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INSTITUTIONUM CALCULI INTEGRALIS

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SUPPLEMENTA

ET

ADDITIONES

AD

INSTITUTIONES CALCULI INTEGRALIS.

SUPPLEMENTUM I.

AD TOM. I. CAP. II.

DE

INTEGRATIONE FORMULARUM DIFFERENTIALIUM IRRATIONALIUM.

1.) De integratione formularum differentialium irrationalium. *Acta Academiae Scientiar. Petropolitanae. Tom. IV. Pars I. Pag.* 4 — 31.

Problema 1.

§. 1. *Si functio X praeter ipsam variabilem x etiam formulam irrationalem $s = \surd(a + bx)$ involvat: ita tamen, ut X sit functio rationalis binarum quantitatum x et s, formulam differentialem $X\partial x$ ab irrationalitate liberare.*

Solutio.

Cum irrationalitas tantum in formula $s = \surd(a + bx)$ insit, hanc tantum ita per idoneam substitutionem tolli oportet, ut inde valor ipsius x non fiat irrationalis. Hoc autem praestabitur, ponendo $a + bx = zz$, ut fiat $s = z$ et $x = \frac{zz - a}{b}$, hincque $\partial x = \frac{2}{b} z\partial z$; quibus valoribus substitutis, tota formula differentialis $X\partial x$ ad rationalem, novam variabilem z complectens, perducitur.

Exemplum 1.

§. 2. Si fuerit $\partial y = \frac{\partial x}{\sqrt{(a+bx)}}$, seu $\partial y = \frac{\partial x}{s}$, posito $\sqrt{(a+bx)} = z$, fiet $\partial y = \frac{2}{b}\partial z$, et integrando $y = \frac{2z}{b}$, unde facta substitutione colligitur $y = \frac{2}{b}\sqrt{(a+bx)} + C$.

Exemplum 2.

§. 3. Si fuerit $\partial y = \partial x \sqrt{(a+bx)} = s\partial x$, sumto $\sqrt{(a+bx)} = z$, erit $\partial y = z\partial x = \frac{2}{b} zz\partial z$, unde integrando fit $y = \frac{2}{3b} z^3$, et facta substitutione prodit

$$y = \frac{2}{3b}(a+bx)^{\frac{3}{2}} + C.$$

Quod integrale si debeat evanescere facto $x = 0$, fiet

$$C = -\frac{2a\sqrt{a}}{3b},$$

ideoque

$$y = \frac{2(a+bx)^{\frac{3}{2}} - 2a\sqrt{a}}{3b}.$$

Exemplum 3.

§. 4. Si fuerit $\partial y = \frac{x\partial x}{\sqrt{(a+bx)}}$, facta substitutione $\sqrt{(a+bx)} = z$, erit

$$\partial y = \frac{2(zz-a)\partial z}{bb} = \frac{2zz\partial z - 2a\partial z}{bb},$$

unde fit integrando

$$y = \frac{2}{3bb}z^3 - \frac{2a}{bb}z + C,$$

et facta restitutione

$$y = \frac{2}{3bb}(a+bx)^{\frac{3}{2}} - \frac{2a}{bb}\sqrt{(a+bx)} + C$$
$$= \frac{2\sqrt{(a+bx)}}{bb}\left(\tfrac{1}{3}bx - \tfrac{2}{3}a\right) + C.$$

Exemplum 4.

§. 5. Si fuerit $\partial y = \frac{\partial x}{(a+bx)^{\frac{3}{2}}}$, facta substitutione $\sqrt{(a+bx)} = z$, erit $\partial y = \frac{\partial x}{z^3}$; quae formula porro ob $\partial x = \frac{2z\partial z}{b}$ abit in $\partial y = \frac{2\partial z}{bzz}$, qua integrata fit $y = -\frac{2}{bz}$, seu facta restitutione, $y = \frac{-2}{b\sqrt{(a+bx)}} + C$. Ubi notetur, pro C sumi debere $\frac{2}{b\sqrt{a}}$, casu quo integrale evanescere debeat facto $x = 0$.

Problema 2.

§. 6. *Si fuerit X functio quaecunque rationalis binarum quantitatum x et s, existente $s = \sqrt[3]{(a+bx)}$, formulam differentialem $X\partial x$ ab irrationalitate liberare.*

Solutio.

Ponatur $\sqrt[3]{(a+bx)} = z$, ut sit $s = z$, erit $a+bx = z^3$, hincque $x = \frac{z^3-a}{b}$, et $\partial x = \frac{3zz\partial z}{b}$; quibus valoribus substitutis tota formula fiet rationalis.

Exemplum 1.

§. 7. Si fuerit

$$\partial y = \frac{\partial x}{\sqrt[3]{(a+bx)}} = \frac{\partial x}{s},$$

posito $\sqrt[3]{(a+bx)} = z$ et substituto valore hinc nato

$$\partial x = \frac{3zz\partial z}{b}, \text{ erit } \partial y = \frac{3z\partial z}{b},$$

unde integrando fit

$$y = \frac{3}{2b}zz = \frac{3}{2b}\sqrt[3]{(a+bx)^2} + C.$$

Exemplum 2.

§. 8. Si fuerit

$$\partial y = \frac{\partial x}{\sqrt[3]{(a+bx)^2}} = \frac{\partial x}{ss},$$

posito $\sqrt[3]{(a+bx)} = z$ fiet $\partial y = \frac{3\partial z}{b}$, hinc integrando

$$y = \tfrac{3}{b}z = \tfrac{3}{b}\sqrt[3]{(a+bx)} + \mathrm{C}.$$

Exemplum 3.

§. 9. Si fuerit $\partial y = \partial x \sqrt[3]{(a+bx)} = s\partial x$, facta substitutione fit $\partial y = \frac{3z^3\partial z}{b}$, hinc integrando

$$y = \tfrac{3}{4b}z^4 = \tfrac{3}{4b}(a+bx)\sqrt[3]{(a+bx)} + \mathrm{C}.$$

Problema 3.

§. 10. *Si fuerit X functio rationalis binarum quantitatum x et s, existente $s = \sqrt[n]{(a+bx)}$, formulam differentialem $X\partial x$ ab irrationalitate liberare.*

Solutio.

Ponatur $\sqrt[n]{(a+bx)} = z$, ut sit $s = z$, erit $a+bx = z^n$, hinc

$$x = \frac{z^n - a}{b} \text{ et } \partial x = \frac{nz^{n-1}\partial z}{b};$$

quibus valoribus substitutis formula proposita $X\partial x$ certe fiet rationalis, si modo numerus exponentialis n fuerit integer.

Exemplum 1.

§. 11. Si fuerit

$$\partial y = \frac{\partial x}{\sqrt[n]{(a + bx)}} = \frac{\partial x}{s},$$

posito $\sqrt{(a + bx)} = z$, ob valorem inde natum

$$\partial x = \frac{nz^{n-1}}{b} \partial z$$

habebitur

$$\partial y = \frac{nz^{n-2}}{b} \partial z;$$

unde integrando colligimus

$$y = \frac{n}{b\,(n-1)}\, z^{n-1} + \mathrm{C}.$$

sive restitutis valoribus

$$y = \frac{n}{b\,(n-1)} (a + bx)^{\frac{n-1}{n}} + \mathrm{C} = \frac{n}{b\,(n-1)} \cdot \frac{a + bx}{\sqrt[n]{(a+bx)}} + \mathrm{C}.$$

Exemplum 2.

§. 12. Si fuerit

$$\partial y = \frac{\partial x}{\sqrt[n]{(a + bx)^{\lambda}}} = \frac{\partial x}{s^{\lambda}},$$

posito $\sqrt[n]{(a + bx)} = z$, et substituto valore

$$\partial x = \frac{nz^{n-1}\,\partial z}{b}, \text{ fiet}$$

$$\partial y = \frac{nz^{n-1}\,\partial z}{bz^{\lambda}} = \frac{n}{b} z^{n-\lambda-1}\, \partial z,$$

cujus integrale dat

$$y = \frac{n}{b(n-\lambda)}(a+bx)^{\frac{n-\lambda}{n}} + C, \text{ sive}$$

$$y = \frac{n}{b(n-\lambda)} \cdot \frac{a+bx}{\sqrt[n]{(a+bx)^\lambda}}.$$

Ex his autem exemplis jam apparet, integrationem non impediri, etiamsi exponentes n et λ non fuerint numeri integri.

Problema 4.

§. 13. *Si fuerit X functio rationalis binarum quantitatum x et s, existente $s = \sqrt{[a + b\sqrt{(f+gx)}]}$, quae formula ergo duplicem irrationalitatem involvit, formulam differentialem $X\partial x$ ab hac duplici irrationalitate liberare.*

Solutio.

Ponatur iterum $\sqrt{[a + b\sqrt{(f+gx)}]} = z$, ut sit $s = z$, erit sumtis quadratis $a + b\sqrt{(f+gx)} = zz$, hinc

$$b\sqrt{(f+gx)} = zz - a:$$

ac sumtis denuo quadratis

$$bb(f+gx) = (zz-a)^2,$$

unde colligitur

$$x = \frac{(zz-a)^2}{bbg} - \frac{f}{g}, \text{ hincque}$$

$$\partial x = \frac{4z\partial z(zz-a)}{bbg}.$$

Quibus valoribus substitutis tota formula reddetur rationalis.

Corollarium.

§. 14. Perspicuum est, eodem modo irrationalitatem tolli posse, si fuerit multo generalius

$$s = \sqrt[n]{[a + b\sqrt[m]{(f+gx)}]}.$$

Posita enim hac formula $= z$, fiet

$$a + b\sqrt[m]{(f+gx)} = z^n \text{ et } b\sqrt[m]{(f+gx)} = z^n - a.$$

Porro $b^m(f+gx) = (z^n - a)^m$, et hinc colligitur

$$x = \frac{(z^n - a)^m}{b^m g} - \frac{f}{g}, \text{ ideoque}$$

$$\partial x = \frac{mnz^{n-1}\partial z\,(z^n - a)^{m-1}}{b^m g}.$$

Sicque etiam hoc modo tota formula rationalis evadet.

Problema 5.

§. 15. *Si fuerit X functio rationalis binarum quantitatum s et x, existente $s = \sqrt{\frac{a+bx}{f+gx}}$, formulam differentialem $X\partial x$ ab irrationalitate liberare.*

Solutio.

Ponatur $\sqrt{\frac{a+bx}{f+gx}} = z$, et sumtis quadratis erit

$$\frac{a+bx}{f+gx} = zz, \text{ hincque } x = \frac{fzz - a}{b - gzz},$$

unde differentiando colligitur

$$\partial x = \frac{2bfz\partial z - 2agz\partial z}{(b - gzz)^2}.$$

Hisque valoribus substitutis formula proposita $X\partial x$ ad rationalitatem erit perducta.

Exemplum 1.

§. 16. Si fuerit $\partial y = \frac{\partial x}{s} = \frac{\partial x\sqrt{(f+gx)}}{\sqrt{(a+bx)}}$, posito

$$\sqrt{\frac{a+bx}{f+gx}} = z \text{ erit } \partial y = \frac{\partial x}{z},$$

et substituto loco ∂x valore supra invento colligitur

$$\partial y = \frac{2(bf-ag)\partial z}{(b - gzz)^2};$$

quae formula, uti jam satis constat, reduci potest ad talem $\int\frac{\partial z}{b-gzz}$, cujus autem integratio vel per logarithmos vel per arcus circulares expedietur.

Exemplum 2.

§. 17. Sit specialius $\partial y = \frac{\partial x \surd(1-x)}{\surd(1+x)}$, ubi $f = 1$, $g = -1$, $a = 1$ et $b = 1$, ideoque

$$z = \frac{\surd(1+x)}{\surd(1-x)}, \quad \text{et} \quad \partial x = \frac{4z\partial z}{(1+zz)^2};$$

quibus valoribus substitutis fiet $\partial y = \frac{4\partial z}{(1+zz)^2}$. Statuatur ergo

$$\int \frac{4\partial z}{(1+zz)^2} = \frac{Az}{1+zz} + B\int \frac{\partial z}{1+zz} = y,$$

unde sumtis differentialibus fiet

$$\frac{4}{(1+zz)^2} = \frac{A - Azz}{(1+zz)^2} + \frac{B}{1+zz} = \frac{A+B+(B-A)zz}{(1+zz)^2}.$$

Oportet igitur sit $A + B = 4$ et $B - A = 0$, ideoque $A = 2$ et $B = 2$; et quia $\int \frac{\partial z}{1+zz} = \text{Arc. tang.}\, z$, adipiscimur

$$y = \frac{2z}{1+zz} + 2\,\text{Arc. tang.}\, z;$$

quocirca facta restitutione, ob $1 + zz = \frac{2}{1-x}$, obtinebitur

$$y = \surd(1-xx) + 2\,\text{Arc. tang.} \surd\frac{1+x}{1-x}.$$

Cum igitur hujus arcus tangens sit $\surd\frac{1+x}{1-x}$, erit ejus sinus $= \surd\frac{1+x}{2}$ et cosinus $= \surd\frac{1-x}{2}$; anguli vero dupli sinus erit $\surd(1-xx)$ et cosinus $= -x$, unde fiet

$$2\,\text{Arc. tang.} \surd\frac{1+x}{1-x} = \text{Arc. cos.} -x = \frac{\pi}{2} + \text{Arc. sin.}\, x;$$

quocirca integrale quaesitum erit

$$y = \surd(1-xx) + \frac{\pi}{2} + \text{Arc. sin.}\, x + C,$$

quod si ita capi debeat, ut evanescat posito $x = 0$, erit

$$C = -1 - \frac{\pi}{2}, \text{ ideoque}$$

$$y = \surd(1-xx) - 1 + \text{Arc. sin.}\, x.$$

Tum igitur, si sumatur $x = 1$, fiet $y = \frac{\pi}{2} - 1$, qui valor in fractionibus decimalibus dat 0,5707963.

Problema 6.

§. 18. *Si fuerit X functio rationalis binarum variabilium x et s, existente* $s = \sqrt[n]{\frac{a+bx}{f+gx}}$, *formulam differentialem* $X\partial x$ *ad rationalitatem perducere.*

Solutio.

Posito $s = \sqrt[n]{\frac{a+bx}{f+gx}} = z$, erit $\frac{a+bx}{f+gx} = z^n$, hincque

$$x = \frac{fz^n - a}{b - gz^n}, \text{ consequenter } \partial x = \frac{n(bf-ag)z^{n-1}\partial z}{(b-gz)^{2n}};$$

hisque valoribus substitutis tota formula proposita $X\partial x$ ad rationalitatem erit perducta.

Problema 7.

§. 19. *Si fuerit X functio binarum quantitatum x et s, existente* $s = \sqrt{(a+bxx)}$, *formulam differentialem* $\frac{X\partial x}{x}$ *ab irrationalitate liberare.*

Solutio.

Ponamus $s = \sqrt{(a+bxx)} = z$, erit $a+bxx = zz$, hinc $xx = \frac{zz-a}{b}$, et quia in functione X tantum quadratum xx, ejusque ergo potestates pares occurrunt: hac substitutione jam functio X evadet rationalis. Sumtis vero logarithmis

$$2lx = l(zz-a) - lb,$$

differentiando fit

$$\frac{2\partial x}{x} = \frac{2z\partial z}{zz-a}, \text{ ideoque } \frac{\partial x}{x} = \frac{z\partial z}{zz-a}.$$

Hoc ergo modo formula proposita $X \cdot \frac{\partial x}{x}$ prorsus reddetur rationalis.

Exemplum 1.

§. 20. Si fuerit

$$\partial y = \frac{x\partial x}{\sqrt{(a+bxx)}}, \text{ erit } \partial y = \frac{\partial x}{x} \cdot \frac{xx}{\sqrt{(a+bxx)}} = \frac{xx}{s} \cdot \frac{\partial x}{x}.$$

Posito ergo $\sqrt{(a+bxx)} = z$ erit $\partial y = \frac{\partial z}{b}$, unde colligitur integrando $y = \frac{z}{b} = \frac{\sqrt{(a+bxx)}}{b}$.

Exemplum 2.

§. 21. Si fuerit

$$\partial y = \frac{x^3 \partial x}{\sqrt{(a+bxx)}} = \frac{\partial x}{x} \cdot \frac{x^4}{s},$$

ponendo $\sqrt{(a+bxx)} = z$, ut sit

$$xx = \frac{zz-a}{b} \quad \text{et} \quad \frac{\partial x}{x} = \frac{z\partial z}{zz-a},$$

erit $\partial y = \frac{1}{bb}\partial z(zz-a)$, hincque integrando adipiscimur $y = \frac{z}{3bb}(zz-3a)$; unde facta restitutione prodibit integrale quaesitum $y = \frac{bxx-2a}{3bb}\sqrt{(a+bxx)} + C$.

Exemplum 3.

§. 22. Si fuerit

$$\partial y = \frac{x^3 \partial x}{\sqrt{(a+bxx)^3}}, \text{ erit } \partial y = \frac{\partial x}{x} \cdot \frac{x^4}{s^3};$$

hinc posito

$$\sqrt{(a+bxx)} = s = z \text{ fiet } \partial y = \frac{\partial z}{bb}\left(\frac{zz-a}{zz}\right),$$

unde sumto integrali fiet $y = \frac{1}{bb}\left(\frac{zz+a}{z}\right)$, quocirca facta restitutione resultat $y = \frac{2a+bxx}{bb\sqrt{(a+bxx)}} + C$.

Problema 8.

§. 23. *Si fuerit X functio rationalis binarum quantita-*

tum x^n *et* s, *existente* $s = \sqrt[m]{(a + bx^n)}$, *formulam differentialem* $X \frac{\partial x}{x}$ *ad rationalitatem perducere.*

Solutio.

Posito $s = \sqrt[m]{(a + bx^n)} = z$, fiet $a + bx^n = z^m$ et $x^n = \frac{z^m - a}{b}$. Quia igitur in functione X tantum potestas x^n occurrit, ea rationalis reddetur, si hi valores substituantur. Tum vero sumtis logarithmis habebitur

$$n\, l x = l\,(z^m - a) - l\, b,$$

et differentiando

$$\frac{\partial x}{x} = \frac{m z^{m-1} \partial z}{n\,(z^m - a)},$$

sicque tota formula proposita fiet rationalis.

Exemplum.

§. 24. Sit

$$\partial y = \frac{x^{n-1} \partial x}{\sqrt[m]{(a + bx^n)}} = \frac{\partial x}{x} \cdot \frac{x^n}{s},$$

factaque substitutione orietur haec aequatio

$$\partial y = \frac{m z^{m-2} \partial z}{n b},$$

qua integrata prodibit

$$y = \frac{m z^{m-1}}{nb\,(m-1)} = \frac{m}{nb\,(m-1)} \sqrt[m]{(a + bx^n)^{m-1}} + C, \text{ sive}$$

$$y = \frac{m}{nb\,(m-1)} \cdot \frac{a + bx^n}{\sqrt[n]{(a + bx^n)}} + C.$$

Problema 9.

§. 25. *Si fuerit X functio rationalis quantitatum xx et s, existente $s = \sqrt{\frac{a+bxx}{f+gxx}}$, formulam differentialem $X\frac{\partial x}{x}$ ab irrationalitate liberare.*

Solutio.

Ponatur $s = \sqrt{\frac{a+bxx}{f+gxx}} = z$, eritque $\frac{a+bxx}{f+gxx} = zz$, hinc $xx = \frac{fzz-a}{b-gzz}$, unde functio X penitus fit rationalis. Porro sumtis logarithmis

$$2lx = l(fzz - a) - l(b - gzz),$$

differentietur, ut prodeat

$$\frac{2\partial x}{x} = \frac{2fz\partial z}{fzz-a} + \frac{2gz\partial z}{b-gzz} = \frac{2(bf-ag)z\partial z}{(fzz-a)(b-gzz)},$$

unde fit

$$\frac{\partial x}{x} = \frac{(bf-ag)z\partial z}{(fzz-a)(b-gzz)};$$

sicque tota formula differentialis fiet rationalis.

Exemplum.

§. 26. Si fuerit $\partial y = \frac{\partial x}{\sqrt{(f+gxx)}}$, repraesentemus hanc formulam ita

$$\partial y = \frac{\partial x}{x} \cdot \frac{x}{\sqrt{(f+gxx)}} = \frac{\partial x}{x}\sqrt{\frac{xx}{f+gxx}}.$$

Hic ergo erit $a = 0$, $b = 1$, et

$$z = \frac{x}{\sqrt{f+gxx}}, \text{ ita ut } \partial y = \frac{z\partial x}{x};$$

erit autem

$$\frac{\partial x}{x} = \frac{\partial z}{z(1-gzz)}, \text{ unde fit } \partial y = \frac{\partial z}{1-gzz},$$

cujus formulae integratio per logarithmos expedietur, si fuerit g numerus positivus: sin autem fuerit negativus per arcus circulares absolvetur. Sit igitur 1°.) $g = + hh$, erit

$$\partial y = \frac{\partial z}{1 - hhzz}, \text{ ideoque}$$

$$y = \frac{1}{2h} l \frac{1 + hz}{1 - hz};$$

et restitutis valoribus supra indicatis, erit

$$y = \frac{1}{2h} l \left(\frac{\sqrt{(f + hhxx)} + hx}{\sqrt{(f + hhxx)} - hx}\right) = \frac{1}{h} l \frac{\sqrt{(f + hhxx)} + hx}{\sqrt{f}}.$$

Sit 2°.) g quantitas negativa, puta $g = - hh$, erit

$$\partial y = \frac{\partial z}{1 + hhzz} = \frac{1}{h} \cdot \frac{h\partial z}{1 + hhzz},$$

unde colligitur

$$y = \frac{1}{h} \text{Arc. tang. } hz = \frac{1}{h} \text{Arc. tang. } \frac{hx}{\sqrt{(f - hhxx)}}.$$

Ubi manifestum est, f esse debere quantitatem positivam, quia alioquin formula differentialis esset imaginaria.

Corollarium.

§. 27. Hinc ergo si proponatur formula

$$\partial y = \sqrt{(1 + xx)}, \text{ ubi } f = 1 \text{ et } g = 1,$$

ex casu priore ob $h = + 1$ erit

$$\int \frac{\partial x}{\sqrt{(1 + xx)}} = l\, [\sqrt{(1 + xx)} + x].$$

At si fuerit

$$\partial y = \frac{\partial x}{\sqrt{(1 - xx)}}, \text{ ubi } f = 1 \text{ et } g = - 1,$$

colligitur ex casu posteriore $x =$ Arc. tang. $\frac{x}{\sqrt{(1 - xx)}}$, unde concluditur

$$\int \frac{\partial x}{\sqrt{(1 - xx)}} = \text{Arc. sin. } x = \text{Arc. cos. } \sqrt{(1 - xx)}.$$

Problema 10.

§. 28. *Si fuerit X functio rationalis quantitatum x^n et s, existente $s = \sqrt[n]{\left(\frac{a + bx^n}{f + gx^n}\right)}$, formulam differentialem $X \frac{\partial x}{x}$ rationalem efficere.*

Solutio.

Ponatur $s = \sqrt[n]{\left(\frac{a + bx^n}{f + gx^n}\right)} = z$, eritque

$$\frac{a + bx^n}{f + gx^n} = z^n, \text{ hinc } x^n = \frac{fz^n - a}{b - gz^n},$$

tum autem sumtis logarithmis, erit

$$nlx = l(fz^n - a) - l(b - gz^n),$$

et differentiando

$$\frac{\partial x}{x} = \frac{fz^{n-1}\partial z}{fz^n - a} + \frac{gz^{n-1}\partial z}{b - gz^n} = \frac{(bf - ag)z^{n-1}\partial z}{(fz^n - a)(b - gz^n)};$$

quibus valoribus substitutis formula proposita fit rationalis.

Problema 11.

§. 29. *Si fuerit X functio rationalis binarum quantitatum x^n et s, existente $s = \sqrt[m]{\left(\frac{a + bx^n}{f + gx^n}\right)}$, formulam differentialem $X \frac{\partial x}{x}$ ab omni irrationalitate liberare.*

Solutio.

Statuatur $s = \sqrt[m]{\left(\frac{a + bx^n}{f + gx^n}\right)} = z$, eritque

$$\frac{a + bx^n}{f + gx^n} = z^m, \text{ unde fit } x^n = \frac{fz^m - a}{b - gz^m};$$

hinc sumtis logarithmis erit

$$nlx = l(fz^m - a) - l(b - gz^m);$$

hinc differentiando

$$\frac{n\partial x}{x} = \frac{m(bf - ag)\, z^{m-1}\, \partial z}{(fz^m - a)(b - gz^m)},$$

ideoque

$$\frac{\partial x}{x} = \frac{m(bf - ag)\, z^{m-1}\, \partial z}{n(fz^m - a)(b - gz^m)},$$

quibus valoribus substitutis irrationalitas formulae propositae penitus tollitur.

Problema 12.

§. 30. *Si fuerit X functio rationalis quaecunque binarum quantitatum x et s, existente $s = \sqrt{(\alpha + \beta x + \gamma xx)}$, formulam differentialem $X\partial x$ ad rationalitatem perducere.*

Solutio.

Hic duos casus a se invicem distingui convenit, prout γ fuerit vel quantitas positiva vel negativa.

I. Sit γ quantitas positiva, ac ponatur $\gamma = cc$ et $\beta = 2bc$, ut habeatur

$$s = \sqrt{(\alpha + 2bcx + ccxx)} = \sqrt{[\alpha - bb + (b + cx)^2]}$$

ubi loco $\alpha - bb$ brevitatis ergo scribatur e, ut sit

$$s = \sqrt{[e + (b + cx)^2]}.$$

Jam statuatur $s = b + cx + z$, eritque

$$ss = e + (b + cx)^2 = (b + cx)^2 + 2(b + cx)z + zz,$$

unde sequitur

$$e - zz = 2z(b + cx), \quad \text{sive} \quad b + cx = \frac{e - zz}{2z};$$

hincque colligitur

$$x = \frac{e - zz}{2cz} - \frac{b}{c}, \quad \text{seu} \quad x = \frac{e - 2bz - zz}{2cz}.$$

Aequatio autem $b + cx = \frac{e - zz}{2z}$ differentiata praebet

$$c\partial x = -\frac{e\partial z}{2zz} - \frac{\partial z}{2} = -\frac{e\partial z - zz\partial z}{2zz},$$

unde deducitur

$$\partial x = -\frac{\partial z(e + zz)}{2czz}, \quad \text{at ob}$$

$$b + cx = \frac{e - zz}{2z} \quad \text{fiet} \quad s = \frac{e + zz}{2z}.$$

His ergo valoribus substitutis formula nostra $X\partial x$ reddetur rationalis. Postquam igitur ejus integrale fuerit inventum, loco z valor ante inventus $\sqrt{[e + (b + cx)^2]} - b - cx$ erit substituendus.

II. Sin autem γ fuerit quantitas negativa, ponatur

$$\gamma = -cc \quad \text{et} \quad \beta = -2bc,$$

ut habeatur

$$s = \sqrt{(\alpha - 2bcx - ccxx)} = \sqrt{[\alpha + bb - (b + cx)^2]},$$

ubi evidens est, quantitatem $\alpha + bb$ necessario esse debere positivam, quia alioquin s evaderet imaginarium. Quamobrem ponamus brevitatis gratia $\alpha + bb = aa$, ut fiat

$$s = \sqrt{[aa - (b + cx)^2]},$$

ad quam formam rationalem efficiendam statuamus

$$\sqrt{[aa - (b + cx)^2]} = a - (b + cx)z,$$

unde sumtis quadratis erit

$$aa - (b + cx)^2 = aa - 2az(b + cx) + (b + cx)^2 zz$$

quae aequatio reducitur ad hanc:

$$-(b + cx) = -2az + (b + cx)zz,$$

unde reperitur

$$b + cx = \frac{2az}{1+zz}, \quad \text{ideoque}$$

$$x = \frac{2az - b - bzz}{c(1+zz)}.$$

Illa autem aequatio differentiata dat

$$c\partial x = \frac{2a\partial z(1+zz) - 4azz\partial z}{(1+zz)^2} = \frac{2a\partial z(1-zz)}{(1+zz)^2};$$

unde fit

$$\partial x = \frac{2a\partial z(1-zz)}{c(1+zz)^2}.$$

Porro autem, cum sit

$$s = a - (b+cx)z, \quad \text{ob } b+cx = \frac{2az}{1+zz}$$

erit $s = \frac{a(1-zz)}{1+zz}$, quocirca, si loco x, s et ∂x inventi hi valores substituantur, formula proposita differentialis $X\partial x$ evadet rationalis, et per variabilem z exprimetur, cujus integrale postquam fuerit inventum, loco z ubique ejus restituatur valor assumtus

$$z = a - \sqrt{[aa - (b+cx)^2]},$$

et integrale obtinebitur per solam variabilem x expressum.

Exemplum 1.

§. 31. Si fuerit

$$\partial y = \frac{\partial x}{\sqrt{[e+(b+cx)^2]}},$$

quae formula ad casum priorem pertinet, erit

$$\partial y = \frac{\partial x}{s} = -\frac{\partial z}{cz}, \text{ ob } \partial x = -\frac{\partial z(e+zz)}{2ezz} \text{ et } s = \frac{e+zz}{2z};$$

cujus integrale est $y = -\frac{1}{c}lz$; restituto ergo valore

$$z = \sqrt{[e+(b+cx)^2]} - b - cx, \quad \text{erit}$$

$$y = -\frac{1}{c}l[\sqrt{[e+(b+cx)^2]} - b - cx] + C,$$

quod integrale si evanescere debeat posito $x = 0$, fiet

$$C = \frac{1}{c}l[\sqrt{(e+bb)} - b].$$

Corollarium.

§. 32. Si ponatur $b = 0$ et $c = 1$, sive

$$\partial y = \frac{\partial x}{\sqrt{(e + xx)}}, \text{ erit integrale}$$

$$y = -l[\sqrt{(e + xx)} - x] + l\sqrt{e} = l\frac{\sqrt{e}}{\sqrt{(e + xx)} - x},$$

quae formula reducitur ad hanc

$$y = l\frac{\sqrt{(e + xx)} + x}{\sqrt{e}}.$$

Cum vero porro sit

$$\partial . \sqrt{(e + xx)} = \frac{x\partial x}{\sqrt{(e + xx)}}, \text{ erit}$$

$$\int \frac{x\partial x}{\sqrt{(e + xx)}} = \sqrt{(e + xx)}.$$

Si igitur hae duae formulae combinentur, habebitur ista integratio notatu digna

$$\int \frac{A\partial x + Bx\partial x}{\sqrt{(e + xx)}} = A\,l\frac{\sqrt{(e + xx)} + x}{\sqrt{e}} + B\sqrt{(e + xx)}.$$

Exemplum 2.

§. 33. Sit $\partial y = \frac{\partial x}{\sqrt{[aa - (b + cx)^2]}}$, quae formula ad casum secundum est referenda, ita ut sit $\partial y = \frac{\partial x}{s}$. Cum igitur sit

$$\partial x = \frac{2a\partial z\,(1 - zz)}{c\,(1 + zz)^2} \text{ et } s = \frac{a\,(1 - zz)}{1 + zz}, \text{ erit}$$

$$y = \frac{\partial x}{s} = \frac{2}{c} \cdot \frac{\partial z}{1 + zz},$$

unde fit integrando $y = \frac{2}{c}$ Arc. tang. z. Quia igitur est

$$z = \frac{a - \sqrt{[aa - (b + cx)^2]}}{b + cx}, \text{ erit}$$

$$y = \frac{2}{c}\text{ Arc. tang. } \frac{a - \sqrt{[aa - (b + cx)^2]}}{b + cx} + C.$$

Corollarium.

§. 34. Sit igitur $b = 0$ et $c = 1$, seu formula differentialis proposita $\partial y = \frac{\partial x}{\sqrt{(aa - xx)}}$, reperieturque

$$y = 2 \text{ Arc. tang. } \frac{a - \sqrt{(aa - xx)}}{x} + C.$$

Quia igitur tangens hujus arcus est $\frac{a - \sqrt{(aa - xx)}}{x}$; tangens dupli arcus erit $= \frac{x}{\sqrt{(aa - xx)}}$, ita ut sit

$$y = \text{Arc. tang. } \frac{x}{\sqrt{(aa - xx)}}:$$

hujus autem arcus sinus erit $\frac{x}{a}$, sicque integrale quaesitum

$$\int \frac{\partial x}{\sqrt{(aa - xx)}} = \text{Arc. sin. } \frac{x}{a}.$$

Quia porro

$$\partial \,.\, \sqrt{(aa - xx)} = - \frac{x \partial x}{\sqrt{(aa - xx)}}, \text{ erit}$$

$$\int \frac{x \partial x}{\sqrt{(aa - xx)}} = - \sqrt{(aa - xx)}:$$

quocirca ista generalior conficitur integratio

$$\int \frac{A \partial x + B x \partial x}{\sqrt{(aa - xx)}} = A \,.\, \text{Arc. sin. } \frac{x}{a} - B \sqrt{(aa - xx)}.$$

Problema 13.

§. 35. *Si fuerit V functio rationalis binarum quantitatum v^n et s, existente*

$$s = \sqrt{(\alpha + \beta v^n + \gamma v^{2n})},$$

formulam differentialem $V v^{n-1} \partial v$ ab irrationalitate liberare.

Solutio.

Ponatur $v^n = x$, erit

$$s = \sqrt{(\alpha + \beta x + \gamma xx)} \text{ et } v^{n-1} \partial v = \frac{\partial x}{n};$$

hic ergo jam erit V functio rationalis binarum quantitatum x et s, existente

$$s = \sqrt{(\alpha + \beta x + \gamma xx)}$$

et formula ab irrationalitate liberanda erit $\frac{\text{v}\partial x}{n}$; qui casus prorsus convenit cum problemate praecedente, ideoque eandem habebit solutionem.

Scholion.

§. 36. Praecepta hactenus tradita ad omnes fere formulas differentiales, quae quidem adhuc tractari potuerunt, extenduntur. Interim tamen ejusmodi casus occurrere possunt, quibus idonea substitutio, ad irrationalitatem tollendam necessaria, non tam facile perspicitur, sed acri judicio demum investigare licet, in quo negotio cum praecepta generalia tradere nondum liceat, exempla quaedam particularia speciminis loco in medium afferamus.

Exemplum 1.

§. 37. *Si proposita fuerit haec formula irrationalis*

$$\partial P = \frac{\partial x (1 + xx)}{(1 - xx)\sqrt{(1 + x^4)}},$$

ejus integrale P investigare.

Si quis hic ejusmodi uti vellet substitutione, qua formula $\sqrt{(1 + x^4)}$ ad rationalitatem perduceretur, oleum et operam esset perditurus, interim tamen singulari artificio sequens substitutio negotium conficere poterit. Statuatur

$$\frac{x\sqrt{2}}{1 - xx} = p, \text{ eritque}$$

$$1 + pp = \frac{1 + x^4}{(1 - xx)^2}, \text{ hinc}$$

$$\sqrt{(1 + pp)} = \frac{\sqrt{(1 + x^4)}}{1 - xx};$$

tum vero erit differentiando

$$\partial p = \frac{\partial x \sqrt{2}(1 + xx)}{(1 - xx)^2},$$

ex quibus valoribus colligitur

$$\frac{\partial p}{\sqrt{(1+pp)}} = \frac{\partial x \sqrt{2}(1+xx)}{(1-xx)\sqrt{(1+x^4)}},$$

quae feliciter cum formula ipsa proposita convenit, ita ut sit

$$\frac{\partial p}{\sqrt{(1+pp)}} = \partial P \sqrt{2}, \quad \text{sive } \partial P = \frac{1}{\sqrt{2}} \cdot \frac{\partial p}{\sqrt{(1+pp)}},$$

unde colligitur integrando

$$P = \frac{1}{\sqrt{2}} l[\sqrt{(1+pp)} + p].$$

Quare si loco p et $\sqrt{(1+pp)}$ valores dati substituantur, haec obtinetur integratio satis memorabilis

$$P = \int \frac{\partial x(1+xx)}{(1-xx)\sqrt{(1+x^4)}} = \frac{1}{\sqrt{2}} l \frac{\sqrt{(1+x^4)}+\sqrt{2}}{1+xx}.$$

Exemplum 2.

§. 38. *Si proposita fuerit haec formula irrationalis* $\partial Q = \frac{\partial x(1-xx)}{(1+xx)\sqrt{(1+x^4)}}$, *ejus integrale* Q *investigare.*

Ad hoc praestandum fiat $\frac{x\sqrt{2}}{1+xx} = q$, eritque

$$\sqrt{(1-qq)} = \frac{\sqrt{(1+x^4)}}{1+xx};$$

tum vero erit $\partial q = \frac{\partial x(1-xx)\sqrt{2}}{(1+xx)^2}$, atque hinc colligitur

$$\frac{\partial q}{\sqrt{(1-qq)}} = \frac{\partial x(1-xx)\sqrt{2}}{(1+xx)\sqrt{(1+x^4)}} = \partial Q \sqrt{2},$$

unde fit

$$Q = \frac{1}{\sqrt{2}} \int \frac{\partial q}{\sqrt{(1-qq)}} = \frac{1}{\sqrt{2}} \text{Arc. sin. } q.$$

Restituto ergo pro q valore assumto, ista obtinebitur integratio

$$Q = \int \frac{\partial x(1-xx)}{(1+xx)\sqrt{(1+x^4)}} = \frac{1}{\sqrt{2}} \text{ Arc. sin. } \frac{x\sqrt{2}}{1+xx}.$$

Scholion.

§. 39. Cum istae duae formulae

$$\frac{\partial x(1+xx)\sqrt{2}}{(1-xx)\sqrt{(1+x^4)}} \text{ et } \frac{\partial x(1-xx)\sqrt{2}}{(1+xx)\sqrt{(1+x^4)}}$$

perductae sint ad has simplices

$$\frac{\partial p}{\sqrt{(1+pp)}} \text{ et } \frac{\partial q}{\sqrt{(1-qq)}},$$

quarum utraque facile ab irrationalitate liberatur, istae ipsae formulae propositae ope idoneae substitutionis ab irrationalitate liberari possunt; unde mirum non est, earum integralia sive per logarithmum sive per arcum circularem exhiberi potuisse. Satis enim jam est ostensum, omnium formularum differentialium rationalium integralia semper vel per logarithmos et arcus circulares, vel adeo algebraice exhiberi posse; quod igitur etiam de illis formulis irrationalibus est tenendum, quas certae substitutionis ope ad rationalitatem perducere licet. Unde vicissim plures Geometrae concluserunt: si quae formula differentialis nullo plane modo ab irrationalitate liberari queat, tum ejus integrale etiam neque per logarithmos nec arcus circulares, multo minus algebraice exprimi posse, sed ad aliud genus quantitatum transcendentium referri oportere. Caeterum combinatio duorum praecedentium exemplorum manuducit ad solutionem sequentium.

Exemplum 3.

§. 40. *Si proposita fuerit haec formula differentialis*

$$\partial y = \frac{\partial x\sqrt{(1+x^4)}}{1-x^4},$$

ejus integrale invenire.

Hanc formulam per neutram substitutionem ante usurpatam rationalem reddere licet: utraque tamen juncta negotium confici poterit, namque ejus integrale per logarithmos et arcus circulares sequenti artificio expedietur. Formula enim proposita in binas sequentes partes discerpi potest, quae sunt

$$\partial y = \frac{\frac{1}{2}\partial x(1+xx)}{(1-xx)\sqrt{(1+x^4)}} + \frac{\frac{1}{2}\partial x}{(1+xx)\sqrt{(1+x^4)}},$$

quippe quarum summa ipsam formulam nostram propositam producit; prodit enim

$$\partial y = \frac{\frac{1}{2}\partial x(1+xx)^2 + \frac{1}{2}\partial x(1-xx)^2}{(1-x^4)\sqrt{(1+x^4)}} = \frac{\partial x(1+x^4)}{(1-x^4)\sqrt{(1+x^4)}}$$
$$= \frac{\partial x\sqrt{(1+x^4)}}{1-x^4}.$$

Quod si ergo duo praecedentia exempla in subsidium vocentur, manifesto fiet $\partial y = \frac{1}{2}\partial P + \frac{1}{2}\partial Q$, consequenter integrale quaesitum erit $y = \frac{1}{2}P + \frac{1}{2}Q$, quod sequenti modo exprimere licebit

$$\int \frac{\partial x\sqrt{(1+x^4)}}{1-x^4} = \frac{1}{2\sqrt{2}}\, l\, \frac{\sqrt{(1+x^4)}+x\sqrt{2}}{1-xx} + \frac{1}{2\sqrt{2}}\,\text{Arc. sin.}\,\frac{x\sqrt{2}}{1+xx}.$$

Exemplum 4.

§. 41. *Si proposita fuerit haec formula differentialis* $\partial y = \frac{xx\partial x}{(1-x^4)\sqrt{(1+x^4)}}$, *ejus integrale investigare.*

Haec formula simili modo ac praecedens tractari potest; discerpatur enim in sequentes duas partes:

$$\frac{\frac{1}{4}\partial x(1+xx)}{(1-xx)\sqrt{(1+x^4)}} - \frac{\frac{1}{4}\partial x(1-xx)}{(1+xx)\sqrt{(1+x^4)}},$$

quippe quae conjunctae producunt

$$\partial y = \frac{\frac{1}{4}\partial x(1+xx)^2 - \frac{1}{4}\partial x(1-xx)^2}{(1-x^4)\sqrt{(1+x^4)}}$$
$$= \frac{\frac{1}{4}\partial x\,.\,4xx}{(1+x^4)\sqrt{(1+x^4)}} = \frac{xx\partial x}{(1-x^4)\sqrt{(1+x^4)}},$$

quae cum sit ipsa formula proposita, erit ex praecedentibus exemplis $\partial y = \frac{1}{4}\partial P - \frac{1}{4}\partial Q$, consequenter $y = \frac{1}{4}P - \frac{1}{4}Q$, hinc integrale quaesitum ita reperietur expressum.

$$\int \frac{xx\,\partial x}{(1-x^4)\sqrt{(1+x^4)}} = \frac{1}{4\sqrt{2}}\, l\, \frac{\sqrt{(1+x^4)}+x\sqrt{2}}{1-xx} - \frac{1}{4\sqrt{2}} \text{ Arc. sin. } \frac{x\sqrt{2}}{1+xx}.$$

Scholion.

§. 42. Haec duo postrema exempla si nullo plane modo ope cujuspiam substitutionis ad rationalitatem perduci possent, insigne praeberent documentum, quod conclusio supra memorata quandoque fallere possit: Re autem attentius perpensa inveni, omnia haec quatuor exempla ope unicae substitutionis immediate ad rationalitatem perduci ideoque integrari posse; id quod ostendisse utique operae erit pretium.

Alia resolutio quatuor postremorum exemplorum.

§. 43. Statuatur pro primo exemplo

$$v = \frac{x\sqrt{2}}{\sqrt{(1+x^4)}}, \quad \text{eritque } \sqrt{(1+vv)} = \frac{1+xx}{\sqrt{(1+x^4)}};$$

tum vero

$$\sqrt{(1-vv)} = \frac{1-xx}{\sqrt{(1+x^4)}},$$

unde fit

$$\sqrt{\frac{1+vv}{1-vv}} = \frac{1+xx}{1-xx} \quad \text{et} \quad \sqrt{(1-v^4)} = \frac{1-x^4}{1+x^4}.$$

At differentiando adipiscimur

$$\partial v = \frac{\partial x\,(1-x^4)\sqrt{2}}{(1+x^4)\sqrt{(1+x^4)}}.$$

Cum nunc sit $\frac{1-x^4}{1+x^4} = \sqrt{(1-v^4)}$, erit

$$\partial v = \frac{\partial x\sqrt{2}.\sqrt{(1-v^4)}}{\sqrt{(1+x^4)}}, \quad \text{sive} \quad \frac{\partial v}{\sqrt{(1-v^4)}} = \frac{\partial x\sqrt{2}}{\sqrt{(1+x^4)}};$$

quae aequalitas maxime est notatu digna. Quod si jam haec aequatio multiplicetur per $\sqrt{\frac{1+vv}{1-vv}} = \frac{1+xx}{1-xx}$, nascetur haec aequatio

$$\frac{\partial v}{1-vv} = \frac{\partial x(1+xx)\sqrt{2}}{(1-xx)\sqrt{(1+x^4)}},$$

sicque erit

$$\int\frac{\partial x(1+xx)}{(1-xx)\sqrt{(1+x^4)}} = \frac{1}{\sqrt{2}}\int\frac{\partial v}{1-vv} = \frac{1}{2\sqrt{2}}\,l\,\frac{1+v}{1-v}.$$

Deinde aequatio

$$\frac{1}{\sqrt{2}}\cdot\frac{\partial v}{\sqrt{(1-v^4)}} = \frac{\partial x}{\sqrt{(1+x^4)}}$$

multiplicetur per

$$\sqrt{\frac{1-vv}{1+vv}} = \frac{1-xx}{1+xx},$$

ac prodibit formula exempli secundi

$$\int\frac{\partial x(1-xx)}{(1+xx)\sqrt{(1+x^4)}} = \frac{1}{\sqrt{2}}\int\frac{\partial v}{1+vv} = \frac{1}{\sqrt{2}}\,\text{Arc. tang. } v.$$

Porro eadem aequatio

$$\frac{1}{\sqrt{2}}\cdot\frac{\partial v}{\sqrt{(1-v^4)}} = \frac{\partial x}{\sqrt{(1+x^4)}}$$

dividatur per

$$\sqrt{(1-v^4)} = \frac{1-x^4}{1+x^4},\ \text{et prodibit}$$

$$\frac{1}{\sqrt{2}}\cdot\frac{\partial v}{1-v^4} = \frac{\partial x\sqrt{(1+x^4)}}{1-x^4};$$

quae est ipsa formula exempli tertii, ita ut jam sit

$$\int\frac{\partial x\sqrt{(1+x^4)}}{1-x^4} = \frac{1}{\sqrt{2}}\int\frac{\partial v}{1-v^4} = \frac{1}{2\sqrt{2}}\int\frac{\partial v}{1+vv} + \frac{1}{2\sqrt{2}}\int\frac{\partial v}{1-vv},$$

quod integrale cum ante invento egregie convenit. Tandem postrema aequatio hic inventa

$$\frac{1}{\sqrt{2}}\cdot\frac{\partial v}{1-v^4} = \frac{\partial x\sqrt{(1+x^4)}}{1-x^4}.$$

ducatur in $vv = \frac{2xx}{1+x^4}$, ut prodeat

$$\frac{1}{\sqrt{2}}\cdot\frac{vv\partial v}{1-v^4} = \frac{2xx\partial x\sqrt{(1+x^4)}}{(1-x^4)(1+x^4)} = \frac{2xx\partial x}{(1-x^4)\sqrt{(1+x^4)}},$$

unde pro exemplo quarto colligitur

$$\int\frac{xx\,\partial x}{(1-x^4)\sqrt{(1+x^4)}} = \frac{1}{2\sqrt{2}}\int\frac{vv\,\partial v}{1-v^4} = -\frac{1}{4\sqrt{2}}\int\frac{\partial v}{1+vv} + \frac{1}{4\sqrt{2}}\int\frac{\partial v}{1-vv},$$

unde cum sit $v = \frac{x\sqrt{2}}{\sqrt{(1+x^4)}}$, erit

$$\int \frac{\partial v}{1-vv} = \frac{1}{2} l \frac{1+v}{1-v} = \frac{1}{2} l \frac{\sqrt{(1+x^4)}+x\sqrt{2}}{\sqrt{(1+x^4)}-x\sqrt{2}}$$
$$= \frac{1}{2} l \frac{[\sqrt{(1+x^4)}+x\sqrt{2}]^2}{(1-xx)^2} = l \frac{\sqrt{(1+x^4)}+x\sqrt{2}}{1-xx}.$$

Deinde vero est

$$\int \frac{\partial v}{1+vv} = \text{Arc. tang. } v = \text{Arc. sin. } \frac{v}{\sqrt{(1+vv)}} = \text{Arc. sin. } \frac{x\sqrt{2}}{1+xx}.$$

Scholion.

§. 44. Quanquam autem haec quatuor exempla ad rationalitatem reducere licuit, tamen conclusio supra memorata, quod omnes formulae integrales, quae nullo modo rationales effici queant, ad aliud pertineant transcendentium genus, neque per solos logarithmos et arcus circulares expediri possint, non solum manet suspecta, sed etiam falsitas ejus evidenter ob oculos poni potest. Sit enim functio

$$X = \frac{a}{\sqrt{(1+xx)}} + \frac{b}{\sqrt[3]{(1+x^3)}} + \frac{c}{\sqrt[4]{(1+x^4)}};$$

tum certe formula differentialis $X\partial x$ nullo modo ad rationalitatem perduci poterit; interim tamen singulos ejus partes

$$\frac{a\partial x}{\sqrt{(1+xx)}}, \quad \frac{b\partial x}{\sqrt[3]{(1+x^3)}} \quad \text{et} \quad \frac{c\partial x}{\sqrt[4]{(1+x^4)}}$$

seorsim rationales effici et integralia per logarithmos et arcus circulares exhiberi possunt. Corodinis loco hic sequens problema notatu dignum adjungamus.

Problema 14.

§. 45. *Formularum integralium* $\int \frac{\partial x}{\sqrt{(1+x^4)}}$ *et* $\int \frac{\partial v}{\sqrt{(1-v^4)}}$ *valores per series investigare, pro casibus, quibus ponitur tam* $v = 1$ *quam* $x = 1$.

Solutio.

Cum posito $v = \frac{x\sqrt{2}}{\sqrt{(1+x^4)}}$, ut supra fecimus, evidens sit, sumto $x = 0$ fore etiam $v = 0$, et sumto $x = 1$ fore $v = 1$, ita ut hae duae quantitates x et v simul evanescant et simul unitati aequentur; hinc deducimus istam aequationem differentialem attentione dignissimam

$$\frac{1}{\sqrt{2}} \cdot \frac{\partial v}{\sqrt{(1-v^4)}} = \frac{\partial x}{\sqrt{(1+x^4)}},$$

quas ergo ambas formulas in series converti oportet; erit autem

$$\frac{1}{\sqrt{(1-v^4)}} = (1-v^4)^{-\frac{1}{2}} = 1 + \tfrac{1}{2}v^4 + \frac{1.3}{2.4}v^8 + \frac{1.3.5}{2.4.6}v^{12} + \text{etc. et}$$

$$\frac{1}{\sqrt{(1+x^4)}} = (1+x^4)^{-\frac{1}{2}} = 1 - \tfrac{1}{2}x^4 + \frac{1.3}{2.4}x^8 - \frac{1.3.5}{2.4.6}x^{12} + \text{etc.}$$

Illa jam per ∂v multiplicata et integrata praebet

$$\int \frac{\partial v}{\sqrt{(1-v^4)}} = v + \frac{1}{2.5}v^5 + \frac{1.3}{2.4.9}v^9 + \frac{1.3.5}{2.4.6.13}v^{13} + \text{etc.}$$

unde posito $v = 1$, valor hujus integralis erit

$$1 + \frac{1}{2.5} + \frac{1.3}{2.4.9} + \frac{1.3.5}{2.4.6.13} + \frac{1.3.5.7}{2.4.6.8.17} + \text{etc.}$$

quam seriem littera A indicemus. Simili modo altera series in ∂x ducta et integrata producit

$$\int \frac{\partial x}{\sqrt{(1+x^4)}} = x - \frac{1}{2.5}x^5 + \frac{1.3}{2.4.9}x^9 - \frac{1.3.5}{2.4.6.13}x^{13} + \text{etc.}$$

cujus valor facto $x = 1$ erit

$$1 - \frac{1}{2.5} + \frac{1.3}{2.4.9} - \frac{1.3.5}{2.4.6.13} + \frac{1.3.5.7}{2.4.6.8.17} - \text{etc.}$$

quem littera B designemus, ita ut sit $B = \frac{A}{\sqrt{2}}$, sive $A = B\sqrt{2}$; unde patet, priorem seriem se habere ad posteriorem ut $\sqrt{2} : 1$.

Scholion.

§. 46. Valor formulae integralis $\int \frac{\partial v}{\sqrt{(1-v^4)}}$ etiam hoc modo per seriem investigari potest. Cum sit

$$\frac{1}{\sqrt{(1-v^4)}} = \frac{(1+vv)^{-\frac{1}{2}}}{\sqrt{(1-vv)}}, \quad \text{et}$$

$$(1+vv)^{-\frac{1}{2}} = 1 - \tfrac{1}{2}vv + \tfrac{1.3}{2.4}v^4 - \tfrac{1.3.5}{2.4.6}v^6 + \text{etc.}$$

notetur esse $\int \frac{\partial v}{\sqrt{(1-vv)}} = \frac{\pi}{2}$. Deinde pro integratione reliquorum terminorum ponatur

$$\int \frac{v^{n+2}\partial v}{\sqrt{(1-vv)}} = Av^{n+1}\sqrt{(1-vv)} + B\int \frac{v^n \partial v}{\sqrt{(1-vv)}},$$

quae aequatio differentiata dat

$$\frac{v^{n+2}}{\sqrt{(1-vv)}} = (n+1)Av^n\sqrt{(1-vv)} - \frac{Av^{n+2}}{\sqrt{(1-vv)}} + \frac{Bv^n}{\sqrt{(1-vv)}},$$

unde per $\sqrt{(1-vv)}$ multiplicando prodit

$$v^{n+2} = (n+1)Av^n - (n+1)Av^{n+2} - Av^{n+2} + Bv^n.$$

Hinc termini in quibus inest v^{n+2}, inter se aequati praebent $1 = -(n+2)A$, ideoque $A = -\frac{1}{n+2}$; termini vero v^n continentes praebent $0 = (n+1)A + B$, unde fit $B = \frac{n+1}{n+2}$, ita ut in genere sit

$$\int \frac{v^{n+2}\partial v}{\sqrt{(1-vv)}} = -\frac{1}{n+2}v^{n+1}\sqrt{(1-vv)} + \frac{n+1}{n+2}\int \frac{v^n\partial v}{\sqrt{(1-vv)}},$$

quod integrale uti requiritur evanescit posito $v = 0$. Ponatur nunc $v = 1$, eritque

$$\int \frac{v^{n+2}\partial v}{\sqrt{(1-vv)}} = \frac{n+1}{n+2}\int \frac{v^n\partial v}{\sqrt{(1-vv)}};$$

hinc ergo pro n scribendo successive valores 0, 2, 4, 6, 8, etc. erit

$$\text{I.} \quad \int \frac{vv\partial v}{\sqrt{(1-vv)}} = \tfrac{1}{2}\cdot\tfrac{\pi}{2};$$

$$\text{II.} \quad \int \frac{v^4\partial v}{\sqrt{(1-vv)}} = \tfrac{3}{4}\cdot\tfrac{1}{2}\cdot\tfrac{\pi}{2};$$

$$\text{III.} \quad \int \frac{v^6 \partial v}{\sqrt{(1-vv)}} = \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2};$$

etc. etc.

quibus valoribus adhibitis, erit casu $v = 1$

$$\int \frac{\partial v}{\sqrt{(1-v^4)}} = \frac{\pi}{2} - \frac{1}{2^2} \cdot \frac{\pi}{2} + \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2} \cdot \frac{\pi}{2} - \frac{1^2 \cdot 3^2 \cdot 5^2}{2^2 \cdot 4^2 \cdot 6^2} \cdot \frac{\pi}{2} + \text{etc.}$$

$$= \frac{\pi}{2} \left(1 - \frac{1^2}{2^2} + \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2} - \frac{1^2 \cdot 3^2 \cdot 5^2}{2^2 \cdot 4^2 \cdot 6^2} + \frac{1^2 \cdot 3^2 \cdot 5^2 \cdot 7^2}{2^2 \cdot 4^2 \cdot 6^2 \cdot 8^2} - \text{etc.}\right)$$

ita ut sit ex problemate praecedente

$$1 - \frac{1}{2 \cdot 5} + \frac{1 \cdot 3}{2 \cdot 4 \cdot 9} - \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 9 \cdot 13} + \text{etc.}$$

$$= \frac{\pi}{2} \left(1 - \frac{1^2}{2^2} + \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2} - \frac{1^2 3^2 \cdot 5^2}{2^2 \cdot 4^2 \cdot 6^2} + \text{etc.}\right)$$

unde fit

$$\frac{\pi}{2} = \frac{1 - \frac{1}{2 \cdot 5} + \frac{1 \cdot 3}{2 \cdot 4 \cdot 9} - \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 13} + \text{etc.}}{1 - \frac{1^2}{2^2} + \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2} - \frac{1^2 \cdot 3^2 \cdot 5^2}{2^2 \cdot 4^2 \cdot 6^2} + \text{etc.}}$$

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2) De integratione formulae irrationalis

$$\int \frac{x^n \partial x}{\sqrt{(aa - 2bx + cxx)}}.$$

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Tom. VI. Pars II. Pag. 62 — 67.

Problema 15.

Invenire integrale hujus formulae irrationalis

$$\int \frac{x^n \partial x}{\sqrt{(aa - 2bx + cxx)}}.$$

Solutio.

§. 47. Incipiamus a casu simplicissimo, quo $n = 0$, et quaeramus integrale formulae $\frac{\partial x}{\sqrt{(aa - 2bx + cxx)}}$, quae posito

$x = \frac{b+z}{c}$ transit in hanc $\frac{\partial z}{\sqrt{(aacc - bbc + czz)}}$, ubi duo casus distingui convenit, prout c fuerit vel quantitas positiva vel negativa. Sit igitur primo $c = +ff$, et formula nostra fiet $\frac{\partial z}{f\sqrt{(aaff - bb + zz)}}$, cujus integrale est $\frac{1}{f}\, l\, \frac{z + \sqrt{(aaff - bb + zz)}}{C}$, ideoque erit nostrum integrale

$$\frac{1}{\sqrt{c}}\, l\, \frac{cx - b + \sqrt{(aac - 2bcx + ccxx)}}{C},$$

quod ergo ita sumtum, ut evanescat posito $x = 0$, evadet

$$\frac{1}{\sqrt{c}}\, l\, \frac{cx - b + \sqrt{c}\,(aa - 2bx + cxx)}{-b + a\sqrt{c}}.$$

At vero si c fuerit quantitas negativa, puta $c = -gg$, formula differentialis per z expressa erit $\frac{\partial z}{g\sqrt{(aagg + bb - zz)}}$, cujus integrale est $\frac{1}{g}$ Arc. sin. $\frac{z}{\sqrt{(aagg + bb)}} + C$; quare integrale ita sumtum, ut evanescat posito $x = 0$, fiet

$$-\frac{1}{g}\,\text{Arc. sin.}\,\frac{cx - b}{\sqrt{(aagg + bb)}} + \frac{1}{g}\,\text{Arc. sin.}\,\frac{b}{\sqrt{(aagg + bb)}}.$$

§. 48. Denotet nunc Π valorem formulae integralis $\int \frac{\partial x}{\sqrt{(aa - 2bx + cxx)}}$ ita sumtum, ut evanescat posito $x = 0$, sive c fuerit quantitas positiva sive negativa; ac si sit $c = +ff$ erit uti vidimus

$$\Pi = \frac{1}{f}\, l\, \frac{ffx - b + f\sqrt{(aa - 2bx + ffxx)}}{af - b};$$

altero vero casu, quo $c = -gg$, erit

$$\Pi = -\frac{1}{g}\,\text{Arc. sin.}\,\frac{ggx + b}{\sqrt{(aagg + bb)}} + \frac{1}{g}\,\text{Arc. sin.}\,\frac{b}{\sqrt{(aagg + bb)}},$$

sive ambobus arcubus contractis habebimus

$$\Pi = \frac{1}{g}\,\text{Arc. sin.}\,\frac{bg\sqrt{(aa - 2bx - ggxx)} - abg - ag^3x}{aagg + bb}.$$

Quoniam igitur mox ostendemus, integrationem formulae generalis $\int \frac{x^n \partial x}{\sqrt{(aa - 2bx + cxx)}}$ semper reduci posse ad casum $n = 0$, si modo fuerit n numerus integer positivus, omnia haec integralia per istum valorem Π exprimi poterunt.

§. 49. Jam post integrationem quantitati variabili x ejusmodi valorem constantem tribuamus, quo formula irrationalis

$$\sqrt{(aa - 2bx + cxx)}$$

ad nihilum redigatur, id quod fit, si sumatur $x = \frac{b \pm \sqrt{(bb - aac)}}{c}$, ideoque duobus casibus. Ponamus pro utroque casu functionem Π abire in Δ, ita ut casu $c = ff$ sit

$$\Delta = \frac{1}{f}\, l \frac{\sqrt{(bb - aaff)}}{af - b} = \frac{1}{f}\, l \sqrt{\frac{b + af}{b - af}};$$

pro altero autem casu, quo $c = -gg$

$$\Delta = \frac{1}{g} \text{ Arc. sin. } \frac{\pm ag\sqrt{(bb + aagg)}}{aagg + bb} = \frac{1}{g} \text{ Arc. sin. } \frac{ag}{\sqrt{(bb + aagg)}}.$$

Hos autem valores Δ in sequentibus casibus, quibus ipsa formula radicalis $\sqrt{(aa - 2bx + cxx)}$ evanescit, potissimum sumus contemplaturi.

§. 50. Nunc ad sequentem casum progressuri, consideremus formulam $s = \sqrt{(aa - 2bx + cxx)} - a$, ut scilicet evanescat facto $x = 0$, et quoniam est

$$\partial s = \frac{-b\partial x + cx\partial x}{\sqrt{(aa - 2bx + cxx)}},$$

erit vicissim integrando

$$c\int \frac{x\partial x}{\sqrt{(aa - 2bx + cxx)}} = b\int \frac{\partial x}{\sqrt{(aa - 2bx + cxx)}} + s,$$

unde colligimus

$$\int \frac{x\partial x}{\sqrt{(aa + 2bx + cxx)}} = \frac{b}{c}\,\Pi + \frac{\sqrt{(aa - 2bx + xx)} - a}{c};$$

quare si post integrationem statuamus $x = \frac{b \pm \sqrt{(bb - aac)}}{c}$, quippe quibus casibus fit $\sqrt{(aa - 2bx + cxx)} = 0$ et $\Pi = \Delta$ fiet

$$\int \frac{x\partial x}{\sqrt{(aa - 2bx + cxx)}} = \frac{b}{c}\,\Delta - \frac{a}{c}.$$

§. 51. Sumamus porro $s = x\sqrt{(aa - 2bx + cxx)}$, fiet $\partial s = \frac{aa\partial x - 3bx\partial x + 2cxx\partial x}{\sqrt{(aa - 2bx + cxx)}}$, unde vicissim integrando colligitur

$$2c\int\frac{xx\partial x}{\sqrt{(aa-2bx+cxx)}}=3b\int\frac{x\partial x}{\sqrt{(aa-2bx+cxx)}}-aa\int\frac{\partial x}{\sqrt{(aa-2bx+cxx)}}+s,$$

unde statim pro casu $\sqrt{(aa-2bx+cxx)}=0$ deducimus

$$\int\frac{xx\partial x}{\sqrt{(aa-2bx+cxx)}}=\frac{(3bb-aac)}{2cc}\Delta-\frac{3ab}{2cc}.$$

§. 52. Jam ad altiores potestates ascensuri statuamus $s=xx\sqrt{(aa-2bx+cxx)}$, et quia hinc fit

$$\partial s=\frac{2aax\partial x-5bxx\partial x+3cx^3\partial x}{\sqrt{(aa-2bx+cxx)}},$$ erit

$$3c\int\frac{x^3\partial x}{\sqrt{(aa-2bx+cxx)}}=5b\int\frac{xx\partial x}{\sqrt{(aa-2bx+cxx)}}$$
$$-2aa\int\frac{x\partial x}{\sqrt{(aa-2bx+cxx)}}+s,$$

hincque porro pro casu quo post integrationem statuitur

$x=\frac{b\pm\sqrt{(bb-aac)}}{c}$, habebitur

$$\int\frac{x^3\partial x}{\sqrt{(aa-2bx+cxx)}}=\left(\frac{5b^3-3aabc}{2c^3}\right)\Delta-\frac{15abb}{6c^3}+\frac{2a^3}{3cc}$$
$$\text{vel }=\left(\frac{5b^3}{2c^3}-\frac{3aab}{2cc}\right)\Delta-\frac{5abb}{2c^3}+\frac{2a^3}{3cc}.$$

§. 53. Simili modo sit $s=x^3\sqrt{(aa-2bx+cxx)}$, et quia hinc fit

$$\partial s=\frac{3aaxx\partial x-7bx^3\partial x+4c^4\partial x}{\sqrt{(aa-2bx+cxx)}},$$

erit vicissim integrando

$$4c\int\frac{x^4\partial x}{\sqrt{(aa-2bx+cxx)}}=7b\int\frac{x^3\partial x}{\sqrt{(aa-2bx+cxx)}}$$
$$-3aa\int\frac{xx\partial x}{\sqrt{(aa-2bx+cxx)}}+s;$$

tum igitur pro casu quo fit $\sqrt{(aa-2bx+cxx)}=0$, habebimus

$$\int\frac{x^4\partial x}{\sqrt{(aa-2bx+cxx)}}=\left(\frac{35b^4}{8c^4}-\frac{15aabb}{4c^3}+\frac{3a^4}{8cc}\right)\Delta-\frac{35ab^3}{8c^4}+\frac{55a^3b}{24c^3}.$$

§. 54. Quo autem ordo in his formulis melius explorari possit, singulas exhibeamus per factores, quemadmodum ordine oriuntur, sine ulla abbreviatione, atque hoc modo formulae integrales inventae ita repraesententur

$$\int \frac{\partial x}{\sqrt{(aa - 2bx + cxx)}} = \Delta,$$

$$\int \frac{x\,\partial x}{\sqrt{(aa - 2bx + cxx)}} = \frac{b}{c}\Delta - \frac{a}{c},$$

$$\int \frac{xx\,\partial x}{\sqrt{(aa - 2bx + cxx)}} = \left(\frac{1.3bb}{1.2cc} - \frac{aa}{1.2c}\right)\Delta - \frac{1.3.ab}{1.2.cc},$$

$$\int \frac{x^3\,\partial x}{\sqrt{(aa - 2bx + cxx)}} = \left(\frac{1.3.5b^3}{1.2.3c^3} - \frac{1.3.5aab}{1.2.3cc}\right)\Delta - \frac{1.3.5abb}{1.2.3c^3} + \frac{1.2.2a^3}{1.2.3cc},$$

$$\int \frac{x^4\,\partial x}{\sqrt{(aa - 2bx + cxx)}} = \left(\frac{1.3.5.7b^4}{1.2.2.4c^4} - \frac{1.3.5.6aabb}{1.2.3.4c^4} + \frac{1.3.3a^4}{1.2.3.4cc}\right)\Delta$$
$$- \frac{1.3.5.7ab^3}{1.2.3.4c^4} + \frac{1.5.11a^3b}{1.2.3.4c^3}.$$

§. 55. Instituamus nunc in genere istam evolutionem, sumendo $s = x^n \sqrt{(aa - 2bx + cxx)}$, et quia hinc fit

$$\partial s = \frac{naax^{n-1}\partial x - (2n+1)bx^n\partial x + (n+1)cx^{n+1}\partial x}{\sqrt{(aa - 2bx + cxx)}},$$

inde vicissim integrando colligitur

$$(n+1)c\int \frac{x^{n+1}\partial x}{\sqrt{(aa - 2bx + cxx)}} = (2n+1)b\int \frac{x^n\partial x}{\sqrt{(aa - 2bx + cxx)}}$$
$$- naa\int \frac{x^{n-1}\partial x}{\sqrt{(aa - 2bx - cxx)}} + x^n\sqrt{(aa - 2bx + cxx)}.$$

Quod si vero jam ante elicuerimus

$$\int \frac{x^{n-1}\partial x}{\sqrt{(aa - 2bx + cxx)}} = \mathrm{M}\Delta - \mathfrak{M} \quad \text{et}$$

$$\int \frac{x^n\partial x}{\sqrt{(aa - 2bx + cxx)}} = \mathrm{N}\Delta - \mathfrak{N},$$

ita ut hae duae formulae sint cognitae, sequens ex iis ita determinabitur, ut sit

$$\int \frac{x^{n+1}\partial x}{\sqrt{(aa-2bx+cxx)}} = \left[\frac{(2n+1)bN}{(n+1)c} - \frac{naaM}{(n+1)c}\right]\Delta$$
$$- \frac{(2n+1)b\mathfrak{N}}{(n+1)c} + \frac{naa\mathfrak{M}}{(n+1)c}.$$

Hoc igitur modo has integrationes, quousque libuerit, continuare licet, dum ex binis quibusque sequens ope hujus regulae formatur, ita ut omnia haec integralia vel a logarithmis vel ab arcubus circularibus pendeant, prouti coëfficiens c fuerit vel positivus vel negativus. Manifestum autem est istos valores assignari non posse, nisi exponens n fuerit numerus integer positivus.

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3) De integratione formulae $\int \frac{\partial x\sqrt{(1+x^4)}}{1-x^4}$, aliarumque ejusdem generis, per logarithmos et arcus circulares. *M. S. Academiae exhib. die* 16 *Sept.* 1776.

§. 56. Cum mihi non ita pridem contigisset, integrale hujus formulae $\int \frac{\partial x\sqrt{(1+x^4)}}{1-x^4}$ per arcum circularem et logarithmum exprimere, haec integratio eo magis mihi visa est notatu digna, quod nullo modo perspiciebam, eam ad rationalitatem perduci posse, quandoquidem certum est, istam formulam, quae simplicior videatur, $\int \partial x \sqrt{(1+x^4)}$, neutiquam ad rationalitatem revocari posse, neque enim videbam, accessionem denominatoris $1-x^4$ hanc reductionem promovere posse, hincque concludebam dari ejusmodi formulas differentiales irrationales, quarum integralia per logarithmos et arcus circulares exhibere liceat, etiamsi nulla substitutione ab irrationalitate liberari queant: quaequidem conclusio utique valet pro formulis compositis, quanquam enim istae formulae

$$\int \frac{\partial x}{\sqrt[3]{(1+x^3)}} \quad \text{et} \quad \int \frac{\partial x}{\sqrt[4]{(1+x^4)}}$$

ad rationalitatem reduci possunt, tamen formula ex iis composita

$$\int \partial x \left[\frac{A}{\sqrt[3]{(1+x^3)}} + \frac{B}{\sqrt[4]{(1+x^4)}} \right]$$

per nullam plane substitutionem ad aliam formulam rationalem reduci potest; propterea quod utraque pars peculiarem substitutionem postulat.

§. 57. Interim tamen cum formulam propositam

$$\int \frac{\partial x \sqrt{(1+x^4)}}{1-x^4} = S$$

attentius essem contemplatus, inveni, eam ab irrationalitate liberari posse, ope hujus substitutionis prorsus singularis

$$x = \frac{\sqrt{(1+tt)} + \sqrt{(1-tt)}}{t\sqrt{2}}.$$

Hinc enim fit

$$\partial x = -\frac{\partial t}{tt\sqrt{2(1+tt)}} - \frac{\partial t}{tt\sqrt{2(1-tt)}},$$

quae duae partes ad eundem denominatorem reductae dant

$$\partial x = -\frac{\partial t}{tt\sqrt{2(1-t^4)}} [\sqrt{(1-tt)} + \sqrt{(1+tt)}].$$

Cum igitur sit

$$\sqrt{(1+tt)} + \sqrt{(1-tt)} = tx\sqrt{2},$$

hoc valore substituto fiet

$$\partial x = -\frac{x\,\partial t}{t\sqrt{(1-t^4)}},$$

ita ut sit

$$\partial S = -\frac{x\,\partial t\sqrt{(1+x^4)}}{t(1-x^4)\sqrt{(1-t^4)}}.$$

§. 58. Porro autem sumtis quadratis erit

$$xx = \frac{1 + \surd(1 - t^4)}{tt},$$

unde colligimus

$$1 + xx = \frac{1 + tt + \surd(1 - t^4)}{tt} = \frac{\surd(1 + tt)}{tt}[\surd(1 + tt) + \surd(1 - tt)],$$

sicque ob

$$\surd(1 + tt) + \surd(1 - tt) = tx\surd 2, \text{ erit}$$

$$1 + xx = \frac{x\surd 2(1 + tt)}{t}.$$

Simili modo erit

$$1 - xx = -\left(\frac{1 - tt + \surd(1 - t^4)}{tt}\right)$$

$$= -\frac{\surd(1 - tt)}{tt}[\surd(1 - tt) + \surd(1 + tt)] = -\frac{x\surd 2(1 - tt)}{t}.$$

Hinc igitur sequitur fore

$$1 - x^4 = -\frac{2xx\surd(1 - t^4)}{tt},$$

qui valor in nostra formula substitutus praebet

$$\partial S = +\frac{t\partial t\surd(1 + x^4)}{2x(1 - t^4)}.$$

§. 59. Deinde sumtis quadratis habebimus

$$(1 + xx)^2 = \frac{2xx(1 + tt)}{tt} \text{ et}$$

$$(1 - xx)^2 = \frac{2xx(1 - tt)}{tt},$$

quibus additis prodibit

$$(1 + xx)^2 + (1 - xx)^2 = 2(1 + x^4) = \frac{4xx}{tt},$$

unde fit

$$\surd(1 + x^4) = \frac{x\surd 2}{t};$$

quo valore substituto nostra formula abit in hanc

$$\partial S = \frac{1}{\surd 2}\cdot\frac{\partial t}{1 - t^4};$$

quae ergo formula est rationalis et solam variabilem t complectitur.

§. 60. Cum igitur porro sit

$$\frac{1}{1-t^4} = \frac{1}{2} \cdot \frac{1}{1+tt} + \frac{1}{2} \cdot \frac{1}{1-tt},$$

tum vero integrando reperiatur

$$\int \frac{\partial t}{1+tt} = \text{Arc. tang. } t, \quad \text{et}$$

$$\int \frac{\partial t}{1-tt} = \frac{1}{2} l \frac{1+t}{1-t} = l \frac{1+t}{\sqrt{1-tt}},$$

quibus valoribus substitutis reperietur

$$S = \frac{1}{2\sqrt{2}} \text{Arc. tang. } t + \frac{1}{2\sqrt{2}} l \frac{1+t}{\sqrt{(1-tt)}}.$$

Quare cum regrediendo sit $t = \frac{x\sqrt{2}}{\sqrt{(1+x^4)}}$, supra autem invenerimus

$$1 + x^4 = \frac{2xx}{tt}, \quad \text{erit} \quad tt = \frac{2xx}{1+x^4},$$

hincque

$$1 - tt = \frac{(1-xx)^2}{1+x^4}, \quad \text{ideoque} \quad \sqrt{(1-tt)} = \frac{1-xx}{\sqrt{(1+x^4)}},$$

his valoribus substitutis, integrale quaesitum per ipsam variabilem x sequenti modo exprimetur

$$\int \frac{\partial x \sqrt{(1+x^4)}}{1-x^4} = \frac{1}{2\sqrt{2}} \text{Arc. tang.} \frac{x\sqrt{2}}{\sqrt{(1+x^4)}} + \frac{1}{2\sqrt{2}} l \frac{x\sqrt{2}+\sqrt{(1+x^4)}}{1-xx}.$$

§. 61. Hic autem merito quaeretur, quonam artificio ad substitutionem illam, quae primo intuitu a scopo prorsus aliena videtur pertigerim? quandoquidem nemo certe in eam incidisset, neque etiam ipse memini, quanam ratione ad eam sim perductus. Verum postquam omnia momenta accuratius perpendissem, methodum multo planiorem detexi, qua istud negotium sine tot ambagibus absolvi potest, quam igitur hic perspicue proponi conveniet.

Methodus planior et magis naturalis, formulam integralem propositam tractandi.

§. 62. Quo ex formula $\partial S = \frac{\partial x \sqrt{(1 + x^4)}}{1 - x^4}$ irrationalitatem saltem apparenter tollamur, ponamus $\sqrt{(1 + x^4)} = px$, ut fiat $\partial S = \frac{px\,\partial x}{1 - x^4}$. Cum igitur sit $1 + x^4 = ppxx$, erit radicem extrahendo

$$xx = \tfrac{1}{2}pp + \sqrt{(\tfrac{1}{4}p^4 - 1)}.$$

Ponatur hic $\frac{1}{2}pp = q$, ut habeamus

$$xx = q + \sqrt{(qq - 1)}, \text{ et}$$
$$2lx = l\,[q + \sqrt{(qq - 1)}],$$

hincque differentiando $\frac{2\,\partial x}{x} = \frac{\partial q}{\sqrt{(qq - 1)}}$: ergo loco q restituto valore $\frac{1}{2}pp$, erit $\frac{2\,\partial x}{x} = \frac{2p\,\partial p}{\sqrt{(p^4 - 4)}}$, sicque fiet $\partial x = \frac{xp\,\partial p}{\sqrt{(p^4 - 4)}}$, quo valore substituto fit $\partial S = \frac{p^2 x^2\,\partial p}{(1 - x^4)\sqrt{(p^4 - 4)}}$.

§. 63. Ut nunc hinc quantitatem x penitus ejiciamus, quoniam invenimus

$$xx = \frac{pp + \sqrt{(p^4 - 4)}}{2}, \text{ erit}$$
$$x^4 = \frac{p^4 - 2 + pp\sqrt{(p^4 - 4)}}{2}, \text{ hincque}$$
$$1 - x^4 = \frac{4 - p^4 - pp\sqrt{(p^4 - 4)}}{2} = -\frac{\sqrt{(p^4 - 4)}\,[pp + \sqrt{(p^4 - 4)}]}{2}.$$

Unde colligitur fore $\frac{xx}{1 - x^4} = -\frac{1}{\sqrt{(p^4 - 4)}}$, quo valore substituto impetramus formulam differentialem rationalem per novam variabilem p expressam, quae est

$$\partial S = -\frac{pp\,\partial p}{p^4 - 4}, \text{ existente } p = \frac{\sqrt{(1 + x^4)}}{x};$$

unde idem integrale, quod ante nacti sumus, deducitur. Similis autem substitutio cum successu adhiberi potest in formulis integralibus multo magis generalibus; veluti in sequente problemate ostendemus.

Problema 16.

§. 64. *Propositam formulam integralem* $S = \int \frac{\partial x \sqrt{(a + bxx + cx^4)}}{a - cx^4}$ *ope idoneae substitutionis ab omni irrationalitate liberare.*

Solutio.

Ad speciem saltem irrationalitatis tollendam, ponamus

$$\sqrt{(a + bxx + cx^4)} = px,$$

ut habeamus $S = \int \frac{px\,\partial x}{a - cx^4}$. Cum igitur sit

$$p = \frac{\sqrt{(a + bxx + cx^4)}}{x}, \text{ erit}$$

$$\partial p = - \frac{a\partial x + cx^4 \partial x}{xx\sqrt{(a + bxx + cx^4)}} = - \frac{a\partial x + cx^4 \partial x}{px^3},$$

unde erit

$$\partial x = - \frac{px^3 \partial p}{a - cx^4},$$

quo valore substituto fiet

$$\partial S = - \frac{ppx^4 \partial p}{(a - cx^4)^2}.$$

§. 65. Deinde cum sit

$$a + cx^4 = (pp - b)\, xx,$$

hincque porro

$$(a + cx^4)^2 = (pp - b)^2 x^4,$$

aufferatur $4acx^4$, ac remanebit

$$(a - cx^4)^2 = [(pp - b)^2 - 4ac]\, x^4,$$

quo substituto formula nostra fiet

$$\partial S = - \frac{pp\,\partial p}{(pp - b)^2 - 4ac}.$$

Sicque quantitas variabilis x penitus e calculo est extrusa, ac deducti sumus ad formulam differentialem prorsus rationalem, cujus ergo integratio per logarithmos et arcus circulares nulla amplius laborat difficultate. Quin etiam formulae adhuc generaliores eodem modo feliciter tractari poterunt.

Problema 17.

§. 66. *Propositam hanc formulam integralem*

$$S = \int \frac{x^{n-2}\partial x \sqrt[n]{(a + bx^n + cx^{2n})}}{a - cx^{2n}}$$

ope idoneae substitutionis ab omni irrationalitate liberare.

Solutio.

Utamur igitur hac substitutione

$$\sqrt[n]{(a + bx^n + cx^{2n})} = px,$$

ut formula proposita hanc induat formam

$$\partial S = \frac{px^{n-1}\partial x}{a - cx^{2n}};$$

tum vero cum sit

$$p^n = \frac{a + bx^n + cx^{2n}}{x^n},$$

erit differentiando

$$p^{n-1}\partial p = -\frac{\partial x(a - cx^{2n})}{x^{n+1}},$$

unde fit

$$\partial x = -\frac{p^{n-1}x^{n+1}\partial p}{a - cx^{2n}},$$

quo valore substituto formula nostra induet hanc formam

$$\partial S = -\frac{p^n x^{2n}\partial p}{(a - cx^{2n})^2}.$$

§. 67. Deinde cum sit

$$a + cx^{2n} = (p^n - b)\,x^n, \text{ erit}$$

$$(a + cx^{2n})^2 = (p^n - b)^2\,x^{2n};$$

hinc subtrahatur $4\,acx^{2n}$, et remanebit

$$(a - cx^{2n})^2 = [(p^n - b)^2 - 4\,ac]\,x^{2n},$$

substituto igitur hoc valore fiet

$$\partial S = - \frac{p^n \partial p}{(p^n - b)^2 - 4\,ac},$$

quae ergo omnino est rationalis, atque adeo integratio per logarithmos et arcus circulares facile expeditur.

Problema 18.

§. 68. *Invenire formulas integrales adhuc generaliores, quae ope substitutionis*

$$\sqrt[n]{(a + bx^n + cx^{2n})} = px$$

ad rationalitatem perduci queant.

Solutio.

Quoniam in praecedente problemate invenimus, hanc formulam differentialem

$$\frac{x^{n-2}\,\partial x \sqrt[n]{(a + bx^n + cx^{2n})}}{a - cx^{2n}}$$

ope hujus substitutionis reduci ad istam formulam rationalem

$$- \frac{p^n\,\partial p}{(p^n - b)^2 - 4\,ac}, \text{ erit}$$

$$\frac{P\,x^{n-2}\,\partial x \sqrt[n]{(a + bx^n + cx^{2n})}}{a - cx^{2n}} = - \frac{Pp^n\,\partial p}{(p^n - b)^2 - 4\,ac},$$

ubi loco P functiones quaecunque ipsius x accipi possunt ejusmodi, ut facta substitutione praebeant functiones rationales ipsius p, id quod infinitis modis fieri poterit, quorum praecipuos hic percurramus.

§. 69. Cum vi substitutionis sit

$$\frac{\sqrt[n]{(a + bx^n + cx^{2n})}}{x} = p,$$

loco P potestas quaecunque ipsius p assumi poterit, quae sit p^λ. Sumatur igitur $P = p^\lambda Q$, eritque etiam

$$P = \frac{Q \sqrt[n]{(a + bx^n + cx^{2n})^\lambda}}{x^\lambda};$$

quibus valoribus substitutis prodibit ista aequatio

$$\frac{Qx^{n-\lambda-2}\,\partial x \sqrt[n]{(a + bx^n + cx^{2n})^{\lambda+1}}}{a - cx^{2n}} = -\frac{Qp^{n+\lambda}\,\partial p}{(p^n - b)^2 - 4ac}$$

quae posterior formula denuo est rationalis.

§. 70. Deinde in praecedente problemate quoque invenimus esse

$$\frac{(a - cx^{2n})^2}{x^{2n}} = (p^n - b)^2 - 4ac,$$

quam ob rem pro Q sumamus potestatem exponentis i harum quantitatum, vel potius harum quantitatum reciprocam, scilicet capiatur

$$Q = \frac{x^{2in}}{(a - cx^{2n})^{2i}} = \frac{1}{[(p^n - b)^2 - 4ac]^i}.$$

Quibus valoribus substitutis obtinebimus formulam latissime patentem hanc

$$\frac{x^{(2i+1)n-\lambda-2}\partial x \sqrt[n]{(a+bx^n+cx^{2n})^{\lambda+1}}}{(a-cx^{2n})^{2i+1}} = -\frac{p^{n+\lambda}\partial p}{[(p^n-b)^2-4ac]^{i+1}};$$

ubi pro litteris λ et i numeros quoscunque integros sive positivos sive negativos accipere licet, perpetuo enim formula differentialis per p expressa manebit rationalis.

§. 71. Quin etiam haec reductio multo generalior reddi potest, propterea quod necessum non est ut λ sit numerus integer: Quaecunque enim fractio pro λ assumatur, formula per p expressa semper facile ad rationalitatem reduci poterit. Si enim ponamus $\lambda = \frac{\mu}{\nu}$, membrum dextrum fiet

$$-\frac{p^{\frac{\nu n+\mu}{\nu}}\partial p}{[(p^n-b)^2-4ac]^{i+1}},$$

quae rationalis redditur ponendo $p = q^\nu$, erit enim $\partial p = \nu q^{\nu-1}\partial q$, ideoque hoc membrum

$$-\frac{\nu q^{\mu+\nu n+\nu-1}\partial q}{[(q^{\nu n}-b)^2-4ac]^{i+1}}.$$

Nunc autem uti oportebit hac substitutione

$$\sqrt[n]{(a+bx^n+cx^{2n})} = q^\nu x,$$

atque habebitur ista reductio

$$\frac{x^{(2i+1)n-\frac{\mu}{\nu}-2}\partial x \sqrt[n]{(a+bx^n+cx^{2n})^{\frac{\mu+\nu}{\nu}}}}{(a-cx^{2n})^{2i+1}}$$
$$= -\frac{\nu q^{\mu+\nu n+\nu-1}\partial q}{[(q^{n\nu}-b)^2-4ac]^{i+1}},$$

quae postrema formula utique est rationalis.

§. 72. Ut etiam in membro sinistro exponentes fractos ipsius x tollamus, ponamus $x = y^\nu$, eritque

$$\frac{y^{(2i+1)n\nu-\mu-\nu-1}\partial y \sqrt[n\nu]{(a+by^{n\nu}+cy^{2n\nu})^{\mu+\nu}}}{(a-cy^{2n\nu})^{2i+1}}$$

$$= - \frac{q^{\mu+n\nu+\nu-1}\partial q}{[(q^{n\nu}-b)^2 - 4ac]^{i+1}},$$

quae expressio autem multo generalior videtur, quam revera est. Si enim loco $n\nu$ ubique scribamus n resultat ista aequatio

$$\frac{y^{(2i+1)n-\mu-\nu-1}\partial y \sqrt[n]{(a+by^{n}+cy^{2n})^{\mu+\nu}}}{(a-cy^{2n})^{2i+1}}$$

$$= - \frac{q^{\mu+\nu+n-1}\partial q}{[(q^{n}-b)^2 - 4ac]^{i+1}};$$

haec autem aequatio manifesto non discrepat ab illa §. 70. allata; si enim hic loco $\mu+\nu-1$ scribamus λ et loco y et q ut ante x et p, ipsa praecedens aequatio reperitur, sicque sufficiet loco λ numeros integros assumere.

Corollarium.

§. 73. Quo clarius indoles harum formularum perspiciatur, sumamus $n = 2$, et formula differentialis variabilem x involvens erit

$$\frac{x^{4i-\lambda}\partial x \sqrt{(a+bxx+cx^4)^{\lambda+1}}}{(a-cx^4)^{2i+1}},$$

quae facta substitutione $\sqrt{(a+bxx+cx^4)} = px$, transmutatur in hanc rationalem

$$- \frac{p^{\lambda+2}\partial p}{[(pp-b)^2 - 4ac]^{i-1}},$$

unde sumendo $\lambda = 4i$ resultat ista aequatio

$$\frac{\partial x \sqrt{(a + bxx + cx^4)^{4i+1}}}{(a - cx^4)^{ci+1}} = - \frac{p^{4i+2}\partial p}{[(pp - b)^2 - 4ac]^{i+1}},$$

in qua si porro ponatur $i = 0$, fiet

$$\frac{\partial x \sqrt{(a + bxx + cx^4)}}{a - cx^4} = - \frac{pp\,\partial p}{(pp - b)^2 - 4ac};$$

quae si insuper ponatur $a = 1$, $b = 0$ et $c = 1$, praebet

$$\frac{\partial x \sqrt{(1 + x^4)}}{1 - x^4} = - \frac{pp\,\partial p}{p^4 - 4},$$

quae est ipsa reductio, quae supra §. 63. fuerat inventa.

Corollarium 2.

§. 74. Si sumamus $n = 3$, prodibit ista reductio generalis

$$\frac{x^{6i-\lambda+1}\partial x \sqrt[3]{(a + bx^3 + cx^6)^{\lambda+1}}}{(a - cx^6)^{2i+1}} = - \frac{p^{\lambda+3}\partial p}{[(p^3 - b)^2 - 4ac]^{i+1}},$$

quae ponendo $i = 0$ migrat in hanc

$$\frac{x^{-\lambda+1}\partial x \sqrt[3]{(a + bx^3 + cx^6)^{\lambda+1}}}{a - cx^6} = - \frac{p^{\lambda+3}\partial p}{(p^3 - b)^2 - 4ac};$$

posito vero $b = 0$, haec prodit formula concinnior

$$\frac{x^{-\lambda+1}\partial x \sqrt[3]{(a+cx^6)^{\lambda+1}}}{a - cx^6} = - \frac{p^{\lambda+3}\partial p}{p^6 - 4ac},$$

cujus duos casus evolvisse juvabit.

I. Sit $\lambda = 0$, eritque

$$\frac{x\partial x \sqrt[3]{(a + cx^6)}}{a - cx^6} = - \frac{p^3\partial p}{p^6 - 4ac};$$

quae concinnior redditur ponendo $xx = y$, reperietur enim

$$\frac{\partial y \sqrt[3]{(a + cy^3)}}{a - cy^3} = - \frac{2p^3 \partial p}{p^6 - 4ac}.$$

II. Sumto autem $\lambda = 1$, ista prodit expressio

$$\frac{\partial x \sqrt[3]{(a + cx^6)^2}}{a - cx^6} = - \frac{p^4 \partial p}{p^6 - 4ac}.$$

Scholion.

§. 75. Ex his exemplis satis intelligitur, quam egregie reductiones ex nostris formulis generalibus deduci queant, quarum resolutio, nisi methodus nostra adhibeatur, omnes vires analyseos superare videatur.

4.) Memorabile genus formularum differentialium maxime irrationalium, quas tamen ad rationalitatem perducere licet. *M. S. Academiae exhib. d. 15. Maii 1777.*

§. 76. Cum nuper hanc formulam differentialem

$$\frac{\partial x}{(1 - xx) \sqrt[4]{(2xx - 1)}}$$

tractassem eamque singulari modo ad rationalitatem perduxissem, mox vidi eandem methodum succedere in hac generaliori $\frac{\partial x}{(a + bxx) \sqrt[4]{(a + 2bxx)}}$, atque adeo in hac multo generaliori

$\frac{\partial x}{(a+bx^n)\sqrt[2n]{(a+2bx^n)}}$, ubi irrationalitas ad ordinem quantumvis altum assurgere potest, cujus resolutio sequenti modo instituitur.

§. 77. Utor scilicet hac substitutione $\frac{x}{\sqrt[2n]{(a+2bx^n)}} = Z$, ut formula nostra integranda, quam per ∂V indicemus, fiat $\partial V = \frac{\partial x}{x} \cdot \frac{Z}{a+bx^n}$, sumtis ergo logarithmis erit

$$lZ = lx - \frac{1}{2n} l(a+2bx^n),$$

unde differentiando fit

$$\frac{\partial Z}{Z} = \frac{\partial x}{x} - \frac{bx^{n-1}\partial x}{a+2bx^n} = \frac{\partial x(a+bx^n)}{x(a+2bx^n)},$$

erit ergo

$$\frac{\partial x}{x} = \frac{\partial Z(a+2bx^n)}{Z(a+bx^n)},$$

hinc ergo nostra formula erit

$$\partial V = \frac{\partial Z(a+2bx^n)}{(a+bx^n)^2}.$$

Cum igitur sit

$$Z^{2n} = \frac{x^{2n}}{a+2bx^n}, \text{ erit } a+2bx^n = \frac{x^{2n}}{Z^{2n}},$$

ideoque

$$\partial V = \frac{x^{2n}\partial Z}{Z^{2n}(a+bx^n)^2}.$$

Cum porro sit $aa + 2abx^n = \frac{ax^{2n}}{Z^{2n}}$, addatur utrinque bbx^{2n}, et prodibit

$$(a+bx^n)^2 = \frac{ax^{2n}}{Z^{2n}} + bbx^{2n} = \frac{x^{2n}(a+bbZ^{2n})}{Z^{2n}},$$

quo valore substituto nostra formula evadet

$$\partial V = \frac{\partial Z}{a + bbZ^{2n}},$$

quae ergo formula est rationalis, ideoque per logarithmos et arcus circulares integrari poterit.

§. 78. Observavi porro, cum hic post signum radicale tantum binomium involvatur, ejus loco quoque trinomia, atque adeo polynomia introduci posse. Pro trinomiis autem formula differentialis talem habebit formam

$$\partial V = \frac{\partial x}{(a + bx^n)\sqrt[3n]{(aa + 3\,abx^n + 3\,bbx^{2n})}},$$

ubi ergo irrationalitas ad ordinem multo altiorem ascendit. Nihilo vero minus etiam ista formula ab irrationalitate liberari poterit ope similis substitutionis

$$Z = \frac{x}{\sqrt[3n]{(aa + 3\,abx^n + 3\,bb\,x^{2n})}};$$

hinc enim sumtis logarithmis per differentiationem nanciscemur

$$\frac{\partial Z}{Z} = \frac{\partial x}{x} - \frac{abx^{n-1}\,\partial x - 2\,bbx^{2n-1}\,\partial x}{aa + 3\,abx^n + 3\,bbx^{2n}}, \quad \text{seu}$$

$$\frac{\partial Z}{Z} = \frac{\partial x\,(a + bx^n)^2}{x\,(aa + 3\,abx^n + 3\,bbx^{2n})},$$

ideoque

$$\frac{\partial x}{x} = \frac{\partial Z}{Z} \cdot \frac{aa + 3\,abx^n + 3\,bbx^{2n}}{(a + bx^n)^2}.$$

Cum igitur nostra formula jam sit $\partial V = \frac{\partial x}{x} \cdot \frac{Z}{a + bx^n}$, introducto elemento ∂Z, obtinebimus

$$\partial V = \frac{\partial Z\,(aa + 3\,abx^n + 3\,bbx^{2n})}{(a + bx^n)^3}.$$

§. 79. Cum igitur vi substitutionis sit

$$\sqrt[3n]{(aa + 3abx^n + 3bbx^{2n})} = \frac{x}{Z}, \quad \text{erit}$$

$$aa + 3abx^n + 3bbx^{2n} = \frac{x^{3n}}{Z^{3n}}.$$

Multiplicetur utrinque per a, et addatur utrinque $b^3 x^{3n}$, eritque

$$(a + bx^n)^3 = \frac{x^{3n}(a + b^2 Z^{3n})}{Z^{3n}}:$$

hoc igitur valore substituto ex formula nostra littera x penitus excludetur, prodibitque $\partial V = \frac{\partial Z}{a + b^3 Z^n}$. Cujus ergo integrale semper per logarithmos et arcus circulares reperire licebit.

§. 80. Pro quadrinomiis autem ponamus brevitatis gratia

$$\sqrt[4n]{(a^3 + 4aabx^n + 6abbx^{2n} + 4b^3 x^{3n})} = S,$$

ac formula ad rationalitatem reducenda proponatur haec

$$\partial V = \frac{\partial x}{(a + bx^n) S},$$

id quod simili modo succedet ope hujus substitutionis $\frac{x}{S} = Z$, unde formula nostra erit $\partial V = \frac{\partial x}{x} \cdot \frac{Z}{a + bx^n}$. Cum nunc sit

$$\frac{\partial S}{S} = \frac{aabx^{n-1}\partial x + 3abbx^{2n-1}\partial x + 3b^3x^{3n-1}\partial x}{a^3 + 4aabx^n + 6abbx^{2n} + 4b^3x^{3n}},$$

sive

$$\frac{\partial S}{S} = \frac{\partial x}{x} \cdot \frac{bx^n(aa + 3abx^n + 3bbx^{2n})}{S^{4n}},$$

erit $\frac{\partial Z}{Z} = \frac{\partial x}{x} - \frac{\partial S}{S}$; consequenter

$$\frac{\partial Z}{Z} = \frac{\partial x}{x} \cdot \frac{(a + bx^n)^3}{S^{4n}}, \quad \text{hincque } \frac{\partial x}{x} = \frac{S^{4n}\partial Z}{Z(a + bx^n)^3},$$

quo valore substituto formula nostra erit

$$\partial V = \frac{S^{4n}\,\partial Z}{(a + bx^n)^4}.$$

§. 81. Cum autem sit

$$S^{4n} = a^3 + 4\,aabx^n + 6\,abbx^{2n} + 4\,b^3x^{3n}, \quad \text{erit}$$
$$a\,S^{4n} + b^4\,x^{4n} = (a + bx^n)^4,$$

quo valore substituto erit

$$\partial V = \frac{S^{4n}\,\partial Z}{aS^{4n} + b^4\,x^{4n}}:$$

quia igitur posuimus $Z = \frac{x}{S}$, erit $S = \frac{x}{Z}$, ideoque $S^{4n} = \frac{x^{4n}}{Z^{4n}}$, qui valor surrogatus dabit

$$\partial V = \frac{\partial Z}{a + b^4\,Z^{4n}},$$

sicque itidem ad rationalitatem est perducta.

§. 82. Hinc jam facile intelligitur, quo modo pro omnibus polynomiis formulae differentiales comparatae esse debeant, ut tali substitutione ad rationalitatem perduci queant, id quod in sequente problemate expediamus

Problema 19.

§. 83. *Si proposita fuerit haec formula differentialis*

$$\partial V = \frac{\partial x}{(a + bx^n)\sqrt[\lambda n]{[(a + bx^n)^\lambda - b^\lambda x^{\lambda n}]}},$$

eam ad rationalitatem reducere, quantumvis magni numeri pro n et λ accipiantur.

Solutio.

Ponamus etiam hic brevitatis gratia

$$\sqrt[\lambda n]{[(a + bx^n)^\lambda - b^\lambda x^{\lambda n}]} = S,$$

ut formula fiat

$$\partial V = \frac{\partial x}{(a + bx^n) S},$$

fiatque insuper $\frac{x}{S} = Z$, ut habeamus

$$\partial V = \frac{\partial x}{x} \cdot \frac{Z}{a + bx^n}.$$

Jam logarithmos differentiando reperietur

$$\frac{\partial S}{S} = \frac{bx^{n-1} \partial x (a + bx^n)^{\lambda - 1} - b^\lambda x^{\lambda n - 1} \partial x}{S^{\lambda n}}, \text{ sive}$$

$$\frac{\partial S}{S} = \frac{\partial x}{x} \cdot \frac{bx^n (a + bx^n)^{\lambda - 1} - b^\lambda x^{\lambda n}}{S^{\lambda n}}.$$

Cum igitur sit $\frac{\partial Z}{Z} = \frac{\partial x}{x} - \frac{\partial S}{S}$, hoc valore substituto erit

$$\frac{\partial Z}{Z} = \frac{\partial x}{x} \cdot \frac{a (a + bx^n)^{\lambda - 1}}{S^{\lambda n}},$$

hincque vicissim erit

$$\frac{\partial x}{x} = \frac{S^{\lambda n} \partial Z}{a Z (a + bx^n)^{\lambda - 1}},$$

quo valore substituto impetramus

$$\partial V = \frac{S^{\lambda n} \partial Z}{a (a + bx^n)^\lambda},$$

quia nunc est $(a + bx^n)^\lambda = S^{\lambda n} + b^\lambda x^{\lambda n}$, erit

$$\partial V = \frac{S^{\lambda n} \partial Z}{a (S^{\lambda n} + b^\lambda x^{\lambda n})}.$$

Denique ob $S = \frac{x}{Z}$, ideoque $S^{\lambda n} = \frac{x^{\lambda n}}{Z^{\lambda n}}$, hoc valore substituto obtinebitur

$$\partial V = \frac{\partial Z}{a(1 + b^{\lambda} Z^{\lambda n})},$$

quae est rationalis unicam variabilem Z involvens, cujus adeo integrale per logarithmos et arcus circulares assignari poterit.

Corollarium 1.

§. 84. Eadem solutio etiam locum habet, si pro λ numeri fracti accipiantur, qua ratione post signum radicale denuo radicalia involvuntur: ita si fuerit $\lambda = \frac{2}{n}$, erit formula radicalis

$$S = \sqrt{[(a + bx^n)^{\frac{2}{n}} - b^{\frac{2}{n}} xx]},$$

et formulae nostrae

$$\partial V = \frac{\partial x}{(a + bx^n) S}$$

integrale erit

$$V = \frac{1}{a} \int \frac{\partial Z}{1 + b^{\frac{2}{n}} ZZ} = \frac{1}{ab^{\frac{1}{n}}} \text{Arc. tang. } b^{\frac{1}{n}} Z.$$

Corollarium 2.

§. 85. Quo haec clariora reddantur, capiamus $a = 1$, $b = 1$, et $n = 4$, ut pro postremo casu sit

$$S = \sqrt{[1 + x^4)^{\frac{1}{2}} - xx]}, \text{ et } \partial V = \frac{\partial x}{(1 + x^4)\sqrt{[(1 + x^4)^{\frac{1}{2}} - xx]}},$$

cujus integrale posito

$$Z = \frac{x}{\sqrt{[(1 + x^4)^{\frac{1}{2}} - xx]}}, \text{ erit}$$

$$V = \text{Arc. tang.} Z, \text{ sive } V = \text{Arc. tang.} \frac{x}{\sqrt{[(1 + x^4)^{\frac{1}{2}} - xx]}}.$$

Sin autem manente $n = 4$ et $a = 1$, fuerit $b = -1$, ideoque

$$S = \sqrt{[(1 - x^4)^{\frac{1}{2}} - xx\sqrt{-1}]},$$

ipsa formula prodiret imaginaria.

Corollarium 3.

§. 86. Pro eodem casu $\lambda = \frac{2}{n}$, sit $n = 6$, $a = 1$ et $b = 1$, eritque

$$S = \sqrt{[(1 + x^6)^{\frac{1}{3}} - xx]}, \text{ ideoque}$$

$$\partial V = \frac{\partial x}{(1 + x^6)\sqrt{[(1 + x^6)^{\frac{1}{3}} - xx]}}.$$

Cujus integrale posito $\frac{x}{S} = Z$, erit

$$V = \text{Arc. tang.} Z = \text{Arc. tang.} \frac{x}{\sqrt{[(1 + x^6)^{\frac{1}{3}} - xx]}}.$$

Similique modo alia hujus generis exempla pro lubitu formari possunt; verum quamquam formula problematis admodum est generalis, tamen adhuc multo magis generalior fieri potest, uti in sequente problemate sumus ostensuri.

Problema 20.

§. 87. *Si proponatur ista formula differentialis multo generalior, quippe in qua tres occurrunt exponentes indeterminati* λ, n, *et* m,

$$\partial V = \frac{x^{m-1}\,\partial x}{(a + bx^n)^\lambda \sqrt[\lambda n]{[(a + bx^n)^\lambda - b^\lambda x^{\lambda n})^m}},$$

eam ab irrationalitate liberare.

Solutio.

Ponatur iterum brevitatis gratia

$$\sqrt[\lambda n]{[(a + bx^n)^\lambda - b^\lambda x^{\lambda n}]} = S,$$

ut formula integranda proposita fiat

$$\partial V = \frac{x^{m-1}\,\partial x}{(a + bx^n)\,S^m} = \frac{\partial x}{x} \cdot \frac{x^m}{(a + bx^n)\,S^m},$$

quae ergo si porro ut ante statuamus $\frac{x}{S} = Z$, fiet

$$\partial V = \frac{\partial x}{x} \cdot \frac{Z^m}{a + bx^n},$$

unde variabilem x penitus eliminari oportet. Quoniam nunc ambae litterae S et Z eosdem habent valores, ut in problemate praecedente atque adeo ipsa formula ∂V oriatur, si praecedens per Z^{m-1} multiplicetur, etiam integrale quaesitum obtinebimus, dum superius integrale per Z^{m-1} multiplicabimus, quo facto erit integrale quaesitum

$$V = \frac{1}{a} \int \frac{Z^{m-1}\,\partial Z}{1 + b^\lambda Z^{\lambda n}}.$$

Corollarium 1.

§. 88. Si exponentem m negativum capiamus, irrationalitas in numeratorem transferetur, ita posita $m = -1$ habebimus

$$\partial V = \frac{\partial x \sqrt[\lambda n]{[(a + bx^n)^\lambda - b^\lambda x^{\lambda n}]}}{xx\,(a + bx^n)},$$

cujus ergo integrale per Z expressum erit

$$V = \frac{1}{a} \int \frac{\partial Z}{ZZ(1 + b^\lambda Z^{\lambda n})}.$$

Quin etiam per hunc exponentem m irrationalitas simplicior reddi poterit, veluti si sumamus $m = \lambda$, erit

$$\partial V = \frac{x^{\lambda - 1}\partial x}{(a + bx^n)\sqrt[n]{[(a + bx^n)^\lambda - b^\lambda x^{\lambda n}]}}.$$

Cujus integrale posito $Z = \frac{x}{S}$, retinente S superiorem valorem erit

$$V = \frac{1}{a} \int \frac{Z^{\lambda - 1}\partial Z}{1 + b^\lambda Z^{\lambda n}}.$$

Corollarium 2.

§. 89. Deinde vero etiam si pro m fractionem assumamus, irrationalitas adhuc magis complicabitur, veluti si sumamus $m = \frac{1}{2}$, formula differentialis jam erit

$$\partial V = \frac{\partial x}{(a + bx^n)\sqrt[2\lambda n]{x^{\lambda n}[(a + bx^n)^\lambda - b^\lambda x^{\lambda n}]}}.$$

Verum hic casus facile ad primum problema revocatur statuendo $x = vv$, ita ut sit

$$\partial V = \frac{2\partial v}{(a + bv^{2n})\sqrt[2\lambda n]{[(a + bv^{2n})^\lambda - b^\lambda v^{2\lambda n}]}},$$

quae formula a primo problemate aliter non discrepat nisi quod hic exponens n duplo sit major.

Scholion.

§. 90. Quamquam binae litterae a et b pro lubitu tam negative, quam positive accipi possunt, tamen occurrunt casus, qui sub hac generali forma non comprehenduntur: veluti si proponatur haec formula $\frac{\partial x}{(1-xx\sqrt[4]{(2xx-1)}}$, haec in problemate primo non continetur, quia fieri deberet $aa=-1$, quod cum in genere evenire posset, etiam problema generale ad hunc casum accommodatum subjungamus.

Problema 21.

§. 91. *Si ponatur ista formula differentialis latissime patens tres exponentes indeterminatos involvens*

$$\partial V = \frac{x^{m-1}\partial x}{(fx^n-g)\sqrt[\lambda n]{[f^\lambda x^{\lambda n}-(fx^n-g)^\lambda]^m}},$$

eam ab omni irrationalitate liberare.

Solutio.

Statuamus ut ante brevitatis gratia

$$\sqrt[\lambda n]{[f^\lambda x^{\lambda n}-(fx^n-g)^\lambda]} = S,$$

tum vero $Z=\frac{x}{S}$, ut formula differentialis fiat

$$\partial V = \frac{\partial x}{x}\cdot\frac{Z^m}{fx^n-g}.$$

Nunc autem sumendo differentialia logarithmica est

$$\frac{\partial S}{S} = \frac{\partial x}{x}\cdot\frac{f^\lambda x^{\lambda n}-fx^n(fx^n-g)^{\lambda-1}}{S^{\lambda n}},$$

atque hinc colligitur fore

$$\frac{\partial Z}{Z} = \frac{\partial x}{x} - \frac{\partial S}{S} = \frac{\partial x}{x}\cdot\frac{g(fx^n-g)^{\lambda-1}}{S^{\lambda n}},$$

sicque habebitur

$$\frac{\partial x}{x} = \frac{\partial Z}{Z}\cdot\frac{S^{\lambda n}}{g(fx^n-g)^{\lambda-1}},$$

quo valore substituto nanciscimur

$$\partial V = \frac{z^{m-1} \partial z s^{\lambda n}}{g\,(f x^n - g)^{\lambda}}.$$

Manifesto autem est $(f x^n - g)^{\lambda} = f^{\lambda} x^{\lambda n} - s^{\lambda n}$, ideoque

$$\partial V = \frac{z^{m-1} s^{\lambda n} \partial z}{g\,(f^{\lambda} x^{\lambda n} + s^{\lambda n})};$$

unde postremo ob $s = \frac{x}{z}$ concluditur haec forma

$$\partial V = \frac{z^{m-1} \partial z}{g\,(f^{\lambda} z^{\lambda n} - 1)},$$

quae formula a praecedentibus tantum signis discrepat.