SUPPLEMENTUM XI.

AD FINEM TOM. III.

DE
CALCULO VARIATIONUM.

Methodus nova et facilis calculum variationum tractandi. *Nov. Comment. Tom. XVI. Pag.* 35 — 70.

§. 1. Si detur aequatio quaecunque inter binas variabiles x et y, seu quod eodem redit, si y fuerit functio quaecunque ipsius x, tum omnes expressiones quomodocunque ex his duabus quantitatibus x et y formatae et compositae, tanquam functiones unius variabilis x spectari poterunt, ita ut pro quovis valore determinato ipsius x, determinatos quoque valores sortiantur.

§. 2. Hujusmodi autem expressionum ex quantitatibus x et y formatarum, tria genera constitui convenit; ad quorum primum referimus omnes illas expressiones, in quibus tantum ipsae quantitates x et y occurrunt et per operationes quascunque sive algebraicas sive etiam transcendentes inter se sunt complicatae, cujusmodi sunt $\alpha x^3 + \beta x y + \gamma y^3$, item $e^{\alpha x}$ Arc. sin. y, in qua posteriore operationes transcendentes cernuntur. Secundum autem genus eas complectitur expressiones, in quibus praeter ipsas quantitates x et y etiam ratio differentialium occurrit, quam rationem adeo ad differentialia cujusque gradus extendimus, cujusmodi expressionum indolem quo clarius perspiciamus, ponatur more solito

$$\partial y = p\,\partial x;\ \partial p = q\,\partial x;\ \partial q = r\,\partial x;\ \text{etc.}$$

ac tales expressiones erunt functiones quantitatum x, y, p, q, r, etc. Tertium denique genus ejusmodi expressiones continet in quibus praeterea formulae integrales involvuntur, quorsum pertinent expressiones illae in calculo variationum imprimis consideratae, quae hac forma sunt repraesentatae $\int V\,\partial x$, ubi V est functio quaecunque non solum ipsarum x et y; sed etiam quantitatum p, q, r, etc., quin etiam ea alias insuper formulas integrales involvere potest.

§. 3. His circa terna hujusmodi expressionum genera constitutis, facilius indolem calculi variationum explicare poterimus. Totum enim negotium huc redit, ut si proposita fuerit relatio quaecunque inter x et y, eaque aliquantillum varietur, seu ejus loco alia quaepiam relatio inter x et y ab illa infinite parum quomodocunque discrepans adhibeatur, investigari oporteat, quantam mutationem omnes illae expressiones, tam primi, quam secundi et tertii generis sint subiturae, ad quod inveniendum in calculo variationum prouti equidem eum olim tractavi, praeter differentiale ∂y, quo quantitas y augetur dum x in $x + \partial x$ abit, ipsi quantitati y aliud incrementum δy tribuitur, penitus ab arbitrio nostro pendens neque per x determinatum, cui incremento variationis nomen indideram, atque methodum exposueram, variationes inde in singula expressionum genera redundantes inveniendi.

§. 4. Videbatur igitur calculus variationum omnino singulare calculi genus constituere, verum postquam ejus indolem accuratius essem perscrutatus, universum hunc calculum perspexi levi facta immutatione ad secundam partem calculi integralis, cujus elementa in tertio volumine operis mei de hoc argumento exposui, reduci posse. Pertractavi autem in ista secunda parte eas integrationes, quae circa functiones duarum variabilium versantur, in quo calculi genere etiam nunc vix ultra prima elementa progredi licuit.

§. 5. Illius scilicet incrementi loco, quod variationem appellavi, ipsam quantitatem y non amplius tanquam functionem solius variabilis x considero, sed eam tanquam functionem binarum variabilium x et t in calculum introduco, sic enim dum $\partial x \left(\frac{\partial y}{\partial x}\right)$ significat verum differentiale ipsius y, haec formula $\partial t \left(\frac{\partial y}{\partial t}\right)$ idem significare poterit, quod antea signo δy indicavimus. Quo haec reddantur clariora concipiamus y ut applicatam cujuspiam curvae abscissae x respondentem, atque in calculo variationum alia relatio requiritur, quae omnes alias curvas huic saltem proximas complectatur, omnes autem hujusmodi curvas, si X denotet illam functionem cui y aequatur, tali aequatione contineri posse $y = X + tV$ manifestum est; denotante V functionem quamcunque ipsius x. Sumta enim t infinite parva haec aequatio omnes omnino lineas curvas propositae proximas in se comprehendet, atque adeo hanc formam multo generaliorem reddere licet, ita ut pro y functio quaecunque binarum variabilium x et t usurpari possit, dummodo ea ita fuerit comparata, ut posito $t = 0$, prodeat ipsa functio proposita $y = X$.

§. 6. Pro variatione igitur invenienda, quantitas x ut constans spectari, ipsius vero y differentiale tantum ex variabilitate ipsius t desumi debet; unde si expressio proposita fuerit primi generis, functio scilicet ipsarum x et y tantum, quam littera Z designemus, ponamus differentiatione consueta prodire $M \partial x + N \partial y$, atque nunc pro variatione invenienda fiat $\partial x = 0$, at loco ∂y scribatur $\partial t \left(\frac{\partial y}{\partial t}\right)$, quippe quod est incrementum ex sola variabilitate t oriundum. Quo facto variatio quaesita hujus expressionis Z erit $= N \partial t \left(\frac{\partial y}{\partial t}\right)$. Quare si ipsa variatio simili modo per $\partial t \left(\frac{\partial Z}{\partial t}\right)$ indicetur, habebimus $\left(\frac{\partial Z}{\partial t}\right) = N \left(\frac{\partial y}{\partial t}\right)$.

§. 7. Nunc ad expressiones secundi generis progrediamur, in quibus quum praeter x et y occurrant quantitates p, q, r, etc. harum variationes quatenus y etiam a variabili t pendet, per legem generalem his formulis exprimentur

$$\partial t \left(\frac{\partial p}{\partial t}\right); \quad \partial t \left(\frac{\partial q}{\partial t}\right); \quad \partial t \left(\frac{\partial r}{\partial t}\right); \text{ etc.}$$

Quum autem pro sola variabili x, sit

$$p = \left(\frac{\partial y}{\partial x}\right); \; q = \left(\frac{\partial p}{\partial x}\right) = \left(\frac{\partial \partial y}{\partial x^2}\right); \text{ et}$$

$$r = \left(\frac{\partial q}{\partial x}\right) = \left(\frac{\partial \partial p}{\partial x^2}\right) = \left(\frac{\partial^3 y}{\partial x^3}\right); \text{ etc.}$$

erit per regulas generales differentiandi functiones duarum variabilium

$$\left(\frac{\partial p}{\partial t}\right) = \left(\frac{\partial \partial y}{\partial x \partial t}\right); \; \left(\frac{\partial q}{\partial t}\right) = \left(\frac{\partial^3 y}{\partial x^2 \partial t}\right); \; \left(\frac{\partial r}{\partial t}\right) = \left(\frac{\partial^4 y}{\partial x^3 \partial t}\right); \text{ etc.}$$

ubi meminisse juvabit formulam verbi gratia $\left(\frac{\partial^3 y}{\partial x^2 . \partial t}\right)$ prodire, si functio y ter differentietur, et duabus vicibus sola x, una vice autem sola t variabilis sumatur, tum vero qualibet differentiatione differentialia simplicia ∂x vel ∂t abjiciantur.

§. 8. His expeditis sit jam Z functio quaecunque ipsarum x, y, p, q, r, etc., hic quidem nullo adhuc respectu habito ad variabilem t, quippe quae tantum in subsidium variationis introducitur, atque differentiatione more solito facta prodeat

$$\partial Z = M \partial x + N \partial y + P \partial p + Q \partial q + R \partial r + \text{etc.}$$

nunc igitur pro variatione seu $\partial t\left(\frac{\partial Z}{\partial t}\right)$ invenienda scribi debebit ut sequitur

$$\partial x = 0;\ \partial y = \partial t\left(\frac{\partial y}{\partial t}\right);\ \partial p = \partial t\left(\frac{\partial p}{\partial t}\right) = \partial t\left(\frac{\partial \partial y}{\partial x \partial t}\right);$$

$$\partial q = \partial t\left(\frac{\partial^3 y}{\partial x^2 \partial t}\right);\ \partial r = \partial t\left(\frac{\partial^4 y}{\partial x^3 \partial t}\right);\ \text{etc.}$$

atque variatio quaesita erit

$$\partial t\left(\frac{\partial Z}{\partial t}\right) = N\partial t\left(\frac{\partial y}{\partial t}\right) + P\partial t\left(\frac{\partial \partial y}{\partial x \partial t}\right) + Q\partial t\left(\frac{\partial^3 y}{\partial x^2 \partial t}\right) + R\partial t\left(\frac{\partial^4 y}{\partial x^3 \partial t}\right) + \text{etc.}$$

unde sequitur divisione per ∂t facta fore

$$\left(\frac{\partial Z}{\partial t}\right) = N\left(\frac{\partial y}{\partial t}\right) + P\left(\frac{\partial \partial y}{\partial x \partial t}\right) + Q\left(\frac{\partial^3 y}{\partial x^2 \partial t}\right) + R\left(\frac{\partial^4 y}{\partial x^3 \partial t}\right) + \text{etc.}$$

§. 9. Sit nunc etiam expressio quaecunque tertii generis proposita $\int Z \partial x$, ubi Z sit functio quaecunque ipsarum x, y, p, q, r, etc. ita ut per differentiationem ordinariam habeatur

$$\partial Z = M\partial x + N\partial y + P\partial p + Q\partial q + R\partial r + \text{etc.}$$

ubi quidem hactenus nulla ratio novae variabilis t est habita, atque integratio formulae propositae $\int Z \partial x$ per solam variabilem x est expedienda, quo observato, quaestio huc redit, ut si jam y ut functio binarum variabilium x et t consideretur et ubique quantitas y elemento $\partial t\left(\frac{\partial y}{\partial t}\right)$ augeatur, augmentum quod ipsa formula integralis $\int Z \partial x$ inde capiet definiatur, hoc enim augmentum ipsa erit variatio formulae integralis propositae.

§. 10. Quare ad hanc variationem inveniendam in functione illa Z ubique loco y scribatur ejus valor auctus $y + \partial t\left(\frac{\partial y}{\partial t}\right)$, sicque ut ante vidimus, ipsa functio Z augmentum capiet $\partial t\left(\frac{\partial Z}{\partial t}\right)$ ex quo ipsa formula integralis augmentum capiet hoc $\int \partial t\left(\frac{\partial Z}{\partial t}\right)\partial x$, quod erit ipsa variatio quaesita. Quoniam vero in hac integratione sola x pro variabili habetur elementum ∂t ante signum poni poterit ita ut jam variatio futura sit $= \partial t \int \partial x\left(\frac{\partial Z}{\partial t}\right)$.

§. 11. Quoniam igitur in §. 8. valor ipsius $\left(\frac{\partial Z}{\partial t}\right)$ jam evolutus habetur, si ille hic substituatur, formulae $\int Z \partial x$ variatio prodibit ita expressa

$$\partial t \int \partial x \left[N\left(\frac{\partial y}{\partial t}\right) + P\left(\frac{\partial \partial y}{\partial x \partial t}\right) + Q\left(\frac{\partial^3 y}{\partial x^2 \partial t}\right) + R\left(\frac{\partial^4 y}{\partial x^3 . \partial t}\right) + \text{etc.}\right]$$

quam etiam sequenti modo per partes repraesentasse juvabit

$$\partial t \int N \partial x \left(\frac{\partial y}{\partial t}\right) + \partial t \int P \partial x \left(\frac{\partial \partial y}{\partial x \partial t}\right) + \partial t \int Q \partial x \left(\frac{\partial^3 y}{\partial x^2 . \partial t}\right) + \partial t \int R \partial x \left(\frac{\partial^4 y}{\partial x^3 \partial t}\right) + \text{etc.}$$

qua expressione contenti esse possemus, si quaestio circa casum aliquem determinatum institueretur, ubi y non solum functioni cuipiam datae ipsius x aequaretur, sed etiam nova variabilis t modo determinato introduceretur; tum enim omnes istas formulas $\left(\frac{\partial y}{\partial t}\right)$; $\left(\frac{\partial \partial y}{\partial x \partial t}\right)$; $\left(\frac{\partial^3 y}{\partial x^2 \partial t}\right)$; etc. actu evolvere liceret, ita ut tum elementum ∂x per solam functionem ipsius x afficeretur, siquidem uti initio innuimus, evolutione facta, iterum poni debet $t = 0$.

§. 12. At vero tales quaestiones determinatae nunquam occurrere solent; sed potius relatio inter y et x semper incognita esse solet, inde demum determinanda, quod variatio in nihilum abire debeat, quippe in quo methodus maximorum et minimorum versatur. Hujusmodi quaestiones ergo ita enunciari convenit: qualis relatio inter quantitates x et y intercedere debeat, ut formulae integralis propositae $\int Z \partial x$ variatio in nihilum abeat, quomodocunque etiam nova variabilis t in calculum introducatur? Quodsi autem quaestio hac ratione instituatur, perspicuum est formulis $\left(\frac{\partial y}{\partial t}\right)$; $\left(\frac{\partial \partial y}{\partial x \partial t}\right)$; $\left(\frac{\partial^3 y}{\partial x^2 \partial t}\right)$; etc. nullos certos valores tribui posse.

§. 13. Verum hic prorsus singulare artificium in subsidium vocari potest, cujus ope formulas integrales posteriores in §. 11. ad formam priores reducere licet, ita ut in omnibus eadem formula $\left(\frac{\partial y}{\partial t}\right)$ occurrat. Quum enim $\partial x\left(\frac{\partial \partial y}{\partial x \partial t}\right)$ sit differentiale formulae $\left(\frac{\partial y}{\partial t}\right)$ sumta sola x variabili, erit per consuetam integralium reductionem

$$\int P \partial x\left(\frac{\partial \partial y}{\partial x \partial t}\right)=P\left(\frac{\partial y}{\partial t}\right)-\int \partial x\left(\frac{\partial P}{\partial x}\right)\left(\frac{\partial y}{\partial t}\right),$$

simili modo quia $\partial x\left(\frac{\partial^3 y}{\partial x^2 . \partial t}\right)$ est differentiale formulae $\left(\frac{\partial \partial y}{\partial x \partial t}\right)$, habebimus statim hanc reductionem

$$\int Q \partial x\left(\frac{\partial^3 y}{\partial x^2 . \partial t}\right)=Q\left(\frac{\partial \partial y}{\partial x \partial t}\right)-\int \partial x\left(\frac{\partial Q}{\partial x}\right)\left(\frac{\partial \partial y}{\partial x \partial t}\right),$$

nunc vero per praecedentem reductionem fit

$$\int \partial x\left(\frac{\partial Q}{\partial x}\right)\left(\frac{\partial \partial y}{\partial x \partial t}\right)=\left(\frac{\partial Q}{\partial x}\right)\left(\frac{\partial y}{\partial t}\right)-\int \partial x\left(\frac{\partial \partial Q}{\partial x^2}\right)\left(\frac{\partial y}{\partial t}\right),$$

sicque omnino habebimus

$$\int Q \partial x\left(\frac{\partial^3 y}{\partial x^2 . \partial t}\right)=Q\left(\frac{\partial \partial y}{\partial x . \partial t}\right)-\left(\frac{\partial Q}{\partial x}\right)\left(\frac{\partial y}{\partial t}\right)+\int \partial x\left(\frac{\partial \partial Q}{\partial x^2}\right)\left(\frac{\partial y}{\partial t}\right),$$

atque nunc satis perspicuum est, sequentem formulam integralem ita reductam iri

$$\int R \partial x\left(\frac{\partial^4 y}{\partial x^3 \partial t}\right)=R\left(\frac{\partial^3 y}{\partial x^2 . \partial t}\right)-\left(\frac{\partial R}{\partial x}\right)\left(\frac{\partial \partial y}{\partial x \partial t}\right)+\left(\frac{\partial \partial R}{\partial x^2}\right)\left(\frac{\partial y}{\partial t}\right)$$
$$-\int \partial x\left(\frac{\partial^3 R}{\partial x^3}\right)\left(\frac{\partial y}{\partial t}\right),$$

ac si insuper talis formula adesset, foret

$$\int S \partial x\left(\frac{\partial^5 y}{\partial x^4 \partial t}\right)=S\left(\frac{\partial^4 y}{\partial x^3 . \partial t}\right)-\left(\frac{\partial S}{\partial x}\right)\left(\frac{\partial^3 y}{\partial x^2 \partial t}\right)+\left(\frac{\partial \partial S}{\partial x^2}\right)\left(\frac{\partial \partial y}{\partial x \partial t}\right)$$
$$-\left(\frac{\partial^3 S}{\partial x^3}\right)\left(\frac{\partial y}{\partial t}\right)+\int \partial x\left(\frac{\partial^4 S}{\partial x^4}\right)\left(\frac{\partial y}{\partial t}\right).$$

§. 14. Quodsi nunc has formulas reductas substituamus in expressione variationis quaesitae formulae $\int Z \partial x$, tum haec variatio non solum formulis constabit integralibus, sed etiam continebit partes absolutas, quarum aliae formulam $\left(\frac{\partial y}{\partial t}\right)$, aliae hanc $\left(\frac{\partial \partial y}{\partial x \partial t}\right)$, aliae vero hanc $\left(\frac{\partial^3 y}{\partial x^2 \partial t}\right)$ etc. continebunt; dum contra omnes integrales eandem formulam $\left(\frac{\partial y}{\partial t}\right)$ involvunt, quocirca variatio quaesita formulae propositae $\int Z \partial x$, sequenti modo habebitur expressa

$$\partial t \int \partial x \left(\frac{\partial y}{\partial t}\right) \left[N - \left(\frac{\partial P}{\partial x}\right) + \left(\frac{\partial \partial Q}{\partial x^2}\right) - \left(\frac{\partial^3 R}{\partial x^3}\right) + \left(\frac{\partial^4 S}{\partial x^4}\right) - \text{etc.}\right]$$
$$+ \partial t \left(\frac{\partial y}{\partial t}\right) \left[P - \left(\frac{\partial Q}{\partial x}\right) + \left(\frac{\partial \partial R}{\partial x^2}\right) - \left(\frac{\partial^3 S}{\partial x^3}\right) + \text{etc.}\right]$$
$$+ \partial t \left(\frac{\partial \partial y}{\partial x \partial t}\right) \left[Q - \left(\frac{\partial R}{\partial x}\right) + \left(\frac{\partial \partial S}{\partial x^2}\right) - \text{etc.}\right]$$
$$+ \partial t \left(\frac{\partial^3 y}{\partial x^2 \partial t}\right) \left[R - \left(\frac{\partial S}{\partial x}\right) + \text{etc.}\right]$$
$$+ \partial t \left(\frac{\partial^3 y}{\partial x^3 \partial t}\right) [S - \text{etc.}]$$
$$+ \text{etc.}$$

§. 15. Quamquam hic meum institutum non est methodum maximorum et minimorum pertractare, quoniam hoc alibi jam satis copiose est factum; tamen hic praetermittere non possum, quin observem, si variatio formulae $\int Z \partial x$ evanescere debeat, quomodocunque etiam nova variabilis t in calculum ingrediatur, id nullo modo fieri posse, nisi tota pars prima integralis seorsim evanescat, ex quo necesse est, inter x et y hanc aequationem constitui

$$0 = N - \left(\frac{\partial P}{\partial x}\right) + \left(\frac{\partial \partial Q}{\partial x^2}\right) - \left(\frac{\partial^3 R}{\partial x^3}\right) + \left(\frac{\partial^4 S}{\partial x^4}\right) - \text{etc.}$$

et quia nunc variabilis t nulla amplius ratio habetur, sicque tantum unica adhuc variabilis x superest, clausulis omissis hanc habebimus aequationem

$$0 = N - \frac{\partial P}{\partial x} + \frac{\partial \partial Q}{\partial x^2} - \frac{\partial^3 R}{\partial x^3} \frac{\partial^4 S}{\partial x^4} - \text{etc.}$$

qua desiderata relatio inter x et y exprimitur. Partes autem absolutae, tantum ad terminos extremos referuntur, circa quas ea observari debent, quae jam alibi fusius sunt praecepta.

§. 16. Hic etiam non immoror iis casibus, quibus quantitas Z ipsa insuper formulas integrales involvit, quoniam etiam hoc argumentum alibi satis est pertractatum, verum hic opus multo magis arduum molior, dum eandem hanc methodum ad functiones adeo duarum variabilium extendere conabor, quod equidem in dissertatione illa, quam olim de calculo variationum conscripseram, tunc temporis praestare non potui, multitudine tot quantitatum diversi generis deterritus.

Applicatio methodi praecedentis ad functiones duarum variabilium.

§. 17. Si habeatur aequatio quaecunque inter ternas variabiles x, y et z, ea naturam cujuspiam superficiei exprimi censemus, ubi quidem binas coordinatas x et y in plano horizontali constitui intelligamus, tertiam vero z verticalem, sicque haec tertia z, ut functio spectari potest binarum x et y; unde more solito duplicia incrementa consideranda occurrunt, quatenus scilicet a variabilitate ipsius x, vel ipsius y nascuntur. Illud nempe incrementum ipsius z quod ex variatione ipsius x oritur hac formula $\partial x \left(\frac{\partial z}{\partial x}\right)$, hoc vero ex variatione ipsius y oriundum ista $\partial y \left(\frac{\partial z}{\partial y}\right)$ indicari solet.

§. 18. Quodsi jam haec superficies aequatione inter x, y et z expressa, cum aliis quibuscunque superficiebus ipsi proximis comparari debeat, id commodissime fiet novam variabilem t introducendo, ita ut jam z spectanda sit ut functio trium variabilium, x, y et t, quae quidem sumto $t = 0$, in functionem superiorem abeat, at dum ipsi t valores infinite parvi tribuuntur, omnes superficies proximas complectatur, quo posito perspicuum est, quoniam variabiles x et y a nova t neutiquam pendent earum differentialia ∂x et ∂y nullo modo cum ∂t permisceri, sola vero coordinata z triplicis generis incrementa capere potest, praeter bina enim jam ante commemorata, quae vel ab x vel ab y proficiscuntur, accipere poterit incrementum a variabilitate ipsius t oriundum, quod tali formula $\partial t \left(\frac{\partial z}{\partial t}\right)$ est repraesentandum.

§. 19. Ponamus nunc V esse expressionem utcunque ex ipsius coordinatis x, y et z compositam, sive per meras operationes algebraicas, sive etiam transcendentes formatas, quae more solito differentiata praebeat

$$\partial V = L \partial x + M \partial y + N \partial z,$$

atque si ejusdem incrementum desideretur a nova variabili t sola oriundum, manifestum est, statui debere $\partial x = 0$ et $\partial y = 0$, at loco ∂z scribi debere $\partial t \left(\frac{\partial z}{\partial t}\right)$, sicque hoc signandi modo usurpato habebimus

$$\partial t \left(\frac{\partial V}{\partial t}\right) = N \partial t \left(\frac{\partial z}{\partial t}\right) \text{ ideoque } \left(\frac{\partial V}{\partial t}\right) = N \left(\frac{\partial z}{\partial t}\right).$$

Tales autem expressiones ut ante primum genus constituunt.

§. 20. Progrediamur ergo ad secundum genus, quo expressio v praeter ipsas coordinatas x, y, z etiam rationes differentialium earum involvat; atque hic quidem ante omnia formam hujusmodi expressionum accuratius perpendi oportet. Quoniam autem hic statim quantitas z duplicia incrementa capere potest, (hic enim nondum ad novam variabilem t respicimus) ponamus brevitatis gratia

$$\left(\frac{\partial z}{\partial x}\right) = p \text{ et } \left(\frac{\partial z}{\partial y}\right) = p',$$

quae duae litterae differentialia primi gradus comprehendunt deinde pro differentialibus secundi gradus ponamus

$$\left(\frac{\partial \partial z}{\partial x^2}\right) = q; \left(\frac{\partial \partial z}{\partial x \partial y}\right) = q'; \left(\frac{\partial \partial z}{\partial y^2}\right) = q'';$$

unde sequentes relationes inter has litteras et praecedentes notasse juvabit

$$\left(\frac{\partial p}{\partial x}\right) = q; \left(\frac{\partial p}{\partial y}\right) = \left(\frac{\partial p'}{\partial x}\right) = q'; \left(\frac{\partial p'}{\partial y}\right) = q'';$$

simili modo differentialia tertii gradus his formulis complectamur

$$\left(\frac{\partial^3 z}{\partial x^3}\right) = r; \left(\frac{\partial^3 z}{\partial x^2 \partial y}\right) = r'; \left(\frac{\partial^3 z}{\partial x \partial y^2}\right) = r''; \left(\frac{\partial^3 z}{\partial y^3}\right) = r''';$$

ubi hae relationes sunt notandae

$$r=\left(\frac{\partial q}{\partial x}\right);\ r'=\left(\frac{\partial q}{\partial y}\right)=\left(\frac{\partial q'}{\partial x}\right);\ r''=\left(\frac{\partial q'}{\partial y}\right)=\left(\frac{\partial q''}{\partial x}\right);\ r'''=\left(\frac{\partial q''}{\partial y}\right);$$

quarta autem differentialia has formulas praebent

$$s=\left(\frac{\partial^4 z}{\partial x^4}\right);\ s'=\left(\frac{\partial^4 z}{\partial x^3\partial y}\right);\ s''=\left(\frac{\partial^4 z}{\partial x^2\partial y^2}\right);\ s'''=\left(\frac{\partial^4 z}{\partial x\partial y^3}\right);$$
$$s''''=\left(\frac{\partial^4 z}{\partial y^4}\right);$$

et sic ultra quousque libuerit.

§. 21. His explicatis, expressiones secundi generis, praeter ipsas coordinatas x, y et z, etiam quantitates p, p', q, q', q'', r, r', r'', r''', etc. utcunque involvere possunt, ex quo si V denotat quamcunque hujusmodi expressionem, ejus differentiale more solito sumtum sequenti forma exhibeamus

$$\partial V = L\partial x + M\partial y + N\partial z + \left.\begin{array}{l} P\partial p + Q\,\partial q + R\,\partial r \\ + P'\partial p' + Q'\,\partial q' + R'\,\partial r' \\ + Q''\partial q'' + R''\partial r'' \\ + R'''\partial r''' \end{array}\right\}\text{etc.}$$

quam formam animo imprimi conveniet, ne opus sit eam saepius repetere.

§. 22. Quodsi jam hujusmodi expressionem variatio, seu id incrementum inveniri debeat, quod resultat ex variatione novae variabilis t, quam in valorem coordinatae z introducimus, jam vidimus sumi debere $\partial x = 0$ et $\partial y = 0$, tum vero fieri $\partial z = \partial t \left(\frac{\partial z}{\partial t}\right)$, ob eandem vero rationem sequentia differentialia simili modo erunt exprimenda, quae cum suis transformationibus per se satis claris ita se habebunt

$$\partial p = \partial t\left(\frac{\partial p}{\partial t}\right) = \partial t\left(\frac{\partial\partial z}{\partial x\partial t}\right);\ \partial p' = \partial t\left(\frac{\partial p'}{\partial t}\right) = \left(\frac{\partial\partial z}{\partial y\partial t}\right);$$
$$\partial q = \partial t\left(\frac{\partial q}{\partial t}\right) = \partial t\left(\frac{\partial^3 z}{\partial x^2\partial t}\right);\ \partial q' = \partial t\left(\frac{\partial q'}{\partial t}\right) = \partial t\left(\frac{\partial^3 z}{\partial x\partial y\partial t}\right);$$
$$\partial q'' = \partial t\left(\frac{\partial q''}{\partial t}\right) = \partial t\left(\frac{\partial^3 z}{\partial y^2\partial t}\right);$$

$$\partial r = \partial t \left(\frac{\partial r}{\partial t}\right) = \partial t \left(\frac{\partial^4 z}{\partial x^3 \partial t}\right); \ \partial r' = \partial t \left(\frac{\partial r'}{\partial t}\right) = \partial t \left(\frac{\partial^4 z}{\partial x^2 \partial y \partial t}\right);$$

$$\partial r'' = \partial t \left(\frac{\partial r''}{\partial t}\right) = \partial t \left(\frac{\partial^4 z}{\partial x \partial y^2 \partial t}\right);$$

$$\partial r''' = \partial t \left(\frac{\partial r'''}{\partial t}\right) = \partial t \left(\frac{\partial^4 z}{\partial y^3 \partial t}\right); \text{ etc.}$$

§. 23. Totum ergo negotium huc redit, ut in formula illa differentiali pro ∂ V data, loco singulorum differentialium isti valores substituantur, hocque modo prodibit variatio expressionis V ex sola variabilitate ipsius t oriunda, seu valor hujus formulae $\partial t \left(\frac{\partial V}{\partial t}\right)$, quoniam autem singula membra elemento ∂t erunt affecta, eo omisso adipiscimur sequentem formam

$$\begin{aligned}\left(\frac{\partial V}{\partial t}\right) = N\left(\frac{\partial z}{\partial t}\right) + P\left(\frac{\partial \partial z}{\partial x \partial t}\right) + Q\left(\frac{\partial^3 z}{\partial x^2 . \partial t}\right) + R\left(\frac{\partial^4 z}{\partial x^3 . \partial t}\right) \\ + P'\left(\frac{\partial \partial z}{\partial y \partial t}\right) + Q'\left(\frac{\partial^3 z}{\partial x \partial y \partial t}\right) + R'\left(\frac{\partial^4 z}{\partial x^2 \partial y \partial t}\right) \\ + Q''\left(\frac{\partial^3 z}{\partial y^2 . \partial t}\right) + R''\left(\frac{\partial^4 z}{\partial x \partial y^2 \partial t}\right) \\ + R'''\left(\frac{\partial^4 z}{\partial y^3 . \partial t}\right)\end{aligned}$$

quae ad variationes quarumcunque expressionum secundi generis inveniendas sufficit.

§. 24. Nunc expressiones tertii generis aggredi poterimus formulas integrales involventes in quibus potissimum vis hujus methodi cernitur. Quando enim quaestio circa maxima vel minima, quae in superficiebus occurrere possunt, versatur, formula illa, quae maximum vel minimum reddi debet, necessario est formula integralis atque adeo formula integralis duplicata, cujus indolem hic paucis explicari convenit. Quemadmodum enim in praecedente parte formulae integrales simplices sunt consideratae, quae ad datam abscissam x sunt relatae, ita hic in superficiebus, quaestiones semper non ad solam abscissam x, sed ad totum quoddam spatium in plano horizontali tanquam basem sunt referendae, cui portio superficiei quae maximi minimive quadam proprietate gaudere debet, immineat. Quare cum talis basis duplicem habeat dimensionem alteram ab x, alteram vero ab y pendentem, hujusmodi formulae integrales erunt duplicatae, hoc modo exprimi solitae $\int\int V \partial x \partial y$, eae scilicet duplicem integrationem postulant, atque in priore sola coordinata x vel sola y pro variabili habetur, et integratio usque ad terminos basis propositae extenditur, tum vero demum etiam altera variabilis assumitur, atque altera integratio absolvitur. Et quoniam perinde est utra prius pro variabili habeatur, sine discrimine geminam illam integrationem signo duplicato $\int\int$ indicamus; neque vero hic loci est, omnia quae circa hujusmodi integrationes duplicatas sunt observanda, fusius exponere, quippe quod argumentum supra in supplemento VI. pag. 416. seq. jam satis accurate est pertractatum.

§. 25. Quodsi ergo hujusmodi formulae integralis $\int\int V \partial x \partial y$ variatio quaeri debeat, ubi V denotat expressionem quamcunque vel primi vel secundi generis, ex superioribus satis liquet hanc variationem ita expressum iri

$$\partial t \int\int \left(\frac{\partial V}{\partial t}\right) \partial x \partial y,$$

quae forma iterum est integralis duplicata, et prouti vel x vel y priore integratione ut constans spectatur, ea formula vel hoc modo

$$\partial t \int \partial x \int \left(\frac{\partial V}{\partial t}\right) \partial y,$$

vel hoc modo

$$\partial t \int \partial y \int \left(\frac{\partial V}{\partial t}\right) \partial x,$$

exhiberi potest.

§. 26. Sit nunc V talis expressio qualem supra §. 19 descripsimus, et cujus variationem seu valorem $\left(\frac{\partial V}{\partial t}\right)$ in §. 23.

evolvimus, tantum opus erit, singula membra ibi exposita hoc loco $\left(\frac{\partial V}{\partial t}\right)$ substituere; unde sequens congeries formularum nascetur, quibus junctim sumtis variatio quaesita $\partial t \int \int \left(\frac{\partial V}{\partial t}\right) \partial x \partial y$ exprimetur

$$\partial t \int\int N\left(\frac{\partial z}{\partial t}\right)\partial x\partial y + \partial t\int\int P\left(\frac{\partial\partial z}{\partial x\partial t}\right)\partial x\partial y + \partial t\int\int Q\left(\frac{\partial^3 z}{\partial x^2\partial t}\right)\partial x\partial y + \partial t\int\int R\left(\frac{\partial^4 z}{\partial x^3\partial t}\right)\partial x\partial y$$
$$+\partial t\int\int P'\left(\frac{\partial\partial z}{\partial y\partial t}\right)\partial x\partial y + \partial t\int\int Q'\left(\frac{\partial^3 z}{\partial x\partial y\partial t}\right)\partial x\partial y + \partial t\int\int R'\left(\frac{\partial^4 z}{\partial x\partial y\partial t}\right)\partial x\partial y$$
$$+\partial t\int\int Q''\left(\frac{\partial^3 z}{\partial y^2\partial t}\right)\partial x\partial y + \partial t\int\int R''\left(\frac{\partial^4 z}{\partial x\partial y^2\partial t}\right)\partial x\partial y$$
$$+\partial t\int\int R'''\left(\frac{\partial^4 z}{\partial y^3\partial t}\right)\partial x\partial y$$

etc.

§. 27. Nunc singula haec membra post primum peculiares reductiones admittunt, quas probe notasse juvabit. Pro secundo membro sumamus primo x tantum variabile eritque:

$$\int P\left(\frac{\partial\partial z}{\partial x\partial t}\right)\partial x = P\left(\frac{\partial z}{\partial t}\right) - \int\left(\frac{\partial z}{\partial t}\right)\partial x\left(\frac{\partial P}{\partial x}\right),$$

unde etiam alteram integrationem adjiciendo erit

$$\int\int P\left(\frac{\partial\partial z}{\partial x\partial t}\right)\partial x\partial y = \int P\left(\frac{\partial z}{\partial t}\right)\partial y - \int\int\left(\frac{\partial z}{\partial t}\right)\left(\frac{\partial P}{\partial x}\right)\partial x\partial y.$$

Pro tertio membro sumatur primo sola y variabilis eritque

$$\int P'\left(\frac{\partial\partial z}{\partial y.\partial t}\right)\partial y = P'\left(\frac{\partial z}{\partial t}\right) - \int\left(\frac{\partial z}{\partial t}\right)\partial y\left(\frac{\partial P'}{\partial y}\right),$$

unde ipsum tertium membrum transibit in

$$\int\int P'\left(\frac{\partial\partial z}{\partial y\partial t}\right)\partial x\partial y = \int P'\left(\frac{\partial z}{\partial t}\right)\partial x - \int\int\left(\frac{\partial z}{\partial t}\right)\left(\frac{\partial P'}{\partial y}\right)\partial x\partial y.$$

§. 28. Pro sequentibus membris hae ipsae reductiones sequentes dabunt transformationes, pro quarto scilicet habebimus ex secundo

$$\int\int Q\left(\frac{\partial^3 z}{\partial x^2\partial t}\right)\partial x = \int Q\left(\frac{\partial\partial z}{\partial x\partial t}\right)\partial y - \int\int\left(\frac{\partial\partial z}{\partial x\partial t}\right)\left(\frac{\partial Q}{\partial x}\right)\partial x\partial y,$$

at vero hoc membrum posterius ad similitudinem secundi reducitur hoc modo, ubi tantum loco P scribi debet $\left(\frac{\partial Q}{\partial x}\right)$,

$$\int\left(\frac{\partial z}{\partial t}\right)\left(\frac{\partial Q}{\partial x}\right)\partial y-\int\int\left(\frac{\partial z}{\partial t}\right)\left(\frac{\partial\partial Q}{\partial x^2}\right)\partial x\,\partial y,$$

ita ut nunc quartum membrum praebeat hanc formam

$$\int Q\left(\frac{\partial\partial z}{\partial x\,\partial t}\right)\partial y-\int\left(\frac{\partial z}{\partial t}\right)\left(\frac{\partial Q}{\partial x}\right)\partial y+\int\int\left(\frac{\partial z}{\partial t}\right)\left(\frac{\partial\partial Q}{\partial x^2}\right)\partial x\,\partial y.$$

Simili modo quintum membrum ope secundi reducitur, ubi loco P scribitur Q' et loco $\left(\frac{\partial\partial z}{\partial x\,\partial t}\right)$, $\left(\frac{\partial^3 z}{\partial x\,\partial y\,\partial t}\right)$, sive loco $\left(\frac{\partial z}{\partial t}\right)$ scribendo $\left(\frac{\partial\partial z}{\partial y\,\partial t}\right)$, sicque habebitur

$$\int\int Q'\left(\frac{\partial^3 z}{\partial x\,\partial y\,\partial t}\right)\partial x\partial y=\int Q'\left(\frac{\partial\partial z}{\partial y\,\partial t}\right)\partial y-\int\int\left(\frac{\partial\partial z}{\partial y\,\partial t}\right)\left(\frac{\partial Q'}{\partial x}\right)\partial x\,\partial y,$$

quod posterius membrum cum tertio conferatur, ubi tantum loco P' scribi debet $\left(\frac{\partial Q'}{\partial x}\right)$, quo pacto totum membrum induet hanc formam

$$\int Q'\left(\frac{\partial\partial z}{\partial y\,\partial t}\right)\partial y-\int\left(\frac{\partial z}{\partial t}\right)\left(\frac{\partial Q'}{\partial x}\right)\partial x+\int\int\left(\frac{\partial z}{\partial t}\right)\left(\frac{\partial\partial Q'}{\partial x\,\partial y}\right)\partial x\,\partial y,$$

sextum vero membrum bis cum secundo collatum reducitur ad hanc formam

$$\int Q''\left(\frac{\partial\partial z}{\partial y\,\partial t}\right)\partial x-\int\left(\frac{\partial z}{\partial t}\right)\left(\frac{\partial Q''}{\partial y}\right)\partial x+\int\int\left(\frac{\partial z}{\partial t}\right)\left(\frac{\partial\partial Q''}{\partial y^2}\right)\partial x\,\partial y.$$

§. 29. Si hoc modo ulterius progrediamur ad sequentia membra, septimum membrum in sequentes partes resolvitur

$$\int R\left(\frac{\partial^3 z}{\partial x^2\partial t}\right)\partial y-\int\left(\frac{\partial\partial z}{\partial x\,\partial t}\right)\left(\frac{\partial R}{\partial x}\right)\partial y+\int\left(\frac{\partial z}{\partial t}\right)\left(\frac{\partial\partial R}{\partial x^2}\right)\partial y$$
$$-\int\int\left(\frac{\partial z}{\partial t}\right)\left(\frac{\partial^3 R}{\partial x^3}\right)\partial x\,\partial y;$$

deinde octavum membrum

$$\int R'\left(\frac{\partial^3 z}{\partial x\,\partial y\,\partial t}\right)\partial y-\int\left(\frac{\partial\partial z}{\partial x\,\partial t}\right)\left(\frac{\partial R'}{\partial x}\right)\partial y+\int\left(\frac{\partial z}{\partial t}\right)\left(\frac{\partial\partial R'}{\partial x\partial y}\right)\partial y$$
$$-\int\int\left(\frac{\partial z}{\partial t}\right)\left(\frac{\partial^3 R}{\partial x^2\,\partial y}\right)\partial x\,\partial y,$$

tum nonum membrum fiet

$$\int R''\left(\frac{\partial^3 z}{\partial x\,\partial y\,\partial t}\right)\partial x-\int\left(\frac{\partial\partial z}{\partial y\,\partial t}\right)\left(\frac{\partial R''}{\partial y}\right)\partial x+\int\left(\frac{\partial z}{\partial t}\right)\left(\frac{\partial\partial R''}{\partial x\,\partial y}\right)\partial x$$
$$-\int\int\left(\frac{\partial z}{\partial t}\right)\left(\frac{\partial^3 R''}{\partial x\,\partial y^2}\right)\partial x\,\partial y,$$

et decimum

$$\int R''' \left(\frac{\partial^3 z}{\partial y^2 \partial t}\right) \partial x - \int \left(\frac{\partial \partial z}{\partial y \partial t}\right) \left(\frac{\partial R'''}{\partial y}\right) \partial x + \int \left(\frac{\partial z}{\partial t}\right) \left(\frac{\partial \partial R'''}{\partial y^2}\right) \partial x - \int\int \left(\frac{\partial z}{\partial t}\right) \left(\frac{\partial^3 R'''}{\partial y^3}\right) \partial x \, \partial y.$$

§. 30. Colligamus nunc omnes istas formulas in unam summam, atque variatio quaesita pluribus constabit membris, quarum primum formulas integrales duplicatas, reliqua vero simplices complectentur: hoc pacto variatio quaesita sequenti modo erit expressa

$$\partial t \int\int \partial x \, \partial y \left(\frac{\partial z}{\partial t}\right) \left\{ \begin{array}{l} N - \left(\frac{\partial P}{\partial x}\right) + \left(\frac{\partial \partial Q}{\partial x^2}\right) - \left(\frac{\partial^3 R}{\partial x^3}\right) \\ \quad - \left(\frac{\partial P'}{\partial y}\right) + \left(\frac{\partial \partial Q'}{\partial x \partial y}\right) - \left(\frac{\partial^3 R'}{\partial x^2 \partial y}\right) \text{ etc.} \\ \qquad + \left(\frac{\partial \partial Q''}{\partial y^2}\right) - \left(\frac{\partial^3 R''}{\partial x \partial y^2}\right) \\ \qquad\qquad - \left(\frac{\partial^3 R'''}{\partial y^3}\right) \end{array} \right\}$$

$$+ \partial t \left\{ \begin{array}{l} \int \left(\frac{\partial z}{\partial t}\right) P \, \partial y + \int Q \, \partial y \left(\frac{\partial \partial z}{\partial x \partial t}\right) - \int \partial y \left(\frac{\partial Q}{\partial x}\right) \left(\frac{\partial z}{\partial t}\right) + \int R \, \partial y \left(\frac{\partial^3 z}{\partial x^2 \partial t}\right) \\ \int \left(\frac{\partial z}{\partial t}\right) P' \partial x + \int Q' \partial y \left(\frac{\partial \partial z}{\partial y \partial t}\right) - \int \partial x \left(\frac{\partial Q'}{\partial x}\right) \left(\frac{\partial z}{\partial t}\right) + \int R' \, \partial y \left(\frac{\partial^3 z}{\partial x \partial y \partial t}\right) \\ \qquad + \int Q'' \partial x \left(\frac{\partial \partial z}{\partial y \partial t}\right) - \int \partial x \left(\frac{\partial Q''}{\partial y}\right) \left(\frac{\partial z}{\partial t}\right) + \int R'' \, \partial x \left(\frac{\partial^3 z}{\partial x \partial y \partial t}\right) \\ \qquad\qquad + \int R''' \partial x \left(\frac{\partial^3 z}{\partial y^2 \partial t}\right) \end{array} \right.$$

$$\left. \begin{array}{l} - \int \partial y \left(\frac{\partial R}{\partial x}\right) \left(\frac{\partial \partial z}{\partial x \partial t}\right) + \int \partial y \left(\frac{\partial z}{\partial t}\right) \left(\frac{\partial \partial R}{\partial x^2}\right) \\ - \int \partial y \left(\frac{\partial R'}{\partial x}\right) \left(\frac{\partial \partial z}{\partial x \partial t}\right) + \int \partial y \left(\frac{\partial z}{\partial t}\right) \left(\frac{\partial \partial R'}{\partial x \partial y}\right) \text{ etc.} \\ - \int \partial x \left(\frac{\partial R''}{\partial y}\right) \left(\frac{\partial \partial z}{\partial y \partial t}\right) + \int \partial x \left(\frac{\partial z}{\partial t}\right) \left(\frac{\partial \partial R''}{\partial x \partial y}\right) \\ - \int \partial x \left(\frac{\partial R'''}{\partial y}\right) \left(\frac{\partial \partial z}{\partial y \partial t}\right) + \int \partial x \left(\frac{\partial z}{\partial t}\right) \left(\frac{\partial \partial R'''}{\partial y^2}\right) \end{array} \right\}$$

§. 31. Verum quid haec singula membra proprie significent et ad quemnam usum adhiberi queant, neutiquam adhuc perspicere licet, unde hoc argumentum cujus prima fundamenta etiamnunc vix jacta sunt censenda, omnem geometrarum attentionem atque multo accuratiorem investigationem postulare videtur, quod negotium vix ante suscipere licet, quam casus nonnulli particulares omni studio et diligentia fuerint evoluti, quin etiam ipsa pars prior, quae tantum circa functiones unius variabilis versatur neutiquam adhuc satis clare et distincte est enucleata, ita ut perspicue intelligeremus veram indolem atque naturam singularum partium, quibus variationem contineri invenimus, quem in finem dilucidationes sequentes hic adjungere visum est.

Dilucidationes super theoria variationum ad functiones saltem unius variabilis accommodata.

§. 32. Quaestiones quae hic occurrunt ad hoc problema generale revocare licet.

Si y fuerit functio quaecunque ipsius x, indeque definiatur valor cujuspiam formulae integralis datae $\int Z \partial x$, denotante Z expressionem ex ipsis quantitatibus x et y earumque differentialium rationibus utcunque compositam, quaestio est, si loco illius functionis y alia quaecunque illi proxima seu infinite parum tantum ab ea discrepans adhibeatur, quanto majorem minoremve valorem, tum eadem formula integralis $\int Z \partial x$ sit consecutura.

§. 33. At quia hoc modo ista quaestio enunciata nimis videri posset abstracta, eam more soluto ad geometriam re-
Fig. 15 vocemus. Sit igitur super axe A P proposita curva quaecunque A M, aequatione inter abscissam $AP = x$ et applicatam $PM = y$ expressa, pro qua definiri oporteat valorem formulae cujuspiam integralis $\int Z \partial x$, qui sit $= W$, quo posito consideretur alia curva quaecunque $\alpha \mu$ infinite parum a data discrepans, ac si pro hac curva itidem definiatur valor formulae $\int Z \partial x$, quaeritur, quantum

iste valor a praecedente sit discrepaturus: evidens enim est, hoc discrimen praebere ipsam variationem quantitatis W, quam supra ope calculi variationum exhibuimus.

§. 34. Quo haec adhuc clariora evadant, exemplum quodpiam proferamus, quo proposita curva A M ejusque axe A P tanquam verticali considerato, quaeritur tempus quo corpus ex puncto A super hac curva A M descendens usque ad punctum M pertingit. Jam quia celeritas corporis in M est ut $\sqrt{AP} = \sqrt{x}$, et ipsum curvae elementum $= \partial x \sqrt{(1 + pp)}$, posito scilicet $\partial y = p \partial x$ uti in solutione generali est praeceptum, erit tempus per elementum $Mm = \partial x \frac{\sqrt{(1+pp)}}{\sqrt{x}}$, unde formula integralis $\int Z \partial x$ pro hoc casu abit in $\int \partial x \frac{\sqrt{(1+pp)}}{\sqrt{x}}$, ita ut habeatur $Z = \frac{\sqrt{(1+pp)}}{\sqrt{x}}$, quare nunc tempus erit definiendum, quo corpus super curva quacunque proxima $\alpha\, \mu$ descendens ab α usque ad μ perveniet, ubi discrimen dabit ipsam variationem formulae $\int \partial x \frac{\sqrt{(1+pp)}}{\sqrt{x}}$, huic casui convenientem.

§. 35. Quoniam hic formula integralis consideranda venit, ante omnia dispiciendum est, quomodo eam determinari oporteat. In exemplo quidem allato, manifestum est formulae $\int \frac{\partial x \sqrt{(1+pp)}}{\sqrt{x}}$ integrale ita capi debere, ut evanescat posito $x = 0$, unde etiam in genere intelligitur, semper pro integratione formulae $\int Z \partial x$, certum aliquem terminum veluti punctum A, tanquam principium integrationis statui, atque integrale $\int Z \partial x$ evanescere debere posito $x = 0$, vel si forte circumstantiae aliter fuerint comparatae, tribuendo ipsi x valorem quempiam datum, deinde vero initio constituto, valor formulae $\int Z \partial x = W$ abscissae $AP = x$ respondebit.

§. 36. His circa formulam integralem $\int Z \partial x$ observatis, videamus, quamnam ideam nobis de curvis illis proximis $\alpha \mu$ formare debeamus. Ac primo quidem patet, has curvas continuo quodam tractu ductas esse debere, ita ut in iis nusquam anguli aliive saltus deprehendantur; hoc solo notato, perinde est sive istae curvae lege quapiam continuitatis vel aequatione quapiam contineantur, sive sint adeo discontinuae, quasi libero manus motu ductae

§. 37. Hujusmodi lineae curvae commodissime sequenti modo formatae menti repraesentari possunt. Ducatur scilicet pro lubitu linea curva quaecunque B N eidem abscissae A P imminens, ac ductis ad singula axis puncta X applicatis X Y V singula intervalla Y V in ratione finiti ad infinite parvum secentur in v, ita ut Y v sit quasi pars infinitesima intervalli Y V. Hoc enim modo curva $\alpha \nu \mu$ obtinebitur a curva proposita A M in omnibus punctis infinite parum dissita, qualem ad institutum nostrum requirimus. Praeterea tamen notandum est, in curva illa arbitraria B N nusquam tangentem ad axem A P normalem esse debere, quia hoc modo divisio illorum intervallorum turbaretur. Atque nunc evidens est, non solum intervalla Y v esse infinite parva, sed etiam tangentes in punctis Y et v infinite parum a parallelismo deficere.

Explicatio partis primae in variatione.

§. 38. His circa ipsam quaestionis propositionem annotatis, contemplemur nunc accuratius quoque solutionem supra inventam, ejusque singulas partes, ut quid quaelibet earum innuat et ad quemnam usum sit transferenda perspicue intelligamus; solutionem autem in §. 14. datam hic contemplabimur. Statim igitur consideremus primam variationis ibi inventae partem, quae hac formula integrali continetur

$$\partial t \int \partial x \left(\frac{\partial y}{\partial t}\right) \left[N - \left(\frac{\partial P}{\partial x}\right) + \left(\frac{\partial \partial Q}{\partial x^2}\right) - \left(\frac{\partial^3 R}{\partial x^3}\right) + \left(\frac{\partial^4 S}{\partial x^4}\right) - \text{etc.}\right],$$

cujus integratio ita capi debet, ut in ipso termino initiali A evanescat, qua conditione constans arbitraria determinatur, quod si ergo in singulis punctis X Y haec formula applicata intelligatur, aggregatum omnium istarum formularum elementarium ab initio A usque ad terminum M extensum praebebit primam partem variationis quaesitae, atque hic quidem in figura perspicuum est, spatiolum Y v exprimere incrementum applicatae y a sola variabili t oriundum, ita ut sit Y $v = \partial t \left(\frac{\partial y}{\partial t}\right)$.

§. 39. Haec igitur prima pars variationis involvit omnia spatiola Y v intra terminos A et M contenta, quae quum in infinitum variari possint, atque adeo a positivis ad negativa transire queant, maximae variationes hic locum habere possunt. Verum tamen unicus casus hinc debet excipi, quo curva A M ita est comparata, ut sit

$$0 = N - \frac{\partial P}{\partial x} + \frac{\partial \partial Q}{\partial x^2} - \frac{\partial^3 R}{\partial x^3} + \frac{\partial^4 S}{\partial x^4} - \text{etc.}$$

tum enim utcunque curvae proximae fuerint comparatae, ista pars prima variationis, semper in nihilum abit. Neque deviatio curvarum proximarum $\alpha\mu$ a principali A M intra terminos A et M quicquam ad variationem confert; ex quo haec curva respectu formulae integralis $\int Z \partial x$ imprimis est memorabilis, quandoquidem in ea haec formula integralis vel maximum vel minimum obtinet valorem.

Explicatio partis secundae in variatione.

§. 40. Progrediamur nunc ad secundam partem variationis supra inventae, quae est

$$\partial t \left(\frac{\partial y}{\partial t}\right) \left(P - \frac{\partial Q}{\partial x} + \frac{\partial \partial R}{\partial x^2} - \frac{\partial^3 S}{\partial x^3} + \text{etc.}\right)$$

circa quam primum observo, quoniam ea ad terminum M refertur, per integrationem rite institutam insuper adjici debere similem expressionem ad terminum priorem A relatam, at vero signo contrario affectam, id quod ideo est necessarium, ut facto $x = 0$, etiam haec expressio penitus tollatur. Refertur autem ista pars

$$\partial t \left(\frac{\partial y}{\partial t}\right) \left(P - \frac{\partial Q}{\partial x} + \frac{\partial \partial R}{\partial x^2} - \text{etc.}\right)$$

unice ad ultimum terminum M, ubi $\partial t \left(\frac{\partial y}{\partial t}\right)$ ipsum spatiolum M μ exprimit, similique modo in alteram partem pro initio A spatiolum A α ingredietur. Hinc patet si omnes curvae proximae $\alpha\,\mu$ per ipsos ambos terminos A et M ducantur tum variationem secundae partis in nihilum abire.

§. 41. Consideremus autem casum, quo curva proxima $\alpha\,\mu$ per primum quidem terminum A transit non vero quoque per alterum M, sed sit punctum μ ejus terminus, atque variatio ex secunda parte nata erit

$$M\mu \left(P - \frac{\partial Q}{\partial x} + \frac{\partial \partial R}{\partial x^2} - \text{etc.}\right)$$

Atque hinc etiam definire poterimus variationem ex eodem fonte oriundam, si curva proxima A μ, non in ipso puncto μ sed alio quocunque ω terminetur, existente semper intervallo $\mu\,\omega$ infinite parvo. Ducta enim applicata $\omega\,m\,p$, variatio modo inventa insuper augeri debet particula formulae $\int Z\,\partial x$, quae elemento $Pp = \partial x$ respondet, quae particula quum sit $= Z \,.\, Pp$, pro arcu curvae proximae A ω erit variatio ex secunda parte oriunda

$$M\mu \left(P - \frac{\partial Q}{\partial x} + \frac{\partial \partial R}{\partial x^2} - \text{etc.}\right) + Z \,.\, Pp.$$

§. 42. Ducatur recta M ω, et quaeramus angulum $\omega\,M\,m$, quem haec recta M ω cum curva principali constituit, ponatur iste angulus $\omega M m = \omega$, et ducta MO ipsi Pp parallela, quia est proxime $m\omega = M\mu$ et anguli mMo tangens $= p$, ideoque $om = p.Pp$, habebitur $O\omega = M\mu + p.Pp$, unde fit

$$\text{tang. } \omega M o = \frac{M\mu}{Pp} + p,$$

atque hinc colligitur

$$\omega M m = \text{tang. } \omega = \frac{M\mu}{Pp(1+pp) + M\mu . p}.$$

Servemus nunc in calculo hunc ipsum angulum ω atque hinc habebimus spatiolum

$$M\mu = \frac{Pp(1+pp)\,\text{tang. }\omega}{1 - p\,\text{tang. }\omega},$$

quo valore substituto variatio pro arcu $A\omega$ erit

$$Pp\left[Z + \frac{(1+pp)\,\text{tang. }\omega}{1 - p\,\text{tang. }\omega}.\left(P - \frac{\partial Q}{\partial x} + \frac{\partial\partial R}{\partial x^2} - \text{etc.}\right)\right].$$

§. 43. Nunc operae pretium erit eum angulum ω definire, ut ista variatio in nihilum abeat, id quod eveniet, si capiatur

$$\text{tang. } \omega = \frac{Z}{pZ - (1+pp)\left(P - \frac{\partial Q}{\partial x} + \frac{\partial\partial R}{\partial x^2} - \text{etc.}\right)},$$

quare hoc angulo ita constituto pro omnibus lineis proximis ubicunque in recta $M\omega$ terminatis variatio ex secunda parte oriunda evanescet. Hic casus prae caeteris omnino notatu dignus considerari meretur, quo recta $M\omega$ fit ad curvam principalem in puncto M normalis, quod evenit, si fuerit

$$pZ - (1+pp)\left(P - \frac{\partial Q}{\partial x} + \frac{\partial\partial R}{\partial x^2} - \text{etc.}\right) = 0,$$

qua aequatione certa conditio ipsius formulae integralis $\int Z\,\partial x$ sive indoles expressionis Z definitur.

§. 44. Non igitur pigebit in talem expressionem Z inquisivisse, ac primo quidem patet eam praeter coordinatas x et y

etiam quantitatem p involvere debere. Sumamus autem praeterea in Z non ingredi litteras q, r, etc. ita ut sit $Q = 0$, $R = 0$, ac nostra aequatio resolvenda erit

$$p Z = (1 + p p) P,$$

ubi notandum est esse

$$\partial Z = M \partial x + N \partial y + P \partial p,$$

quare si ambae coordinatae x et y tanquam constantes tractentur, erit

$$\partial Z = P \partial p, \text{ ideoque } P = \frac{\partial Z}{\partial p},$$

quo valore ibi introducto haec prodibit aequatio

$$\frac{\partial Z}{Z} = \frac{p \partial p}{1 + p p},$$

quae integrata dat

$$l. Z = l. \sqrt{(1 + p p)} + l. C,$$

quae constans functio quaecunque ipsarum x et y esse potest, talis functio sit V, atque habebimus

$$Z = V \sqrt{(1 + p p)},$$

ideoque formula integralis

$$\int V \partial x \sqrt{(1 + p p)}.$$

Hujus formulae significatum satis eleganter per tempus, quo corpus quodpiam per curvam A M promovetur exprimi potest. Si enim celeritas in puncto M, fuerit $= \frac{1}{V}$, hoc est, si celeritas in singulis punctis proportionalis fuerit functioni cuicunque binarum variabilium x et y, tum

$$V \partial x \sqrt{(1 + p p)}$$

exprimit elementum temporis, ideoque formula

$$\int V \partial x \sqrt{(1 + p p)}$$

totum tempus quo corpus ab A ad M pervenit

Explicatio partis tertiae in variatione.

§. 45. Quod ad tertiam partem variationis attinet, scilicet

$$\partial t \left(\frac{\partial \partial y}{\partial x \partial t}\right)\left(Q - \frac{\partial R}{\partial x} + \frac{\partial \partial S}{\partial x^2} - \text{etc.}\right)$$

ea locum non habet, nisi expressio Z etiam differentialia secundi gradus involvat, quod quidem rarissime usu venire solet. Hic autem observandum est, quoniam $M\mu = \partial t \left(\frac{\partial y}{\partial t}\right)$ fore pro sequenti elemento

$$m\omega = \partial t \left(\frac{\partial y}{\partial t}\right) + \partial t \, \partial x \left(\frac{\partial \partial y}{\partial x \partial t}\right),$$

unde colligitur

$$\partial t \left(\frac{\partial \partial y}{\partial x \partial t}\right) = \frac{m\omega - M\mu}{\partial x} = \frac{M\omega - M\mu}{Pp},$$

hac autem formula exprimitur declinatio directionis $\mu\,\omega$ a directione $M\,m$, quae quidem, ut jam ante observavimus, semper est quam minima.

§. 46. Quodsi ergo tangens in μ perfecte fuerit parallela tangenti in M, quod evenit, si etiam in curva generatrice B N, tangens ad N huic fuerit parallela, tum variatio ex tertia parte oriunda prorsus evanescit, quod etiam de termino initiali A est intelligendum, si tangentes in A et B inter se fuerint parallelae: atque hinc jam perspicitur, ut variationes ex quarta parte oriundae evanescant, necesse esse, ut praeterea etiam radii osculi in punctis M et μ fiant aequales.

§. 47. Atque ex his jam satis perspicuum est, variationes ex secunda parte oriundas evanescere, si omnes curvae proximae $\alpha\,\mu$ per utrumque terminum M et A ducantur. Deinde vero insuper etiam variationes tertiae partis, si omnes curvae proximae simul in utroque termino A et M cum curva principali A M communes habeat tangentes. Praeterea vero quoque variationes quartae partis in nihilum abire, si omnes curvae proximae in terminis A et M insuper ratione curvaturae cum curva principali conveniant. Hic autem probe meminisse juvabit, variationes tertiae partis per se evanescere, si modo quantitas Z non differentialia secundi gradus involvat; quartae vero partis semper evanescere nisi differentialia tertii gradus in quantitatem Z ingrediantur, et ita porro. Unde quum initio ostenderimus, quomodo variatio primae partis ad nihilum sit redigenda, hunc evidentissime intelligimus sub quibusnam conditionibus, omnes variationis partes simul evanescant.

Dilucidationes circa curvas maximi, minimive proprietate praeditas.

§. 48. Si formula integralis $\int Z\,\partial x$ in curva quaesita debeat esse vel maximum vel minimum, jam supra ostendimus, posito

$$\partial Z = M\,\partial x + N\,\partial y + P\,\partial p + Q\,\partial q + R\,\partial r + \text{etc.}$$

naturam hujus curvae, hac exprimi aequatione

$$0 = N - \frac{\partial P}{\partial x} + \frac{\partial\partial Q}{\partial x^2} - \frac{\partial^3 R}{\partial x^3} + \text{etc.}$$

quae aequatio nisi quantitates P, Q, R evanescant, vel sint constantes, semper est differentialis vel secundi, vel quarti, vel sexti, aliusve gradus paris. Hic ergo statim memoratu dignum occurrit quod ista aequatio nunquam vel simpliciter differentialis, vel tertii, vel quinti, aliusve gradus imparis evadat, id quod mox clarius exponemus.

§. 49. Quaestiones ergo huc pertinentes sponte in varias dividuntur classes, pro gradu differentialium, ad quem aequationes exsurgunt, quandoquidem ab hoc gradu natura solutionis maxime pendet, propterea quod ea semper totidem constantes arbitrarias involvit. Ad primam ergo classem referimus eos casus quibus aequatio pro maximo vel minimo inventa prorsus est finita. Ad secundam autem classem eos, quibus haec aequatio fit differentialis secundi gradus, ad tertiam eos, quibus aequatio ad quartum gradum ascendit et ita porro, quas singulas classes ordine describamus.

Classis I.

§. 50. Ad solutionem ergo primae classis formula $\int Z \partial x$ statim perducit, quando expressio Z tantum per coordinatas x et y exclusis omnium differentialium rationibus determinatur, quia enim hoc casu, simpliciter fit $\partial Z = M \partial x + N \partial y$, aequatio pro curva maximi vel minimi erit $N = 0$, quae ergo aequatio omnino est determinata, atque adeo curva satisfaciens unica in suo genere. Veluti si quaeratur linea, in qua valor formulae $\int \partial x (2xy - yy)$ fiat maximus vel minimus, ob $Z = 2xy - yy$, ideoque $N = 2(x - y)$, aequatio quaesita erit $x - y = 0$, seu linea quaesita erit recta ad axem angulo semirecto inclinata, pro qua ergo valor formulae propositae integralis est $\frac{x^3}{5}$, qui utique minor est, quam si ulla alia linea curva sumeretur pro eadem scilicet abscissa.

§. 51. His autem casibus prima classis nondum exhauritur, sed dantur adhuc alii perinde ad aequationes finitas ducentes, ad quod ostendendum, sit $\mathfrak{Z}$ functio quaecunque ipsarum x et y atque $\partial \mathfrak{Z} = \mathfrak{M} \partial x + \mathfrak{N} \partial y$, jamque ponatur $Z = \mathfrak{Z} p$, eritque $M = \mathfrak{M} p$; $N = \mathfrak{N} p$; $P = \mathfrak{Z}$, quare ut formula $\int Z \partial x$ fiat maximum vel minimum, aequatio reperitur

$$0 = \mathfrak{N} p - \frac{\partial \mathfrak{Z}}{\partial x} = \mathfrak{N} p - \mathfrak{M} - \frac{\mathfrak{N} \cdot \partial y}{\partial x} = - \mathfrak{M},$$

quae itidem est aequatio finita. Quod quidem etiam statim praevidere licuisset, quum enim sit $p \partial x = \partial y$, haec formula integralis $\int \mathfrak{Z}\,\partial y$ a praecedente $\int Z\,\partial x$ aliter non differt, nisi quod coordinatae x et y sint permutatae, unde quod de priore erat affirmatum, etiam de posteriore valet.

Hinc natura primae classis adhuc generalius ita describi potest, ut ea complectatur omnes formulas integrales hujusmodi $\int (Z + \mathfrak{Z}\,p)\,\partial x$, ubi litterae Z et $\mathfrak{Z}$ denotant functiones quascunque ipsarum x et y, tum enim aequatio pro curva maximi vel minimi erit, $0 = \mathfrak{N} - \mathfrak{M}$, quae est aequatio omnino determinata.

Classis II.

§. 52. Ad classem secundam referimus eas formulas integrales $\int Z\,\partial x$, quae deducunt ad aequationem differentialem secundi gradus, huc ergo primo pertinent casus, quibus Z tantum ex litteris x, y et p componitur, ita ut sit

$$\partial Z = M\,\partial x + N\,\partial y + P\,\partial p,$$

unde quidem casum posteriorem primae classis excipere oportet, quippe quod evenit, si P fuerit functio tantum ipsarum x et y, ita ut pro praesenti casu quantitas P praeter x et y etiam litteram p complecti debeat. Tum autem aequatio pro curva quaesita erit $0 = N - \frac{\partial P}{\partial x}$, ubi quum P involvat p, ideoque $\frac{\partial y}{\partial x}$, formula $\frac{\partial P}{\partial x}$ continebit differentialia secundi gradus, haec ergo aequatio neutiquam est determinata, quum duas adeo constantes arbitrarias recipiat, quibus effici potest, ut curva per data duo puncta transeat, atque adeo quaestiones hujus classis ita accuratius sunt definiendae, ut curvae investigentur, quae non inter omnes plane curvas, sed inter eas tantum, quae per eadem duo puncta ducuntur, praescripta maximi minimive proprietate gaudeant; semper autem quaestiones hujus classis ita sunt comparatae, ut per naturam suam hanc restrictionem postulent.

§. 53. Praeterea vero etiam ad secundam classem referri oportet casus, quibus $Z = \mathfrak{Z} q$ existente $\mathfrak{Z}$ functione quacunque ipsarum x, y et p, si enim fuerit

$$\partial \mathfrak{Z} = \mathfrak{M} \partial x + \mathfrak{N} \partial y + \mathfrak{P} \partial p,$$

habebimus

$$M = \mathfrak{M} q; \; N = \mathfrak{N} q; \; P = \mathfrak{P} q; \text{ et } Q = \mathfrak{Z};$$

quare quum aequatio pro curva sit

$$0 = N - \frac{\partial P}{\partial x} + \frac{\partial \partial Q}{\partial x^2}, \text{ sive } 0 = N - \frac{1}{\partial x} \partial . \left(P - \frac{\partial Q}{\partial x}\right),$$

formula haec $P - \frac{\partial Q}{\partial x}$ abit in

$$\mathfrak{P} q - \frac{\partial \mathfrak{Z}}{\partial x} = \mathfrak{P} q - \mathfrak{M} - \mathfrak{N} p - \mathfrak{P} q = - \mathfrak{M} - \mathfrak{N} p;$$

unde aequatio nostra evadet

$$0 = N + \frac{1}{\partial x} \partial (\mathfrak{M} + \mathfrak{N} p) = 2 \mathfrak{N} q + \frac{\partial \mathfrak{M}}{\partial x} + p \frac{\partial \mathfrak{N}}{\partial x},$$

quae manifesto tantum differentialia secundi gradus continet. Generalius ergo adhuc si formula integralis proposita fuerit $\int (Z + \mathfrak{Z} q) \partial x$, ubi Z et $\mathfrak{Z}$ quomodocunque ex quantitatibus x, y et p sint compositae, aequatio pro curva quaesita erit

$$0 = N - \frac{\partial P}{\partial x} + 2 \mathfrak{N} q + \frac{\partial \mathfrak{M}}{\partial x} + p \frac{\partial \mathfrak{N}}{\partial x}.$$

sive etiam

$$0 = N \partial x - \partial P + 2 \mathfrak{N} \partial p + \partial \mathfrak{M} + p \partial \mathfrak{N},$$

quae manifesto tantum est differentialis secundi gradus.

Classis III.

§. 54. At si quantitas Z ita ex litteris x, y, p et q fuerit composita ut posito

$$\partial Z = M \partial x + N \partial y + P \partial p + Q \partial q,$$

etiam quantitas Q involvat litteram q, tum hujusmodi casus ad tertiam classem erunt referendi, et cum aequatio pro curva quaesita reperiatur

$$0 = N - \frac{\partial P}{\partial x} + \frac{\partial \partial Q}{\partial x^2},$$

evidens est terminum $\frac{\partial \partial Q}{\partial x^2}$ involvere differentialia quarti gradus, unde aequatio finita pro curva implicabit quatuor constantes arbitrarias, quibus ergo effici potest, ut curva desiderata non solum per datos duos terminos transeat, sed etiam ejus tangentes in utroque termino datam obtineant positionem, in qua quadruplici determinatione natura quaestionum ad hanc classem pertinentium continetur et accuratissime perspicitur.

§. 55. Reliquis casibus ad hanc classem pertinentibus non immoror, verum potius illustrationis caussa insigne adferam exemplum, quo curvae elasticae investigari solent. Scilicet si

Fig. 15. littera ϱ denotet radium osculi curvae quaesitae in puncto M, omnes hae curvae hac gaudent proprietate, ut in iis haec formula $\int \frac{\partial x \surd(1+pp)}{\varrho \varrho}$ sit minimum, ideoque habeatur $Z = \frac{\surd(1+pp)}{\varrho \varrho}$, cum vero sit

$$\varrho = \frac{(1+pp)^{\frac{3}{2}}}{q}, \text{ habebimus } Z = \frac{qq}{(1+pp)^{\frac{5}{2}}}, \text{ unde fit}$$

$$M = 0, \; N = 0, \; P = \frac{-5pqq}{(1+pp)^{\frac{7}{2}}} \text{ et } Q = \frac{+2q}{(1+pp)^{\frac{5}{2}}},$$

quare cum ob $N = 0$ aequatio pro curvis quaesitis sit

$$0 = -\frac{\partial P}{\partial x} + \frac{\partial \partial Q}{\partial x^2},$$

ejus integrale statim praebet

$$P - \frac{\partial Q}{\partial x} = A,$$

quae adhuc est differentialis tertii gradus.

§. 56. Verum haec aequatio adhuc in genere integrari potest, multiplicetur enim per $q\,\partial x = \partial p$, ut habeatur haec aequatio $P \partial p - q \partial Q = A \partial q$; quum vero sit $\partial Z = P \partial p - Q \partial q$, erit $P \partial p = \partial Z - Q \partial q$, quo valore substituto aequatio resultat haec $\partial Z - Q \partial p - q \partial Q = A \partial p$, cujus integrale manifesto est $Z - Q q = A p + B$; nunc igitur pro Z et Q valores supra dati substituantur, atque nanciscemur sequentem aequationem

$$\frac{-qq}{(1+pp)^{\frac{5}{2}}} = Ap + B,$$

mutatis igitur signis constantium colligemus

$$qq = (Ap + B)(1+pp)^{\frac{5}{2}}, \text{ ideoque}$$

$$q = (1+pp)^{\frac{5}{4}} \sqrt{(Ap+B)} = \frac{\partial p}{\partial x},$$

sicque concludimus

$$\partial x = \frac{\partial p}{(1+pp)^{\frac{5}{4}} \sqrt{(Ap+B)}},$$

hincque porro

$$\partial y = \frac{p \partial p}{(1+pp)^{\frac{5}{4}} \sqrt{(Ap+B)}},$$

quibus duabus aequationibus constructio curvae absolvitur.

§. 57. Cum olim haec methodus maximorum et minimorum tractari est coepta, non solum ejusmodi curvae sunt investigatae, in quibus formula quaepiam integralis $\int Z \partial x$ esset vel maximum vel minimum; sed etiam ejusmodi quaestiones proponebantur, ut non inter omnes plane curvas, sed inter eas tantum, quae habeant eandem longitudinem ea quaeratur, in qua illa formula fiat maxima vel minima, ex quo ipso casu nomen problematis Isoperimetrici est natum; hoc autem nomen non impedivit, quo minus ejusmodi quaestiones generaliores proponerentur, ut inter omnes eas curvas quibus valor certae cujuspiam formulae integralis $\int V \partial x$ aeque conveniat, ea definiatur in qua formula $\int Z \partial x$ maximum minimumve sortiatur valorem, quin etiam conditiones adhuc fuerunt multiplicatae in hunc modum, ut tantum inter omnes eas curvas, quibus non solum formula $\int V \partial x$, sed etiam hae quotcunque $\int V' \partial x$, $\int V'' \partial x$, etc. aequaliter competant, ea definiatur in qua $\int Z \partial x$ sit maximum vel minimum, ejusmodi problemata tum temporis summopere ardua sunt visa. Postquam vero in tractatu meo de hoc argumento ostendissem, hujusmodi problemata semper reduci posse ad hoc problema simplex, quo inter omnes plane lineas, ea investigetur, in qua haec formula integralis

$$\int \partial x (Z + \alpha V + \beta V' + \gamma V'' + \text{etc.})$$

fiat maximum vel minimum, hujus generis problemata nullam amplius habent difficultatem.

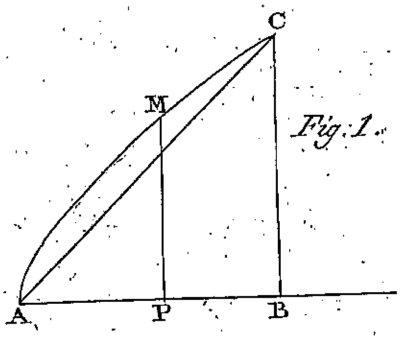


Fig: 1.

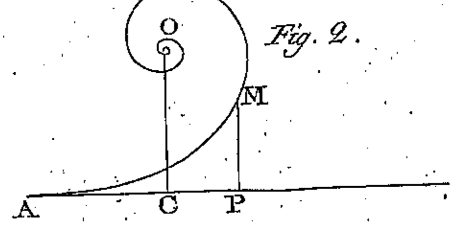


Fig. 2.

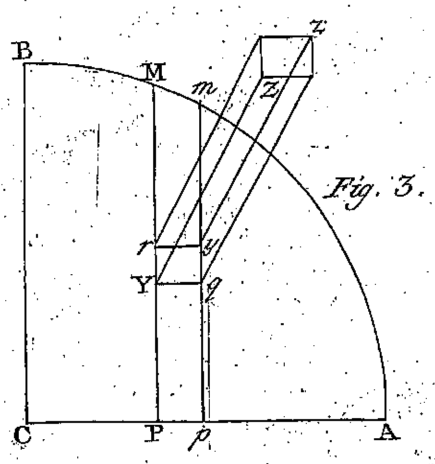


Fig. 3.

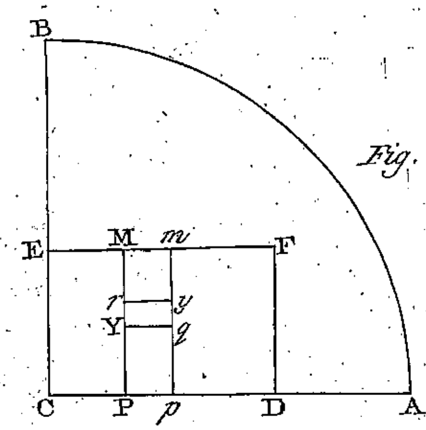


Fig. 4.

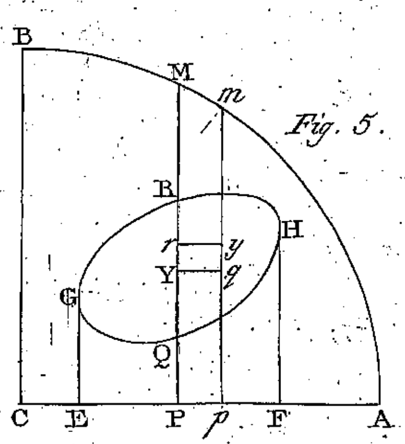


Fig. 5.

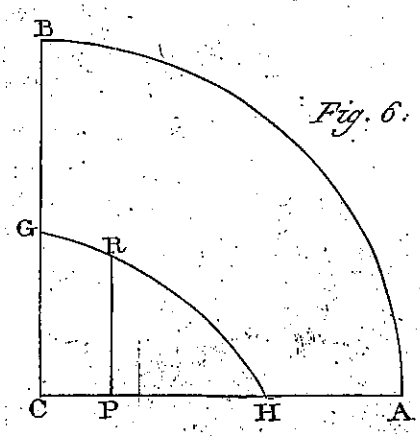


Fig. 6.

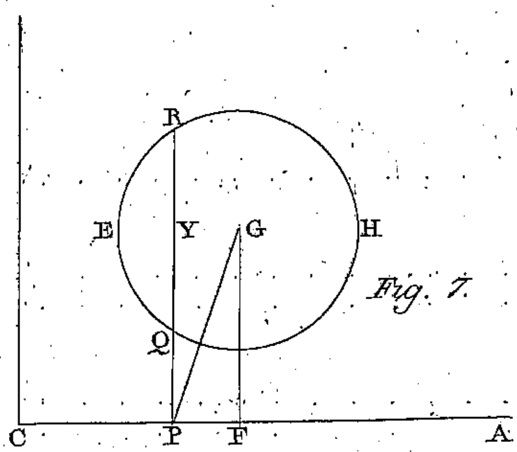


Fig. 7.

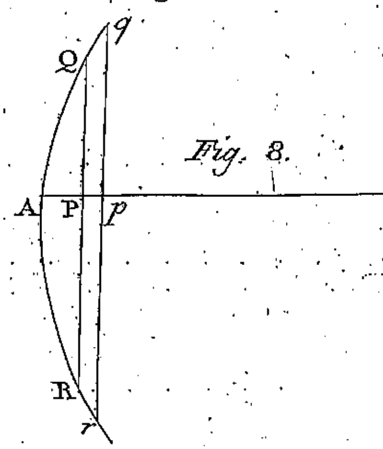


Fig. 8.

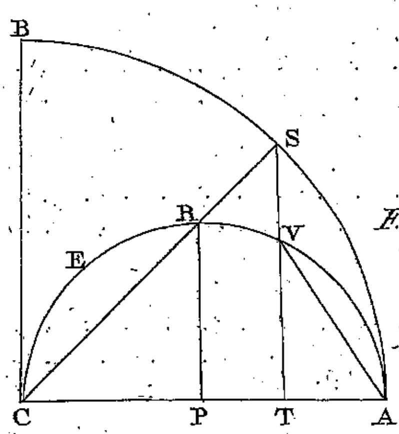


Fig. 9.

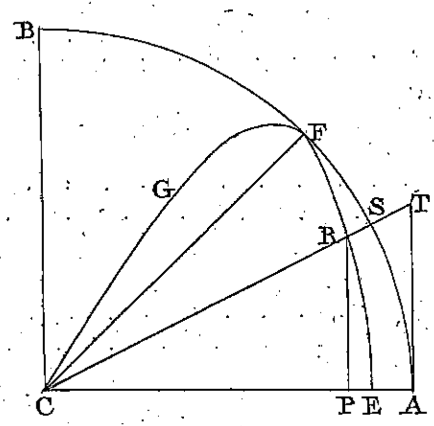


Fig. 10.

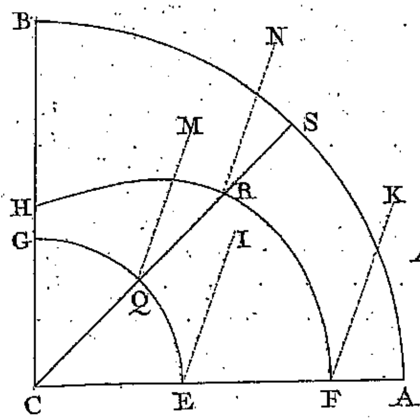
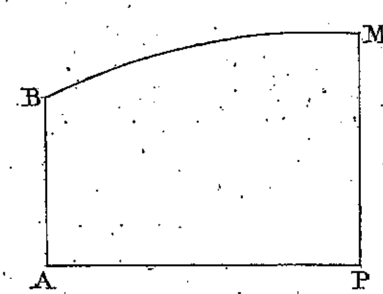


Fig. 11.



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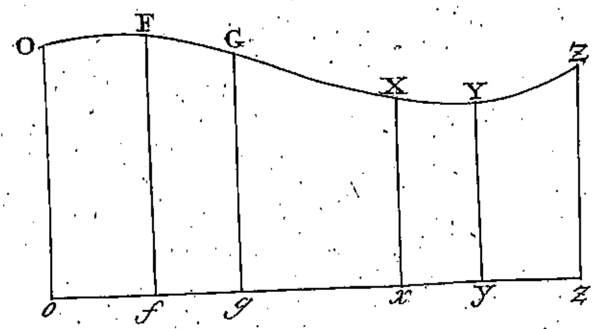

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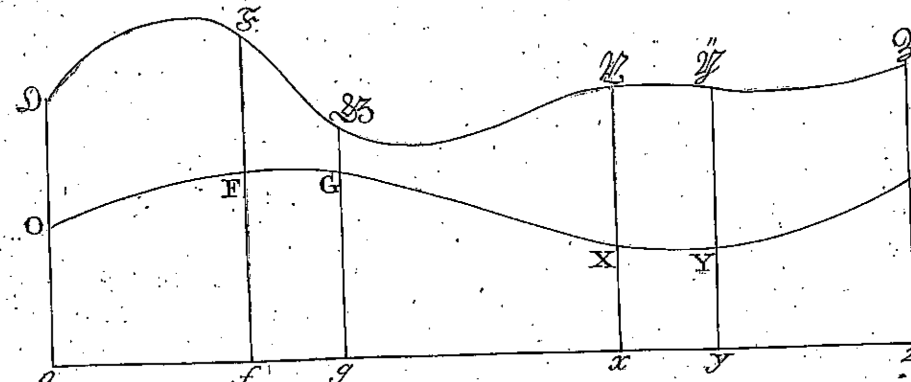

Fig. 14.

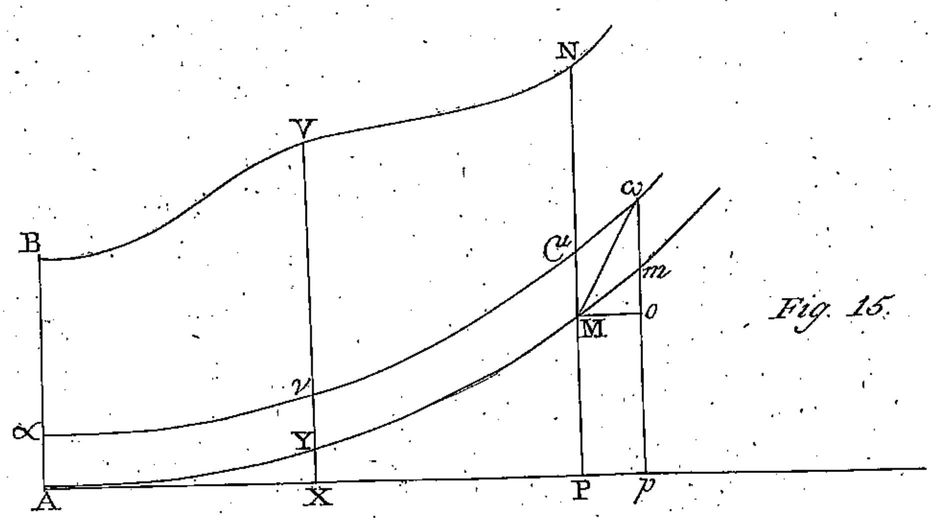

Fig. 15.