.5\varphi + \cos.7\varphi + \cos.9\varphi + \text{etc.} \\ &- \cos.3\varphi - \cos.5\varphi - \cos.7\varphi + \cos.9\varphi - \text{etc.} \end{aligned}$$

Hac igitur serie pro $\frac{\cos.\varphi}{\sin.\varphi}$ in usum vocata, erit

$$s = C - \cos.2\varphi - \tfrac{1}{2}\cos.4\varphi - \tfrac{1}{3}\cos.6\varphi - \tfrac{1}{4}\cos.8\varphi - \tfrac{1}{5}\cos.10\varphi - \text{etc.}$$

ubi cum sit $s = l\sin.\varphi$, ideoque $s = 0$, quando $\sin.\varphi = 1$, ideoque $\varphi = \frac{\pi}{2}$, constantem C ita definire oportet, ut posito $\varphi = \frac{\pi}{2} = 90^0$, evadat $s = 0$, ex quo colligitur fore

$$C = -1 + \tfrac{1}{2} - \tfrac{1}{3} + \tfrac{1}{4} - \tfrac{1}{5} + \text{etc.} = -l\,2.$$

§. 123. Cum igitur sit

$$l\sin.\varphi = -l\,2 - \cos.2\varphi - \tfrac{1}{2}\cos.4\varphi - \tfrac{1}{3}\cos.6\varphi - \tfrac{1}{4}\cos.8\varphi - \text{etc.}$$

erit valor formulae propositae

$$\begin{aligned} \int d\varphi\, l\sin.\varphi = C - \varphi\, l\,2 - \tfrac{1}{2}\sin.2\varphi - \tfrac{1}{8}\sin.4\varphi - \tfrac{1}{18}\sin.6\varphi \\ - \tfrac{1}{32}\sin.8\varphi - \tfrac{1}{50}\sin.10\varphi - \text{etc.} \end{aligned}$$

quae expressio cum evanescere debeat posito $\varphi = 0$, constans hic ingressa erit $C = 0$, ita ut jam in genere sit

$$\int d\varphi\, l \sin.\varphi = -\varphi\, l\, 2 - \frac{2 \sin. 2\varphi}{2^2} - \frac{2 \sin. 4\varphi}{4^2} - \frac{2 \sin. 6\varphi}{6^2} - \frac{2 \sin. 8\varphi}{8^2} - \frac{2 \sin. 10\varphi}{10^2} - \frac{2 \sin. 12\varphi}{12^2} - \text{etc.}$$

Quodsi jam hic capiatur $\varphi = 90^0 = \frac{\pi}{2}$, omnium angulorum 2φ; 4φ, 6φ, 8φ, etc. qui hic occurrunt sinus evanescunt, ideoque valor quaesitus erit

$$S = \int d\varphi\, l \sin.\varphi \left[\begin{smallmatrix} a\ \varphi = 0 \\ a\ d\varphi = 90^0 \end{smallmatrix}\right] = -\frac{\pi}{2}\, l\, 2;$$

quemadmodum etiam in priore demonstratione ostendimus.

§. 124. Ista autem demonstratio praecedenti ideo longe antecellit, quod nobis non solum valorem formulae propositae exhibeat casu quo $\varphi = 90^0$, sed etiam verum ejus valorem ostendat, quicunque angulus pro φ accipiatur, id quod ad ipsam formulam propositam $\int \frac{dx\, lx}{\sqrt{(1-xx)}}$ transferri poterit, cujus adeo valorem pro quolibet valore ipsius x assignare poterimus. Quodsi enim istius formulae valorem desideremus ab $x = 0$ usque ad $x = a$, quaeratur angulus α cujus sinus sit aequalis ipsi a, atque semper habebitur

$$\int \frac{dx\, lx}{\sqrt{(1-xx)}} \left[\begin{smallmatrix} a\,b\,x = 0 \\ a\,d\,x = a \end{smallmatrix}\right] = -\alpha\, l\, 2 - \frac{2 \sin. 2\alpha}{2^2} - \frac{2 \sin. 4\alpha}{4^2} - \frac{2 \sin. 6\alpha}{6^2} - \frac{2 \sin. 8\alpha}{8^2} - \text{etc.}$$

Unde patet, quoties fuerit $\alpha = \frac{i\pi}{2}$, denotante i numerum integrum quemcunque, quoniam omnes sinus evanescunt, valor formulae his casibus finite exprimi per $-\frac{i\pi}{2}\, l\, 2$; aliis vero casibus valor nostrae formulae per seriem infinitam satis concinnam exprimetur. Ita si capiatur $a = \frac{1}{\sqrt{2}}$, ut sit $\alpha = \frac{\pi}{4}$, valor nostrae formulae erit

$$-\frac{\pi}{4}\, l\, 2 - \frac{2}{2^2} + \frac{2}{6^2} - \frac{2}{10^2} + \frac{2}{14^2} - \frac{2}{18^2} + \frac{2}{22^2} - \text{etc.}$$

quae series elegantius ita exprimitur

$$-\frac{\pi}{4}\, l\, 2 - \frac{1}{2}\left(1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \frac{1}{9^2} - \frac{1}{11^2} + \text{etc.}\right);$$

sicque hic occurrit series satis memorabilis

$$1-\frac{1}{9}+\frac{1}{25}-\frac{1}{49}+\frac{1}{81}-\frac{1}{121}+\text{etc.}$$

cujus summam nullo adhuc modo ad mensuras cognitas revocare licuit.

§. 125. Quoniam tam egregia series hic se quasi praeter exspectationem obtulit, etiam alios casus evolvamus notabiliores, sumamusque $a=\frac{1}{2}$, ut sit $\alpha=30^0=\frac{\pi}{6}$, atque nostrae formulae hoc casu valor erit

$$-\frac{\pi}{6}l\,2-\frac{\sqrt{3}}{2^2}-\frac{\sqrt{3}}{4^2}+\frac{\sqrt{3}}{8^2}+\frac{\sqrt{3}}{10^2}-\frac{\sqrt{3}}{14^2}-\frac{\sqrt{3}}{16^2}+\text{etc.}$$

quae expressio ita exhiberi potest

$$-\frac{\pi}{6}l\,2-\frac{\sqrt{3}}{4}\left(1+\frac{1}{2^2}-\frac{1}{4^2}-\frac{1}{5^2}+\frac{1}{7^2}+\frac{1}{8^2}-\frac{1}{10^2}-\frac{1}{11^2}+\text{etc.}\right)$$

in qua serie quadrata multiplorum ternarii deficiunt. Sumamus nunc simili modo $a=\frac{\sqrt{3}}{2}$, ut sit $\alpha=60^0=\frac{\pi}{3}$, ac valor nostrae formulae hoc casu prodibit

$$-\frac{\pi}{3}l\,2-\frac{\sqrt{3}}{2^2}+\frac{\sqrt{3}}{4^2}-\frac{\sqrt{3}}{8^2}+\frac{\sqrt{3}}{10^2}-\frac{\sqrt{3}}{14^2}+\frac{\sqrt{3}}{16^2}-\text{etc.}$$

sive hoc modo exprimetur

$$-\frac{\pi}{3}l\,2-\frac{\sqrt{3}}{4}\left(1-\frac{1}{2^2}+\frac{1}{4^2}-\frac{1}{5^2}+\frac{1}{7^2}-\frac{1}{8^2}+\frac{1}{10^2}-\frac{1}{11^2}+\text{etc.}\right)$$

Adhuc alia demonstratio integrationis propositae.

§. 126. Introducatur in formulam nostram angulus φ, cujus cosinus sit $=x$, sive sit $x=\cos.\varphi$, et formula nostra induet hanc formam $-\int d\varphi\, l\cos.\varphi$, quod integrale a $\varphi=90^0$ usque ad $\varphi=0$ erit extendendum. Quodsi autem hos terminos permutemus, valor S, quem quaerimus, ita exprimetur

$$S=\int d\varphi\, l\cos.\varphi\left[\begin{smallmatrix}a\ \varphi=0\\ ad\ \varphi=90^0\end{smallmatrix}\right].$$

Ut hic $l\cos.\varphi$ in seriem idoneam convertamus, statuamus ut ante $s=l\cos.\varphi$, eritque $d s = -\frac{d\varphi \sin.\varphi}{\cos.\varphi}$. Constat autem per seriem esse

$$\frac{\sin.\varphi}{\cos.\varphi} = 2\sin.2\varphi - 2\sin.4\varphi + 2\sin.6\varphi - 2\sin.8\varphi + \text{etc.}$$

Cum enim in genere sit

$$2\sin.n\varphi\cos.\varphi = \sin.(n+1)\varphi + \sin.(n-1)\varphi,$$

si utrinque per $\cos.\varphi$ multiplicemus, orietur

$$\begin{aligned}\sin.\varphi = & \qquad\qquad \sin.3\varphi - \sin.5\varphi + \sin.7\varphi - \sin.9\varphi + \text{etc.}\\ & + \sin.\varphi - \sin.3\varphi + \sin.5\varphi - \sin.7\varphi + \sin.9\varphi - \text{etc.}\end{aligned}$$

quare cum sit $d s = -\frac{d\varphi\sin.\varphi}{\cos.\varphi}$, erit nunc

$$s = C + \frac{\cos.2\varphi}{1} - \frac{\cos.4\varphi}{2} + \frac{\cos.6\varphi}{3} - \frac{\cos.8\varphi}{4} + \frac{\cos.10\varphi}{5} - \text{etc.}$$

Quia igitur est $s = l\cos.\varphi$, evidens est posito $\varphi = 0$, fieri debere $s = 0$, unde colligitur

$$C = -1 + \tfrac{1}{2} - \tfrac{1}{3} + \tfrac{1}{4} - \tfrac{1}{5} + \text{etc.} = -l2;$$

sicque erit

$$l\cos.\varphi = -l2 + \frac{\cos.2\varphi}{1} - \frac{\cos.4\varphi}{2} + \frac{\cos.6\varphi}{3} - \frac{\cos.8\varphi}{4} + \text{etc.}$$

quae series ducta in $d\varphi$ et integrata praebet

$$S = \int d\varphi\, l\cos.\varphi = C - \varphi l2 + \frac{\sin.2\varphi}{2} - \frac{\sin.4\varphi}{8} + \frac{\sin.6\varphi}{18} - \frac{\sin.8\varphi}{32} + \frac{\sin.10\varphi}{50} - \text{etc.}$$

quae expressio quia sponte evanescit posito $\varphi = 0$, inde patet fore $C = 0$, sicque habebimus

$$\int d\varphi\, l\cos.\varphi = -\varphi l2 + \tfrac{1}{2}\left(\frac{\sin.2\varphi}{1} - \frac{\sin.4\varphi}{2^2} + \frac{\sin.6\varphi}{3^2} - \frac{\sin.8\varphi}{4^2} + \frac{\sin.10\varphi}{5^2} - \text{etc.}\right)$$

Sumto igitur $\varphi = \frac{\pi}{2} = 90^\circ$, oritur ut ante $S = -\frac{\pi}{2}l2$. Praeterea vero etiam hinc integrale ad quemvis terminum usque extendere licet.

§. 127. Quodsi formulam posteriorem a praecedente subtrahamus, adipiscemur in genere hanc integrationem

2 Sheets duplicate
& don't know which is the right one

$$\int d\varphi\, l\,\text{tang.}\,\varphi = -\sin.2\varphi - \tfrac{1}{3^2}\sin.6\varphi - \tfrac{1}{5^2}\sin.10\varphi - \text{etc.}$$

unde patet hoc integrale evanescere casibus $\varphi = 0^0$ et in genere $\varphi = \frac{i\pi}{2}$. Postquam igitur istam integrationem triplici modo demonstravimus, ipsam Analysin, quae me primum huc perduxit, hic delucide sum expositurus.

Analysis ad integrationem formulae $\int \frac{dx\,lx}{\sqrt{(1-xx)}}$ aliarumque similium perducens.

§. 128. Tota haec Analysis innititur sequenti lemmati a me jam olim demonstrato: Posito brevitatis gratia $(1-x^n)^{\frac{m-n}{n}} = X$, si hinc duae formulae integrales formentur $\int X x^{p-1}dx$ et $\int X x^{q-1}dx$, quae a termino $x=0$ usque ad terminum $x=1$ extendantur, ratio horum valorum sequenti modo ad productum ex infinitis factoribus conflatum reduci potest

$$\frac{\int X x^{p-1}dx}{\int X x^{q-1}dx} = \frac{(m+p)q}{p(m+q)} \cdot \frac{(m+p+n)(q+n)}{(p+n)(m+q+n)} \cdot \frac{(m+p+2n)(q+2n)}{(p+2n)(m+q+2n)} \text{ etc.}$$

ubi scilicet singuli factores tam numeratoris, quam denominatoris continuo eadem quantitate n augentur. Hic autem probe tenendum est, veritatem istius lemmatis subsistere non posse, nisi singulae m, n, p et q denotent numeros positivos, quos tamen semper tanquam integros spectare licet.

§. 129. Circa has duas formulas integrales, a termino $x=0$ usque ad $x=1$ extensas, duo casus imprimis seorsim notari merentur, quibus integratio actu succedit, verusque valor absolute assignari potest. Prior casus locum habet, si fuerit $p=n$, ita ut formula sit $\int X x^{n-1}dx$. Posito enim $x^n = y$ fiet

$$X = (1-y)^{\frac{m-n}{n}}, \text{ et } x^{n-1}dx = \frac{1}{n}dy$$

sicque ista formula evadet $\frac{1}{n}\int dy\,(1-y)^{\frac{m-n}{n}}$, pariter a termino $y=0$ usque ad $y=1$ extendenda, quae porro posito $1-y=z$ abit in hanc formulam $-\frac{1}{n}\int z^{\frac{m-n}{n}}\,dz$, a termino $z=1$ usque ad $z=0$ extendendam; ejus ergo integrale manifesto est $-\frac{1}{m}z^{\frac{m}{n}}+\frac{1}{m}$; unde facto $z=0$ valor erit $=\frac{1}{m}$. Consequenter pro casu $p=n$ habebimus

$$\int X x^{n-1}\,dx\begin{bmatrix}a\,b\,x=0\\ a\,d\,x=1\end{bmatrix}=\frac{1}{m};$$

sicque si fuerit vel $p=n$ vel $q=n$; integrale absolute innotescit.

§. 130. Alter casus notatu dignus est, quo $p=n-m$, ita ut formula integranda sit $\int X x^{n-m-1}\,dx$; tum enim, si ponatur

$$x\,(1-x^n)^{\frac{-1}{n}} \text{ sive } \frac{x}{(1-x^n)^{\frac{1}{n}}}=y,$$

posito $x=0$ fiet $y=0$, at posito $x=1$ fiet $y=\infty$; tum autem erit

$$y^{n-m}=\frac{x^{n-m}}{(1-x^n)^{\frac{n-m}{n}}}=X x^{n-m},$$

unde formula integranda erit $\int y^{n-m}\frac{dx}{x}$. Cum igitur sit

$$\frac{x}{(1-x^n)^{\frac{1}{n}}}=y, \text{ erit } \frac{x^n}{1-x^n}=y^n,$$

unde colligitur $x^n=\frac{y^n}{1+y^n}$, ideoque $n\,lx=n\,ly-l(1+y^n)$, cujus differentiatio praebet

$$\frac{dx}{x}=\frac{dy}{y\,(1+y^n)},$$

quo valore substituto formula nostra integranda erit

$$\int \frac{y^{n-m-1}\,dy}{1+y^n},$$

a termino $y=0$ usque ad $y=\infty$ extendenda, quae formula ideo est notatu digna, quod ab omni irrationalitae est liberata.

§. 131. Quoniam igitur hoc casu ad formulam rationalem sumus perducti, ex elementis calculi integralis constat, ejus integrationem semper per logarithmos et arcus circulares absolvi posse, tum vero pro hoc casu non ita pridem ostendi, hujus formulae $\int \frac{x^{m-1}\,dx}{1+x^n}$ integrale, ab $x=0$ usque ad $x=\infty$ extensum, reduci ad valorem $\frac{\pi}{n\sin.\frac{m\pi}{n}}$; facta igitur applicatione pro nostro casu habebimus

$$\int \frac{y^{n-m-1}\,dy}{1+y^n} = \frac{\pi}{n\sin.\frac{(n-m)\pi}{n}} = \frac{\pi}{n\sin.\frac{m\pi}{n}};$$

quamobrem pro casu $p=n-m$ valor integralis sequenti modo absolute exprimi potest, eritque

$$\int X\,x^{n-m-1}\,dx \begin{bmatrix} a\,b\,x=0 \\ a\,d\,x=1 \end{bmatrix} = \frac{\pi}{n\sin.\frac{m\pi}{n}},$$

quod idem manifesto tenendum est, si fuerit $q=n-m$.

§. 132. His praemissis, ponamus porro brevitatis gratia

$$\int X\,x^{p-1}\,dx \begin{bmatrix} a\,b\,x=0 \\ a\,d\,x=1 \end{bmatrix} = P \text{ et}$$

$$\int X\,x^{q-1}\,dx \begin{bmatrix} a\,b\,x=0 \\ a\,d\,x=1 \end{bmatrix} = Q,$$

atque lemma allatum nobis praebet hanc aequationem

$$\frac{P}{Q} = \frac{(m+p)\,q}{p\,(m+q)} \cdot \frac{(m+p+n)(q+n)}{(p+n)(m+q+n)} \cdot \frac{(m+p+2n)(q+2n)}{(p+2n)(m+q+2n)} \text{ etc.}$$

hinc igitur sumendis logarithmis deducimus

$$lP - lQ = l(m+p) - lp + l(m+p+n) - l(p+n) + l(m+p+2n) - l(p+2n) + \text{etc.}$$
$$+ lq - l(m+q) + l(q+n) - l(m+q+n) + l(q+2n) - l(m+q+2n) + \text{etc.}$$

haecque aequalitas semper locum habebit, quicunque valores litteris m, n, p et q tribuantur, dummodo fuerint positivi.

§. 133. Cum igitur haec aequalitas in genere subsistat, etiam veritati erit consentanea, quando quaepiam harum litterarum m, n, p, et q infinite parum immutantur, sive tanquam variabiles spectantur. Hanc ob rem consideremus solam quantitatem p tanquam variabilem, ita ut reliquae litterae m, n et q maneant constantes, ideoque etiam quantitas Q erit constans dum altera P variabitur; ex quo differentiando nanciscemur hanc aequationem

$$\frac{dP}{P} = \frac{dp}{m+p} - \frac{dp}{p} + \frac{dp}{m+p+n} - \frac{dp}{p+n} + \frac{dp}{m+p+2n} - \frac{dp}{p+2n}$$
$$+ \frac{dp}{m+p+3n} - \frac{dp}{p+3n} + \text{etc.}$$

ubi totum negotium eo redit, quemadmodum differentiale formulae P, quae est integralis, exprimi oporteat.

§. 134. Cum igitur P sit formula integralis solam quantitatem x tanquam variabilem involvens, quandoqidem in ejus integratione exponens p ut constans tractari debet, demum post integrationem ipsam quantitatem P tanquam functionem duarum variabilium x et p spectare licebit; unde quaestio huc redit, quomodo valorem, hoc charactere $\left(\frac{dP}{dp}\right)$ exprimi solitum, investigari oporteat, qui si indicetur littera Π, aequatio ante inventa hanc induet formam

$$\frac{\Pi}{P} = \frac{1}{m+p} - \frac{1}{p} + \frac{1}{m+p+n} - \frac{1}{p+n} + \frac{1}{m+p+2n} - \frac{1}{p+2n} + \text{etc.}$$

Hanc vero seriem infinitam haud difficulter ad expressionem finitam revocare licebit hoc modo: Ponatur

$$s = \frac{v^{m+p}}{m+p} - \frac{v^p}{p} + \frac{v^{m+p+n}}{m+p+n} - \frac{v^{p+n}}{p+n} + \frac{v^{m+p+2n}}{m+p+2n} - \frac{v^{p+2n}}{p+2n} + \text{etc.}$$

ita ut facto $v = 1$ littera s nobis exhibeat valorem quaesitum $\frac{\Pi}{P}$; at vero differentiatio nobis dabit

$$\frac{ds}{dv} = v^{m+p-1} - v^{p-1} + v^{m+p+n-1} - v^{p+n-1} + v^{m+p+2n-1} - v^{p+2n-1} + \text{etc.}$$

cujus seriei infinitae summa manifesto est

$$\frac{v^{m+p-1} - v^{p-1}}{1 - v^n} = \frac{v^{p-1}(v^m - 1)}{1 - v^n}.$$

Hinc igitur vicissim concludimus fore

$$s = \int \frac{v^{p-1}(v^m - 1)\,dv}{1 - v^n},$$

quae formula integralis a $v = 0$ usque ad $v = 1$ est extendenda; sicque habebimus

$$\frac{\Pi}{P} = \int \frac{v^{p-1}(v^m - 1)\,dv}{1 - v^n} \begin{bmatrix} a\,v = 0 \\ a\,d\,v = 1 \end{bmatrix}.$$

§. 135. Ad valorem autem $\left(\frac{dP}{dp}\right)$, quem hic littera Π idicavimus, investigandum, ex principiis calculi integralis ad functiones duarum variabilium applicati jam satis notum est, differentiale formulae integralis $P = \int X\,x^{p-1}\,dx$ ex sola variabilitate ipsius p oriundum obtineri, si formula post signum integrationis posita $X\,x^{p-1}$, ex sola variabilitate ipsius p differentietur, atque elementum dp signo integrationis praefigatur; at vero quia X non continet p, hic ut constans tractari debet: potestatis vero x^{p-1} differentiale hinc natum erit $x^{p-1}\,dp\,lx$; quam ob rem ex hac differentiatione orietur $dP = dp \int X\,x^{p-1}\,dx\,lx$, ita ut tantum post signum integrationis factor lx accesserit, ex quo manifestum est, fore

$$\Pi = \int X x^{p-1}\, dx\, lx \begin{bmatrix} ab\, x = 0 \\ ad\, x = 1 \end{bmatrix},$$

hinc igitur sequens theorema generale constituere libcebit.

Theorema generale.

§. 136. Posito brevitatis gratia $X = (1 - x^n)^{\frac{m-n}{n}}$, si sequentes formulae integrales omnes a termino $x = 0$ ad termino $x = 1$ extendantur, sequens aequalitas semper erit veritati consentanea

$$\frac{\int X x^{p-1}\, dx\, lx}{\int X x^{p-1}\, dx} = \int \frac{x^{p-1}(x^m - 1)\, dx}{1 - x^n}$$

nihil enim obstabat, quo minus loco v scriberemus x, quandoquidem isti valores tantum a terminis integrationis pendent.

§. 137. Hoc igitur modo deducti sumus ad integrationem hujusmodi formularum $\int X x^{p-1}\, dx\, lx$, in quibus quantitas logarithmica lx post signum integrationis tanquam factor inest, quarum valorem exprimere licuit per binas formulas integrales ordinarias, cum sit

$$\int X x^{p-1}\, dx\, lx = \int X x^{p-1}\, dx \,.\int \frac{x^{p-1}(x^m - 1)\, dx}{1 - x^n},$$

integralibus scilicet ab $x = 0$ ad $x = 1$ extensis, ubi brevitatis gratia posuimus $(1 - x^n)^{\frac{m-n}{n}} = X$. Hinc igitur pro binis casibus memorabilibus supra expositis bina theoremata particularia derivemus.

Theorema particulare I, quo $p = n$.

§. 138. Quoniam supra vidimus casu $p = n$ fieri $\int X x^{n-1}\, dx = \frac{1}{m}$, hoc valore substituto habebimus istam aequationem satis elegantem

$$\int X x^{n-1} dx\, lx = \frac{1}{m} \int \frac{x^{n-1}(x^m - 1)\,dx}{1 - x^n},$$

dum scilicet ambo integralia ab $x = 0$ ad $x = 1$ extenduntur.

Theorema particulare II, quo $p = n - m$.

§. 139. Quoniam pro hoc casu, quo $p = n - m$ supra ostendimus esse

$$\int X x^{n-m-1} dx = \frac{\pi}{n \sin. \frac{m\pi}{n}},$$

nunc deducimur ad sequentem integrationem maxime notatu dignam

$$\int X x^{n-m-1} dx\, lx = \frac{\pi}{n \sin. \frac{m\pi}{n}} \int \frac{x^{n-m-1}(x^m - 1)\,dx}{1 - x^n},$$

si quidem haec ambo integralia ab $x = 0$ usque ad $x = 1$ extendantur; ubi meminisse oportet esse

$$X = (1 - x^n)^{\frac{m-n}{n}}.$$

§. 140. Hic probe notetur, theorema generale latissime patere, propterea quod in eo insunt tres exponentes indefiniti, scilicet m, n et p, qui penitus arbitrio nostro relinquuntur, quos ergo infinitis modis pro lubitu definire licet, dummodo singulis valores positivi tribuantur, ita ut semper valor hujus formulae integralis $\int X x^{p-1} dx\, lx$, quam ob factorem lx tanquam transcendentem spectari oportet, per formulas integrales ordinarias exprimi queat, quae cum sint generalissima, operae pretium erit nonnullos casus speciales evolvere.

I. Evolutio casus quo $m = 1$ et $n = 2$.

§. 141. Hoc igitur casu erit $X = \frac{1}{\sqrt{(1 - xx)}}$, unde pro hoc casu theorema generale ita se habebit esse

$$\frac{\sin.\phi}{\cos.\phi} = 2\sin.2\phi - 2\sin.4\phi + 2\sin.6\phi - 2\sin.8\phi + \text{etc.}$$

Cum enim in genere sit

$$2\sin.n\phi\cos.\phi = \sin.(n+1)\phi + \sin.(n-1)\phi,$$

si utrinque per $\cos.\phi$ multiplicemus, orietur

$$\begin{aligned}\sin.\phi = &\quad\quad\quad\quad \sin.3\phi - \sin.5\phi + \sin.7.\phi - \sin.9\phi + \text{etc.}\\ &+\sin.\phi - \sin.3\phi + \sin.5\phi - \sin.7.\phi + \sin.9\phi - \text{etc.}\end{aligned}$$

quare cum sit $\partial s = -\frac{\partial\phi\sin.\phi}{\cos.\phi}$, erit nunc

$$S = C + \frac{\cos.2\phi}{1} - \frac{\cos.4\phi}{2} + \frac{\cos.6\phi}{3} - \frac{\cos.8\phi}{4} + \frac{\cos.10\phi}{5} - \text{etc.}$$

Quia igitur est $s = l\cos.\phi$, evidens est posito $\phi = 0$, fieri debere $s = 0$, unde colligitur

$$C = -1 + \tfrac{1}{2} - \tfrac{1}{3} + \tfrac{1}{4} - \tfrac{1}{5} + \text{etc.} = -l2;$$

sicque erit

$$l\cos.\phi = -l2 + \frac{\cos.2\phi}{1} - \frac{\cos.4\phi}{2} + \frac{\cos.6\phi}{3} - \frac{\cos.8\phi}{4} + \text{etc.}$$

quae series ducta in $\partial\phi$ et integrata praebet

$$S = \int\partial\phi\, l\cos.\phi = C - \phi l2 + \frac{\sin.2\phi}{2} - \frac{\sin.4\phi}{8} + \frac{\sin.6\phi}{18} - \frac{\sin.8\phi}{32} + \frac{\sin.10\phi}{50} - \text{etc.}$$

quae expressio quia sponte evanescit posito $\phi = 0$, inde patet fore $C = 0$, sicque habebimus

$$\int\partial\phi\, l\cos.\phi = -\phi l2 + \tfrac{1}{2}\left(\frac{\sin.2\phi}{1} - \frac{\sin.4\phi}{2^2} + \frac{\sin.6\phi}{3^2} - \frac{\sin.8\phi}{4^2} + \frac{\sin.10\phi}{5^2} - \text{etc.}\right)$$

Sumto igitur $\phi = \frac{\pi}{2} = 90^0$, oritur ut ante $S = -\frac{\pi}{2}l2$. Praeterea vero etiam hinc integrale ad quemvis terminum usque extendere licet.

§. 127. Quodsi formulam posteriorem a praecedente subtrahamus, adipiscemur in genere hanc integrationem

$\int \partial \phi\, l\,\mathrm{tang.}\,\phi = -\sin. 2\phi - \frac{1}{3^2}\sin. 6\phi - \frac{1}{5^2}\sin. 10\phi - \text{etc.}$ unde patet hoc integrale evanescere casibus $\phi = 0^0$ et in genere $\phi = \frac{i\pi}{2}$. Postquam igitur istam integrationem triplici modo demonstravimus, ipsam Analysin, quae me primum huc perduxit, hic delucide sum expositurus.

Analysis ad integrationem formulae $\int \frac{\partial x\, l x}{\sqrt{(1-xx)}}$ aliarumque similium perducens.

§. 123. Tota haec Analysis innititur sequenti lemmati a me jam olim demonstrato: Posito brevitatis gratia $(1-x^n)^{\frac{m-n}{n}} = X$, si hinc duae formulae integrales formentur $\int X x^{p-1} \partial x$ et $\int X x^{q-1} \partial x$, quae a termino $x = 0$ usque ad terminum $x = 1$ extendantur, ratio horum valorum sequenti modo ad productum ex infinitis factoribus conflatum reduci potest

$$\frac{\int X x^{p-1}\partial x}{\int X x^{q-1}\partial x} = \frac{(m+p)q}{p(m+q)} \cdot \frac{(m+p+n)(q+n)}{(p+n)(m+q+n)} \cdot \frac{(m+p+2n)(q+2n)}{(p+2n)(m+q+2n)} \text{ etc.}$$

ubi scilicet singuli factores tam numeratoris, quam denominatoris continuo eadem quantitate n augentur. Hic autem probe tenendum est, veritatem istius lemmatis subsistere non posse, nisi singulae litterae m, n, p, et q denotent numeros positivos, quos tamen semper tanquam integros spectare licet.

§. 129. Circa has duas formulas integrales, a termino $x = 0$ usque ad $x = 1$ extensas, duo casus imprimis seorsim notari merentur, quibus integratio actu succedit, verusque valor absolute assignari potest. Prior casus locum habet, si fuerit $p = n$, ita ut formula sit $\int X x^{n-1} \partial x$. Posito enim $x^n = y$ fiet

$$X = (1-y)^{\frac{m-n}{n}}, \text{ et } x^{n-1}\partial x = \frac{1}{n}\partial y$$

sicque ista formula evadet $\frac{1}{n}\int\partial y.(1-y)^{\frac{m-n}{n}}$, pariter a termino $y=0$ usque ad $y=1$ extendenda, quae porro posito $1-y=z$ abit in hanc formulam $-\frac{1}{n}\int z^{\frac{m-n}{n}}\partial z$, a termino $z=1$ usque ad $z=0$ extendendam; ejus ergo integrale manifesto est $-\frac{1}{m}z^{\frac{m}{n}}+\frac{1}{m}$; unde facto $z=0$ valor erit $=\frac{1}{m}$. Consequenter pro casu $p=n$ habebimus

$$\int X x^{n-1}\partial x\left[\begin{smallmatrix}ab\ x=0\\ ad\ x=1\end{smallmatrix}\right]=\frac{1}{m};$$

sicque si fuerit vel $p=n$ vel $q=n$, integrale absolute innotescit.

§. 130. Alter casus notatu dignus est, quo $p=n-m$, ita ut formula integranda sit $\int X x^{n-m-1}\partial x$; tum enim, si ponatur $x(1-x^n)^{\frac{-1}{n}}$ sive $\frac{x}{(1-x^n)^{\frac{1}{n}}}=y$, posito $x=0$ fiet $y=0$, at posito $x=1$ fiet $y=\infty$; tum autem erit

$$y^{n-m}=\frac{x^{n-m}}{(1-x^n)^{\frac{n-m}{n}}}=X x^{n-m},$$

unde formula integranda erit $\int y^{n-m}\frac{\partial x}{x}$. Cum igitur sit

$$\frac{x}{(1-x^n)^{\frac{1}{n}}}=y, \text{ erit } \frac{x^n}{1-x^n}=y^n,$$

unde colligitur $x^n=\frac{y^n}{1+y^n}$, ideoque $n\,l\,x=n\,l\,y-l(1+y^n)$, cujus differentiatio praebet

$$\frac{\partial x}{x}=\frac{\partial y}{y(1+y^n)},$$

quo valore substituto formula nostra integranda erit

$$\int \frac{y^{n-m-1}\partial y}{1+y^n},$$

a termino $y = 0$ usque ad $y = \infty$ extendenda, quae formula ideo est notatu digna, quod ab omni irrationalitate est liberata.

§. 131. Quoniam igitur hoc casu ad formulam rationalem sumus perducti, ex elementis calculi integralis constat, ejus integrationem semper per logarithmos et arcus circulares absolvi posse, tum vero pro hoc casu non ita pridem ostendi, hujus formulae $\int \frac{x^{m-1}\partial x}{1+x^n}$ integrale, ab $x = 0$ usque ad $x = \infty$ extensum, reduci ad valorem $\frac{\pi}{n \sin. \frac{m\pi}{n}}$; facta igitur applicatione pro nostro casu habebimus

$$\int \frac{y^{n-m-1}\partial y}{1+y^n} = \frac{\pi}{n \sin. \frac{(n-m)\pi}{n}} = \frac{\pi}{n \sin. \frac{m\pi}{n}};$$

quamobrem pro casu $p = n - m$ valor integralis sequenti modo absolute exprimi potest, eritque

$$\int X x^{n-m-1} \partial x \left[\begin{smallmatrix} ab\, x = 0 \\ ad\, x = 1 \end{smallmatrix}\right] = \frac{\pi}{n \sin. \frac{m\pi}{n}},$$

quod idem manifesto tenendum est, si fuerit $q = n - m$.

§. 132. His praemissis, ponamus porro brevitatis gratia

$$\int X x^{p-1} \partial x \left[\begin{smallmatrix} ab\, x = 0 \\ ad\, x = 1 \end{smallmatrix}\right] = P \text{ et}$$
$$\int X x^{q-1} \partial x \left[\begin{smallmatrix} ab\, x = 0 \\ ad\, x = 1 \end{smallmatrix}\right] = Q,$$

atque lemma allatum nobis praebet hanc aequationem

$$\frac{P}{Q} = \frac{(m+p)q}{p(m+q)} \cdot \frac{(m+p+n)(p+n)}{(p+n)(m+q+n)} \cdot \frac{(m+p+2n)(p+2n)}{(p+2n)(m+q+2n)} \text{ etc.}$$

Hinc igitur sumendis logarithmis deducimus

$$lP - lQ = l(m+p) - lp + l(m+p+n) - l(p+n) + l(m+p+2n) - l(p+2n) + \text{etc.}$$
$$+ lq - l(m+q) + l(q+n) - l(m+q+n) + l(q+2n) - l(m+q+2n) + \text{etc.}$$

haecque aequalitas semper locum habebit, quicunque valores litteris m, n, p et q tribuantur, dummodo fuerint positivi.

§. 133. Cum igitur haec aequalitas in genere subsistat, etiam veritati erit consentanea, quando quaepiam harum litterarum m, n, p et q infinite parum immutantur, sive tanquam variabiles spectantur. Hanc ob rem consideremus solam quantitatem p tanquam variabilem, ita ut reliquae litterae m, n et q maneant constantes, ideoque etiam quantitas Q erit constans dum altera P variabitur; ex quo differentiando nanciscemur hanc aequationem

$$\frac{\partial P}{P} = \frac{\partial p}{m+p} - \frac{\partial p}{p} + \frac{\partial p}{m+p+n} - \frac{\partial p}{p+n} + \frac{\partial p}{m+p+2n} - \frac{\partial p}{p+2n}$$
$$+ \frac{\partial p}{m+p+3n} - \frac{\partial p}{p+3n} + \text{etc.}$$

ubi totum negotium eo redit, quemadmodum differentiale formulae P, quae est integralis, exprimi oporteat.

§. 134. Cum igitur P sit formula integralis solam quantitatem x tanquam variabilem involvens, quandoquidem in ejus integratione exponens p ut constans tractari debet, demum post integrationem ipsam quantitatem P tanquam functionem duarum variabilium x et p spectare licebit; unde quaestio huc redit, quomodo valorem, hoc charactere $\left(\frac{\partial P}{\partial p}\right)$ exprimi solitum, investigari oporteat, qui si indicetur littera Π, aequatio ante inventa hanc induet formam

$$\frac{\Pi}{P} = \frac{1}{m+p} - \frac{1}{p} + \frac{1}{m+p+n} - \frac{1}{p+n} + \frac{1}{m+p+2n} - \frac{1}{p+2n} + \text{etc.}$$

Hanc vero seriem infinitam haud difficulter ad expressionem finitam revocare licebit hoc modo: Ponatur

$$s = \frac{v^{m+p}}{m+p} - \frac{v^p}{p} + \frac{v^{m+p+n}}{m+p+n} - \frac{v^{p+n}}{p+n} + \frac{v^{m+p+2n}}{m+p+2n} - \frac{v^{p+2n}}{p+2n} + \text{etc.}$$

ita ut facto $v = 1$ littera s nobis exhibeat valorem quaesitum $\frac{\Pi}{\text{P}}$; at vero differentiatio nobis dabit

$$\frac{\partial s}{\partial v} = v^{m+p-1} - v^{p-1} + v^{m+p+n-1} - v^{p+n-1} + v^{m+p+2n-1} - v^{p+2n-1} + \text{etc.}$$

cujus seriei infinitae summa manifesto est

$$\frac{v^{m+p-1} - v^{p-1}}{1 - v^n} = \frac{v^{p-1}(v^m - 1)}{1 - v^n}.$$

Hinc igitur vicissim concludimus fore

$$s = \int \frac{v^{p-1}(v^m - 1)\partial v}{1 - v^n},$$

quae formula integralis a $v = 0$ usque ad $v = 1$ est extendenda; sicque habebimus

$$\frac{\Pi}{\text{P}} = \int \frac{v^{p-1}(v^m - 1)\partial v}{1 - v^n} \left[\begin{smallmatrix} a\ v = 0 \\ ad\ v = 1 \end{smallmatrix}\right].$$

§. 135. Ad valorem autem $\left(\frac{\partial \text{P}}{\partial p}\right)$, quem hic littera Π indicavimus, investigandum, ex principiis calculi integralis ad functiones duarum variabilium applicati jam satis notum est, differentiale formulae integralis $\text{P} = \int \text{X}\, x^{p-1} \partial x$ ex sola variabilitate ipsius p oriundum obtineri, si formula post signum integrationis posita $\text{X}\, x^{p-1}$, ex sola variabilitate ipsius p differentietur, atque elementum ∂p signo integrationis praefigatur; at vero quia X non continet p, hic ut constans tractari debet: potestatis vero x^{p-1} differentiale hinc natum erit $x^{p-1} \partial p\, l x$; quam ob rem ex hac differentiatione orietur $\partial \text{P} = \partial p \int \text{X}\, x^{p-1} \partial x\, l x$, ita ut tantum post signum integrationis factor $l x$ accesserit, ex quo manifestum est, fore

$$\Pi = \int X x^{p-1} \partial x \, lx \left[\begin{smallmatrix} ab\ x = 0 \\ ad\ x = 1 \end{smallmatrix}\right],$$

hinc igitur sequens theorema generale constituere licebit.

Theorema generale.

§. 136. Posito brevitatis gratia $X = (1 - x^n)^{\frac{m-n}{n}}$, si sequentes formulae integrales omnes a termino $x = 0$ ad terminum $x = 1$ extendantur, sequens aequalitas semper erit veritati consentanea

$$\int \frac{X x^{p-1} \partial x \, lx}{\int X x^{p-1} \partial x} = \int \frac{x^{p-1}(x^m - 1)\partial x}{1 - x^n}$$

nihil enim obstabat, quo minus loco v scriberemus x, quandoquidem isti valores tantum a terminis integrationis pendent.

§. 137. Hoc igitur modo deducti sumus ad integrationem hujusmodi formularum $\int X x^{p-1} \partial x \, lx$, in quibus quantitas logarithmica lx post signum integrationis tanquam factor inest, quarum valorem exprimere licuit per binas formulas integrales ordinarias, cum sit

$$\int X x^{p-1} \partial x \, lx = \int X x^{p-1} \partial x \, . \int \frac{x^{p-1}(x^m - 1)\partial x}{1 - x^n},$$

integralibus scilicet ab $x = 0$ ad $x = 1$ extensis, ubi brevitatis gratia posuimus $(1 - x^n)^{\frac{m-n}{n}} = X$. Hinc igitur pro binis casibus memorabilibus supra expositis bina theoremata particularia derivemus.

Theorema particulare I, quo $p = n$.

§. 138. Quoniam supra vidimus casu $p = n$ fieri $\int X x^{n-1} \partial x = \frac{1}{m}$, hoc valore substituto habebimus istam aequationem satis elegantem

$$\int X x^{n-1} \partial x\, l x = \frac{1}{m} \int \frac{x^{n-1}(x^m - 1)\partial x}{1 - x^n},$$

dum scilicet ambo integralia ab $x = 0$ ad $x = 1$ extenduntur.

Theorema particulare II, quo $p = n - m$.

§. 139. Quoniam pro hoc casu, quo $p = n - m$ supra ostendimus esse

$$\int X x^{n-m-1} \partial x = \frac{\pi}{n \sin. \frac{m\pi}{n}},$$

nunc deducimur ad sequentem integrationem maxime notatu dignam

$$\int X x^{n-m-1} \partial x\, l x = \frac{\pi}{n \sin. \frac{m\pi}{n}} \int \frac{x^{n-m-1}(x^m - 1)\partial x}{1 - x^n},$$

si quidem haec ambo integralia ab $x = 0$ usque ad $x = 1$ extendantur; ubi meminisse oportet esse

$$X = (1 - x^n)^{\frac{m-n}{n}}.$$

§. 140. Hic probe notetur, theorema generale latissime patere, propterea quod in eo insunt tres exponentes indefiniti, scilicet m, n et p, qui penitus arbitrio nostro relinquuntur, quos ergo infinitis modis pro lubitu definire licet, dummodo singulis valores positivi tribuantur, ita ut semper valor hujus formulae integralis $\int X x^{p-1} \partial x\, l x$, quam ob factorem $l x$ tanquam transcendentem spectari oportet, per formulas integrales ordinarias exprimi queat, quae cum sint generalissima, operae pretium erit nonnullos casus speciales evolvere.

I. Evolutio casus, quo $m = 1$ et $n = 2$.

§. 141. Hoc igitur casu erit $X = \frac{1}{\sqrt{(1 - xx)}}$, unde pro hoc casu theorema generale ita se habebit

$$\int \frac{x^{p-1}\partial x\, lx}{\sqrt{(1-xx)}} = -\int \frac{x^{p-1}\partial x}{\sqrt{(1-xx)}} \cdot \int \frac{x^{p-1}\partial x}{1+x},$$

siquidem singula haec integralea ab $x = 0$ ad $x = 1$ extendantur. Quoniam igitur hic tantum exponens p arbitrio nostro relinquitur, hinc sequentia exempla perlustremus.

Exemplum I. quo $p = 1$.

§. 142. Hoc igitur casu aequatio superior hanc induet formam

$$\int \frac{\partial x\, lx}{\sqrt{(1-xx)}} = -\int \frac{\partial x}{\sqrt{(1-xx)}} \cdot \int \frac{\partial x}{1+x}$$

ubi, integralibus ab $x = 0$ ad $x = 1$ extensis, notum est fieri

$$\int \frac{\partial x}{\sqrt{(1-xx)}} = \frac{\pi}{2} \text{ et } \int \frac{\partial x}{1+x} = l\,2;$$

ita ut jam habeamus

$$\int \frac{\partial x\, lx}{\sqrt{(1-xx)}} \begin{bmatrix} ab\ x = 0 \\ ad\ x = 1 \end{bmatrix} = -\frac{\pi}{2} l\,2,$$

quae est ea ipsa formula, quam initio hujus dissertationis tractavimus et cujus veritatem jam triplici demonstratione corroboravimus

§. 143. Eundem valorem elicere licet ex theoremate particulari secundo, quo erat $p = n - m$, siquidem nunc ob $n = 2$ et $m = 1$ erit $p = 1$; inde enim ob $X = \frac{1}{\sqrt{(1-xx)}}$, istud theorema praebet

$$\int \frac{\partial x\, lx}{\sqrt{(1-xx)}} = \frac{\pi}{2\sin.\frac{\pi}{2}} \cdot \int \frac{\partial x}{1+x} = -\frac{\pi}{2} l\,2.$$

Exemplum II. quo $p = 2$.

§. 144. Hoc igitur casu aequatio superior hanc induet formam

$$\int \frac{x\partial x\, lx}{\sqrt{(1-xx)}} = -\int \frac{x\partial x}{\sqrt{(1-xx)}} \cdot \int \frac{x\partial x}{1+x}$$

Jam vero integralibus ab $x = 0$ ad $x = 1$ extensis, notum est fore

$$\int \frac{x \partial x}{\sqrt{(1-xx)}} = 1 \text{ et} \int \frac{x \partial x}{1+x} = 1 - l\,2;$$

ita ut habeamus

$$\int \frac{x \partial x\, l x}{\sqrt{(1-xx)}} \left[\begin{matrix} ab\; x = 0 \\ ad\; x = 1 \end{matrix}\right] = l\,2 - 1.$$

§. 145. Quoniam in hac formula integrale $\int \frac{x \partial x}{\sqrt{(1-xx)}}$, algebraice exhiberi potest, cum sit $= 1 - \sqrt{(1-xx)}$, valor quaesitus etiam per reductiones consvetas erui potest, cum sit

$$\int \frac{x \partial x\, l x}{\sqrt{(1-xx)}} = [1 - \sqrt{(1-xx)}]\, l x - \int \frac{\partial x}{x} [1 - \sqrt{(1-xx)}],$$

positoque $x = 1$ erit

$$\int \frac{x \partial x\, l x}{\sqrt{(1-xx)}} = - \int \frac{\partial x}{x} [1 - \sqrt{(1-xx)}],$$

ad quam formam integrandam fiat $1 - \sqrt{(1-xx)} = z$, unde colligitur $xx = 2z - zz$, ergo $2\, l x = l z + l(2-z)$, sicque fiet $\frac{\partial x}{x} = \frac{\partial z (1-z)}{z(2-z)}$, quibus valoribus substitutis erit

$$+ \int \frac{\partial x}{x} [1 - \sqrt{(1-xx)}] = + \int \frac{\partial z (1-z)}{2-z},$$

qui ergo valor erit $= C - z - l(2-z)$. Quia igitur posito $x = 0$ fit $z = 0$, constans erit $C = + l\,2$; facto igitur $x = 1$, quia tum fit $z = 1$, iste valor integralis erit $l\,2 - 1$, prorsus ut ante.

§. 146. Eundem valorem suppeditat theorema prius supra allatum, quo erat $p = n = 2$; inde enim statim fit $\int \frac{x \partial x\, l x}{\sqrt{(1-xx)}} = \int - \frac{x \partial x}{1+x}$. Ante autem vidimus esse $\int \frac{x \partial x}{1+x} = 1 - l\,2$; ita ut etiam hinc prodeat valor quaesitus $l\,2 - 1$.

Exemplum III. quo $p = 3$.

§. 147. Hoc igitur casu aequatio in theoremate generali allata hanc induet formam

$$\int \frac{xx \partial x\, l x}{\sqrt{(1-xx)}} = - \int \frac{xx \partial x}{\sqrt{(1-xx)}} \cdot \int \frac{xx \partial x}{1+x}.$$

Per reductiones autem notissimas constat esse

$$\int \frac{x x \partial x}{\sqrt{(1-xx)}} \left[\begin{matrix} ab\, x=0 \\ ad\, x=1 \end{matrix}\right] = \tfrac{1}{2} \cdot \tfrac{\pi}{2},$$

at vero fractio spuria $\frac{xx}{1+x}$ resolvitur in has partes $x - 1 + \frac{1}{1+x}$, unde erit

$$\int \frac{x x \partial x}{1+x} = \tfrac{1}{2} x x - x + l(1+x),$$

quod integrale jam evanescit posito $x=0$; facto ergo $x=1$ ejus valor erit $= -\frac{1}{2} + l\,2$; quamobrem integrale quod quaerimus, erit

$$\int \frac{x x \partial x\, l x}{\sqrt{(1-xx)}} \left[\begin{matrix} ab\, x=0 \\ ad\, x=1 \end{matrix}\right] = -\tfrac{\pi}{4}\left(l\,2 - \tfrac{1}{2}\right).$$

Exemplum IV. quo $p=4$.

§. 148. Hoc igitur casu aequatio superior hanc induet formam

$$\int \frac{x^3 \partial x\, l x}{\sqrt{(1-xx)}} = -\int \frac{x^3 \partial x}{\sqrt{(1-xx)}} \cdot \int \frac{x^3 \partial x}{1+x}.$$

Per reductiones autem notissimas constat esse

$$\int \frac{x^3 \partial x}{\sqrt{(1-xx)}} \left[\begin{matrix} ab\, x=0 \\ ad\, x=1 \end{matrix}\right] = \tfrac{2}{3},$$

tum vero fractio spuria $\frac{x^3}{1+x}$ resolvitur in has partes $x x - x + 1 - \frac{1}{1+x}$, unde integrando fit

$$\int \frac{x^3 \partial x}{1+x} = \tfrac{1}{3} x^3 - \tfrac{1}{2} x x + x - l(1+x),$$

ex quo valor formulae erit $= \frac{5}{6} - l\,2$. His ergo valoribus substitutis adipiscimur hanc integrationem

$$\int \frac{x^3 \partial x\, l x}{\sqrt{(1-xx)}} \left[\begin{matrix} ab\, x=0 \\ ad\, x=1 \end{matrix}\right] = -\tfrac{2}{3}\left(\tfrac{5}{6} - l\,2\right).$$

Exemplum V. quo $p=5$.

§. 149. Hoc igitur casu aequatio superior hanc induet formam

$$\int \frac{x^4 \partial x\, l x}{\sqrt{(1-xx)}} = -\int \frac{x^4 \partial x}{\sqrt{(1-xx)}} \cdot \int \frac{x^4 \partial x}{1+x}.$$

Constat autem esse

$$\int \frac{x^4 \partial x}{\sqrt{(1-xx)}} \left[\begin{smallmatrix} ab\ x = 0 \\ ad\ x = 1 \end{smallmatrix}\right] = \frac{1.3}{2.4} \cdot \frac{\pi}{2},$$

tum vero fractio spuria $\frac{x^4}{1+x}$ manifesto resolvitur in has partes $x^3 - xx + x - 1 + \frac{1}{x+1}$ unde integrando fit

$$\int \frac{x^4 \partial x}{1+x} = \tfrac{1}{4} x^4 - \tfrac{1}{3} x^3 + \tfrac{1}{2} xx - x + l(1+x),$$

ex quo valor formulae erit $= -\frac{7}{12} + l\,2$. His igitur valoribus substitutis prodibit ista integratio

$$\int \frac{x^4 \partial x\, lx}{\sqrt{(1-xx)}} \left[\begin{smallmatrix} ab\ x = 0 \\ ad\ x = 1 \end{smallmatrix}\right] = -\frac{1.3}{2.4} \cdot \frac{\pi}{2} \left(l\,2 - \tfrac{7}{12}\right).$$

Exemplum VI. quo $p = 6$.

§. 150. Hoc igitur casu aequatio superior induet hanc formam

$$\int \frac{x^6 \partial x\, lx}{\sqrt{(1-xx)}} = - \int \frac{x^6 \partial x}{\sqrt{(1-xx)}} \cdot \int \frac{x^5 \partial x}{1+x}.$$

Constat autem per reductiones notas esse

$$\int \frac{x^6 \partial x}{\sqrt{(1-xx)}} \left[\begin{smallmatrix} ab\ x = 0 \\ ad\ x = 1 \end{smallmatrix}\right] = \frac{2.4}{3.5},$$

tum vero fractio spuria $\frac{x^6}{1+x}$ resolvitur in has partes

$$x^4 - x^3 + xx - x + 1 - \frac{1}{x+1},$$

unde integrando nanciscimur

$$\int \frac{x^5 \partial x}{1+x} = \tfrac{1}{5} x^5 - \tfrac{1}{4} x^4 + \tfrac{1}{3} x^3 - \tfrac{1}{2} xx + x - l(1+x),$$

ex quo valor hujus formulae erit $= \frac{47}{60} - l\,2$; quibus valoribus substitutis prodibit ista integratio

$$\int \frac{x^6 \partial x\, lx}{\sqrt{(1-xx)}} \left[\begin{smallmatrix} ab\ x = 0 \\ ad\ x = 1 \end{smallmatrix}\right] = -\frac{2.4}{3.5} \left(\tfrac{47}{60} - l\,2\right).$$

II. Evolutio casus quo $m = 3$ et $n = 2$.

§. 151. Hic ergo erit $X = \sqrt{(1-xx)}$, unde theorema nostrum generale nobis praebebit hanc aequationem

$$\int x^{p-1}\partial x\, lx\,.\sqrt{(1-xx)} = \int x^{p-1}\partial x\sqrt{(1-xx)}\,.\int\frac{x^{p-1}(x^3-1)\partial x}{1-xx},$$

ubi cum sit

$$\frac{x^3-1}{1-xx} = \frac{-xx-x-1}{x+1} = -x-\frac{1}{x+1},$$

erit postrema formula integralis

$$-\int x^p\partial x - \int\frac{x^{p-1}\partial x}{1+x};$$

quae integrata ab $x=0$ ad $x=1$ dat

$$-\frac{1}{p+1}-\int\frac{x^{p-1}\partial x}{1+x},$$

quamobrem habebimus

$$\int x^{p-1}\partial x\, lx\,.\sqrt{(1-xx)} = -\int x^{p-1}\partial x\sqrt{(1-xx}\left(\frac{1}{p+1}+\int\frac{x^{p-1}\partial x}{1+x}\right).$$

Hinc igitur sequentia exempla notasse juvabit.

Exemplum I. quo $p=1$.

§. 152. Pro hoc igitur casu postremus factor evadet, $\frac{1}{2}+l2$, ita ut sit

$$\int\partial x\, lx\,.\sqrt{(1-xx)} = -(\tfrac{1}{2}+l2)\int\partial x\sqrt{(1-xx)}.$$

Pro formula autem $\int\partial x\sqrt{(1-xx)}$ statuatur $\sqrt{(1-xx)}=1-vx$, fietque

$$x=\frac{2v}{1+vv},\ \text{et}\ \sqrt{(1-xx)}=\frac{1-vv}{1+vv},$$

atque $\partial x=\frac{2\partial v(1-vv)}{(1+vv)^2}$, unde fiet

$$\partial x\sqrt{(1-xx)}=\frac{2\partial v(1-vv)^2}{(1+vv)^3},$$

cujus integrale resolvitur in has partes

$$\frac{2v}{(1+vv)^2}-\frac{v}{1+vv}+\text{Arc. tang. } v;$$

quae expressio, cum extendi debeat ab $x=0$ usque ad $x=1$, prior terminus erit $v = 0$, alter vero terminus est $v = 1$; ita ut integrale illud a $v = 0$ usque ad $v = 1$ extendi debeat. At vero illa expressio sponte evanescit posito $v = 0$, facto autem $v = 1$, valor integralis erit $= \frac{\pi}{4}$, quamobrem habebimus

$$\int \partial x\, l x . \sqrt{(1 - xx)} \left[\begin{smallmatrix} ab\, x = 0 \\ ad\, x = 1 \end{smallmatrix}\right] = -\frac{\pi}{4}(\tfrac{1}{2} + l\,2).$$

§. 153. Hic quidem calculum per longas ambages evolvimus, prouti reductio ad rationalitatem formulae $\sqrt{(1 - xx)}$ manuduxit; at vero solus aspectus formulae $\int \partial x \sqrt{(1 - xx)}$ statim declarat, eam exprimere aream quadrantis circuli, cujus radius $= 1$, quem novimus esse $= \frac{\pi}{4}$. Caeterum adhiberi potuisset ista reductio

$$\int \partial x \sqrt{(1 - xx)} = \tfrac{1}{2} x \sqrt{(1 - xx)} + \tfrac{1}{2} \int \frac{\partial x}{\sqrt{(1 - xx)}}$$

cujus valor ab $x = 0$ ad $x = 1$ extensus manifesto dat $\frac{\pi}{4}$.

Exemplum II. quo $p = 2$.

§. 154. Hoc ergo casu postremus factor fit

$$\tfrac{1}{3} + \int \frac{x \partial x}{1 + x} = \tfrac{4}{3} - l\,2;$$

sicque habebimus

$$\int x \partial x\, l x . \sqrt{(1 - xx)} = -(\tfrac{4}{3} - l\,2) \int x \partial x \sqrt{(1 - xx)}:$$

perspicuum autem est, esse

$$\int x \partial x \sqrt{(1 - xx)} = C - \tfrac{1}{3}(1 - xx)^{\frac{3}{2}},$$

qui valor ab $x = 0$ ad $x = 1$ extensus praebet $\frac{1}{3}$, ita ut habeamus

$$\int x \partial x\, l x . \sqrt{(1 - xx)} \left[\begin{smallmatrix} ab\, x = 0 \\ ad\, x = 1 \end{smallmatrix}\right] = -\tfrac{1}{3}(\tfrac{4}{3} - l\,2).$$

III. Evolutio casus quo $m = 1$ et $n = 3$.

§. 155. Hoc igitur casu erit $X = \frac{1}{\sqrt[3]{(1-x^3)^2}}$, unde theorema generale nobis praebet hanc aequationem

$$\int \frac{x^{p-1}\partial x\, lx}{\sqrt[3]{(1-x^3)^2}} = \int \frac{x^{p-1}\partial x}{\sqrt[3]{(1-x^3)^2}} \cdot \int \frac{x^{p-1}(x-1)\partial x}{1-x^3},$$

ubi postrema formula reducitur ad hanc

$$-\int \frac{x^{p-1}\partial x}{xx+x+1},$$

ita ut habeamus

$$\int \frac{x^{p-1}\partial x\, lx}{\sqrt[3]{(1-x^3)^2}} = -\int \frac{x^{p-1}\partial x}{\sqrt[3]{(1-x^3)^2}} \cdot \int \frac{x^{p-1}\partial x}{xx+x+1}:$$

tequentia igitur exempla adiungamus.

Exemplum I. quo $p = 1$.

§. 156. Hoc igitur casu postremus factor evadit $\frac{\partial x}{xx+x+1}$, cujus integrale indefinitum reperitur $\frac{2}{\sqrt{3}}$ Arc. tang. $\frac{x\sqrt{3}}{2+x}$, qui valor posito $x = 1$ abit in $\frac{\pi}{3\sqrt{3}}$; quocirca hoc casu habebimus

$$\int \frac{\partial x \,.\, lx}{\sqrt[3]{(1-x^3)^2}} = -\frac{\pi}{3\sqrt{3}} \int \frac{\partial x}{\sqrt[3]{(1-x^3)^2}};$$

at vero formula integralis $\int \frac{\partial x}{\sqrt[3]{(1-x^3)^2}}$ peculiarem quantitatem transcendentem involvit, quam neque per logarithmos, neque per arcus circulares explicare licet.

Exemplum II. quo $p = 2$.

§. 157. Hoc igitur casu postremus factor erit $\int \frac{x \partial x}{1+x+xx}$, qui in has partes resolvatur

$$\tfrac{1}{2}\int \frac{2x\partial x + \partial x}{1x+x+x} - \tfrac{1}{2}\int \frac{\partial x}{1+x+xx},$$

ubi partis prioris integrale est

$$\tfrac{1}{2}\, l(1+x+xx) = \tfrac{1}{2}\, l\, 3 \;\;(\text{posito scilicet } x = 1);$$

alterius vero partis integrale est $= \frac{1}{2} \cdot \frac{\pi}{3\sqrt{3}}$, quo valore substituto habebimus

$$\int \frac{x \partial x\, l x}{\sqrt[3]{(1-x^3)^2}} = -\tfrac{1}{2}\left(l\,3 - \frac{\pi}{3\sqrt{3}}\right)\int \frac{x \partial x}{\sqrt[3]{(1-x^3)^2}}.$$

Nunc vero istam formulam integralem commode assignare licet per reductionem supra initio indicatam; cum enim hic sit $m = 1$ et $n = 3$, tum vero sumserimus $p = 2$, erit $p = n - m$. Supra autem §. 131. invenimus, hoc casu integrale fore

$$= \frac{\pi}{n \sin.\frac{m\pi}{n}},$$

qui valor nostro casu abit in

$$\frac{\pi}{3 \sin.\frac{\pi}{3}} = \frac{2\pi}{3\sqrt{3}}.$$

Hoc igitur valore substituto, nostram formulam per meras quantitates cognitas exprimere poterimus, hoc modo

$$\int \frac{x \partial x\, l x}{\sqrt[3]{(1-x^3)^2}} \left[\begin{matrix} ab\; x = 0 \\ ad\; x = 1 \end{matrix}\right] = -\frac{\pi}{3\sqrt{3}}\left(l\,3 - \frac{\pi}{3\sqrt{3}}\right)$$

IV. Evolutio casus quo $m=2$ et $n=3$.

§. 158. Hoc igitur casu erit $X=\frac{1}{\sqrt[3]{(1-x^3)}}$, unde theorema generale praebet istam aequationem

$$\int\frac{x^{p-1}\partial x\, lx}{\sqrt[3]{(1-x^3)}}=\int\frac{x^{p-1}\partial x}{\sqrt[3]{(1-x^3)}}\cdot\int\frac{x^{p-1}(xx-1)\partial x}{1-x^3},$$

ubi forma postrema transmutatur in hanc

$$-\int\frac{x^{p-1}\partial x(1+x)}{1+x+xx};$$

unde fiet

$$\int\frac{x^{p-1}\partial x\, lx}{\sqrt[3]{(1-x^3)}}=-\int\frac{x^{p-1}\partial x}{\sqrt[3]{(1-x^3)}}\cdot\int\frac{x^{p-1}\partial x(1+x)}{1+x+xx};$$

unde sequentia exempla expediamus.

Exemplum I. quo $p=1$.

§. 159. Hoc ergo casu membrum postremum erit $\int\frac{\partial x(1+x)}{1+x+xx}$, cujus integrale in has partes distribuatur.

$$\tfrac{1}{2}\int\frac{2x\partial x+\partial x}{1+x+xx}+\tfrac{1}{2}\int\frac{\partial x}{1+x+xx},$$

unde manifesto pro casu $x=1$ prodit $\frac{1}{2}\left(l3+\frac{\pi}{3\sqrt{3}}\right)$; quamobrem nostra aequatio erit

$$\int\frac{\partial x\, lx}{\sqrt[3]{(1-x^3)}}=-\tfrac{1}{2}\left(l3+\frac{\pi}{3\sqrt{3}}\right)\int\frac{\partial x}{\sqrt[3]{(1-x^3)}}.$$

In hac autem formula integrali, ob $m=2$ et $n=3$, quia sumsimus $p=1$, erit $p=n-m$; pro hoc ergo casu per §. 131. valor istius formulae absolute exprimi poterit, eritque

$$\int \frac{\partial x}{\sqrt[3]{(1-x^3)}} = \frac{2\pi}{3\sqrt{3}};$$

consequenter etiam hoc casu per quantitates absolutas consequimur hanc formam

$$\frac{\partial x\, l x}{\sqrt[3]{(1-x^3)}} \left[\begin{smallmatrix} ab\, x = 0 \\ ad\, x = 1 \end{smallmatrix}\right] = -\frac{\pi}{3\sqrt{3}}\left(l\,3 + \frac{\pi}{3\sqrt{3}}\right).$$

§. 160. Quodsi hanc formam cum postrema casus praecedentis, quae itidem absolute prodiit expressa, combinemus, earum summa primo dabit

$$\int \frac{x\,\partial x\, l x}{\sqrt[3]{(1-x^3)^2}} + \int \frac{\partial x\, l x}{\sqrt[3]{(1-x^3)}} = -\frac{2\pi l 3}{3\sqrt{3}}:$$

sin autem posterior a priore subtrahatur, orietur ista aequatio

$$\int \frac{x\,\partial x\, l x}{\sqrt[3]{(1-x^3)^2}} - \int \frac{\partial x\, l x}{\sqrt[3]{(1-x^3)}} = \frac{2\pi\pi}{27}.$$

Quoniam hoc modo ad expressiones satis simplices sumus perducti, operae pretium erit ambas aequationes sub alia forma repraesentare, qua binae partes integrales commode in unam conjungi queant; statuamus scilicet $\frac{x}{\sqrt[3]{(1-x^3)}} = z$, unde fit $\frac{xx}{\sqrt[3]{(1-x^3)^2}} = zz$, sicque prior formula induet hanc speciem $\int \frac{zz\,\partial x\, l x}{x}$, posterior vero istam $\int \frac{z\,\partial x\, l x}{x}$; tum vero habebimus $\frac{x^3}{1-x^3} = z^3$, unde fit $x^3 = \frac{z^3}{1+z^3}$ ideoque

$$l x = l z - \tfrac{1}{3} l (1+z^3) = l \frac{z}{\sqrt[3]{(1+z^3)}},$$

hincque porro

$$\frac{\partial x}{x} = \frac{\partial z}{z} - \frac{zz\partial z}{1+z^3} = \frac{\partial z}{z(1+z^3)};$$

quare his valoribus adhibitis, prior formula integralis evadit

$$\int \frac{z\partial z}{1+z^3} \cdot l \frac{z}{\sqrt[3]{(1+z^3)}};$$

altera vero formula erit

$$\int \frac{\partial z}{1+z^3} \cdot l \frac{z}{\sqrt[3]{(1+z^3)}}.$$

§. 161. Quoniam autem integralia ab $x = 0$ ad $x = 1$ extendi debent, notandum est, casu $x = 0$ fieri $z = 0$, at vero casu $x = 1$ prodire $z = \infty$, ita ut novas istas formas a $z = 0$ ad $z = \infty$ extendi oporteat. Quo observato prior harum formularum dabit

$$\int \frac{z\partial z}{1+z^3} \cdot l \frac{z}{\sqrt[3]{(1+z^3)}} \begin{bmatrix} a\ x = 0 \\ ad\ x = \infty \end{bmatrix} = -\frac{\pi l 3}{3\sqrt{3}} + \frac{\pi\pi}{27},$$

posterior vero

$$\int \frac{\partial z}{1+z^3} \cdot l \frac{z}{\sqrt[3]{(1+z^3)}} \begin{bmatrix} a\ z = 0 \\ ad\ z = \infty \end{bmatrix} = -\frac{\pi l 3}{3\sqrt{3}} - \frac{\pi\pi}{27}.$$

Hinc igitur summa harum formularum erit

$$\int \frac{\partial z(1+z)}{1+z^3} \cdot l \frac{z}{\sqrt[3]{(1+z^3)}} = -\frac{2\pi l 3}{3\sqrt{3}},$$

at vero differentia

$$\int \frac{\partial z(z-1)}{1+z^3} \cdot l \frac{z}{\sqrt[3]{(1+z^3)}} = \frac{2\pi\pi}{27}.$$

§. 162. Hic non inutile erit observasse, istum logarithmum $l\frac{z}{\sqrt[3]{(1+z^3)}}$ commode in seriem infinitam satis simplicem converti posse; cum enim sit

$$l\frac{z}{\sqrt[3]{(1+z^3)}} = \tfrac{1}{3}\, l\frac{z^3}{1+z^3} = -\tfrac{1}{3}\, l\frac{1+z^3}{z^3},$$

erit per seriem

$$l\frac{z}{\sqrt[3]{(1+z^3)}} = -\tfrac{1}{3}\left(\frac{1}{z^3} - \frac{1}{2z^6} + \frac{1}{3z^9} - \frac{1}{4z^{12}} + \frac{1}{5z^{15}} - \text{etc.}\right)$$

verum ista resolutio nullum usum praestare potest ad integralia haec per series evolvenda, propterea quod potestates ipsius z in denominatoribus occurrunt, ideoque singulae partes non ita integrari possunt, ut evanescant posito $z = 0$.

Exemplum II. quo $p = 2$.

§. 163. Hoc igitur casu factor postremus evadit $\int\frac{x\partial x(1+x)}{1+x+xx}$, qui in has duas partes discerpitur $\int\partial x - \int\frac{\partial x}{1+x+xx}$, cujus ergo integrale ab $x=0$ ad $x=1$ extensum est $= 1 - \frac{\pi}{3\sqrt{3}}$. Hinc igitur deducimur ad hanc aequationem

$$\int\frac{x\partial x\, lx}{\sqrt[3]{(1-x^3)}} = -\left(1 - \frac{\pi}{3\sqrt{3}}\right)\int\frac{x\partial x}{\sqrt[3]{(1-x^3)}}.$$

Hic autem notandum, istam formulam integralem nullo modo absolute exhiberi posse, sed peculiarem quandam quantitatem transcendentem involvere.

V. Evolutio casus, quo $m = 2$ et $n = 4$.

§. 164. Hoc igitur casu erit $X = \frac{1}{\sqrt{(1-x^4)}}$, unde theorema nostrum generale nobis dabit hanc aequationem

$$\int \frac{x^{p-1}\partial x\, l x}{\sqrt{(1-x^4)}} = -\int \frac{x^{p-1}\partial x}{\sqrt{(1-x^4)}} \cdot \int \frac{x^{p-1}\partial x}{1+xx};$$

at vero problema particulare prius pro hoc casu praebet

$$\int \frac{x^3\partial x\, l x}{\sqrt{(1-x^4)}} = -\tfrac{1}{2}\int \frac{x^3\partial x}{1+xx}.$$

Cum autem sit

$$\int \frac{x^3\partial x}{1+xx} = \tfrac{1}{2} - \tfrac{1}{2} l\, 2,$$

erit absolute

$$\int \frac{x^3\partial x\, l x}{\sqrt{(1-x^4)}} \left[\begin{matrix} ab\ x=0 \\ ad\ x=1 \end{matrix}\right] = -\tfrac{1}{4}(1 - l\, 2),$$

at vero hic casus congruit cum supra §. 144. tractato. Si enim hic ponamus $xx = y$, quo facto termini integrationis manent $y = 0$ et $y = 1$, erit $l x = \frac{1}{2} l y$ et $x \partial x = \frac{1}{2}\partial y$; quibus valoribus substitutis nostra aequatio abibit in hanc formam

$$\tfrac{1}{4}\int \frac{y\partial y\, l y}{\sqrt{(1-yy)}} = -\tfrac{1}{4}(1 - l\, 2), \text{ sive } \int \frac{y\partial y}{\sqrt{(1-yy)}} = l\, 2 - 1,$$

prorsus ut supra.

§. 165. Alterum vero theorema particulare ad praesentem casum accommodatum dabit

$$\int \frac{x\partial x\, l x}{\sqrt{(1-x^4)}} = -\tfrac{\pi}{4}\int \frac{x\partial x}{1+xx};$$

est vero

$$\int \frac{x\partial x}{1+xx} = l\sqrt{(1+xx)} = \tfrac{1}{2} l\, 2,$$

ita ut habeamus

$$\int \frac{x\partial x\, l x}{\sqrt{(1-x^4)}} \left[\begin{matrix} ab\ x=0 \\ ad\ x=1 \end{matrix}\right] = -\tfrac{\pi}{8} l\, 2.$$

Quodsi vero hic ut ante statuamus $xx = y$, obtinebitur

$$\int \frac{\partial y\, l y}{\sqrt{(1-yy)}} = -\tfrac{\pi}{2} l\, 2,$$

qui est casus supra §. 142. tractatus. His duobus casibus exponens p erat numerus par, unde casus impares evolvi conveniet.

Exemplum I. quo $p = 1$.

§. 166. Hoc igitur casu formula integralis postrema fiet $\int \frac{\partial x}{1+xx} = \text{Arc. tang. } x$, ita ut posito $x = 1$ prodeat Arc. tang. $x = \frac{\pi}{4}$; tum vero aequatio nostra erit

$$\int \frac{\partial x\, lx}{\sqrt{(1-x^4)}} = -\frac{\pi}{4} \int \frac{\partial x}{\sqrt{(1-x^4)}},$$

integralibus scilicet ab $x = 0$ ad $x = 1$ extensis; ubi formula $\int \frac{\partial x}{\sqrt{(1-x^4)}}$ arcum curvae elasticae rectangulae exprimit, ideoque absolute exhiberi nequit.

Exemplum II. quo $p = 3$.

§. 167. Hoc ergo casu formula integralis postrema erit

$$\int \frac{xx\partial x}{1+xx} = \int \partial x - \int \frac{\partial x}{1+xx},$$

cujus integrale posito $x = 1$ fit $= 1 - \frac{\pi}{4}$, ita ut nunc aequatio nostra evadat

$$\int \frac{xx\partial x\, lx}{\sqrt{(1-x^4)}} = -\left(1 - \frac{\pi}{4}\right) \int \frac{xx\partial x}{\sqrt{(1-x^4)}},$$

quae formula integralis pariter absolute exhiberi nequit; exprimit enim applicatam curvae elasticae rectangulae.

§. 168. Quanquam autem haec duo exempla ad formulas inextricabiles perduxerunt, tamen jam pridem demonstravi, productum horum duorum integralium

$$\int \frac{\partial x}{\sqrt{(1-x^4)}} \cdot \int \frac{xx\partial x}{\sqrt{(1-x^4)}}$$

aequari areae circuli, cujus diameter $= 1$, sive esse $= \frac{\pi}{4}$; quamobrem, binis exemplis conjungendis, hoc insigne theorema adipiscimur

$$\int \frac{\partial x\, lx}{\sqrt{(1-x^4)}} \cdot \int \frac{xx\partial x\, lx}{\sqrt{(1-x^4)}} = \frac{\pi^2}{16}\left(1 - \frac{\pi}{4}\right).$$

Facile autem patet, innumera alia hujusmodi theoremata ex hoc fonte hauriri posse, quae, per se spectata, profundissimae indaginis sunt censenda.