DE TERMINO GENERALI SERIERVM HYPERGEOMETRICARVM.

Auctore

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§. 1.

Series hypergeometricas hic cum Wallisio appello, quarum termini secundum factores continuo multiplicatos procedunt, cudum ipsi factores progressionem arithmeticam constituunt, cuiusmodi est notissima series: 1, 2, 6, 24, 120, 720 etc. cuius terminus indici n respondens est $1. 2. 3. 4 \ldots\ldots n$. Generaliter igitur ex progressione arithmetica quacunque formabitur talis series hypergeometrica:

$$a, a(a+b), a(a+b)(a+2b), a(a+b)(a+2b)(a+3b), \text{ etc.}$$

cuius ergo terminus indici n respondens, qui constat n factoribus, erit

$$a(a+b)(a+2b)(a+3b)\ldots\ldots[a+(n-1)b];$$

vnde quoties n fuerit numerus integer positiuus, terminus ipsi respondens per huiusmodi factores facillime assignatur, ita vt, si index n exprimat abscissam cuiuspiam lineae curuae, cuius applicatae per terminos ipsius seriei exprimantur; nullum plane est dubium, quin etiam abscissis per numeros fractos vel etiam surdos

ſurdos expreſſis determinatae reſpondeant applicatae, quarum autem quantitatem hinc neutiquam definire licet, ſed ad hoc requiritur eiusmodi expreſſio ex quantitatibus a, b et n formata, quae ſemper valorem determinatum exhibeat, ſiue index n fuerit numerus integer, ſiue fractus, ſiue adeo ſurdus.

§. 2. Huiusmodi ſeries hypergeometricas iam ſaepius ſum perſerutatus, vbi potiſſimum ex doctrina interpolationum ſeriei Wallisianae 1, 2, 6, 24, 120 etc. terminum indici indefinito n reſpondentem ita exprimi poſſe inueni, vt ſit

$$= \frac{1^{1-n} \cdot 2^n}{1+n} \cdot \frac{2^{1-n} \cdot 3^n}{2+n} \cdot \frac{3^{1-n} \cdot 4^n}{3+n} \cdot \frac{4^{1-n} \cdot 5^n}{4+n} \cdot \frac{5^{1-n} \cdot 6^n}{5+n} \text{ etc.}$$

quae quidem expreſſio in infinitum excurrit, verum tamen ſemper valorem determinatum exprimit, quicunque valor indici n tribuatur. Simili modo pro ſerie generali ſupra allata terminum generalem, ſiue indici indefinito n reſpondentem, ſequenti producto in infinitum excurrente repraeſentari oſtendi:

$$\frac{a^n \cdot a^{1-n}(a+b)^n}{a+nb} \cdot \frac{(a+b)^{1-n}(a+2b)^n}{a+(n+1)b} \cdot \frac{(a+2b)^{1-n}(a+3b)^n}{a+(n+2)b} \text{ etc.}$$

Ratiocinia autem, quae me tum temporis ad has formulas perduxerunt, ad theoriam interpolationum erant adſtricta, neque fortaſſe ita enucleata, vt ſatis clare intelligi queant; quamobrem conſtitui iſtam inueſtigationem ex ipſa natura harum ſerierum denuo repetere ac perſpicue explicare.

§. 3. Incipiam igitur ab ipſa ſerie Wallisiana, quandoquidem vim ratiociniorum in caſu ſpeciali multo clarius perſpicere licebit, quam ſi ſtatim ea ad ſeriem generalem accommodare vellem. Cum igitur terminus generalis indici n reſpondens tanquam functio ipſius indicis n ſpectari poſſit, eum more ſatis recepto per $\Delta : n$ exprimam, vbi Δ non quantita-

 tem,

tem, sed characterem functionis denotat. Hinc ergo, quoties n fuerit numerus integer positiuus, erit

$$\Delta : n = 1 . 2 . 3 . 4 \ldots\ldots n;$$

ex quo intelligitur, pro terminis sequentibus fore

$$\Delta : (n + 1) = (n + 1)\, \Delta : n;$$

$$\Delta : (n + 2) = (n + 1)(n + 2)\, \Delta : n;$$

$$\Delta : (n + 3) = (n + 1)(n + 2)(n + 3)\, \Delta : n.$$

Et quoniam in hac continua nouorum factorum accessione ipsa seriei natura contineri est censenda, hae posteriores formulae etiam veritati debent esse consentaneae, quicunque valores indici n tribuantur. Ita cum $\Delta : \frac{1}{2}$ designet terminum indici $\frac{1}{2}$ respondentem, quem per quadraturam circuli exprimi notum est, ex eo sequentes ita assignari poterunt:

$$\Delta : 1 + \tfrac{1}{2} = \tfrac{3}{2}\, \Delta : \tfrac{1}{2};\ \Delta : 2\tfrac{1}{2} = \tfrac{3}{2} \cdot \tfrac{5}{2} \cdot \Delta : \tfrac{1}{2};\ \Delta : 3\tfrac{1}{2} = \tfrac{3}{2} \cdot \tfrac{5}{2} \cdot \tfrac{7}{2} \cdot \Delta : \tfrac{1}{2};\ \text{etc.}$$

Hocque modo res se habebit, quicunque alius numerus pro n accipiatur, quamuis fortasse valorem $\Delta : n$ nullo modo per mensuras cognitas exprimere liceat.

§. 4. Constituto igitur fundamento, cui tota natura harum serierum innititur, aliud principium stabilire necesse est, in hoc consistens, quod huiusmodi series in infinitum continuatae tandem cum progressione geometrica confundantur; propterea quod noui factores vlterius accedentes tanquam inter se aequales spectari possunt. Ita si i denotet numerum infinitum, sicque $\Delta : i$ terminum ab initio infinite remotum designet, termini eum sequentes ita exhiberi poterunt:

$$\Delta : (i + 1) = (i + 1)\, \Delta : i = i\, \Delta : i;$$

$$\Delta : (i + 2) = (i + 1)(i + 2)\, \Delta : i = i\, i\, \Delta : i;$$

$$\Delta : (i + 3) = (i + 1)(i + 2)(i + 3)\, \Delta : i = i^3\, \Delta : i;$$

sicque

ſicque in genere ſtatui poterit $\Delta : (i + n) = i^n \Delta : i$. Haec quidem primo aſpectu paradoxa videri poſſunt, quoniam formula $\Delta : i$ iam habet valorem infinite magnum; verum vbi tantum de ratione inter duas pluresue huiusmodi expreſſiones agitur, vtique loco $i + 1$ et $i + 2$ ſcribere licebit i. Quin etiam vice verſa loco i ſcribere licebit $i + 1$, vel $i + 2$, vel in genere $i + \alpha$, denotante α numerum quemcunque finitum, unde etiam generaliter ſtatui poterit

$$\Delta : (i + n) = (i + \alpha)^n \Delta : i$$

§. 5. Cum igitur in genere ſit

$$\Delta : i = 1 . 2 . 3 . 4 \ldots\ldots i,$$

ſi i denotet numerum infinitum, etiamſi fortaſſe fractum, iſta expreſſio nihilominus tanquam determinata ſpectari poterit; tum autem, vt vidimus, erit

$$\Delta : (i + n) = (i + \alpha)^n \Delta : n.$$

Eodem autem modo a formula $\Delta : n$ pariter in infinitum progrediamur, et cum ſit

$$\Delta : (n + 1) = (n + 1) \Delta : n \text{ et}$$

$$\Delta : (n + 2) = (n + 1)(n + 2) \Delta : n; \text{ erit}$$

$$\Delta : (n + i) = (n + 1)(n + 2)(n + 3) \ldots\ldots (n + i) \Delta : n;$$

vbi factorum, quibus formula $\Delta : n$ multiplicatur, numerus eſt i; at vero expreſſio ſupra pro $\Delta : (i + n)$ data, ſi loco $\Delta : i$ eius valor naturalis ſcribatur, erit

$$\Delta : (i + n) = 1 . 2 . 3 . 4 \ldots\ldots i (i + \alpha)^n$$

vbi factorum, quibus formula $(i + \alpha)^n$ multiplicatur, numerus pariter eſt $= i$.

§. 6. Cum igitur manifeſto debeat eſſe $\Delta : (n + i) = \Delta : (i + n)$, ſi altera formularum inuentarum per alteram diuidatur, et fractiones, quia numerus factorum vtrinque eſt

 idem,

idem, feorfim exprimantur, quotus vtique unitati aequalis prodire debet, fcilicet erit

$$1 = \frac{1}{n+1} \cdot \frac{2}{n+2} \cdot \frac{3}{n+3} \cdot \frac{4}{n+4} \cdots \frac{i}{n+i} \cdot \frac{(i+\alpha)^n}{\Delta : n}.$$

Ex hac igitur aequatione deriuare licet verum valorem formulae $\Delta : n$, qui adeo femper fubfiftere debet, fiue index n fit numerus integer, fiue fecus; habebimus fcilicet

$$\Delta : n = \frac{1}{n+1} \cdot \frac{2}{n+2} \cdot \frac{3}{n+3} \cdot \frac{4}{n+4} \cdots \frac{i}{n+i} \cdot (i+\alpha)^n.$$

quae expreffio, quomodo noftrum negotium conficiat, in aliquot cafibus, vbi loco n numeros integros, faltem minores, affumimus, oftendiffe iuuabit.

I. Sit igitur $n = 1$, eritque hinc

$$\Delta : 1 = \tfrac{1}{2} \cdot \tfrac{2}{3} \cdot \tfrac{3}{4} \cdot \tfrac{4}{5} \cdots \tfrac{i}{1+i} (i+\alpha);$$

vbi deletis factoribus fe deftruentibus prodibit $\Delta : 1 = \frac{1}{1+i}(i+\alpha)$; vbi manifefto eft $\frac{i+\alpha}{1+i} = 1$, ob i numerum infinitum, quicunque etiam valor pro α affumatur.

II. Sit $n = 2$, et noftra expreffio dabit

$$\Delta : 2 = \tfrac{1}{3} \cdot \tfrac{2}{4} \cdot \tfrac{3}{5} \cdot \tfrac{4}{6} \cdot \tfrac{5}{7} \cdot \tfrac{6}{8} \cdots \tfrac{i}{2+i} (i+\alpha)^2;$$

vbi deletis terminis fe deftruentibus, in numeratore tantum duo primi, in denominatore autem duo vltimi relinquuntur, ita vt prodeat $\Delta : 2 = \frac{1.2(i+\alpha)^2}{(1+i)(2+i)}$; vbi manifefto fit $\frac{(i+\alpha)^2}{(1+i)(2+i)} = 1$, ita vt fit $\Delta : 2 = 1.2$.

III. Sit $n = 3$, et noftra expreffio dabit

$$\Delta : 3 = \tfrac{1}{4} \cdot \tfrac{2}{5} \cdot \tfrac{3}{6} \cdot \tfrac{4}{7} \cdot \tfrac{5}{8} \cdot \tfrac{6}{9} \cdot \tfrac{7}{10} \cdots \tfrac{i}{3+i} (i+\alpha)^3;$$

vbi in numeratore tantum tres priores, in denominatore vero tantum tres pofteriores factores relinquuntur, ita vt fit

$$\Delta : 3 = \frac{1.2.3(i+\alpha)^3}{(1+i)(2+i)(3+i)}, \text{ ideoque } \Delta : 3 = 1.2.3.$$

Hoc

Hoc igitur modo veritas expreſſionis demonſtrari poteſt pro omnibus numeris integris loco n aſſumtis, hincque ſimul ratio intelligitur, cur numerus ille arbitrarius α tuto in calculum introduci potuerit, quia hic tantum ratio inter bina infinita in computum ingreditur.

§. 7. Verum ſi loco n numeros non integros aſſumeremus, ex hac forma nihil plane pro valore $\Delta : n$ cognoſci poſſet, quia tam in numeratore quam denominatore innumerabiles factores relinquerentur, inter quos adeo innumeri forent ipſi infiniti. Quo igitur huic incommodo occurramus, quanquam i in ſe denotat numerum infinitum, nihilominus eius loco ſucceſſiue numeros naturales 1, 2, 3, 4 etc. ſcribamus, et nanciſcemur ſequentes formulas:

$$\text{I.}\quad \Delta : n = \frac{1}{n+1}(1+\alpha)^n.$$

$$\text{II.}\quad \Delta : n = \frac{1}{n+1}\cdot\frac{2}{n+2}(2+\alpha)^n.$$

$$\text{III.}\quad \Delta : n = \frac{1}{n+1}\cdot\frac{2}{n+2}\cdot\frac{3}{n+3}(3+\alpha)^n.$$

$$\text{IV.}\quad \Delta : n = \frac{1}{n+1}\cdot\frac{2}{n+2}\cdot\frac{3}{n+3}\cdot\frac{4}{n+4}(4+\alpha)^n.$$

etc. etc.

vbi euidens eſt has formulas continuo propius ad veritatem accedere debere, quo ulterius continuentur, quandoquidem iis in infinitum continuatis tandem ad verum valorem ipſius $\Delta : n$ perueniri oportet.

§. 8. Quia quaelibet harum formularum antecedentem inuoluit, ſiue totam, ſiue ex parte, diuidamus quamlibet per ſuam praecedentem ac obtinebimus

$$\frac{\text{II}}{\text{I}}\cdot\frac{2}{n+2}\cdot\frac{(2+\alpha)^n}{(1+\alpha)^n}$$

$$\frac{\text{III}}{\text{II}} \cdot \frac{3}{n+3} \cdot \frac{(3+\alpha)^n}{(2+\alpha)^n}$$

$$\frac{\text{IV}}{\text{III}} \cdot \frac{4}{n+4} \cdot \frac{(4+\alpha)^n}{(3+\alpha)^n}$$

etc.

Hoc igitur modo valores praecedentes in ſequentes inuoluamus, et cum ſit I.

$$\Delta : n = \frac{1}{n+1}(1+\alpha)^n,$$

pro numero II habebimus

$$\Delta : n = \frac{1}{n+1}(1+\alpha)^n \frac{2}{n+2} \cdot \frac{(2+\alpha)^n}{(1+\alpha)^n}.$$

Ex hoc porro pro numero III prodit

$$\Delta : n = \frac{1}{n+1}(1+\alpha)^n \frac{2}{n+2} \cdot \frac{(2+\alpha)^n}{(1+\alpha)^n} \cdot \frac{3}{n+3} \cdot \frac{(3+\alpha)^n}{(2+\alpha)^n}.$$

Ex hoc pari modo pro numero IV prodibit

$$\Delta : n = \frac{1}{n+1}(1+\alpha)^n \frac{2}{n+2} \cdot \frac{(2+\alpha)^n}{(1+\alpha)^n} \cdot \frac{3}{n+3} \cdot \frac{(3+\alpha)^n}{(2+\alpha)^n} \cdot \frac{4}{n+4} \cdot \frac{(4+\alpha)^n}{(3+\alpha)^n}.$$

§. 9. Quodſi ergo has expreſſiones in infinitum continuemus, tandem ipſam veritatem aſſequemur; verum quia ratio primi membri a ſequentibus recedit, id facile ad vniformitatem perducere licebit, dum ſcilicet per α^n diuiditur, tum vero tota expreſſio per α^n multiplicatur; ſicque perueniemus ad ſequens productum in infinitum excurrens, quo verus valor functionis $\Delta : n$ exprimetur, erit nimirum

$$\Delta : n = \alpha^n \frac{1}{n+1}\left(\frac{1+\alpha}{\alpha}\right)^n \frac{2}{n+2}\left(\frac{2+\alpha}{1+\alpha}\right)^n \frac{3}{n+3}\left(\frac{3+\alpha}{2+\alpha}\right)^n \frac{4}{n+4}\left(\frac{4+\alpha}{3+\alpha}\right)^n \text{etc.}$$

quae

quae ergo expreſſio manifeſto determinatum valorem indicat, quicunque numerus, ſiue integer, ſiue fractus, pro n accipiatur, propterea quod hi factores continuo propius ad unitatem accedunt, id quod clariſſime patebit ex forma factoris infiniteſimi, quae eſt

$$\frac{i}{n+i} \cdot \left(\frac{i+\alpha}{i-1+\alpha}\right)^n$$

cuius valor, ob $i = \infty$, manifeſto eſt $= 1$, quandoquidem prae i abiiciuntur adiecta n, α et $\alpha - 1$.

§. 10. Haec forma iam egregie conuenit cum ea, quam initio commemorauimus, quippe quae, ſi poteſtates exponentis n coniungantur, reducitur ad hanc formam:

$$\Delta : n = \frac{1}{n+1}\left(\frac{2}{1}\right)^n \cdot \frac{2}{n+2}\left(\frac{3}{2}\right)^n \cdot \frac{3}{n+3}\left(\frac{4}{3}\right)^n \cdot \text{ etc.}$$

ad quam modo inuenta redigitur, ſumendo $\alpha = 1$. Ex quo intelligitur, formulam, quam nunc inuenimus, multo generaliorem eſſe, dum loco α alios quoſuis numeros accipere licet. Interim tamen nullum eſt dubium, quin ambae formulae pro omnibus valoribus ipſius n eosdem valores exhibeant. Id ſaltem ex ſupra factis euolutionibus, pro numeris integris, ſatis eſt euictum, dum verbi gratia prodiit $\Delta : 3 = 1 . 2 . 3$, quicquid etiam pro α acciperetur.

§. 11. Quod autem quantitas litterae α plane non afficiat valorem ipſius $\Delta : n$, id primo inde intelligi poteſt, quod omnes poteſtates ipſius α, vsque ad infiniteſimam, ſe mutuo tollunt; tum vero etiam hoc modo oſtendi poteſt, ſi loco α ſcribatur alius quicunque valor β, tum pariter fore

$$\Delta : n = \beta^n \frac{1}{n+1}\left(\frac{1+\beta}{\beta}\right)^n \cdot \frac{2}{n+2}\left(\frac{2+\beta}{1+\beta}\right)^n \cdot \frac{3}{n+3}\left(\frac{3+\beta}{2+\beta}\right)^n \text{etc.}$$

 Quo-

Quotus ex diuifione illius per hanc ortus prodit

$$1 = \left(\frac{\alpha}{\beta}\right)^n \left[\frac{(\alpha + 1)\,\beta}{\alpha\,(\beta + 1)}\right]^n \cdot \left[\frac{(\alpha + 2)\,(\beta + 1)}{(\alpha + 1)\,(\beta + 2)}\right]^n \text{ etc.}$$

vnde fi radix poteftatis n extrahatur, prodit

$$1 = \frac{\alpha}{\beta} \cdot \frac{(\alpha + 1)\,\beta}{\alpha\,(\beta + 1)} \cdot \frac{(\alpha + 2)(\beta + 1)}{(\alpha + 1)(\beta + 2)} \cdot \frac{(\alpha + 3)(\beta + 2)}{(\alpha + 2)(\beta + 3)} \text{ etc.}$$

quae fractio in infinitum continuata manifefto reducitur ad $\frac{\alpha + i}{\beta + i}$, cuius valor, ob i infinitum, vtique eft unitas; ficque demonftratum eft hos ambos valores ipfius $\Delta : n$ effe inter fe aequales. Eadem autem demonftratio quoque valebit pro fequenti euolutione, vbi omnes feries hypergeometricas in genere fumus contemplaturi.

Explicatio Analyfeos pro ferie hypergeometrica generali.

§. 13. Confideremus nunc fimili modo hanc feriem hypergeometricam generalem:

$$a,\ a\,(a + b),\ a\,(a + b)\,(a + 2\,b),\ a\,(a + b)\,(a + 2\,b)\,(a + 3\,b), \text{ etc.}$$

cuius terminus indici n respondens, qui fcilicet componitur ex n factoribus, ponatur

$$a\,(a + b)\,(a + 2\,b)\,(a + 3\,b) \ldots . [a + (n - 1)\,b] = \Delta : n,$$

quandoquidem, ob litteras a et b conftantes, fpectari poterit tanquam functio quantitatis variabilis n. Hinc igitur ex indole formationis erit

$$\Delta : (n + 1) = \Delta : n\,(a + n\,b);$$

$$\Delta : (n + 2) = \Delta : n\,(a + n\,b)\,[a + (n + 1)\,b];$$

$$\Delta : (n + 3) = \Delta : n\,(a + n\,b)\,[a + (n + 1)\,b]\,[a + (n + 2)\,b];$$

atque adeo, fi i denotet numerum infinitum, erit

$$\Delta : (n + i) = \Delta : n\,(a + n\,b)\,[a + (n + 1)\,b] \ldots . [a + (n + i - 1)\,b]$$

§. 14. Ex indole autem ipfius formae eft

$\Delta : i$

$$\Delta : i = a\,(a+b)\,(a+2\,b)\,\ldots\ldots\,[a+(i-1)\,b],$$

vnde pro ſequentibus fiet

$$\Delta : (i+1) = \Delta : i\,(a+i\,b)$$
$$\Delta : (i+2) = \Delta : i\,(a+i\,b)\,[a+(i+1)\,b]$$
$$\Delta : (i+3) = \Delta : i\,(a+i\,b)\,[a+(i+1)\,b]\,[a+(i+2)\,b]$$
$$\text{etc.} \qquad\qquad \text{etc.}$$

vbi factores de nouo accedentes tanquam aequales inter ſe ſpectari poſſunt, ita vt ſit

$$\Delta : (i+3) = \Delta : i\,(a+i\,b)^3,$$

vnde pro numero indefinito n habebimus

$$\Delta : (i+n) = \Delta : i\,(a+i\,b)^n$$

atque adeo, quoniam aequo iure loco $a+i\,b$ ſumere potuiſſemus $a+b+i\,b$, vel etiam $a+2\,b+i\,b$, generaliter ſtatuere poterimus

$$\Delta : (i+n) = \Delta : i\,(\alpha+i\,b)^n$$

denotante α quantitatem quamcunque finitam, prae $i\,b$ euaneſcentem.

§. 15. Quoniam igitur duplicem nacti ſumus expreſſionem pro eadem functione $\Delta : (i+n)$, comparatione inſtituta inde eliciemus

$$\Delta : n = \frac{\Delta : i\,(\alpha+i\,b)^n}{(a+n\,b)\,[a+(n+1)\,b]\,\ldots\ldots\,[a+(n+i-1)\,b]}$$

vbi in denominatore habebimus i factores. Reſtituamus igitur loco $\Delta : i$ ipſum productum, quod pariter conſtat ex i factoribus, unde reſultabit ſequens valor:

$$\Delta : n = \frac{a}{a+n\,b}\cdot\frac{a+b}{a+(n+1)\,b}\cdot\frac{a+2\,b}{a+(n+2)\,b}\cdots\cdots\frac{a+(i-1)\,b}{a+(n+i-1)\,b}\,(\alpha+i\,b)^n;$$

hicque eſt verus valor ipſius $\Delta : n$, dummodo pro i accipiatur numerus infinite magnus, nullo habito reſpectu ad indicem n, ſiue is ſit numerus integer, ſiue fractus, ſiue adeo ſurdus.

§. 16. Ex his intelligitur, ſi loco i finitos accipiamus valores, errorem huius expreſſionis eo fieri minorem, quo maior ſumtus fuerit numerus i; quae appropinquatio ad veritatem quo clarius perſpici queat, loco i ordine ſcribamus numeros 1, 2, 3, 4 etc. et expreſſiones inde ortas deſignemus ſignis I, II, III, etc. prodibitque

$$\text{I. } \frac{a}{a+nb}(\alpha+b)^n.$$

$$\text{II. } \frac{a}{a+nb}\cdot\frac{a+b}{a+(n+1)b}(\alpha+2b)^n.$$

$$\text{III. } \frac{a}{a+nb}\cdot\frac{a+b}{a+(n+1)b}\cdot\frac{a+2b}{a+(n+2)b}(\alpha+3b)^n.$$

$$\text{IV. } \frac{a}{a+nb}\cdot\frac{a+b}{a+(n+1)b}\cdot\frac{a+2b}{a+(n+2)b}\cdot\frac{a+3b}{a+(n+3)b}(\alpha+4b)^n.$$

etc.

Hinc autem porro colligitur

$$\frac{\text{II.}}{\text{I.}}=\frac{a+b}{a+(n+1)b}\left(\frac{\alpha+2b}{\alpha+b}\right)^n;$$

$$\frac{\text{III.}}{\text{II.}}=\frac{a+2b}{a+(n+2)b}\left(\frac{\alpha+3b}{\alpha+2b}\right)^n;$$

$$\frac{\text{IV.}}{\text{III.}}=\frac{a+3b}{a+(n+3)b}\left(\frac{\alpha+4b}{\alpha+3b}\right)^n.$$

etc. etc.

Quodſi ergo hoc modo in infinitum progrediamur, perueniemus ad verum valorem ipſius $\Delta:n$, qui ſequentibus factoribus in ſe inuicem ducendis conſtabit, poſtquam ſcilicet primum factorem ad formam ſequentium perduxerimus:

$$\Delta:n=\alpha^n\frac{a}{a+nb}\left(\frac{\alpha+b}{\alpha}\right)^n\cdot\frac{a+b}{a+(n+1)b}\left(\frac{\alpha+2b}{\alpha+b}\right)^n\cdot\frac{a+2b}{a+(n+2)b}\left(\frac{\alpha+3b}{\alpha+2b}\right)^n$$
$$\frac{a+3b}{a+(n+3)b}\left(\frac{\alpha+4b}{\alpha+3b}\right)^n \text{ etc.}$$

§. 17.

§. 17. Hoc igitur modo nacti sumus pro $\Delta : n$ productum in infinitum excurrens, cuius singuli factores secundum legem satis regularem procedunt; vbi imprimis obseruari conuenit, singulos factores totos, siue membra, continuo propius ad veritatem accedere, quandoquidem membrum infinitesimum erit

$$\frac{a + (i - 1)\,b}{a + (n + i - 1)\,b}\left(\frac{\alpha + i\,b}{\alpha + (i - 1)\,b}\right)^n$$

quae expressio, deletis partibus quae prae infinito euanescunt, manifesto ad unitatem redigitur. Tum vero iam obseruauimus quantitatem α penitus arbitrio nostro relinqui, neque inde valorem $\Delta : n$ affici, unde quouis casu eum ita accipere licebit, vt calculus commodior euadat; quamobrem vtique operae pretium erit obseruasse, pro huiusmodi seriebus terminum generalem multo uniuersaliori forma repraesentari posse ea, quam initio adduximus, quippe quae ex praesenti nascitur, ponendo $\alpha = a$.

Applicatio huius formae generalis ad casum $n = \frac{1}{2}$.

§. 18. Facile intelligitur, expressionem infinitam pro $\Delta : n$ inuentam imprimis summum vsum praestare posse, quando termini seriei desiderantur, quorum indices sunt numeri fracti, quandoquidem termini indicibus integris respondentes per se sunt cogniti. Quaeramus igitur primo eum nostrae seriei terminum, qui indici $n = \frac{1}{2}$ respondeat, qui ergo per $\Delta : \frac{1}{2}$ exprimetur, ita vt sit

$$\Delta : \tfrac{1}{2} = \sqrt{\alpha}\,\frac{a}{a + \frac{1}{2}b}\sqrt{\frac{\alpha + b}{\alpha}}\cdot\frac{a + b}{a + \frac{3}{2}b}\sqrt{\frac{\alpha + 2b}{\alpha + b}}\cdot\frac{a + 2b}{a + \frac{5}{2}b}\sqrt{\frac{\alpha + 3b}{\alpha + 2b}}\ \text{etc.}$$

in infinitum. Inuento autem isto valore simul facile innotescunt omnes termini intermedii a binis contiguis aequidistantes; erit enim

 $\Delta : 1\frac{1}{2}$

$$\Delta : 1\tfrac{1}{2} = \Delta : \tfrac{1}{2}(a + \tfrac{1}{2}b)$$

qui terminus inter primum a et ſecundum $\tfrac{1}{a}(a+b)$ medium interiacet; ſimilique modo erit

$$\Delta : 2\tfrac{1}{2} = \Delta : \tfrac{1}{2}(a + \tfrac{1}{2}b)(a + \tfrac{3}{2}b)$$

qui inter ſecundum et tertium medium interiacet. Praeterea vero erit

$$\Delta : 3\tfrac{1}{2} = \Delta : \tfrac{1}{2}(a + \tfrac{1}{2}b)(a + \tfrac{3}{2}b)(a + \tfrac{5}{2}b)$$

$$\Delta : 4\tfrac{1}{2} = \Delta : \tfrac{1}{2}(a + \tfrac{1}{2}b)(a + \tfrac{3}{2}b)(a + \tfrac{5}{2}b)(a + \tfrac{7}{2}b)$$

etc. etc.

Nunc autem hic quaeri ſolet, quomodo iſta producta in infinitum excurrentia ad expreſſiones finitas reuocari conueniat, quandoquidem illa producta in infinitum extenſa tantum inſeruire poſſunt valori ipſius $\Delta : \tfrac{1}{2}$ vero proxime inueniendo. Imprimis autem deſiderari ſolet ſpecies quantitatum transcendentium, ad quam iſte valor $\Delta : \tfrac{1}{2}$ ſit referendus, id quod per ea, quae a me paſſim circa huiusmodi producta infinita ſunt expoſita, haud difficulter praeſtari poterit. Ante omnia autem neceſſe eſt factores radicales e medio tolli, quod fit quadratis ſumendis, unde habebimus

$$(\Delta : \tfrac{1}{2})^2 = \alpha \frac{aa}{(a+\tfrac{1}{2}b)^2} \cdot \frac{\alpha+b}{\alpha} \left(\frac{a+b}{a+\tfrac{3}{2}b}\right)^2 \frac{\alpha+2b}{\alpha+b} \left(\frac{a+2b}{a+\tfrac{5}{2}b}\right)^2 \frac{\alpha+3b}{\alpha+2b} \text{ etc.}$$

§. 19. Quoniam autem hic littera α ab arbitrio noſtro pendet, eam ita accipiamus, vt numerus factorum in ſingulis membris imminuatur, quod fiet ſumendo $\alpha = a$; tum enim erit

$$(\Delta : \tfrac{1}{2})^2 = a \frac{a(a+b)}{(a+\tfrac{1}{2}b)(a+\tfrac{1}{2}b)} \cdot \frac{(a+b)(a+2b)}{(a+\tfrac{3}{2}b)(a+\tfrac{3}{2}b)} \cdot \frac{(a+2b)(a+3b)}{(a+\tfrac{5}{2}b)(a+\tfrac{5}{2}b)} \cdot \frac{(a+3b)(a+4b)}{(a+\tfrac{7}{2}b)(a+\tfrac{7}{2}b)} \cdot$$

etc.

Vt

Vt autem hinc fractiones partiales ex denominatoribus tollamus, singulos factores tam numeratoris quam denominatoris duplicemus, vt prodeat ista forma:

$$(\Delta:\tfrac{1}{2})^2 = a \cdot \frac{2a(2a+2b)}{(2a+b)(2a+b)} \cdot \frac{(2a+2b)(2a+4b)}{(2a+3b)(2a+3b)} \cdot \frac{(2a+4b)(2a+6b)}{(2a+5b)(2a+5b)} \text{ etc.}$$

vbi singuli factores cuiusque membri pro membro sequenti augmentum capiunt $= 2b$. Nunc autem ista forma facile ad expressiones finitas reuocari poterit, per ea quae passim sunt explicata.

§. 20. Si enim litteris P et Q istae formulae integrales designentur:

$$P = \int \frac{x^{p-1}\partial x}{(1-x^n)^{1-\frac{m}{n}}} \text{ et } Q = \int \frac{x^{q-1}\partial x}{(1-x^n)^{1-\frac{m}{n}}}$$

quae scilicet integralia ab $x = 0$ ad $x = 1$ extendi sunt intelligenda, ostendi fractionem $\frac{P}{Q}$ in sequens productum infinitum conuerti posse:

$$\frac{P}{Q} = \frac{q(m+p)}{p(m+q)} \cdot \frac{(q+n)(m+p+n)}{(p+n)(m+q+n)} \cdot \frac{(q+2n)(m+p+2n)}{(p+2n)(m+q+2n)} \text{ etc.}$$

vbi singuli factores cuiusque membri continuo quantitate $= n$ increscunt; unde statim patet, vt ista forma ad eam, quae nobis est proposita, redigatur, sumi debere $n = 2b$; tum vero sufficit prima membra vtrinque inter se aequari, scilicet

$$\frac{q(m+p)}{p(m+q)} = \frac{2a(2a+2b)}{(2a+b)(2a+b)}$$

id quod fit sumendo $q = 2a$ et $p = 2a + b$, tum vero $m = b$, quibus valoribus substitutis fractio $\frac{P}{Q}$, ducta in a, ipsum valorem $(\Delta : \frac{1}{2})^2$, quem quaerimus, modo finito exprimet.

§. 21. Facta autem substitutione modo inuenta, ob $\frac{m}{n} = \frac{1}{2}$ fit

$$P = \int \frac{x^{2a+b-1}\partial x}{\sqrt{(1-x^{2b})}} \text{ et } Q = \int \frac{x^{2a-1}\partial x}{\sqrt{(1-x^{2b})}}$$

quae

quae integralia perpetuo ab $x = 0$ ad $x = 1$ ſunt extendenda, quo facto erit $(\Delta : \frac{1}{2})^2 = a \frac{P}{Q}$, hincque radice extracta erit

$$\Delta : \tfrac{1}{2} = \sqrt{a} \cdot \int \frac{x^{2a+b-1} \partial x}{\sqrt{(1-x^{2b})}} : \int \frac{x^{2a-1} \partial x}{\sqrt{(1-x^{2b})}},$$

ex qua formula ſtatim patebit quouis caſu, a quanam quantitatum tranſcendentium ſpecie valor quaeſitus $\Delta : \frac{1}{2}$ pendeat, id quod operae pretium erit nonnullis exemplis illuſtrare.

Exemplum 1.

§. 22. Sumatur $a = 1$ et $b = 1$, vt prodeat ipſa ſeries hypergeometrica Walliſiana

$$1, 1.2, 1.2.3, 1.2.3.4, 1.2.3.4.5, \text{ etc.}$$

cuius terminus indici $\frac{1}{2}$ reſpondens per $\Delta : \frac{1}{2}$ deſignatus requiritur. Per formulam igitur inuentam erit

$$\Delta : \tfrac{1}{2} = \sqrt{\int \frac{xx \partial x}{\sqrt{1-xx}} : \int \frac{x \partial x}{\sqrt{(1-xx)}}}.$$

Conſtat autem, his integralibus ab $x = 0$ ad $x = 1$ extenſis, eſſe primo $\int \frac{x \partial x}{\sqrt{(1-xx)}} = 1$, tum vero

$$\int \frac{xx \partial x}{\sqrt{(1-xx)}} = \tfrac{1}{2} \int \frac{\partial x}{\sqrt{(1-xx)}} = \frac{\pi}{4},$$

vnde patet fore

$$\Delta : \tfrac{1}{2} = \sqrt{\tfrac{\pi}{4}} = \tfrac{1}{2} \sqrt{\pi};$$

reliqui autem termini intermedii huius ſeriei erunt

$$\Delta : 1\tfrac{1}{2} = \tfrac{1}{2} \cdot \tfrac{3}{2} \cdot \sqrt{\pi}.$$
$$\Delta : 2\tfrac{1}{2} = \tfrac{1}{2} \cdot \tfrac{3}{2} \cdot \tfrac{5}{2} \cdot \sqrt{\pi}.$$
$$\Delta : 3\tfrac{1}{2} = \tfrac{1}{2} \cdot \tfrac{3}{2} \cdot \tfrac{5}{2} \cdot \tfrac{7}{2} \cdot \sqrt{\pi}.$$

etc. etc.

vnde patet terminum ordine praecedentem, qui reſpondet indici $-\frac{1}{2}$ fore $\Delta : -\frac{1}{2} = \sqrt{\pi}$, prorſus vti iam a Walliſio eſt obſeruatum.

Exem-

Exemplum 2.

§. 23. Sumatur $a = 1$ et $b = 2$, vnde ista progressio hypergeometrica oritur: $1,\ 1.3,\ 1.3.5,\ 1.3.5.7,\ 1.3.5.7.9$ etc. Alius terminus indici $\frac{1}{2}$ respondens quaeritur, is ergo erit $\Delta : \frac{1}{2} = \sqrt{\int \frac{x^3 \partial x}{\sqrt{(1 - x^4)}} : \int \frac{x \partial x}{\sqrt{(1 - x^4)}}}$. Quodsi iam hic loco xx scribamus y, habebimus $\int \frac{x^3 \partial x}{\sqrt{(1 - x^4)}} = \frac{1}{2} \int \frac{y \partial y}{\sqrt{(1 - yy)}}$, cuius valor ab $y = 0$ ad $y = 1$ extensus est $= \frac{1}{2}$; altera vero formula $\int \frac{x \partial x}{\sqrt{(1 - x^4)}}$ abit in hanc: $\frac{1}{2} \int \frac{\partial y}{\sqrt{(1 - yy)}} = \frac{1}{2} \cdot \frac{\pi}{2}$. His ergo valoribus substitutis erit $\Delta : \frac{1}{2} = \sqrt{\frac{2}{\pi}}$, qui valor etiam a quadratura circuli pendet. Tum autem termini sequentes intermedii erunt:

$$\Delta : 1\tfrac{1}{2} = 2 \sqrt{\tfrac{2}{\pi}},$$
$$\Delta : 2\tfrac{1}{2} = 2.4 \sqrt{\tfrac{2}{\pi}},$$
$$\Delta : 3\tfrac{1}{2} = 2.4.6 \sqrt{\tfrac{2}{\pi}},$$
$$\Delta : 4\tfrac{1}{2} = 2.4.6.8 \sqrt{\tfrac{2}{\pi}},$$

etc. etc.

hincque patet terminum indici $-\frac{1}{2}$ respondentem $\Delta : -\frac{1}{2}$ fore infinitum.

Applicatio pro termino harum serierum inueniendo cuius index $= \frac{1}{3}$.

§. 24. Ponamus igitur hic $n = \frac{1}{3}$, et formula generalis inuenta nobis praebebit

$$\Delta : \tfrac{1}{3} = \sqrt[3]{\alpha} \cdot \frac{a}{a + \frac{1}{3} b} \sqrt[3]{\frac{\alpha + b}{\alpha}} \cdot \frac{a + b}{a + \frac{4}{3} b} \sqrt[3]{\frac{\alpha + 2b}{\alpha + b}} \cdot \frac{a + 2b}{a + \frac{7}{3} b} \sqrt[3]{\frac{\alpha + 3b}{\alpha + 2b}} \text{ etc.}$$

Hinc ergo sumendis cubis erit

$$(\Delta : \tfrac{1}{3})^3 = \alpha \cdot \frac{a^3}{(a + \frac{1}{3} b)^3} \cdot \frac{\alpha + b}{\alpha} \cdot \frac{(a + b)^3}{(a + \frac{4}{3} b)^3} \cdot \frac{\alpha + 2b}{\alpha + b} \text{ etc.}$$

 Jam

Jam ad fractiones tollendas ponamus $b = 3c$, fietque

$$(\Delta : \tfrac{1}{3})^3 = \alpha \cdot \frac{\alpha^3}{(\alpha + c)^3} \cdot \frac{\alpha + 3c}{\alpha} \cdot \frac{(\alpha + 3c)^3}{(\alpha + 4c)^3} \cdot \frac{\alpha + 6c}{\alpha + 3c} \cdot \frac{(\alpha + 6c)^3}{(\alpha + 7c)^3} \cdot \frac{\alpha + 9c}{\alpha + 6c} \text{ etc.}$$

vbi $3c$ eſt incrementum, quod ſinguli factores accipiunt, dum a quouis membro ad ſequens progredimur. Quodſi ergo capiamus $\alpha = a$, ad ſequentem formam ſimpliciorem peruenimus:

$$(\Delta : \tfrac{1}{3})^3 = a \cdot \frac{aa(a + 3c)}{(a + c)(a + c)(a + c)} \cdot \frac{(a + 3c)(a + 3c)(a + 6c)}{(a + 4c)(a + 4c)(a + 4c)} \cdot \frac{(a + 6c)(a + 6c)(a + 9c)}{(a + 7c)(a + 7c)(a + 7c)} \text{ etc.}$$

§. 25. Quoniam hic in quouis membro terni occurrunt factores, comparationem cum forma pro $\frac{P}{Q}$ exhibita immediate inſtituere non licet. Verum hic binas huiusmodi fractiones $\frac{P}{Q}$ et $\frac{P'}{Q'}$, quarum productum aequatur formae inuentae, in ſubſidium vocari oportet; et quoniam primum noſtrum membrum eſt $= \frac{aa(a + 3c)}{(a + c)(a + c)(a + c)}$, bina autem membra ex multiplicatione illa orta quatuor producunt factores, in ſingulis membris tam ſupra quam infra nouum factorem f adiungamus, vt in duas partes diſcerpi poſſint, quae pro primo ſint $\frac{aa}{(a + c)f} \cdot \frac{f(a + 3c)}{(a + c)(a + c)}$, et nunc vtramque partem cum $\frac{q(m + p)}{p(m + q)}$ comparemus. Pro priore autem parte ſtatuamus $q = a$ et $p = f$, fietque $m + p = m + f = a$ et $m + q = m + a = a + c$, vnde colligitur $m = c$ et $f = a - c$; tum autem ad ſequentia membra progrediendo fiet $n = 3c$.

§. 26. Quodſi iam ſingula membra noſtrae expreſſionis in binas huiusmodi partes reſoluamus, introducta noua littera $f = a - c$, quae pariter in ſequentibus membris augmentum $3c$ accipiet, omnes partes priores ſeorſim conſideremus, quarum productum aequabitur fractioni $\frac{P}{Q}$, eritque ex valoribus iam erutis

$$P = \int \frac{x^{a - c - 1} \partial x}{\sqrt[3]{(1 - x^{3c})^2}} \text{ et } Q = \int \frac{x^{a - 1} \partial x}{\sqrt[3]{(1 - x^{3c})^2}}.$$

§. 27.

§. 27. Pro partibus autem posterioribus in subsidium vocemus fractionem $\frac{P'}{Q'}$, dum etiam litteras minusculas p, q, et m simili apice notabimus. Hinc igitur comparatio primorum membrorum dabit

$$\frac{q'(m'+p')}{p'(m'+q')} = \frac{f(a+3c)}{(a+c)(a+c)},$$

hanc ob rem sumamus

$$q' = f = a - c \text{ et } p' = a + c$$

tum autem erit

$$m' + p' = m' + a + c = a + 3c \text{ et}$$
$$m' + q' = m' + a - c = a + c$$

vtrinque autem fit $m' = 2c$, tum vero manet vt ante $n = 3c$, ex quo nouae istae formulae ita determinabuntur:

$$P' = \int \frac{x^{a+c-1}\,\partial x}{\sqrt[3]{(1 - x^{3c})}} \text{ et } Q' = \int \frac{x^{a-c-1}\,\partial x}{\sqrt[3]{(1 - x^{3c})}}.$$

§. 28. Cum igitur fractio $\frac{P}{Q}$ exprimat productum omnium partium priorum, at vero $\frac{P'}{Q'}$ productum omnium partium posteriorum, habebimus

$$(\Delta : \tfrac{1}{3})^3 = a\,\frac{P}{Q}\cdot\frac{P'}{Q'}, \text{ ideoque}$$
$$\Delta : \tfrac{1}{3} = \sqrt[3]{\frac{a\,P\,P'}{Q\,Q'}},$$

sicque ad hunc terminum interpolatum $\Delta : \frac{1}{3}$ definiendum quatuor formulis integralibus opus erit, inter quas in genere nulla relatio perspicitur: erit enim facta substitutione

$$\Delta : \tfrac{1}{3} = \sqrt[3]{a \int \frac{x^{a-c-1}\,\partial x}{\sqrt[3]{(1-x^{3c})^2}} \times \int \frac{x^{a+c-1}\,\partial x}{\sqrt[3]{(1-x^{3c})}}} : \sqrt[3]{\int \frac{x^{a-1}\,\partial x}{\sqrt[3]{(1-x^{3c})^2}} \times \int \frac{x^{a-c-1}\,\partial x}{\sqrt[3]{(1-x^{3c})}}}$$

vbi meminisse oportet loco litterae b hic scriptum esse $3c$, ita vt sit $c = \frac{1}{3}b$. Hanc expressionem exemplo seriei Wallisianae illustrasse sufficiat.

Exemplum.

§. 29. Sit igitur $a = 1$ et $b = 1$, ideoque $c = \frac{1}{3}$, et quatuor formulae integrales erunt:

$$P = \int \frac{x^{-\frac{1}{3}} \partial x}{\sqrt[3]{(1-x)^2}} \text{ et } Q = \int \frac{\partial x}{\sqrt[3]{(1-x)^2}}$$

$$P' = \int \frac{x^{\frac{1}{3}} \partial x}{\sqrt[3]{(1-x)}} \text{ et } Q' = \int \frac{x^{-\frac{1}{3}} \partial x}{\sqrt[3]{(1-x)}}$$

quae formulae vt ab exponentibus fractis liberentur, statuatur $x = y^3$, eritque

$$P = 3\int \frac{y \partial y}{\sqrt[3]{(1-y^3)^2}}, \quad Q = 3\int \frac{y y \partial y}{\sqrt[3]{(1-y^3)^2}};$$

$$P' = 3\int \frac{y^3 \partial y}{\sqrt[3]{(1-y^3)}}, \quad Q' = 3\int \frac{y \partial y}{\sqrt[3]{(1-y^3)}};$$

ex quibus valoribus fit

$$\Delta : \tfrac{1}{3} = \sqrt[3]{\frac{a P P'}{Q Q'}}$$

vbi notetur esse

$$\int \frac{y^3 \partial y}{\sqrt[3]{(1-y^3)}} = \tfrac{1}{3} \int \frac{\partial y}{\sqrt[3]{(1-y^3)}}.$$

§. 30. Iam singulas istas formulas accuratius euoluamus, ac primo quidem formula $\int \frac{\partial y}{\sqrt[3]{(1-y^3)}}$ ad circulum reduci potest. Posito enim $\frac{y}{\sqrt[3]{(1-y^3)}} = z$, formula nostra fit

fit $\frac{z\partial y}{y}$, tum autem erit

$$y^3 = \frac{z^3}{1+z^3} \text{ et } 3\,ly = 3\,lz - l(1+z^3)$$

hincque

$$\frac{\partial y}{y} = \frac{\partial z}{z} - \frac{zz\partial z}{1+z^3} = \frac{\partial z}{z(1+z^3)},$$

ficque formula noftra fiet $=\int\frac{\partial z}{1+z^3}$, cuius integrale ab $y=0$ ad $y=1$, hoc eft a $z=0$ ad $z=\infty$, eft $=\frac{\pi}{3\,\text{fin.}\,\frac{1}{3}\pi} = \frac{2\pi}{3\sqrt{3}}$, ficque erit $P' = \frac{2\pi}{3\sqrt{3}}$. At pro P formula $\int\frac{y\partial y}{\sqrt[3]{(1-y^3)^2}}$ per eandem fubftitutionem $\frac{y}{\sqrt[3]{(1-y^3)}} = z$ reducitur ad hanc: $\int\frac{zz\partial y}{y}$ $=\int\frac{z\partial z}{1+z^3}$, cuius integrale eft $\frac{\pi}{3\,\text{fin.}\,\frac{2}{3}\pi} = \frac{2\pi}{3\sqrt{3}}$, vnde fit $P = \frac{2\pi}{\sqrt{3}}$.

Porro vero pro Q eft $\int\frac{yy\partial y}{\sqrt[3]{(1-y^3)^2}} = 1 - \sqrt[3]{(1-y^3)}$, vnde pofito $y=1$ fiet $Q = 3$. Denique formula $Q' = 3\int\frac{y\partial y}{\sqrt[3]{(1-y^3)}}$ nullo modo ad menfuras cognitas fe reduci patitur, fed quadraturam fingularem inuoluit. Ex his formulis conficitur

$$\Delta : \tfrac{1}{3} = \sqrt[3]{\frac{4\pi\pi}{81.\int\frac{y\partial y}{\sqrt[3]{(1-y^3)}}}},$$

ex quo valore porro deducuntur fequentes:

$$\Delta : 1\tfrac{1}{3} = \tfrac{4}{3}\,\Delta : \tfrac{1}{3}.$$

$$\Delta : 2\tfrac{1}{3} = \tfrac{4}{3}\,\tfrac{7}{3}\,\Delta : \tfrac{1}{3}.$$

$$\Delta : 3\tfrac{1}{3} = \tfrac{4}{3} \cdot \tfrac{7}{3} \cdot \tfrac{10}{3}\, \Delta : \tfrac{1}{3}.$$

etc. etc.

Hinc autem facile intelligitur quomodo inueſtigationem tractari oporteat, ſi loco n aliae fractiones proponantur.

Concluſio.

§. 31. Quemadmodum hic pro ſerie

$$a,\ a(a+b),\ a(a+b)(a+2b),\ a(a+b)(a+2b)(a+3b),\ \text{etc.}$$

terminum indici indefinito n reſpondentem $\Delta : n$ ita expreſſum inuenimus, vt ſit:

$$\Delta : n = \alpha^n \cdot \frac{a}{a+nb}\left(\frac{\alpha+b}{\alpha}\right)^n \cdot \frac{a+b}{a+(n+1)b}\left(\frac{\alpha+2b}{\alpha+b}\right)^n \text{etc.}$$

ſi loco a alium quemcunque numerum c accipiamus, vt ſeries ſit

$$c,\ c(c+b),\ c(c+b)(c+2b),\ c(c+b)(c+2b)(c+3b)\ \text{etc.}$$

et eius terminum indici n reſpondentem deſignemus per $\Gamma : n$, tum erit ſimili modo

$$\Gamma : n = \alpha^n \cdot \frac{c}{c+nb}\left(\frac{\alpha+b}{\alpha}\right)^n \cdot \frac{c+b}{c+(n+1)b}\left(\frac{\alpha+2b}{\alpha+b}\right)^n \text{etc.}$$

vbi quidem α alium quemcunque valorem ſignificare poſſet atque in prima. Quodſi nunc poſteriorem ſeriem per priorem diuidamus, naſcetur inde ſequens ſeries:

$$\frac{a}{c},\ \frac{a(a+b)}{c(c+b)},\ \frac{a(a+b)(a+2b)}{c(c+b)(c+2b)},\ \frac{a(a+b)(a+2b)(a+3b)}{c(c+b)(c+2b)(c+3b)}\ \text{etc.}$$

atque manifeſtum eſt terminum indici n reſpondentem fore $\frac{\Delta : n}{\Gamma : n}$; vnde ſi vtrinque pro α eundem accipiamus numerum, poteſtates exponentis n omnes ſe mutuo tollent, ita vt pro hac ſerie terminus generalis, ſeu indici n reſpondens, ſit

$$= \frac{a(c+nb)}{(a+nb)c} \cdot \frac{(a+b)[c+(n+1)b]}{[a+(n+1)b](c+b)} \cdot \frac{(a+2b)[c+(n+2)b]}{[a+(n+2)b](c+2b)}\ \text{etc.}$$

vbi

vbi ergo interpolatio ſine vlla difficultate inſtitui poteſt. Quia etiam in genere ipſe hic terminus generalis per fractionem $\frac{P}{Q}$ commode exprimi poterit, ſumendo $q = a$; $p = c$; $m = n b$; ita vt incrementum ſucceſſiuum, quod erat n, nunc ſit b, vnde ambae formulae integrales ita ſe habebunt:

$$P = \int \frac{x^{c-1}\partial x}{(1-x^b)^{1-n}} \text{ et } Q = \int \frac{x^{a-1}\partial x}{(1-x^b)^{1-n}};$$

ſicque his caſibus ſemper terminum generalem per binas formulas integrales, ideoque modo finito ac determinato, exprimere licebit, neque interpolatio nouas quadraturas poſtulat, quemadmodum in caſibus ſupra tractatis vſu venit.

DE

DE ITERATA INTEGRATIONE FORMVLARVM INTEGRALIVM DVM ALIQVIS EXPONENS PRO VARIABILI ASSVMITVR.

Auctore

L. EVLERO.

Conuent. exhib. die 19 *Aug.* 1776.

Problema 1.

Cum sit $\int x^{\theta-1}\partial x \begin{bmatrix}\text{ab } x=0\\ \text{ad } x=1\end{bmatrix} = \frac{1}{\theta}$, *hanc formulam denuo integrare, sumto exponente* θ *variabili.*

Solutio.

§. 1. Quoniam hic de integratione agitur, ut ea determinetur, integrale ita capi assumamus, ut evanescat certo casu, posito scilicet $\theta=\alpha$. Multiplicetur ergo utrinque per elementum $\partial\theta$, et integratione juxta hanc legem instituta pro parte dextra habebimus $\int\frac{\partial\theta}{\theta}=l\,\theta-l\,\alpha=l\,\frac{\theta}{\alpha}$. At pro parte sinistra notum est, hanc integrationem a signo summatorio $\int$ penitus non turbari, et quia jam sola littera θ pro variabili habetur, $\frac{\partial x}{x}$ vero ut constans spectatur, ob $x^{\theta-1}\partial x=\frac{\partial x}{x}x^{\theta}$, habebimus

$\int$

$$\int x^\theta \partial \theta = \frac{x^\theta}{l\,x} - \frac{x^\alpha}{l\,x};$$

quo valore ſubſtituto membrum ſiniſtrum erit

$$\int \frac{\partial x}{x} \cdot \frac{x^\theta - x^\alpha}{l\,x},$$

quamobrem iſta integratio iterata nos perducit ad hanc aequationem:

$$\int \frac{x^{\theta - 1} - x^{\alpha - 1}\,\partial x}{l\,x} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = l\frac{\theta}{\alpha}.$$

Corollarium 1.

§. 2. Si eodem modo formula integralis

$$\int x^{n + \theta - 1}\,\partial x \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = \frac{1}{n + \theta}$$

denuo integretur, ſumto θ variabili, reperietur haec aequatio integrata:

$$\int (x^{n + \theta - 1} - x^{n + \alpha - 1}) \frac{\partial x}{l\,x} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = l\frac{n + \theta}{n + \alpha}.$$

At ſi θ negatiue capiatur, tum etiam α negatiue accipi debebit, vnde aequatio denuo integrata haec prodibit:

$$\int (x^{n - \theta - 1} - x^{n - \alpha - 1}) \frac{\partial x}{l\,x} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = l\frac{n - \theta}{n - \alpha}.$$

Corollarium 2.

§. 3. Hic igitur notentur iſtae integrationes, quas in parte ſiniſtra inſtitui oportet, et quibus pro aliis formulis in poſterum erit vtendum, vbi ſemper aſſumamus, integralia ita capi debere, vt euaneſcant poſito $\vartheta = \alpha$. Primo ſcilicet erit

$$\int x^\theta \partial \theta = \frac{x^\theta - x^\alpha}{l\,x}.$$

Praeterea vero ſimili modo

$$\int x^{n+\theta} \partial \theta = \frac{x^{n+\theta} - x^{n+\alpha}}{l\,x};$$

atque hinc porro intelligitur fore

$$\int x^{n+\lambda\theta} \partial \theta = \frac{x^{n+\lambda\theta} - x^{n+\lambda\alpha}}{\lambda\,l\,x},$$

vnde patet, ſi λ capiatur negative, fore

$$\int x^{n-\lambda\theta} \partial \theta = \frac{x^{n-\lambda\theta} - x^{n-\lambda\alpha}}{-\lambda\,l\,x}.$$

Problema 2.

Cum ſit, vti jam ſaepius eſt oſtenſum,

$$\int \frac{x^{\theta-1}\partial x}{1+x^\nu}\left[\begin{array}{l}\text{ab } x=0\\ \text{ad } x=\infty\end{array}\right] = \frac{\pi}{\nu\,\text{ſin.}\,\frac{\theta\pi}{\nu}},$$

hanc aequationem denuo integrare, ſumto exponente θ pro variabili.

Solutio.

§. 4. Perpetuo hic, vt hactenus, integralia ita accipi ſtatuamus, vt evaneſcant poſito $\theta=\alpha$; quo obſervato pro parte dextra habebimus $\int \frac{\pi\,\partial\theta}{\nu\,\text{ſin.}\,\frac{\theta\pi}{\nu}}$, quae formula poſito $\frac{\theta\pi}{\nu}=\Phi$ abit in hanc: $\int \frac{\partial\Phi}{\text{ſin.}\,\Phi}$, cujus integrale novimus eſſe $l\,\text{tang.}\,\frac{1}{2}\Phi$; quamobrem adjecta debita conſtante pro hac parte habebimus

$$\int \frac{\pi\partial\theta}{\nu\,\text{ſin.}\,\frac{\theta\pi}{\nu}} = l\,\text{tang.}\,\frac{\theta\pi}{2\nu} - l\,\text{tang.}\,\frac{\alpha\pi}{2\nu} = l\,\frac{\text{tang.}\,\frac{\theta\pi}{2\nu}}{\text{tang.}\,\frac{\alpha\pi}{2\nu}}.$$

Pro

Pro parte autem finiftra, vbi folus factor $x^{\theta-1}$ eft variabilis, erit

$$\int x^{\theta-1}\partial\theta = \frac{x^{\theta-1}-x^{\alpha-1}}{l\,x}.$$

Hoc igitur valore introducto formula noftra integralis denuo integrata erit

$$\int\frac{\partial x\,(x^{\theta-1}-x^{\alpha-1})}{(1+x^{\nu})\,l\,x}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=\infty\end{bmatrix} = l\,\frac{\text{tang.}\,\frac{\theta\pi}{2\nu}}{\text{tang.}\,\frac{\alpha\pi}{2\nu}}.$$

Corollarium.

§. 5. Quodfi ergo fumamus $\alpha=\frac{1}{2}\nu$, quoniam tang. $\frac{\pi}{4}=1$, hoc cafu, ponendo potius $\nu=2\alpha$, habebimus hanc aequationem integralem fatis memorabilem:

$$\int\frac{\partial x\,(x^{\theta-1}-x^{\alpha-1})}{(1+x^{2\alpha})\,l\,x}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=\infty\end{bmatrix} = l\,\text{tang.}\,\frac{\theta\pi}{4\alpha}.$$

Problema 3.

Cum fit, vti jam fatis conftat:

$$\int\frac{(x^{\theta-1}+x^{\nu-\theta-1})\,\partial x}{1+x^{\nu}}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=1\end{bmatrix} = \frac{\pi}{\nu\,\text{fin.}\,\frac{\theta\pi}{\nu}},$$

hanc aequationem denuo integrare per exponentem variabilem θ, *ita vt integralia evanefcant pofito* $\theta=\alpha$.

Solutio.

§. 6. Multiplicando igitur per $\partial\theta$ et integrando, pro parte dextra, prorfus vt in praecedente problemate, habebimus

$$l\,\frac{\text{tang.}\,\frac{\theta\pi}{2\nu}}{\text{tang.}\,\frac{\alpha\pi}{2\nu}}.$$

Pro

Pro parte autem finiſtra, quia formula $\frac{\partial x}{1+x^\nu}$ eſt conſtans, et exponens θ in duobus terminis occurrit, pro priore termino habebimus

$$\int x^{\theta-1}\partial\theta = \frac{x^{\theta-1}-x^{\alpha-1}}{lx},$$

pro altero vero termino ex §. 3. habebimus

$$\int x^{\nu-\theta-1}\partial\theta = \frac{x^{\nu-\alpha-1}-x^{\nu-\theta-1}}{lx},$$

quibus valoribus ſubſtitutis orietur iſta noua integratio:

$$\int\frac{\partial x}{lx}\cdot\frac{x^{\theta-1}-x^{\alpha-1}+x^{\nu-\alpha-1}-x^{\nu-\theta-1}}{1+x^\nu}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=1\end{bmatrix} = l\frac{\text{tang.}\frac{\theta\pi}{2\nu}}{\text{tang.}\frac{\alpha\pi}{2\nu}}.$$

Corollarium 1.

§. 7. Iſta aequatio aliquanto ſuccinctius ita repraeſentari poteſt:

$$\int\frac{\partial x}{x\,lx}\,\frac{(x^\theta-x^\alpha+x^{\nu-\alpha}-x^{\nu-\theta})}{1+x^\nu}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=1\end{bmatrix} = l\frac{\text{tang.}\frac{\theta\pi}{2\nu}}{\text{tang.}\frac{\alpha\pi}{2\nu}}.$$

vbi cum ſit $x^{\nu-\alpha}-x^{\nu-\theta} = x^{\nu-\alpha-\theta}(x^\theta-x^\alpha)$, iſta aequatio ita commodius per factores repraeſentari poterit:

$$\int\frac{\partial x}{x\,lx}\,\frac{(x^\theta-x^\alpha)(1+x^{\nu-\alpha-\theta})}{1+x^\nu}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=1\end{bmatrix} = l\frac{\text{tang.}\frac{\theta\pi}{2\nu}}{\text{tang.}\frac{\alpha\pi}{2\nu}}.$$

Corollarium 2.

§. 8. Quodſi hic capiamus $\theta=\nu-\alpha$, vt fiat $x^{\nu-\alpha-\theta}=1$, pro parte dextra erit tang. $\frac{(\nu-\alpha)\pi}{2\nu}$ = cotang. $\frac{\alpha\pi}{2\nu}$, vnde totum hoc membrum erit $2\,l\,\text{cot.}\frac{\alpha\pi}{2\nu}$; quare cum pro parte finiſtra factor

factor $1+x^{\nu-\alpha-\theta}$ evadat $=2$, vtrinque per 2 dividendo habebimus

$$\int \frac{\partial x}{x\,l\,x}\cdot\frac{x^{\nu-\alpha}-x^{\alpha}}{1+x^{\nu}}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=1\end{bmatrix}=l\,\cot.\frac{\alpha\pi}{2\nu}.$$

Corollarium 3.

§. 9. Quodsi sumamus $\nu=2\alpha$, vt fiat tang. $\frac{\alpha\pi}{2\nu}=1$, pro parte sinistra factor $1+x^{\nu-\alpha-\theta}$ abit in $1+x^{\alpha-\theta}$, dum prior factor $x^{\theta}-x^{\alpha}$ ita repraesentari potest: $x^{\theta}(1-x^{\alpha-\theta})$; vnde amborum productum erit $x^{\theta}(1-x^{2\alpha-2\theta})$, quamobrem integratio nostra ita se habebit:

$$\int \frac{x^{\theta-1}\,\partial x}{l\,x}\cdot\frac{1-x^{2\alpha-2\theta}}{1+x^{2\alpha}}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=1\end{bmatrix}=l\,\text{tang.}\frac{\theta\pi}{4\alpha}.$$

Scholion.

§. 10. Istae integrationes eo majorem attentionem merentur, quod in iis tres exponentes α, θ, ν indefiniti occurrunt, quos singulos pro lubitu vtcunque determinare licet, ita vt istae formulae multo latius pateant, quam eae quas non ita pridem ex iisdem fundamentis derivavi.

Problema 4.

Cum sit, vti jam abunde est demonstratum,

$$\int \frac{x^{\theta-1}-x^{\nu-\theta-1}}{1-x^{\nu}}\,\partial x\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=1\end{bmatrix}=\frac{\pi}{\nu\,\text{tang.}\frac{\theta\pi}{\nu}},$$

hanc formulam denuo integrare, sumto exponente θ variabili, ita vt integralia evanescant posito $\theta=\alpha$.

Solutio.

§. 11. Quodsi ergo hic per $\partial\theta$ multiplicemus, pro parte dextra habebimus $\frac{\pi\,\partial\theta}{\nu\,\text{tang.}\,\frac{\theta\pi}{\nu}}$, quae formula, posito $\frac{\pi\theta}{\nu}=\Phi$, abit in $\frac{\partial\Phi}{\text{tang.}\,\Phi}=\frac{\partial\Phi\,\text{cos.}\,\Phi}{\text{sin.}\,\Phi}$, cujus integrale manifesto est $l\,\text{sin.}\,\Phi$; quamobrem constanti debita adjecta, pro parte dextra habebimus

$$l\,\text{sin.}\,\frac{\theta\pi}{\nu}-l\,\text{sin.}\,\frac{\alpha\pi}{\nu}=l\,\frac{\text{sin.}\,\frac{\theta\pi}{\nu}}{\text{sin.}\,\frac{\alpha\pi}{\nu}}.$$

Pro parte autem sinistra, quae ita repraesentetur:

$$\int\frac{\partial x}{x}\cdot\frac{x^{\theta}-x^{\nu-\theta}}{1-x^{\nu}},$$

habebimus

$$\int x^{\theta}\,\partial\theta=\frac{x^{\theta}-x^{\alpha}}{l\,x}\quad\text{et}$$

$$\int x^{\nu-\theta}\,\partial\theta=\frac{x^{\nu-\alpha}-x^{\nu-\theta}}{l\,x},$$

quibus valoribus substitutis orietur sequens aequatio integrata:

$$\int\frac{\partial x}{x\,l\,x}\cdot\frac{(x^{\theta}-x^{\alpha}-x^{\nu-\alpha}+x^{\nu-\theta})}{1-x^{\nu}}\begin{bmatrix}\text{ab }x=0\\ \text{ad }x=1\end{bmatrix}=l\,\frac{\text{sin.}\,\frac{\theta\pi}{\nu}}{\text{sin.}\,\frac{\alpha\pi}{\nu}}$$

vbi iterum tres exponentes indefiniti occurrunt, α, θ, ν.

Corollarium 1.

§. 12. Cum sit, vti jam ante obseruauimus,

$$x^{\nu-\alpha}-x^{\nu-\theta}=x^{\nu-\alpha-\theta}(x^{\theta}-x^{\alpha}),$$

formula nostra commodius ita per factores exprimi poterit:

$$\int\frac{\partial x}{x\,l\,x}\cdot\frac{(x^{\theta}-x^{\alpha})(1-x^{\nu-\alpha-\theta})}{1-x^{\nu}}\begin{bmatrix}\text{ab }x=0\\ \text{ad }x=1\end{bmatrix}=l\,\frac{\text{sin.}\,\frac{\theta\pi}{\nu}}{\text{sin.}\,\frac{\alpha\pi}{\nu}},$$

vbi

vbi si sumeremus $\nu = \alpha + \theta$, membrum sinistrum evanesceret, dextrum autem manifesto quoque evanesceret.

Corollarium 2.

§. 13. Quodsi autem hic sumamus $\nu = 2\,\alpha$, pro dextra foret sin. $\frac{\alpha\pi}{\nu} = 1$, vnde hoc casu formula nostra integralis erit

$$\int \frac{\partial x}{x\, l\, x} \cdot \frac{(x^{\theta} - x^{\alpha})(1 - x^{\alpha - \theta})}{1 - x^{2\alpha}} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = l \text{ sin. } \frac{\theta\pi}{2\alpha},$$

quae forma evidenter in hanc contrahitur:

$$\int \frac{x^{\theta - 1}\partial x}{l\, x} \cdot \frac{(1 - x^{\alpha - \theta})^2}{1 - x^{2\alpha}} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = l \text{ sin. } \frac{\theta\pi}{2\alpha}.$$

Scholion.

§. 14. Has igitur egregias integrationes deduximus ex formulis integralibus jam pridem erutis, quatenus in iis exponentes indefiniti occurrunt; quod si ergo aliae hujusmodi formulae integrales insuper innotescerent, eas simili modo tractare liceret; verum hactenus nullae tales formulae sunt inventae quae ad hunc scopum accommodari possunt, quam ob causam integrationes hic exhibitae summa attentione Geometrarum dignae sunt existimandae.

Additamentum.

§. 15. Cum nuper ostendissem hujus formulae integralis

$$\int \frac{x^{a-1}\partial x}{l\, x} \cdot \frac{(1 - x^{b})(1 - x^{c})}{1 - x^{n}}$$

a termino $x = 0$ ad terminum $x = 1$ extensae valorem ita exprimi, vt sit $l\frac{P}{Q}$, existente

P

$$P = \int \frac{x^{a+b-1}\partial x}{(1-x^n)^{1-\frac{c}{n}}} \text{ et } Q = \int \frac{x^{a-1}\partial x}{(1-x^n)^{1-\frac{c}{n}}}$$

quae integralia denuo ab $x = 0$ ad $x = 1$ ſunt extendenda: manifeſtum eſt in hac forma generali pleraſque integrationes ſupra inuentas contineri; quamobrem cum illis caſibus valores integralium abſolute exprimantur, operae pretium erit iſtam formam generalem ad illos caſus applicare, vt relatio inter binas formulas integrales P et Q inde innoteſcat. Problema quidem primum et ſecundum huc plane non pertinent. Ex problemate igitur tertio et quarto eos perſcrutemur caſus, quos ad formam noſtram generalem reuocare licet.

Euolutio formulae integralis ſupra §. 8. inuentae.

$$\int \frac{\partial x}{x\, l\, x} \cdot \frac{x^{\nu-\alpha} - x^{\alpha}}{1+x^{\nu}} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = \nu \end{bmatrix} = l \operatorname{cof.} \frac{\alpha\pi}{2\nu}.$$

§. 16. Quoniam hic denominator eſt $1 + x^{\nu}$, vt is ad formam generalem reducatur, multiplicetur fractio ſupra et infra per $1 - x^{\nu}$, et formula iſta integralis hanc induet formam:

$$\int \frac{\partial x}{x\, l\, x} \cdot \frac{(x^{\nu-\alpha} - x^{\alpha})(1 - x^{\nu})}{1 - x^{2\nu}}.$$

Hic ante omnia diſpiciendum eſt, vter exponentium $\nu - \alpha$ et α ſit major, vnde duos caſus evolvi conveniet, prouti fuerit vel $\nu - \alpha < \alpha$, hoc eſt $\nu < 2\alpha$, vel $\nu - \alpha > \alpha$, hoc eſt $\nu > 2\alpha$.

§. 17. Sit igitur primo $\nu < 2\alpha$, ſeu $\alpha > \frac{1}{2}\nu$, atque formula integralis ita repraeſentari poterit:

$$\int \frac{x^{\nu-\alpha-1}\partial x}{l\, x} \cdot \frac{(1 - x^{2\alpha-\nu})(1 - x^{\nu})}{1 - x^{2\nu}}.$$

Hinc jam comparatione cum forma generali inſtituta manifeſto habebimus $a = \nu - \alpha$, $b = 2\alpha - \nu$ et $c = \nu$, denique $n = 2\nu$, ex quibus valoribus formabuntur ſequentes formulae:

P

$$P = \int \frac{x^{\alpha - 1} \partial x}{\sqrt{(1 - x^{2\nu})}} \text{ et } Q = \int \frac{x^{\nu - \alpha - 1} \partial x}{\sqrt{(1 - x^{2\nu})}}.$$

Ponere etiam potuissemus $b = \nu$ et $c = 2\alpha - \nu$, manentibus $a = \nu - \alpha$ et $n = 2\nu$, hincque prodiissent valores

$$P = \int \frac{x^{2\nu - \alpha - 1} \partial x}{(1 - x^{2\nu})^{\frac{3\nu - 2\alpha}{2\nu}}} \text{ et } Q = \int \frac{x^{\nu - \alpha - 1} \partial x}{(1 - x^{2\nu})^{\frac{3\nu - 2\alpha}{2\nu}}},$$

vtrinque autem erit $l\frac{P}{Q} = l \cot. \frac{\alpha\pi}{2\nu}$.

§. 18. Hinc igitur duas nanciscimur integrationes notatu dignissimas. Cum enim sit $\frac{P}{Q} = \cot. \frac{\alpha\pi}{2\nu}$, hae duae integrationes ita se habebunt:

$$\text{I. } \int \frac{x^{\alpha - 1} \partial x}{\sqrt{1 - x^{2\nu}}} : \int \frac{x^{\nu - \alpha - 1} \partial x}{\sqrt{1 - x^{2\nu}}} = \cot. \frac{\alpha\pi}{2\nu};$$

$$\text{II. } \int \frac{x^{2\nu - \alpha - 1} \partial x}{(1 - x^{2\nu})^{\frac{3\nu - 2\alpha}{2\nu}}} : \int \frac{x^{\nu - \alpha - 1} \partial x}{(1 - x^{2\nu})^{\frac{3\nu - 2\alpha}{2\nu}}} = \cot. \frac{\alpha\pi}{2\nu}.$$

§. 19. Sin autem fuerit $\nu > 2\alpha$, ipsa formula generalis mutatis signis ita debet repraesentari:

$$\int \frac{\partial x}{x\, l x} \cdot \frac{(x^{\alpha} - x^{\nu - \alpha})(1 - x^{\nu})}{1 - x^{2\nu}} = l \text{ tang. } \frac{\alpha\pi}{2\nu},$$

cui aequationi nunc induamus hanc formam:

$$\int \frac{x^{\alpha - 1} \partial x}{l x} \cdot \frac{(1 - x^{\nu - 2\alpha})(1 - x^{\nu})}{1 - x^{2\nu}}$$

vnde iam manifesto habemus $a = \alpha$, $b = \nu - 2\alpha$, $c = \nu$, atque $n = 2\nu$, vnde deducuntur isti valores:

$$P = \int \frac{x^{\nu - \alpha - 1} \partial x}{\sqrt{(1 - x^{2\nu})}} \text{ et } Q = \int \frac{x^{\alpha - 1} \partial x}{\sqrt{(1 - x^{2\nu})}}.$$

Sin autem sumamus $c = \nu - 2\alpha$ et $b = \nu$, manente $a = \alpha$ et

$n = 2\nu$, reperietur

$$P = \int \frac{x^{\alpha+\nu-1}\partial x}{(1-x^{2\nu})^{\frac{\nu+2\alpha}{2\nu}}} \text{ et } Q = \int \frac{x^{\alpha-1}\partial x}{(1-x^{2\nu})^{\frac{\nu+2\alpha}{2\nu}}}.$$

§. 20. Cum nunc utrinque fit $l\frac{P}{Q} = l \text{ tang. } \frac{\alpha\pi}{2\nu}$ ideoque $\frac{P}{Q} = \text{tang. } \frac{\alpha\pi}{2\nu}$, hinc adipifcimur iterum has duas integrationes:

$$\text{III. } \int \frac{x^{\nu-\alpha-1}\partial x}{\sqrt{1-x^{2\nu}}} : \int \frac{x^{\alpha-1}\partial x}{\sqrt{1-x^{2\nu}}} = \text{tang. } \frac{\alpha\pi}{2\nu},$$

quae quidem conuenit cum priore antecedentium, fiquidem formulae P et Q tantum inter fe permutantur; altera vero integratio eft noua, fcilicet

$$\text{IV. } \int \frac{x^{\alpha+\nu-1}\partial x}{(1-x^{2\nu})^{\frac{\nu+2\alpha}{2\nu}}} : \int \frac{x^{\alpha-1}\partial x}{(1-x^{2\nu})^{\frac{\nu+2\alpha}{2\nu}}} = \text{tang. } \frac{\alpha\pi}{2\nu}.$$

Euolutio formulae integralis §. 9. allatae:

$$\int \frac{x^{\theta-1}\partial x}{l x} \cdot \frac{1-x^{2\alpha-2\theta}}{1+x^{2\alpha}} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = l \text{ tang. } \frac{\theta\pi}{4\alpha}.$$

§. 21. Quo haec expreffio ad formam praefcriptam reducatur, multiplicetur fupra et infra per $1 - x^{2\alpha}$, vt habeamus hanc formam:

$$\int \frac{x^{\theta-1}\partial x}{l x} \cdot \frac{(1-x^{2\alpha-2\theta})(1-x^{2\alpha})}{1-x^{4\alpha}} = l \text{ tang. } \frac{\theta\pi}{4\alpha},$$

quae fponte ad formam generalem reuocatur, fumendo $\alpha = \theta$, $b = 2\alpha - 2\theta$, $c = 2\alpha$ et $n = 4\alpha$, fi modo fuerit $\alpha > \theta$. Si enim fuerit $\theta > \alpha$, alio modo comparatio inftitui debet, vti deinceps videbimus. Ex his autem valoribus conficietur

$$P = \int \frac{x^{2\alpha-\theta-1}\partial x}{\sqrt{(1-x^{4\alpha})}} \text{ et } Q = \int \frac{x^{\theta-1}\partial x}{\sqrt{(1-x^{4\alpha})}},$$

vnde

vnde ergo deducitur

$$\text{V.} \int \frac{x^{2\alpha-\theta-1}\partial x}{\sqrt{(1-x^{4\alpha})}} : \int \frac{x^{\theta-1}\partial x}{\sqrt{(1-x^{4\alpha})}} = \text{tang.} \frac{\theta\pi}{4\alpha}.$$

§. 22. Possumus etiam valores litterarum b et c inter se permutare, vt sit $b = 2\alpha$ et $c = 2\alpha - 2\theta$, manentibus $a = \theta$ et $n = 4\alpha$; tum autem fiet

$$P = \int \frac{x^{2\alpha+\theta-1}\partial x}{(1-x^{4\alpha})^{\frac{\alpha+\theta}{2\alpha}}} \text{ et } Q = \int \frac{x^{\theta-1}\partial x}{(1-x^{4\alpha})^{\frac{\alpha+\theta}{2\alpha}}},$$

hincque deducitur reductio

$$\text{VI.} \int \frac{x^{2\alpha+\theta-1}\partial x}{(1-x^{4\alpha})^{\frac{\alpha+\theta}{2\alpha}}} : \int \frac{x^{\theta-1}\partial x}{(1-x^{4\alpha})^{\frac{\alpha+\theta}{2\alpha}}} = \text{tang.} \frac{\theta\pi}{4\alpha},$$

quae autem, aeque ac praecedens, locum non habet, nisi sit $\alpha > \theta$.

§. 23. Quod si autem θ superet α, aequationem nostram in aliam formam transfundi oportet, signa vtrinque mutando, vnde prodibit

$$\int \frac{x^{2\alpha-\theta-1}\partial x}{l\,x} \cdot \frac{(1-x^{2\theta-2\alpha})(1-x^{2\alpha})}{1-x^{4\alpha}} = l \cot. \frac{\theta\pi}{4\alpha}.$$

Hic iam iterum duplex comparatio institui potest: primo scilicet sumamus $a = 2\alpha - \theta$, $b = 2\theta - 2\alpha$, $c = 2\alpha$ et $n = 4\alpha$, vnde formamus

$$P = \int \frac{x^{\theta-1}\partial x}{\sqrt{(1-x^{4\alpha})}} \text{ et } Q = \int \frac{x^{2\alpha-\theta-1}\partial x}{\sqrt{(1-x^{4\alpha})}}$$

hincque oritur septima relatio haec:

$$\text{VII.} \int \frac{x^{\theta-1}\partial x}{\sqrt{(1-x^{4\alpha})}} : \int \frac{x^{2\alpha-\theta-1}\partial x}{\sqrt{(1-x^{4\alpha})}} = \cot. \frac{\theta\pi}{4\alpha},$$

quae manifesto cum quinta congruit.

§. 24. Noua autem reductio obtinebitur, ſi ſtatuamus $b = 2\alpha$ et $c = 2\theta - 2\alpha$, manentibus $a = 2\alpha - \theta$ et $n = 4\alpha$; tum igitur erit

$$P = \int \frac{x^{4\alpha-\theta-1}\,\partial x}{(1-x^{4\alpha})^{\frac{3\alpha-\theta}{2\alpha}}} \text{ et } Q = \int \frac{x^{2\alpha-\theta-1}\,\partial x}{(1-x^{4\alpha})^{\frac{3\alpha-\theta}{2\alpha}}}.$$

Hinc vero colligitur reductio octaua

$$\text{VIII.} \int \frac{x^{4\alpha-\theta-1}\,\partial x}{(1-x^{4\alpha})^{\frac{3\alpha-\theta}{2\alpha}}} : \int \frac{x^{2\alpha-\theta-1}\,\partial x}{(1-x^{4\alpha})^{\frac{3\alpha-\theta}{2\alpha}}} = \cot.\frac{\theta\pi}{4\alpha}.$$

§. 25. Hic autem probe notandum eſt, quaternas poſteriores reductiones ex quatuor prioribus oriri, ſi in iſtis loco α ſcribatur θ, at 2α loco ν, ita vt quatuor poſteriores reductiones iam in prioribus contineantur; quamobrem ſiue quatuor priores, ſiue poſteriores, penitus omittere licebit, ita vt nobis tantum quatuor relinquantur, inter quas porro, quoniam tertia non diſcrepat a prima, tantum tres ſupererunt huiusmodi reductiones, quae quidem ex problemate tertio ſunt natae.

Euolutio formulae integralis §. 12. allatae:

$$\int \frac{\partial x}{x\,l\,x} \cdot \frac{(x^{\theta}-x^{\alpha})(1-x^{\nu-\alpha-\theta})}{1-x^{\nu}} \begin{bmatrix}\text{ab } x=0\\ \text{ad } x=1\end{bmatrix} = l\,\frac{\text{ſin.}\frac{\theta\pi}{\nu}}{\text{ſin.}\frac{\alpha\pi}{\nu}}.$$

§. 26. Iſta expreſſio iam congruit cum forma noſtra generali, neque idcirco vlteriori transformatione indiget. Hic quidem duo caſus eſſent diſtinguendi, prouti fuerit vel $\theta > \alpha$, vel $\theta < \alpha$; verum hac etiam diſtinctione carere poſſumus, propterea quod binae litterae α et θ inter ſe ſunt permutabiles: iis enim permutatis ſigna vtrinque inuertuntur. Hanc ob cauſſam, quoscunque valores habuerint ambae litterae α et θ, minorem ſemper littera θ, maiorem vero littera α deſignare licebit, vnde aequatio noſtra ita repraeſentabitur:

$\int$

$$\int \frac{x^{\theta - 1} \partial x}{l x} \cdot \frac{(1 - x^{\alpha - \theta})(1 - x^{\nu - \alpha - \theta})}{1 - x^{\nu}} = l \frac{\text{fin.} \frac{\theta \pi}{\nu}}{\text{fin.} \frac{\alpha \pi}{\nu}}.$$

§. 27. Nihilo vero minus duo casus distinguendi etiam hic occurrunt, prouti fuerit vel $\nu > \alpha + \theta$, vel $\nu < \alpha + \theta$. Sit igitur primo $\nu > \alpha + \theta$, et forma exposita manebit inuariata, quae denuo duplicem comparationem cum generali admittit. Primo igitur statuamus $a = \theta$, $b = \alpha - \theta$, $c = \nu - \alpha - \theta$ et $n = \nu$, qui valores nobis suppeditant

$$P = \int \frac{x^{\alpha - 1} \partial x}{(1 - x^{\nu})^{\frac{\alpha + \theta}{\nu}}} \text{ et } Q = \int \frac{x^{\theta - 1} \partial x}{(1 - x^{\nu})^{\frac{\alpha - \theta}{\nu}}}$$

sicque ex hac euolutione habebimus sequentem reductionem:

$$\text{I. } \int \frac{x^{\alpha - 1} \partial x}{(1 - x^{\nu})^{\frac{\alpha + \theta}{\nu}}} : \int \frac{x^{\theta - 1} \partial x}{(1 - x^{\nu})^{\frac{\alpha + \theta}{\nu}}} = \frac{\text{fin.} \frac{\theta \pi}{\nu}}{\text{fin.} \frac{\alpha \pi}{\nu}}.$$

§. 28. Secunda nascetur reductio permutandis litteris b et c, ita vt sit $a = \theta$, $b = \nu - \alpha - \theta$, $c = \alpha - \theta$, et $n = \nu$, vnde formantur hae formulae:

$$P = \int \frac{x^{\nu - \alpha - 1} \partial x}{(1 - x^{\nu})^{\frac{\nu - \alpha + \theta}{\nu}}} \text{ et } Q = \int \frac{x^{\theta - 1} \partial x}{(1 - x^{\nu})^{\frac{\nu - \alpha + \theta}{\nu}}}$$

quare secunda reductio hinc orta erit

$$\text{II. } \int \frac{x^{\nu - \alpha - 1} \partial x}{(1 - x^{\nu})^{\frac{\nu - \alpha + \theta}{\nu}}} : \int \frac{x^{\theta - 1} \partial x}{(1 - x^{\nu})^{\frac{\nu - \alpha + \theta}{\nu}}} = \frac{\text{fin.} \frac{\theta \pi}{\nu}}{\text{fin.} \frac{\alpha \pi}{\nu}},$$

quae duae reductiones postulant vt sit $\nu > \alpha + \theta$.

§. 29. Sin autem fuerit $\nu < \alpha + \theta$, ipsa aequationis forma hoc modo immutari debebit:

$$\int \frac{x^{\nu - \alpha - 1} \partial x}{l x} \cdot \frac{(1 - x^{\alpha - \theta})(1 - x^{\alpha + \theta - \nu})}{1 - x^{\nu}} = l \frac{\text{fin.} \frac{\alpha \pi}{\nu}}{\text{fin.} \frac{\theta \pi}{\nu}}$$

 vbi

vbi iterum gemina comparatio inſtitui poteſt. Sit igitur primo $a = \nu - \alpha$, $b = \alpha - \theta$, $c = \alpha + \theta - \nu$ et $n = \nu$, vnde oriuntur hae formulae:

$$P = \int \frac{x^{\nu - \theta - 1} \partial x}{(1 - x^\nu)^{\frac{2\nu - \alpha - \theta}{\nu}}} \text{ et } Q = \int \frac{x^{\nu - \alpha - 1} \partial x}{(1 - x^\nu)^{\frac{2\nu - \alpha - \theta}{\nu}}}.$$

Hinc igitur concluditur tertia reductio:

$$\text{III. } \int \frac{x^{\nu - \theta - 1} \partial x}{(1 - x^\nu)^{\frac{2\nu - \alpha - \theta}{\nu}}} : \int \frac{x^{\nu - \alpha - 1} \partial x}{(1 - x^\nu)^{\frac{2\nu - \alpha - \theta}{\nu}}} = \frac{\text{ſin.} \frac{\alpha \pi}{\nu}}{\text{ſin.} \frac{\theta \pi}{\nu}}.$$

§. 30. Denique ſtatuamus $a = \nu - \alpha$, $b = \alpha + \theta - \nu$, $c = \alpha - \theta$ et $n = \nu$, et formulae hinc ſequentes naſcentur:

$$P = \int \frac{x^{\theta - 1} \partial x}{(1 - x^\nu)^{\frac{\nu - \alpha + \theta}{\nu}}} \text{ et } Q = \int \frac{x^{\nu - \alpha - 1} \partial x}{(1 - x^\nu)^{\frac{\nu - \alpha + \theta}{\nu}}}$$

ita vt quarta hinc oriatur reductio:

$$\text{IV. } \int \frac{x^{\theta - 1} \partial x}{(1 - x^\nu)^{\frac{\nu - \alpha + \theta}{\nu}}} : \int \frac{x^{\nu - \alpha - 1} \partial x}{(1 - x^\nu)^{\frac{\nu - \alpha + \theta}{\nu}}} = \frac{\text{ſin.} \frac{\alpha \pi}{\nu}}{\text{ſin.} \frac{\theta \pi}{\nu}}.$$

§. 31. Quatuor igitur hic nacti ſumus formularum integralium paria, quae eandem inter ſe tenent rationem ac ſinus duorum angulorum; dum euolutiones praecedentes tantum tria huiusmodi paria praebuerant, quarum ratio P:Q tangenti cuiuspiam anguli aequatur, vbi quidem euidens eſt ſecundam et quartam inter ſe conuenire. Cum igitur huiusmodi reductiones altioris ſint indaginis, ac ſine dubio inſignem vſum habere queant, opere pretium erit eas clarius ob oculos exponere.

Problema.

§. 32. *Inuenire binas formulas integrales P et Q ab $x = 0$ ad $x = 1$ extenſas, vt fiat* $\frac{P}{Q} = \text{tang.} \frac{m \pi}{2 n}$.

So-

Solutio.

Triplici igitur modo hoc fieri poteſt, ſecundum euolutionem primam ſupra inſtitutam. I. Ex prima enim reductione, cum ſit cot. $\frac{\alpha\pi}{2\nu}$ = tang. $\frac{(\nu-\alpha)\pi}{2\nu}$, fiet $\nu-\alpha=m$ et $\nu=n$, ita vt ſit $\alpha=n-m$. Hinc igitur erit

$$P=\int\frac{x^{n-m-1}\partial x}{\sqrt{(1-x^{2n})}} \text{ et } Q=\int\frac{x^{m-1}\partial x}{\sqrt{(1-x^{2n})}},$$

quae ergo eſt ſolutio prima. II. Secunda reductio ſupra allata erat $\frac{P}{Q}$ = cot. $\frac{\alpha\pi}{2\nu}$ = tang. $\frac{(\nu-\alpha)\pi}{2\nu}$, vbi ergo iterum eſt $\alpha=n-m$ et $\nu=n$, ſicque ſecunda ſolutio huius problematis conſtabit his formulis:

$$P=\int\frac{x^{m+n-1}\partial x}{(1-x^{2n})^{\frac{2m+n}{2n}}} \text{ et } Q=\int\frac{x^{m-1}\partial x}{(1-x^{2n})^{\frac{2m+n}{2n}}}.$$

Hae autem formulae tantum valent, quando fuerit $m<\frac{1}{2}a$, ideoque ipſe angulus $\frac{m\pi}{2n}$ minor ſemirecto. III. Quoniam tertia reductio ibi allata cum prima conuenit, ex quarta, vbi erat $\frac{P}{Q}$ = tang. $\frac{\alpha\pi}{2\nu}$, ideoque pro noſtro caſu $\alpha=m$ et $\nu=n$, tertia ſolutio ita ſe habebit:

$$P=\int\frac{x^{m+n-1}\partial x}{(1-x^{2n})^{\frac{2m+n}{2n}}} \text{ et } Q=\int\frac{x^{m-1}\partial x}{(1-x^{2n})^{\frac{2m+n}{2n}}},$$

qui valores quoniam a praecedentibus non ſunt diuerſi, duas tantum adipiſcimur ſolutiones noſtri problematis, quarum ſecunda limitatione quadam indiget, ſcilicet $m<\frac{1}{2}n$, prior vero ad omnes angulos recto non maiores patet. Hae ergo duae ſolutiones ita repraeſententur:

$$\text{I. } P=\int\frac{x^{n-m-1}\partial x}{\sqrt{(1-x^{2n})}},\quad Q=\int\frac{x^{m-1}\partial x}{\sqrt{(1-x^{2n})}},$$

II.

$$\text{II. } P = \int \frac{x^{m+n-1}\,\partial x}{(1-x^{2n})^{\frac{2m+n}{2n}}}, \quad Q = \int \frac{x^{m-1}\,\partial x}{(1-x^{2n})^{\frac{2m+n}{2n}}};$$

ex vtraque igitur erit $\frac{P}{Q} = \text{tang.}\, \frac{m\pi}{2n}$.

Problema.

§. 33. *Inuenire binas formulas integrales* P *et* Q, *vt fiat* $\frac{P}{Q} = \frac{\text{fin.}\, \frac{p\pi}{2n}}{\text{fin.}\, \frac{q\pi}{2n}}$, *fiquidem ambo illa integralia ab* $x = 0$ *ad* $x = 1$ *extendantur.*

Solutio.

Ad hanc igitur formam transferamus quatuor illas reductiones in euolutione tertia traditas, et cum pro prima et fecunda effet $\frac{P}{Q} = \frac{\text{fin.}\, \frac{\theta\pi}{\nu}}{\text{fin.}\, \frac{\alpha\pi}{\nu}}$, pro forma hic praefcripta erit $\theta = p$, $\alpha = q$ et $\nu = 2n$, quamobrem hinc nancifcimur duas fequentes folutiones:

$$\text{I. } P = \int \frac{x^{q-1}\,\partial x}{(1-x^{2n})^{\frac{p+q}{2n}}} \text{ et } Q = \int \frac{x^{p-1}\,\partial x}{(1-x^{2n})^{\frac{p+q}{2n}}},$$

$$\text{II. } P = \int \frac{x^{2n-q-1}\,\partial x}{(1-x^{2n})^{\frac{2n-q+p}{2n}}} \text{ et } Q = \int \frac{x^{p-1}\,\partial x}{(1-x^{2n})^{\frac{2n-q+p}{2n}}}.$$

Tertia vero et quarta reductio habebant $\frac{P}{Q} = \frac{\text{fin.}\, \frac{\alpha\pi}{\nu}}{\text{fin.}\, \frac{\theta\pi}{\nu}}$, pro qua igitur erit $\alpha = p$, $\theta = q$, $\nu = 2n$, vnde ambae folutiones fequentes deducuntur:

$$\text{III. } P = \int \frac{x^{2n-q-1}\,\partial x}{(1-x^{2n})^{\frac{4n-p-q}{2n}}} \text{ et } Q = \int \frac{x^{2n-p-1}\,\partial x}{(1-x^{2n})^{\frac{4n-p-q}{2n}}},$$

IV.

$$\text{IV. } P = \int \frac{x^{q-1}\partial x}{(1-x^{2n})^{\frac{2n-p+q}{2n}}} \text{ et } Q = \int \frac{x^{2n-p-1}\partial x}{(1-x^{2n})^{\frac{2n-p+q}{2n}}}.$$

Hinc igitur patet quadruplici modo fieri posse $\frac{P}{Q} = \frac{\text{fin. } \frac{p\pi}{2n}}{\text{fin. } \frac{q\pi}{2n}}$.

Corollarium 1.

§. 34. Si assumamus $q = n$, vt fiat fin. $\frac{q\pi}{2n} = 1$, ideoque prodire debeat $\frac{P}{Q} =$ fin. $\frac{p\pi}{2n}$; pro hoc casu quatuor inuentae solutiones dabunt

$$\text{I. } P = \int \frac{x^{n-1}\partial x}{(1-x^{2n})^{\frac{p+n}{2n}}} \text{ et } Q = \int \frac{x^{p-1}\partial x}{(1-x^{2n})^{\frac{p+n}{2n}}},$$

$$\text{II. } P = \int \frac{x^{n-1}\partial x}{(1-x^{2n})^{\frac{n+p}{2n}}} \text{ et } Q = \int \frac{x^{p-1}\partial x}{(1-x^{2n})^{\frac{n+p}{2n}}},$$

$$\text{III. } P = \int \frac{x^{n-1}\partial x}{(1-x^{2n})^{\frac{3n-p}{2n}}} \text{ et } Q = \int \frac{x^{2n-p-1}\partial x}{(1-x^{2n})^{\frac{3n-p}{2n}}},$$

$$\text{IV. } P = \int \frac{x^{n-1}\partial x}{(1-x^{2n})^{\frac{3n-p}{2n}}} \text{ et } Q = \int \frac{x^{2n-p-1}\partial x}{(1-x^{2n})^{\frac{3n-p}{2n}}},$$

vbi ergo solutio prima cum secunda et tertia cum quarta conuenit.

Corollarium 2.

§. 35. Sumamus nunc esse $q = n - p$, vt fiat fin. $\frac{q\pi}{2n} =$ cos. $\frac{p\pi}{2n}$, ideoque prodire debeat $\frac{P}{Q} =$ tang. $\frac{p\pi}{2n}$. Pro hoc ergo casu quatuor solutiones inuentae euadent

$$\text{I. } P = \int \frac{x^{n-p-1}\partial x}{\sqrt{(1-x^{2n})}} \text{ et } Q = \int \frac{x^{p-1}\partial x}{\sqrt{(1-x^{2n})}},$$

$$\text{II. } P = \int \frac{x^{n+p-1}\,\partial x}{(1-x^{2n})^{\frac{n+2p}{2n}}} \text{ et } Q = \int \frac{x^{p-1}\,\partial x}{(1-x^{2n})^{\frac{n+2p}{2n}}},$$

$$\text{III. } P = \int \frac{x^{n+p-1}\,\partial x}{(1-x^{2n})^{\frac{3}{2}}} \text{ et } Q = \int \frac{x^{2n-p-1}\,\partial x}{(1-x^{2n})^{\frac{3}{2}}},$$

$$\text{IV. } P = \int \frac{x^{n-p-1}\,\partial x}{(1-x^{2n})^{\frac{3n-2p}{2n}}} \text{ et } Q = \int \frac{x^{2n-p-1}\,\partial x}{(1-x^{2n})^{\frac{3n-2p}{2n}}},$$

hincque erit $\frac{P}{Q} = \text{tang.} \frac{p\pi}{2n}$,

vbi prima et fecunda forma cum iis quas in praecedente problemate inuenimus prorfus conueniunt; tertia autem forma, ob $(1-x^{2n})^{\frac{3}{2}}$, fit incongrua, quia inde P et Q in infinitum excrefcerent; quarta autem nouam formam dare videtur.

METHO-