SVPPLEMENTVM

ad differtationem praecedentem, circa integrationem formulae $\int \frac{z^{m-1}\partial z}{1-z^n}$, cafu quo ponitur

$$z = v(\text{cof.}\,\phi + \sqrt{-1}\,\text{fin.}\,\phi).$$

§. 1. Refolutio formulae $\int \frac{z^{m-1}\partial z}{1+z^n}$, quam fupra in problemate, pro cafu quo $z = v(\text{cof.}\,\phi + \sqrt{-1}\,\text{fin.}\,\phi)$ dedimus, eximia et notatu digniffima artificia complectitur, quae animo firmius imprimere haud inutile erit. Cum igitur formula, quam hic tractandam fufcipimus, non minore attentione fit digna quam ea quam fupra tractauimus, eius integrale per eandem methodum exhibere conftitui; vbi fimul occafionem inueniemus nouum compendium in calculo adhibendi.

Problema.

Si ponatur $z = v(\text{cof.}\,\phi + \sqrt{-1}\,\text{fin.}\,\phi)$, *inueftigare integrale huius formulae:* $\int \frac{z^{m-1}\partial z}{1-z^n}$.

Solutio.

§. 2. Cum ob valorem ipfius z imaginarium integrale quaefitum etiam effe debeat imaginarium, id fub forma $P + Q\sqrt{-1}$ complectamur, ita vt P et Q fint quantitates reales. Hanc ob rem erit facta fubftitutione

$$\int \frac{z^{m-1}\partial z}{1-z^n} = P + Q\sqrt{-1}.$$

§. 3. Cum porro fit $z = v(\text{cof.}\,\phi + \sqrt{-1}\,\text{fin.}\,\phi)$, hincque $\frac{\partial z}{z} = \frac{\partial v}{v} + \partial\phi\sqrt{-1}$, erit numerator

z

$$z^{m-1}\partial z = v^m(\text{cos.}\, m\Phi + \sqrt{-1}\,\text{sin.}\, m\Phi)\left(\frac{\partial v}{v} + \partial\Phi\sqrt{-1}\right),$$

denominator vero erit

$$1 - v^n(\text{cos.}\, n\Phi + \sqrt{-1}\,\text{sin.}\, n\Phi),$$

qui ergo euanescit ponendo

$$v^n = \frac{1}{\text{cos.}\, n\Phi + \sqrt{-1}\,\text{sin.}\, n\Phi} = \text{cos.}\, n\Phi - \sqrt{-1}\,\text{sin.}\, n\Phi.$$

§. 4. Iam vt imaginaria ex denominatore tollantur, supra et infra multiplicemus per

$$1 - v^n(\text{cos.}\, n\Phi - \sqrt{-1}\,\text{sin.}\,\Phi),$$

sicque fractio nostra euoluenda erit

$$\partial V = \frac{z^{m-1}\partial z(1 - v^n\,\text{cos.}\, n\Phi + v^n\sqrt{-1}\,\text{sin.}\, n\Phi)}{1 - 2v^n\,\text{cos.}\, n\Phi + v^{2n}}.$$

Quod si iam hic loco $z^{m-1}\partial z$ valor modo assignatus substituatur et partes reales ab imaginariis segregentur, ob

$$(\text{cos.}\, m\Phi + \sqrt{-1}\,\text{sin.}\, m\Phi)(\text{cos.}\, n\Phi - \sqrt{-1}\,\text{sin.}\, n\Phi)$$
$$= \text{cos.}\,(m-n)\Phi + \sqrt{-1}\,\text{sin.}\,(m-n)\Phi$$

prodibit pars realis ita expressa:

$$v^{m-1}\partial v\,[\text{cos.}\, m\Phi - v^n\,\text{cos.}\,(m-n)\Phi]$$
$$- v^m\partial\Phi\,[\text{sin.}\, m\Phi - v^n\,\text{sin.}\,(m-n)\Phi],$$

pars vero imaginaria per $\sqrt{-1}$ diuisa:

$$v^{m-1}\partial v\,[\text{sin.}\, m\Phi - v^n\,\text{sin.}\,(m-n)\Phi]$$
$$+ v^m\partial\Phi\,(\text{cos.}\, m\Phi - v^n\,\text{cos.}\,(m-n)\Phi].$$

§. 5. Quod si iam breuitatis gratia statuamus

$$R = v^{m-1}[\text{cos.}\, m\Phi - v^n\,\text{cos.}\,(m-n)\Phi] \text{ et}$$
$$S = v^{m-1}[\text{sin.}\, m\Phi - v^n\,\text{cos.}\,(m-n)\Phi],$$

ambae litterae quaesitae P et Q per sequentes formulas integrales exprimentur:

P =

$$P = \int \frac{R \partial v - S v \partial \phi}{1 - 2 v^n \cos. n \phi + v^{2n}} \text{ et}$$

$$Q = \int \frac{S \partial v + R v \partial \phi}{1 - 2 v^n \cos. n \phi + v^{2n}}.$$

Has igitur duas formulas integrare oportebit, quod fiet, dum denominatoris ſingulos factores trinomiales inueſtigabimus et ex ſingulis fractiones partiales inde oriundas definiemus.

§. 6. Conſideremus igitur in genere hanc fractionem: $\frac{N}{1 - 2 v^n \cos. n \phi + v^{2n}}$, et fingamus denominatoris factorem eſſe $1 - 2 v \cos. \omega + v v$, vbi angulus ω ita debet eſſe comparatus, vt poſito

$$1 - 2 v \cos. \omega + v v = 0, \text{ ſiue}$$

$$v = \cos. \omega + \sqrt{-1} \sin. \omega,$$

ſimul quoque denominator euaneſcat, id quod fit, vti vidimus, quando $v^n = \cos. n \phi - \sqrt{-1} \sin. n \phi$. At vero ex factore ſuppoſito fit $v^n = \cos. n \omega + \sqrt{-1} \sin. n \omega$, vnde ſtatui debebit $\cos. n \omega = \cos. n \phi$ et $\sin. n \omega = - \sin. n \phi$, id quod euenit in genere quando $n \omega + n \phi = i \pi$, denotante i omnes numeros pares, ſicque erit $n \omega = i \pi - n \phi$, ideoque $\omega = \frac{i \pi}{n} - \phi$, vnde n diuerſi valores pro angulo ω deducuntur, dum ſcilicet loco i ſcribuntur ſucceſſiue numeri 0, 2, 4, 6, etc. vsque ad $2 n$, excluſo poſtremo.

§. 7. Ponamus nunc fractionem partialem ex iſto factore oriundam eſſe $\frac{F}{1 - 2 v \cos. \omega + v v}$, atque ex ſuperioribus patet ſtatui debere

$$F = \frac{N (1 - 2 v \cos. \omega + v v)}{1 - 2 v^n \cos. n \phi + v^{2n}},$$

vnde-

vnde ſcilicet ope aequationis $v\,v - 2\,v\cos.\,\omega + 1 = 0$ pro F huiusmodi forma $A\,v + B$ elici debet. Quoniam vero hoc caſu tam numerator quam denominator euaneſcit, differentialibus in ſubſidium vocatis fiet

$$F = \frac{N\,(v - \cos.\,\omega)}{n\,v^{n-1}\,(v^n - \cos.\,n\,\Phi)}.$$

§. 8. Cum nunc caſu quo $v\,v - 2\,v\cos.\,\omega + 1 = 0$ fit

$v - \cos.\,\omega = \sqrt{-1}\,\sin.\,\omega$ et

$v^n - \cos.\,n\,\Phi = -\sqrt{-1}\,\sin.\,n\,\Phi$, erit

$$F = -\frac{N\,v\,\sin.\,\omega}{n\,v^n\,\sin.\,n\,\Phi},$$

qui valor prorſus conuenit cum eo qui ſupra eſt repertus. Hic igitur tantum opus eſt, vt loco N ſiue R ſiue S ſubſtituatur, indeque forma praeſcripta pro iſto numeratore F deriuetur, in vſum vocando lemma ſupra allatum

Euolutio fractionis

$$\frac{R\,v\,\sin.\,\omega}{n\,v^n\,\sin.\,n\,\Phi} \text{ ſiue } \frac{v^m\,\sin.\,\omega\,[\cos.\,m\,\Phi - v^n\cos.\,(m-n)\,\Phi]}{n\,v^n\,\sin.\,n\,\Phi}.$$

§. 9. Hinc ergo erit

$$F = -\frac{v^{m-n}\,\sin.\,\omega\cos.\,m\,\Phi + v^m\,\sin.\,\omega\cos.\,(m-n)\,\Phi}{n\,\sin.\,n\,\Phi}.$$

Per lemma autem memoratum habebitur

$$\sin.\,\omega\,v^{m-n} = v\,\sin.\,(m-n)\,\omega - \sin.\,(m-n-1)\,\omega.$$

Cum igitur ſit $n\,\omega = i\,\pi - n\,\Phi$, erit

$\sin.\,(m-n)\,\omega = \sin.\,(m\,\omega + n\,\Phi)$ et

$\sin.\,(m-n-1)\,\omega = \sin.\,(m-1)\,\omega + n\,\Phi]$.

Deinde vero eſt

$$\text{ſin.}\,\omega.\,v^m = v\,\text{ſin.}\,m\,\omega - \text{ſin.}\,(m-1)\,\omega,$$

quibus valoribus ſubſtitutis erit

$$F = -\frac{1}{n\,\text{ſin.}\,n\,\Phi}\left[\begin{array}{l} v\,\text{coſ.}\,m\,\Phi\,\text{ſin.}\,(m\,\omega + n\,\Phi) - \text{coſ.}\,m\,\Phi\,\text{ſin.}\,[(m-1)\,\omega + n\,\Phi] \\ - v\,\text{ſin.}\,m\,\omega\,\text{coſ.}\,(m-n)\,\Phi] + \text{ſin.}\,(m-1)\,\omega\,\text{coſ.}\,(m-n)\,\Phi \end{array}\right].$$

Facta iam euolutione formularum

$$\text{ſin.}\,(m\,\omega + n\,\Phi) = \text{ſin.}\,m\,\omega\,\text{coſ.}\,n\,\Phi + \text{coſ.}\,m\,\omega\,\text{ſin.}\,n\,\Phi \text{ et}$$

$$\text{coſ.}\,(m-n)\,\Phi = \text{coſ.}\,m\,\Phi\,\text{coſ.}\,n\,\Phi + \text{ſin.}\,m\,\Phi\,\text{ſin.}\,n\,\Phi,$$

littera v hic multiplicatur per hanc formam:

$$\text{ſin.}\,n\,\Phi\,\text{coſ.}\,m\,\Phi\,\text{coſ.}\,m\,\omega - \text{ſin.}\,n\,\Phi\,\text{ſin.}\,m\,\Phi\,\text{ſin.}\,m\,\omega$$

$$= \text{ſin.}\,n\,\Phi\,\text{coſ.}\,(m\,\Phi + m\,\omega),$$

reliqui vero termini, quia ab his tantum in eo differunt vt loco $m\,\omega$ ſcribi debeat $(m-1)\,\omega$, erunt:

$$-\text{ſin.}\,n\,\Phi\,\text{coſ.}\,[m\,(\omega + \Phi) - \omega]$$

ſicque pro numeratore quem quaerimus erit

$$F = -\frac{1}{n}\,v\,\text{coſ.}\,m\,(\omega + \Phi) + \frac{1}{n}\,\text{coſ.}\,[m\,(\omega + \Phi) - \omega].$$

Euolutio fractionis

$$\frac{S\,v\,\text{ſin.}\,\omega}{n\,v^n\,\text{ſin.}\,n\,\Phi} = \frac{v^m\,\text{ſin.}\,\omega\,[\text{ſin.}\,m\,\Phi - v^n\,\text{ſin.}\,(m-n)\,\Phi]}{n\,v^n\,\text{ſin.}\,n\,\Phi}.$$

§. 10. Hoc caſu erit

$$F = -\frac{v^{m-n}\,\text{ſin.}\,\omega\,\text{ſin.}\,m\,\Phi + v^m\,\text{ſin.}\,\omega\,\text{ſin.}\,(m-n)\,\Phi}{n\,\text{ſin.}\,n\,\Phi}.$$

Hic igitur eodem lemmate in ſubſidium vocato erit

$$F = -\frac{1}{n\,\text{ſin.}\,n\,\Phi}\left[\begin{array}{l} v\,\text{ſin.}\,m\,\Phi\,\text{ſin.}\,(m\,\omega + n\,\Phi) - \text{ſin.}\,m\,\Phi\,\text{ſin.}\,[(m-1)\,\omega + n\,\Phi] \\ - v\,\text{ſin.}\,(m-n)\,\Phi\,\text{ſin.}\,m\,\omega + \text{ſin.}\,(m-n)\,\Phi\,\text{ſin.}\,(m-1)\,\omega \end{array}\right];$$

vbi per ſimilem euolutionem quantitas, qua v multiplicatur, inuenitur $= \text{ſin.}\,n\,\Phi\,\text{ſin.}\,[m\,(\omega + \Phi)]$; reliqua vero pars erit

$- \sin. n \Phi \sin. [m(\omega + \Phi) - \omega]$,

hinc igitur pro littera S valor quaesitus numeratoris erit

$$F = -\frac{1}{n} v \sin. m(\omega + \Phi) + \frac{1}{n} \sin. [m(\omega + \Phi) - \omega].$$

§. 11. Cum igitur sit $\omega + \Phi = \frac{i\pi}{n}$, ponamus breuitatis gratia angulum $m(\omega + \Phi) = \frac{mi\pi}{n} = \zeta$, atque pro littera R erit

$$F = -\frac{1}{n}[v \cos. \zeta - \cos. (\zeta - \omega)]$$

at vero pro S erit

$$F = -\frac{1}{n}[v \sin. \zeta - \sin. (\zeta - \omega)],$$

quibus valoribus inuentis pro denominatoris factore $1 - 2v\cos.\omega + vv$ partes, ex quibus litterae P et Q componuntur, per sequentes formulas integrales exprimentur:

$$P = -\frac{1}{n}\int \frac{[v \cos. \zeta - \cos. (\zeta - \omega)]\,\partial v - v\,\partial\Phi\,[v \sin. \zeta - \sin. (\zeta - \omega)]}{1 - 2v\cos.\omega + vv}$$

$$Q = -\frac{1}{n}\int \frac{[v \sin. \zeta - \sin. (\zeta - \omega)]\,\partial v + v\,\partial\Phi\,[v \cos. \zeta - \cos. (\zeta - \omega)]}{1 - 2v\cos.\omega + vv}.$$

§. 12. Quoniam hae formulae prorsus conueniunt cum iis, quas supra sumus nacti, et ne signa quidem sunt immutata, peculiari integratione non indigemus, sed pro quantitatibus P et Q sequentes habebimus valores integratos:

$$P = -\frac{\cos. \zeta}{n} l \sqrt{(1 - 2v\cos.\omega + vv)} + \frac{\sin. \zeta}{n} \text{A tang.} \frac{v \sin. \omega}{1 - v\cos.\omega} \text{ et}$$

$$Q = -\frac{\sin. \zeta}{n} l \sqrt{(1 - 2v\cos.\omega + vv)} - \frac{\cos. \zeta}{n} \text{A tang.} \frac{v \sin. \omega}{1 - v\cos.\omega}.$$

Tales scilicet formulae ex singulis factoribus denominatoris formae $1 - 2v\cos.\omega + vv$ deriuari et in vnam summam colligi debent, vt veri valores pro P et Q obtineantur, vbi tantum recordari oportet esse $\omega = \frac{i\pi}{n} - \Phi$ et $\zeta = \frac{mi\pi}{n}$; pro i autem hic numeros pares accipi oportet.

Exemplum 1.

§. 13. *Sit* $m = 1$ *et* $n = 1$, *ita vt quaeri debeat* $\int \frac{\partial z}{1-z} = P + Q\sqrt{-1}$, *posito scilicet* $z = v(\text{cos.}\,\Phi + \sqrt{-1}\,\text{sin.}\,\Phi)$.

Quia hic $n = 1$, vnicus valor pro ω locum habet, resultans ex $i = 0$, eritque ergo $\omega = -\Phi$ et $\zeta = 0$, vnde statim colligimus

$$P = -l\sqrt{(1 - 2v\,\text{cos.}\,\Phi + vv)} \text{ et } Q = -\text{A tang.}\,\frac{v\,\text{sin.}\,\Phi}{1 - v\,\text{cos.}\,\Phi}.$$

Exemplum 2.

§. 14. *Sit* $m = 1$ *et* $n = 2$, *ideoque formula integranda* $\int \frac{\partial z}{1-zz} = P + Q\sqrt{-1}$, *posito* $z = v(\text{cos.}\,\Phi + \sqrt{-1}\,\text{sin.}\,\Phi)$.

Quia hic est $n = 2$, pro ω duos habebimus valores ex $i = 0$ et $i = 2$ oriundos, vnde

Si $i = 0$, erit $\omega = -\Phi$ et $\zeta = 0$

Si $i = 2$, erit $\omega = \pi - \Phi$ et $\zeta = \pi$.

Hinc igitur statim colligemus

$$P = \begin{cases} -\frac{1}{2}l\sqrt{(1 - 2v\,\text{cos.}\,\Phi + vv)} + 0 \\ +\frac{1}{2}l\sqrt{(1 + 2v\,\text{cos.}\,\Phi + vv)} + 0. \end{cases}$$

$$Q = \begin{cases} 0 + \frac{1}{2}\,\text{A tang.}\,\frac{v\,\text{sin.}\,\Phi}{1 - v\,\text{cos.}\,\Phi} \\ 0 + \frac{1}{2}\,\text{A tang.}\,\frac{v\,\text{sin.}\,\Phi}{1 + v\,\text{cos.}\,\Phi}. \end{cases}$$

Exemplum 3.

§. 15. *Sit nunc* $m = 2$ *et* $n = 2$, *ideoque formula integranda* $\int \frac{z\partial z}{1-zz} = P + Q\sqrt{-1}$, *posito scilicet* $z = v(\text{cos.}\,\Phi + \sqrt{-1}\,\text{sin.}\,\Phi)$.

Hic ergo primo sumi debet $i = 0$, tum vero $i = 2$, vnde

Si $i = 0$, erit $\omega = -\Phi$ et $\zeta = 0$

Si $i = 2$, erit $\omega = \pi - \Phi$ et $\zeta = 2\pi$

vnde valores pro P et Q eruuntur sequentes

P =

$$P = \begin{cases} -\tfrac{1}{2} l \sqrt{(1 - 2v \operatorname{cof.} \Phi + vv)} - 0 \\ -\tfrac{1}{2} l \sqrt{(1 + 2v \operatorname{cof.} \Phi + vv)} - 0. \end{cases}$$

$$Q = \begin{cases} 0 + \tfrac{1}{2} \text{A tang.} \frac{v \sin. \Phi}{1 - v \operatorname{cof.} \Phi} \\ 0 - \tfrac{1}{2} \text{A tang.} \frac{v \sin. \Phi}{1 - v \operatorname{cof.} \Phi}. \end{cases}$$

Exemplum 4.

§. 16. *Sit* $m = 1$ *et* $n = 3$, *ideoque formula integranda* $\int \frac{\partial z}{1 - z^3} = P + Q\sqrt{-1}$, *posito* $z = v(\operatorname{cof.} \Phi + \sqrt{-1} \sin. \Phi)$.

Hic igitur ternos valores pro angulo ω habebimus, quos sequenti modo repraesentemus:

i	0	2	4
ω	$-\Phi$	$120° - \Phi$	$240° - \Phi$
fin. ω	$-$ fin. Φ	$+$ fin. $(60° + \Phi)$	$-$ fin. $(60° - \Phi)$
cof. ω	$+$ cof. Φ	$-$ cof. $(60° + \Phi)$	$+$ cof. $(60° - \Phi)$
ζ	0	$120°$	$240.$
fin. ζ	0	$+\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$
cof. ζ	$+1$	$-\frac{1}{2}$	$-\frac{1}{2}$

Hinc ergo inueniemus

$$P = \begin{cases} -\tfrac{1}{3} l \sqrt{(1 - 2v \operatorname{cof.} \Phi + vv)} + 0 \\ +\tfrac{1}{6} l \sqrt{[1 + 2v \operatorname{cof.} (60° + \Phi) + vv]} + \frac{1}{2\sqrt{3}} \text{A tang.} \frac{v \sin. (60° + \Phi)}{1 + v \operatorname{cof.} (60° + \Phi)} \\ +\tfrac{1}{6} l \sqrt{[1 + 2v \operatorname{cof.} (60° - \Phi) + vv]} + \frac{1}{2\sqrt{3}} \text{A tang.} \frac{v \sin. (60° - \Phi)}{1 + v \operatorname{cof.} (60° - \Phi)}. \end{cases}$$

$$Q = \begin{cases} 0 \qquad\qquad + \tfrac{1}{3} \text{A tang.} \frac{v \sin. \Phi}{1 - v \operatorname{cof.} \Phi} \\ -\frac{1}{2\sqrt{3}} l \sqrt{[1 + 2v \operatorname{cof.} (60° + \Phi) + vv]} + \tfrac{1}{6} \text{A tang.} \frac{v \sin. (60° + \Phi)}{1 + v \operatorname{cof.} (60° + \Phi)} \\ +\frac{1}{2\sqrt{3}} l \sqrt{[1 + 2v \operatorname{cof.} (60° - \Phi) + vv]} - \tfrac{1}{6} \text{A tang.} \frac{v \sin. (60° - \Phi)}{1 + v \operatorname{cof.} (60° - \Phi)}. \end{cases}$$

Exemplum 5.

§. 17. *Sumatur nunc* $m = 2$, *manente* $n = 3$, *vt formula integranda fit* $\int \frac{z\partial z}{1-z^3} = P + Q\sqrt{-1}$, *posito* $z = v(\cos.\Phi + \sqrt{-1}\sin.\Phi)$.

Hic notetur, valores ipsius ω prorsus eosdem manere vt ante, sicque etiam logarithmi et arcus circulares iidem manebunt; valores autem pro ζ erunt sequentes:

Si $i = 0$, erit $\zeta = 0$, $\sin.\zeta = 0$ et $\cos.\zeta = +1$.
Si $i = 2$, erit $\zeta = \frac{4}{3}\pi$, $\sin.\zeta = -\frac{\sqrt{3}}{2}$ et $\cos.\zeta = -\frac{1}{2}$.
Si $i = 4$, erit $\zeta = \frac{8}{3}\pi$, $\sin.\zeta = +\frac{\sqrt{3}}{2}$ et $\cos.\zeta = -\frac{1}{2}$.

Hinc igitur fiet

$$P = \begin{cases} -\frac{1}{3}l\sqrt{(1-2v\cos.\Phi+vv)} + 0 \\ +\frac{1}{6}l\sqrt{[1-2v\cos.(60^\circ+\Phi)+vv]} - \frac{1}{2\sqrt{3}}\,\text{A tang.}\frac{v\sin.(60^\circ+\Phi)}{1+v\cos.(60^\circ\ \Phi)} \\ +\frac{1}{6}l\sqrt{[1+2v\cos.(60^\circ-\Phi)+vv]} - \frac{1}{2\sqrt{3}}\,\text{A tang.}\frac{v\sin.(60^\circ-\Phi)}{1+v\cos.(60^\circ-\Phi)}. \end{cases}$$

$$Q = \begin{cases} 0 \qquad\qquad + \frac{1}{3}\,\text{A tang.}\frac{v\sin.\Phi}{1-v\cos.\Phi} \\ +\frac{1}{2\sqrt{3}}l\sqrt{[1+2v\cos.(60^\circ+\Phi)+vv]} + \frac{1}{6}\,\text{A tang.}\frac{v\sin.(60^\circ+\Phi)}{1+v\cos.(60^\circ+\Phi)} \\ -\frac{1}{2\sqrt{3}}l\sqrt{[1+2v\cos.(60^\circ-\Phi)+vv]} + \frac{1}{6}\,\text{A tang.}\frac{v\sin.(60^\circ-\Phi)}{1+v\cos.(60^\circ-\Phi)}. \end{cases}$$

Exemplum 6.

§. 18. *Sit nunc* $m = 1$ *et* $n = 4$, *vt formula integranda fiat* $\int \frac{\partial z}{1-z^4} = P + Q\sqrt{-1}$, *posito* $z = v(\cos.\Phi + \sqrt{-1}\sin.\Phi)$.

Quia hic $n = 4$, pro angulis ω et ζ quaternos valores adipiscimur, scilicet

i	0	2	4	6
ω	$-\Phi$	$\frac{1}{2}\pi-\Phi$	$\pi-\Phi$	$\frac{3}{2}\pi-\Phi$
fin. ω	$-$ fin. Φ	$+$ cof. Φ	$+$ fin. Φ	$-$ cof. Φ
cof. ω	$+$ cof. Φ	$+$ fin. Φ	$-$ cof. Φ	$-$ fin. Φ
ζ	0	90°	180°	270°
fin. ζ	0	$+1$	0	-1
cof. ζ	$+1$	0	-1	0

Hinc jam litterae P et Q fequenti modo exprimentur:

$$P=\begin{cases}-\frac{1}{4}l\sqrt{(1-2v\,\mathrm{cof.}\Phi+vv)}+0\\ 0\qquad\qquad\qquad\qquad+\frac{1}{4}\,\mathrm{A\,tang.}\frac{v\,\mathrm{cof.}\Phi}{1-v\,\mathrm{fin.}\Phi}\\ +\frac{1}{4}l\sqrt{(1+2v\,\mathrm{cof.}\Phi+vv)}+0\\ 0\qquad\qquad\qquad\qquad+\frac{1}{4}\,\mathrm{A\,tang.}\frac{v\,\mathrm{cof.}\Phi}{1+v\,\mathrm{fin.}\Phi}\end{cases}$$

$$Q=\begin{cases}0\qquad\qquad\qquad\qquad+\frac{1}{4}\,\mathrm{A\,tang.}\frac{v\,\mathrm{fin.}\Phi}{1-v\,\mathrm{cof.}\Phi}\\ -\frac{1}{4}l\sqrt{(1-2v\,\mathrm{fin.}\Phi+vv)}+0\\ 0\qquad\qquad\qquad\qquad+\frac{1}{4}\,\mathrm{A\,tang.}\frac{v\,\mathrm{fin.}\Phi}{1+v\,\mathrm{cof.}\Phi}\\ +\frac{1}{4}l\sqrt{(1+2v\,\mathrm{fin.}\Phi+vv)}+0.\end{cases}$$

§. 19. Super hoc exemplo notaffe juuabit effe

$$\int\frac{\partial z}{1-z^4}=\frac{1}{2}\int\frac{\partial z}{1-zz}+\frac{1}{2}\int\frac{\partial z}{1+zz}.$$

Modo autem vidimus pro formula $\int\frac{\partial z}{1-zz}$ effe

$$P=-\tfrac{1}{2}l\sqrt{(1-2v\,\mathrm{cof.}\Phi+vv)}+\tfrac{1}{2}l\sqrt{(1+2v\,\mathrm{cof.}\Phi+vv)}\ \text{et}$$
$$Q=+\tfrac{1}{2}\,\mathrm{A\,tang.}\frac{v\,\mathrm{fin.}\Phi}{1-v\,\mathrm{cof.}\Phi}\qquad+\tfrac{1}{2}\,\mathrm{A\,tang.}\frac{v\,\mathrm{fin.}\Phi}{1+v\,\mathrm{cof.}\Phi}.$$

Pro altera vero formula $\int\frac{\partial z}{1+zz}$ in fuperiore differtatione §. 30. et feqq. inuenimus

$$P=\tfrac{1}{2}\,\mathrm{A\,tang.}\frac{2v\,\mathrm{cof.}\Phi}{1-vv}\ \text{et}\ Q=\tfrac{1}{4}l\frac{1+2v\,\mathrm{fin.}\Phi+vv}{1-2v\,\mathrm{fin.}\Phi+vv}$$

quos autem valores ob arcum circuli hic contractum potius ex formulis problematis generalis §. 54. et feqq. deriuemus.

Erit

Erit enim, pofito ibi $m = 1$, $n = 2$, pro forma integrali $\int \frac{\partial z}{1 + zz}$ valor

$$P = \tfrac{1}{2} \text{A tang.} \frac{v \, cof. \Phi}{1 - v \, fin. \Phi} + \tfrac{1}{2} \text{A tang.} \frac{v \, cof. \Phi}{1 + v \, fin. \Phi}$$

$$Q = -\tfrac{1}{2} l \sqrt{(1 + 2 v \, \text{fin.} \Phi + vv)} - \tfrac{1}{2} l \sqrt{(1 - 2 v \, \text{fin.} \Phi + vv)}.$$

Additis ergo binis P et Q per binarium diuifis prodit pro forma integrali $\int \frac{\partial z}{1 - z^4}$ valor

$$P = \begin{cases} +\tfrac{1}{4} l \sqrt{(1 + 2 v \, \text{cof.} \Phi + vv)} + \tfrac{1}{4} \text{A tang.} \frac{v \, cof. \Phi}{1 - v \, fin. \Phi} \\ -\tfrac{1}{4} l \sqrt{(1 - 2 v \, \text{cof.} \Phi + vv)} + \tfrac{1}{4} \text{A tang.} \frac{v \, cof. \Phi}{1 + v \, fin. \Phi} \end{cases}$$

$$Q = \begin{cases} +\tfrac{1}{4} l \sqrt{(1 + 2 v \, \text{fin.} \Phi + vv)} + \tfrac{1}{4} \text{A tang.} \frac{v \, fin. \Phi}{1 - v \, cof. \Phi} \\ -\tfrac{1}{4} l \sqrt{(1 - 2 v \, \text{fin.} \Phi + vv)} + \tfrac{1}{4} \text{A tang.} \frac{v \, fin. \Phi}{1 + v \, cof. \Phi} \end{cases}$$

prorfus vti fupra inuenimus.

§. 20. Quanquam haec folutio fatis eft commoda et fine multis ambagibus ad optatum finem perducit, tamen aliam hic fubjungam, quae quidem multo fimplicior et breuior, ita tamen eft comparata, vt ejus bonitas nequidem perfpici queat, atque eatenus tantum admitti poffit, quatenus ad veritatem jam aliunde cognitam perducit. In eo autem ifta folutio a praecedente folutione recedit, quod primo denominatorem $1 - z^n$ ab imaginariis liberare non eft opus; deinde etiam numerator ita tractari poteft, vt quantitas v inde penitus elidatur, neque permixtio quantitatum realium et imaginariarum vllam moram faceffat.

Alia folutio Problematis.

§. 21. Cum pofito $z = v (\text{cof.} \Phi + \sqrt{-1} \, \text{fin.} \Phi)$ effe debeat

$$\int \frac{z^{m-1} \partial z}{1 - z^n} = P + Q \sqrt{-1},$$

ftatim

statim considero denominatoris factorem $1 - 2v \cos. \omega + vv$, quo ergo posito $= 0$ etiam ipse denominator euanescere debet; inde autem fit $v = \cos. \omega + \sqrt{-1} \sin. \omega$, et cum sit

$$z = v (\cos. \Phi + \sqrt{-1} \sin. \Phi), \text{ erit}$$

$$z^n = v^n (\cos. n\Phi + \sqrt{-1} \sin. n\Phi).$$

Quare cum sit $v^n = \cos. n\omega + \sqrt{-1} \sin. n\omega$, hinc fiet

$$z^n = \cos. (n\omega + n\Phi) + \sqrt{-1} \sin. (n\omega + n\Phi),$$

quae expressio cum vnitati debeat esse aequalis, erit $\cos. (n\omega + n\Phi) = 1$, vnde fit $n\omega + n\Phi = i\pi$, denotante i numerum parem quemqunque, sicque altera pars $\sqrt{-1}\sin.(n\omega + n\Phi)$ sponte euanescit. Cum igitur hinc fit $n\omega = i\pi - n\Phi$, erit $\omega = \frac{i\pi}{n} - \Phi$, vnde n diuersi valores pro ω eliciuntur.

§. 22. Statuamus nunc fractionem partialem ex hoc factore oriundam esse $= \frac{F}{1 - 2v \cos. \omega + vv}$, atque vt supra vidimus, statui debet

$$F = z^{m-1} \partial z \cdot \frac{1 - 2v \cos. \omega + vv}{1 - z^n},$$

vnde ope aquationis $vv - 2v \cos. \omega + 1 = 0$ iste valor F penitus a litteris z et v debet liberari. Quoniam autem hinc fractionis illius tam numerator quam denominator euanescit, sumtis differentialibus, ob $\partial . z^n = nz^n \frac{\partial z}{z} = nz^n \frac{\partial v}{v}$, quandoquidem in hac reductione anguli ω et Φ vt constantes spectari possunt, illa fractio induet hanc formam: $\frac{2(v - \cos. \omega)v}{nz^n}$. Quoniam igitur $v - \cos. \omega = \sqrt{-1} \sin. \omega$ et $z^n = 1$, erit ista fractio $= -\frac{2v\sqrt{-1} \sin. \omega}{n}$, sicque habebimus

$$F = -\frac{2v}{n} z^{m-1} \partial z \sqrt{-1} \sin. \omega.$$

§. 23. Cum nunc, sumto etiam angulo Φ variabili, fit $\frac{\partial z}{z} = \frac{\partial v}{v} + \partial \Phi \sqrt{-1}$, ideoque

$$\frac{2 v \sqrt{-1}}{n} \cdot \frac{\partial z}{z} = \frac{2}{n} \partial v \sqrt{-1} - 2 v \frac{\partial \Phi}{n},$$

habebimus

$$F = -\frac{2}{n} z^m \partial v \sqrt{-1} \text{ fin. } \omega + \frac{2}{n} v z^m \partial \Phi \text{ fin. } \omega, \text{ fiue}$$

$$F = \frac{2}{n} z^m \text{ fin. } \omega (v \partial \Phi - \partial v \sqrt{-1}).$$

Nunc vero, vti ante euoluimus poteſtatem z^n, hic ſimili modo euoluamus poteſtatem z^m, eritque

$$z^m = \text{cof.} (m \omega + m \Phi) + \sqrt{-1} \text{ fin.} (m \omega + m \Phi),$$

quo valore introducto fiet

$$F = \frac{2}{n} \text{ fin. } \omega (v \partial \Phi - \partial v \sqrt{-1}) [\text{cof.} (m \omega + m \Phi) + \sqrt{-1} \text{ fin.} (m \omega + m \Phi)].$$

Cum denique fit $\omega = \frac{i \pi}{n} - \Phi$, erit $m \omega + m \Phi = \frac{m i \pi}{n}$, quem ergo angulum ſi vocemus $= \zeta$, valor litterae F quaeſitus erit

$$F = \frac{2}{n} \text{ fin. } \omega (v \partial \Phi - \partial v \sqrt{-1}) (\text{cof. } \zeta + \sqrt{-1} \text{ fin. } \zeta),$$

quem partiamur in has partes:

$$F = + \frac{2}{n} \partial v \text{ fin. } \omega (\text{fin. } \zeta - \sqrt{-1} \text{ cof. } \zeta)$$
$$+ \frac{2}{n} v \partial \Phi \text{ fin. } \omega (\text{cof. } \zeta + \sqrt{-1} \text{ fin. } \zeta).$$

§. 24. Quia haec expreſſio ex partibus realibus et imaginariis conſtat, videri poſſet partes reales ſumi debere pro valore litterae P, imaginarias pro $Q \sqrt{-1}$; verum hinc in craſſiſſimum errorem illaberemur, quemadmodum ex collatione cum ſuperiore ſolutione manifeſtum eſt. Interim tamen obſeruaui, ex hac ipſa formula veros valores pro P et Q elici poſſe. Scilicet pro valore ipſius P inueniendo haec tota formula ex realibus et imaginariis permixta in valorem realem transformetur; tum enim eius ſemiſſis pro littera P valebit. Simili modo pro littera Q eandem expreſſionem totam in formam ſimplicipliciter imaginariam transfundi oportet, cuius pariter ſemiſſis pro valore litterae Q adhiberi debebit; ſcilicet cum valor ipſius F coëfficientem habeat 2, ex altera ſemiſſi littera P, ex altera vero littera Q formari debet.

§. 25. Hinc ergo omiſſo factore formulam pro F inventam primo ad litteram P accomodemus, qui valor cum debeat eſſe realis, ſtatuatur $= A v + B$, et loco v valorem $\text{cof.}\,\omega + \sqrt{-1}\,\text{ſin.}\,\omega$ ſubſtituendo habebimus hanc aequationem:

$$\left\{\begin{array}{l} +\tfrac{1}{n}\partial v\,\text{ſin.}\,\omega(\text{ſin.}\,\zeta - \sqrt{-1}\,\text{cof.}\,\zeta) \\ +\tfrac{1}{n}v\,\partial\Phi\,\text{ſin.}\,\omega(\text{cof.}\,\zeta + \sqrt{-1}\,\text{ſin.}\,\zeta) \end{array}\right\} = A\,\text{cof.}\,\omega + B + A\sqrt{-1}\,\text{ſin.}\,\omega.$$

Hinc iam partibus realibus et imaginariis ſeorſim aequatis primo ex imaginariis elicitur:

$$A\,\text{ſin.}\,\omega = \tfrac{1}{n}\,\text{ſin.}\,\omega\,(-\partial v\,\text{cof.}\,\zeta + v\,\partial\Phi\,\text{ſin.}\,\zeta),$$

vnde fit

$$A = -\tfrac{1}{n}(\partial v\,\text{cof.}\,\zeta - v\,\partial\Phi\,\text{ſin.}\,\zeta).$$

Hic iam valor in aequalitate partium realium ſubſtitutus dabit

$$\tfrac{1}{n}\text{ſin.}\,\omega(\partial v\,\text{ſin.}\,\zeta + v\,\partial\Phi\,\text{cof.}\,\zeta) = -\tfrac{\text{cof.}\,\omega}{n}(\partial v\,\text{cof.}\,\zeta - v\,\partial\Phi\,\text{ſin.}\,\zeta) + B$$

vnde colligitur

$$B = \tfrac{1}{n}\partial v\,\text{cof.}\,(\zeta - \omega) - \tfrac{1}{n}v\,\partial\Phi\,\text{ſin.}\,(\zeta - \omega).$$

Hinc ergo pro littera P erit

$$F = -\tfrac{1}{n}\partial v\,[v\,\text{cof.}\,\zeta - \text{cof.}\,(\zeta - \omega)]$$
$$+ \tfrac{1}{n}v\,\partial\Phi\,[v\,\text{ſin.}\,\zeta - \text{ſin.}\,(\zeta - \omega)],$$

ſicque ex factore denominatoris $1 - 2v\,\text{cof.}\,\omega + vv$ habebimus

$$P = -\tfrac{1}{n}\int\frac{\partial v\,[v\,\text{cof.}\,\zeta - \text{cof.}\,(\zeta-\omega)] - v\,\partial\Phi\,[v\,\text{ſin.}\,\zeta - \text{ſin.}\,(\zeta-\omega)]}{1 - 2v\,\text{cof.}\,\omega + vv}.$$

§. 26. Pro littera Q altera ſemiſſis litterae F aequetur huic quantitati ſimpliciter imaginariae: $(Cv + D)\sqrt{-1}$, vnde exorietur iſta aequatio:

$$\left.\begin{array}{l}+\frac{1}{n}\partial v\,\text{sin.}\,\omega(\text{sin.}\,\zeta-\sqrt{-1}\,\text{cos.}\,\zeta)\\+\frac{1}{n}v\,\partial\Phi\,\text{sin.}\,\omega\,(\text{cos.}\,\zeta+\sqrt{-1}\,\text{sin.}\,\zeta)\end{array}\right\}=C\,\text{cos.}\,\omega\sqrt{-1}+D\sqrt{-1}-C\,\text{sin.}\,\omega.$$

Hinc ex partibus realibus concluditur

$$C=-\tfrac{1}{n}(\partial v\,\text{sin.}\,\zeta+v\,\partial\Phi\,\text{cos.}\,\zeta),$$

quo valore substituto ex partibus imaginariis haec emerget aequatio:

$$-\tfrac{1}{n}\text{sin.}\,\omega(\partial v\,\text{cos.}\,\zeta-v\,\partial\Phi\,\text{sin.}\,\zeta)=-\tfrac{\text{cos.}\,\omega}{n}(\partial v\,\text{sin.}\,\zeta+v\,\partial\Phi\,\text{cos.}\,\zeta)+D,$$

vnde eruitur

$$D=\tfrac{1}{n}\partial v\,\text{sin.}\,(\zeta-\omega)+\tfrac{1}{n}v\,\partial\Phi\,\text{cos.}\,(\zeta-\omega).$$

Hinc ergo pro littera Q habemus:

$$F=-\tfrac{1}{n}\partial v\,[v\,\text{sin.}\,\zeta-\text{sin.}\,(\zeta-\omega)]$$
$$-\tfrac{1}{n}v\,\partial\Phi\,[v\,\text{cos.}\,\zeta-\text{cos.}\,(\zeta-\omega)],$$

vnde valor ipsius Q ex factore $1-2v\,\text{cos.}\,\omega+vv$ oriundus erit:

$$Q=-\tfrac{1}{n}\int\frac{\partial v\,[v\,\text{sin.}\,\zeta-\text{sin.}\,(\zeta-\omega)]+v\,\partial\Phi\,[v\,\text{cos.}\,\zeta-\text{cos.}\,(\zeta-\omega)]}{1-2v\,\text{cos.}\,\omega+vv}.$$

§. 27. Quoniam haec solutio tam egregie cum praecedente conuenit, id profecto casui fortuito tribui nequit; quam ob rem mihi quidem haec solutio prorsus singularis haud parum in recessu habere videtur, vnde eam Geometris perscrutandam proponere non dubito, vt eius soliditatem ex firmis principiis deriuare conentur.

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