

DE  
MULTIPLICATIONE  
ANGVLORVM PER FACTORES  
EXPEDIENDA.

Auctore  
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§. 1.

Denotet  $\Phi$  angulum quemcunque propositum, sitque  $n\Phi$  eius  
multipulum, cuius tam sinum quam cosinum per factores  
exprimere oporteat. Ad hoc praestandum in subsidium vocetur  
formula imaginaria  $u = \cos. \Phi + \sqrt{-1} \sin. \Phi$ , eritque  
 $\frac{1}{u} = u^{-1} = \cos. \Phi - \sqrt{-1} \sin. \Phi$ , vnde ergo sequitur fore  
 $u + u^{-1} = 2 \cos. \Phi$  et  $u - u^{-1} = 2 \sqrt{-1} \sin. \Phi$ . Constat  
autem similem fore rationem omnium potestatum huius for-  
mulae, siquidem erit  $u^n = \cos. n\Phi + \sqrt{-1} \sin. n\Phi$  et  
 $u^{-n} = \cos. n\Phi - \sqrt{-1} \sin. n\Phi$ , atque hinc colligimus

$$u^n + u^{-n} = 2 \cos. n\Phi \text{ et } u^n - u^{-n} = 2 \sqrt{-1} \sin. n\Phi.$$

Hinc igitur sequitur fore

$$\cos. n\Phi = \frac{1}{2}(u^n + u^{-n}) \text{ et } \sin. n\Phi = \frac{1}{2\sqrt{-1}}(u^n - u^{-n}).$$

Per huiusmodi ergo formulas etiam tangentem et cotangen-  
tem, item secantem et cosecantem anguli multipli  $n\Phi$  expri-  
mere licebit, ita vt sufficiat eius tantum sinum et cosinum in  
factores resoluisse, quorum vtrumque hic seorsim euoluamus.

## I. Resolutio cosinus anguli multipli $n\phi$ in factores.

§. 2. Cum fit, vt modo vidimus,

$$\cos. n\phi = \frac{1}{2}(u^n + u^{-n}),$$

totum negotium huc redit, vt formula  $u^n + u^{-n}$  in suos factores resoluatur. Facile autem perspicitur, singulos huius formulae factores talem formam esse habituros:  $u - 2 \cos. \omega + u^{-1}$ , ac numerum talium factorum esse debere  $= n$ ; quandoquidem si  $n$  huiusmodi factores in se inuicem multiplicentur, summa potestas positua ipsius  $u$  vtique prodibit  $u^n$ , summa autem potestas negatiua  $= u^{-n}$ ; ex quo necesse est, vt omnes potestates intermediae ipsius  $u$  se mutuo destruant. Ex hac igitur conditione singulos angulos  $\omega$  definire oportebit, id quod vtique insignem laborem postularet; vnde hanc inuestigationem ex ipsa indole factorum deriuemus.

§. 3. Hic autem perspicuum est, si haec formula:  $u - 2 \cos. \omega + u^{-1}$ , fuerit factor formae propositae  $u^n + u^{-n}$ , angulum  $\omega$  ita comparatum esse debere, vt si statueretur iste factor  $u - 2 \cos. \omega + u^{-1} = 0$ , tum etiam ipsa forma proposita  $u^n + u^{-n}$  ad nihilum redigeretur. Neque vero hic putandum est, istam formulam reuera nihilo aequalem supponi, sed tantum hypothetice asseritur, si ista formula, esset  $= 0$ , etiamsi id fieri nequeat, tum etiam ipsam formam propositam in nihilum esse abituram; quemadmodum scilicet nouimus, formulae  $1 - x x$  factorem esse  $1 - x$ , propterea quod posito  $1 - x = 0$ , siue  $x = 1$ , etiam fiat  $1 - x x = 0$ , etiamsi quantitas  $x$  fortasse nunquam vnitati fiat aequalis.

§. 4. Iam vero si esset  $u - 2 \cos. \omega + u^{-1} = 0$ , inde foret  $u = \cos. \omega + \sqrt{-1} \sin. \omega$ , quamobrem hic valor, in formam

nam  $u^n + u^{-n}$  introductus, eam ad nihilum redigere deberet; tum autem foret, vt supra notauimus:

$$u^n = \cos. n\omega + \sqrt{-1} \sin. n\omega \text{ et}$$

$$u^{-n} = \cos. n\omega - \sqrt{-1} \sin. n\omega,$$

ficque forma proposita fiet  $= 2 \cos. n\omega$ , quae ergo quantitas necessario euanescere debet, quia aliter formula assumpta factor formae propositae esse non posset.

§. 5. Hinc igitur patet angulum  $\omega$  ita accipi debere, vt eius multipli  $n\omega$  cosinus euanescat; vnde manifestum est, si  $\rho$  designet angulum rectum, tum illi conditioni satisfieri, si statuatur vel  $n\omega = \rho$ ; vel  $n\omega = 3\rho$ ; vel  $n\omega = 5\rho$ ; vel  $n\omega = 7\rho$ ; etc. ex quo valores idonei pro angulo quaesito  $\omega$  erunt sequentes:  $1^\circ. \frac{\rho}{n}$ ;  $2^\circ. \frac{3\rho}{n}$ ;  $3^\circ. \frac{5\rho}{n}$ ;  $4^\circ. \frac{7\rho}{n}$ ; etc. et in genere  $\frac{i\rho}{n}$ , denotante  $i$  numerum imparem quemcunque; ita vt horum valorum numerus reuera sit infinitus, cum tamen pro nostro instituto tantum  $n$  valoribus diuersis indigeamus.

§. 6. Quanquam autem numerus valorum pro  $\omega$  inuentorum sit infinitus, tamen inter eos innumerabiles dantur, qui pro  $\cos. \omega$  eundem valorem producent. Cum enim omnibus istis angulis:  $\psi$ ,  $4\rho + \psi$ ,  $8\rho + \psi$ , etc. idem cosinus conueniat, etiam omnes isti anguli:  $\frac{\rho}{n}$ ,  $\frac{(4n+1)\rho}{n}$ ,  $\frac{(8n+1)\rho}{n}$ , etc. communi gaudebunt cosinu, similique modo etiam omnes hi anguli:  $\frac{3\rho}{n}$ ,  $\frac{(4n+3)\rho}{n}$ ,  $\frac{(8n+3)\rho}{n}$ , etc., tum vero etiam isti:  $\frac{5\rho}{n}$ ,  $\frac{(4n+5)\rho}{n}$ ,  $\frac{(8n+5)\rho}{n}$ , etc. communi cosinu erunt praediti; vnde ex quolibet ordine vnicum tantum valorem, eumque simplicissimum, sumi conueniet, donec eorum numerus fiat  $= n$ .

§. 7. Ex his igitur colligimus, omnes valores idoneos pro angulo  $\omega$  assumendos ordine ita progredi:

$$\begin{array}{cccc}
 1. & 2. & 3. & 4. \\
 \frac{\rho}{n}; & \frac{3\rho}{n}; & \frac{5\rho}{n}; & \frac{7\rho}{n}; \text{ etc.}
 \end{array}$$

quorum numerus cum debeat esse  $= n$ , ultimus eorum erit  $\frac{(2n-1)\rho}{n}$ . Si enim ulterius progredi vellemus, ad eiusmodi angulos perueniremus, quorum cosinus iam in praecedentibus occurrerent. Anguli enim proxime sequentis  $\frac{(2n+1)\rho}{n}$  cosinus foret ipsius ultimi  $\frac{(2n-1)\rho}{n}$  cosinui aequalis: in genere enim hi duo anguli  $\psi$  et  $4\rho - \psi$  communem habent cosinum; vnde si fuerit  $\psi = \frac{(2n+1)\rho}{n}$ , erit  $4\rho - \psi = \frac{(2n-1)\rho}{n}$ , qui est terminus ultimus. Simili modo sequentium secundus  $\frac{(2n+3)\rho}{n}$ , eundem habet cosinum quem angulus  $\frac{(2n-3)\rho}{n}$  habet, qui in nostro ordine est penultimus. Eodem modo sequentium tertius  $\frac{(2n+5)\rho}{n}$  cum nostro antepenultimo  $\frac{(2n-5)\rho}{n}$  communem habebit cosinum; quod idem de reliquis sequentibus est tenendum, quippe quorum omnium cosinus iam in nostro ordine occurrunt; vnde patet omnes valores idoneos anguli  $\omega$ , quorum numerus est  $= n$ , contineri in hac serie:

$$\begin{array}{cccccccc}
 1. & 2. & 3. & 4. & . & . & . & n \\
 \frac{\rho}{n}; & \frac{3\rho}{n}; & \frac{5\rho}{n}; & \frac{7\rho}{n}; & . & . & . & \frac{(2n-1)\rho}{n} .
 \end{array}$$

§. 8. Cum igitur, si pro  $\omega$  accipiatur valor quicumque huius progressionis, ista formula  $u - 2 \cos. \omega + u^{-1}$  certe sit factor formae  $u^n + u^{-n}$ , restituamus nunc valorem initio assumptum  $u = \cos. \Phi + \sqrt{-1} \sin. \Phi$ , et quia hinc  $u + u^{-1} = 2 \cos. \Phi$ , huius formae  $u^n + u^{-n}$  factor quicumque erit  $2 (\cos. \Phi - \cos. \omega)$ , quare si loco  $\omega$  successive omnes eius valores scribamus, omnes factores obtinebimus, quorum numerus cum sit  $= n$ , forma nostra  $u^n + u^{-n}$  aequabitur huic producto ex  $n$  factoribus constanti:

$$2^n (\cos. \Phi - \cos. \frac{\rho}{n}) (\cos. \Phi - \cos. \frac{3\rho}{n}) (\cos. \Phi - \cos. \frac{5\rho}{n}) \times \\ \times (\cos. \Phi - \cos. \frac{7\rho}{n}) . . . . (\cos. \Phi - \cos. \frac{(2n-1)\rho}{n}).$$

§. 9. Cum igitur sit  $\cos. n\Phi = \frac{1}{2}(u^n + u^{-n})$ , si hoc productum substituamus, cosinus anguli  $n\Phi$  sequenti modo per productum ex  $n$  factoribus compositum exprimetur:

$$\cos. n\Phi = 2^{n-1} (\cos. \Phi - \cos. \frac{\rho}{n}) (\cos. \Phi - \cos. \frac{3\rho}{n}) \times \\ \times (\cos. \Phi - \cos. \frac{5\rho}{n}) . . . . (\cos. \Phi - \cos. \frac{(2n-1)\rho}{n}),$$

unde sequentes deducimus resolutiones speciales, dum loco  $n$  ordine numeros 1, 2, 3, 4, etc. assumimus:

$$\cos. 1\Phi = \cos. \Phi - \cos. \frac{\rho}{1} = \cos. \Phi.$$

$$\cos. 2\Phi = 2 (\cos. \Phi - \cos. \frac{1}{2}\rho) (\cos. \Phi - \cos. \frac{3}{2}\rho).$$

$$\cos. 3\Phi = 4 (\cos. \Phi - \cos. \frac{1}{3}\rho) (\cos. \Phi - \cos. \frac{2}{3}\rho) (\cos. \Phi - \cos. \frac{5}{3}\rho).$$

$$\cos. 4\Phi = 8 (\cos. \Phi - \cos. \frac{1}{4}\rho) (\cos. \Phi - \cos. \frac{3}{4}\rho) (\cos. \Phi - \cos. \frac{5}{4}\rho) \times \\ \times (\cos. \Phi - \cos. \frac{7}{4}\rho).$$

$$\cos. 5\Phi = 16 (\cos. \Phi - \cos. \frac{1}{5}\rho) (\cos. \Phi - \cos. \frac{3}{5}\rho) (\cos. \Phi - \cos. \frac{4}{5}\rho) \times \\ \times (\cos. \Phi - \cos. \frac{7}{5}\rho) (\cos. \Phi - \cos. \frac{9}{5}\rho)$$

$$\cos. 6\Phi = 32 (\cos. \Phi - \cos. \frac{1}{6}\rho) (\cos. \Phi - \cos. \frac{5}{6}\rho) (\cos. \Phi - \cos. \frac{7}{6}\rho) \times \\ \times (\cos. \Phi - \cos. \frac{11}{6}\rho) (\cos. \Phi - \cos. \frac{13}{6}\rho) (\cos. \Phi - \cos. \frac{17}{6}\rho).$$

etc.

etc.

§. 10. Quodsi omnes istos factores attentius consideremus, reperiemus binos factores ab extremis aequae distantes commode in vnum contrahi posse. Cum enim duorum angulorum, quorum summa est  $180^\circ = 2\rho$ , cosinus sint aequales, sed contrario signō affecti, ob angulum  $\frac{(2n-1)\rho}{n} = 2\rho - \frac{\rho}{n}$ , eius cosinus erit  $= -\cos. \frac{\rho}{n}$ , ideoque vltimus factor  $= \cos. \Phi$   
+  $\cos.$

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+  $\cos. \frac{\rho}{n}$ , qui ductus in primum praebet productum  $\cos. \Phi^2 - (\cos. \frac{\rho}{n})^2$ . Simili modo pro penultimo factore, ob  $\frac{(2n-3)\rho}{n} = 2\rho - \frac{3\rho}{n}$ , huius anguli cosinus erit  $= -\cos. \frac{3\rho}{n}$ , ideoque ipse factor penultimus  $= \cos. \Phi + \cos. \frac{3\rho}{n}$ , qui ductus in secundum dabit productum  $\cos. \Phi^2 - (\cos. \frac{3\rho}{n})^2$ . Eodem modo factor tertius cum antepenultimo coniunctus dabit hoc productum  $\cos. \Phi^2 - (\cos. \frac{5\rho}{n})^2$ . Hoc igitur modo numerus factorum ad semissem reducetur, si numerus  $n$  fuerit par, sin autem fuerit impar, facta hac contractione solitarius relinquetur terminus medius, qui semper erit  $\cos. \Phi$ , ob  $\cos. \rho = 0$ .

§. 11. Quodsi ergo hoc modo resolutionem in factores exhibere velimus, ea pro casibus simplicioribus ita se habebit:

$$\cos. \Phi = \cos. \Phi.$$

$$\cos. 2\Phi = 2[\cos. \Phi^2 - (\cos. \frac{1}{2}\rho)^2].$$

$$\cos. 3\Phi = 4[\cos. \Phi^2 - (\cos. \frac{1}{3}\rho)^2] \cos. \Phi.$$

$$\cos. 4\Phi = 8[\cos. \Phi^2 - (\cos. \frac{1}{4}\rho)^2][\cos. \Phi^2 - (\cos. \frac{3}{4}\rho)^2].$$

$$\cos. 5\Phi = 16[\cos. \Phi^2 - (\cos. \frac{1}{5}\rho)^2][\cos. \Phi^2 - (\cos. \frac{3}{5}\rho)^2] \cos. \Phi.$$

$$\cos. 6\Phi = 32[\cos. \Phi^2 - (\cos. \frac{1}{6}\rho)^2][\cos. \Phi^2 - (\cos. \frac{3}{6}\rho)^2] \times$$

$$\times [\cos. \Phi^2 - (\cos. \frac{5}{6}\rho)^2].$$

$$\cos. 7\Phi = 64[\cos. \Phi^2 - (\cos. \frac{1}{7}\rho)^2][\cos. \Phi^2 - (\cos. \frac{3}{7}\rho)^2] \times$$

$$\times [\cos. \Phi^2 - (\cos. \frac{5}{7}\rho)^2] \cos. \Phi.$$

$$\cos. 8\Phi = 128[\cos. \Phi^2 - (\cos. \frac{1}{8}\rho)^2][\cos. \Phi^2 - (\cos. \frac{3}{8}\rho)^2] \times$$

$$\times [\cos. \Phi^2 - (\cos. \frac{5}{8}\rho)^2][\cos. \Phi^2 - (\cos. \frac{7}{8}\rho)^2].$$

etc.

etc.

§. 12. Istos autem factores in formam adhuc concinniorem transmutare licet. Cum enim in genere fit

$$\text{cof. } \psi^2 = \frac{1}{2} + \frac{1}{2} \text{cof. } 2\psi, \text{ erit}$$

$$\text{cof. } \Phi^2 - \text{cof. } \psi^2 = \frac{1}{2} \text{cof. } 2\Phi - \frac{1}{2} \text{cof. } 2\psi;$$

tum vero etiam constat esse in genere

$$\text{cof. } 2\Phi - \text{cof. } 2\psi = 2 \text{fin. } (\Phi + \psi) \text{fin. } (\psi - \Phi),$$

consequenter

$$\text{cof. } \Phi^2 - \text{cof. } \psi^2 = \text{fin. } (\Phi + \psi) \text{fin. } (\psi - \Phi).$$

§. 13. Quodsi ergo ista reductione in superioribus formulis utamur, singuli factores ad simplices sinus reuocabuntur; habebimus enim:

$$\text{cof. } \Phi = \text{cof. } \Phi.$$

$$\text{cof. } 2\Phi = 2 \text{fin. } \left(\frac{1}{2}\varrho + \Phi\right) \text{fin. } \left(\frac{1}{2}\varrho - \Phi\right)$$

$$\text{cof. } 3\Phi = 4 \text{fin. } \left(\frac{1}{3}\varrho + \Phi\right) \text{fin. } \left(\frac{1}{3}\varrho - \Phi\right) \text{cof. } \Phi.$$

$$\text{cof. } 4\Phi = 8 \text{fin. } \left(\frac{1}{4}\varrho + \Phi\right) \text{fin. } \left(\frac{1}{4}\varrho - \Phi\right) \text{fin. } \left(\frac{3}{4}\varrho + \Phi\right) \text{fin. } \left(\frac{3}{4}\varrho - \Phi\right).$$

$$\text{cof. } 5\Phi = 16 \text{fin. } \left(\frac{1}{5}\varrho + \Phi\right) \text{fin. } \left(\frac{1}{5}\varrho - \Phi\right) \text{fin. } \left(\frac{3}{5}\varrho + \Phi\right) \text{fin. } \left(\frac{3}{5}\varrho - \Phi\right) \times \\ \times \text{cof. } \Phi$$

$$\text{cof. } 6\Phi = 32 \text{fin. } \left(\frac{1}{6}\varrho + \Phi\right) \text{fin. } \left(\frac{1}{6}\varrho - \Phi\right) \text{fin. } \left(\frac{3}{6}\varrho + \Phi\right) \text{fin. } \left(\frac{3}{6}\varrho - \Phi\right) \times \\ \times \text{fin. } \left(\frac{5}{6}\varrho + \Phi\right) \text{fin. } \left(\frac{5}{6}\varrho - \Phi\right).$$

etc.

etc.

vnde in genere concludimus fore

$$\text{cof. } n\Phi = 2^{n-1} \text{fin. } \left(\frac{\varrho}{n} + \Phi\right) \text{fin. } \left(\frac{\varrho}{n} - \Phi\right) \text{fin. } \left(\frac{3\varrho}{n} + \Phi\right) \times \\ \times \text{fin. } \left(\frac{3\varrho}{n} - \Phi\right) \text{fin. } \left(\frac{5\varrho}{n} + \Phi\right) \text{fin. } \left(\frac{5\varrho}{n} - \Phi\right) \text{ etc.}$$

Vbi obseruetur, si  $n$  fuerit numerus impar, tum ultimo factori accedere cof.  $\Phi$ . Caeterum hos factores eo vsque produci oportet, donec eorum numerus fiat  $= n$ .

§. 14. Si quis maluerit omnes istos factores etiam per cosinus exhibere, talis transformatio in promptu est. Cum enim sit  $\sin. \psi = \text{cof.} (\varrho - \psi)$ , nostrae resolutiones sequenti modo procedent:

$$\text{cof.} \Phi = \text{cof.} \Phi.$$

$$\text{cof.} 2 \Phi = 2 \text{cof.} \left(\frac{1}{2} \varrho - \Phi\right) \text{cof.} \left(\frac{1}{2} \varrho + \Phi\right).$$

$$\text{cof.} 3 \Phi = 4 \text{cof.} \left(\frac{2}{3} \varrho - \Phi\right) \text{cof.} \left(\frac{2}{3} \varrho + \Phi\right) \text{cof.} \Phi.$$

$$\text{cof.} 4 \Phi = 8 \text{cof.} \left(\frac{3}{4} \varrho - \Phi\right) \text{cof.} \left(\frac{3}{4} \varrho + \Phi\right) \text{cof.} \left(\frac{1}{4} \varrho - \Phi\right) \text{cof.} \left(\frac{1}{4} \varrho + \Phi\right)$$

$$\text{cof.} 5 \Phi = 16 \text{cof.} \left(\frac{4}{5} \varrho - \Phi\right) \text{cof.} \left(\frac{4}{5} \varrho + \Phi\right) \text{cof.} \left(\frac{2}{5} \varrho - \Phi\right) \times \\ \times \text{cof.} \left(\frac{2}{5} \varrho + \Phi\right) \text{cof.} \Phi.$$

$$\text{cof.} 6 \Phi = 32 \text{cof.} \left(\frac{5}{6} \varrho - \Phi\right) \text{cof.} \left(\frac{5}{6} \varrho + \Phi\right) \text{cof.} \left(\frac{3}{6} \varrho - \Phi\right) \text{cof.} \left(\frac{3}{6} \varrho + \Phi\right) \times \\ \times \text{cof.} \left(\frac{1}{6} \varrho - \Phi\right) \text{cof.} \left(\frac{1}{6} \varrho + \Phi\right).$$

$$\text{cof.} 7 \Phi = 64 \text{cof.} \left(\frac{6}{7} \varrho - \Phi\right) \text{cof.} \left(\frac{6}{7} \varrho + \Phi\right) \text{cof.} \left(\frac{4}{7} \varrho - \Phi\right) \text{cof.} \left(\frac{4}{7} \varrho + \Phi\right) \times \\ \times \text{cof.} \left(\frac{2}{7} \varrho - \Phi\right) \text{cof.} \left(\frac{2}{7} \varrho + \Phi\right) \text{cof.} \Phi.$$

$$\text{cof.} 8 \Phi = 128 \text{cof.} \left(\frac{7}{8} \varrho - \Phi\right) \text{cof.} \left(\frac{7}{8} \varrho + \Phi\right) \text{cof.} \left(\frac{5}{8} \varrho - \Phi\right) \text{cof.} \left(\frac{5}{8} \varrho + \Phi\right) \times \\ \times \text{cof.} \left(\frac{3}{8} \varrho - \Phi\right) \text{cof.} \left(\frac{3}{8} \varrho + \Phi\right) \text{cof.} \left(\frac{1}{8} \varrho - \Phi\right) \times \\ \times \text{cof.} \left(\frac{1}{8} \varrho + \Phi\right).$$

§. 15. Quo igitur ordo in his expressionibus clarius ob oculos ponatur, duos casus distingui conueniet, quos hic seorsim referemus, prouti numerus  $n$  fuerit vel par vel impar.

Casu quo  $n = 2i$ , erit:

$$\text{cof.} 2i \Phi = 2^{2i-1} \text{cof.} \left(\frac{\varrho}{2i} + \Phi\right) \text{cof.} \left(\frac{\varrho}{2i} - \Phi\right) \text{cof.} \left(\frac{3\varrho}{2i} + \Phi\right) \times \\ \times \text{cof.} \left(\frac{3\varrho}{2i} - \Phi\right) \text{cof.} \left(\frac{5\varrho}{2i} + \Phi\right) \text{cof.} \left(\frac{5\varrho}{2i} - \Phi\right) \dots \times \\ \times \text{cof.} \left(\frac{(2i-1)\varrho}{2i} + \Phi\right) \text{cof.} \left(\frac{(2i-1)\varrho}{2i} - \Phi\right).$$

Casu



Casu quo  $n = 2i + 1$ , erit

$$\begin{aligned} \text{cof. } (2i + 1) \Phi &= 2^{2i} \text{cof. } \Phi \text{cof. } \left(\frac{2\rho}{2i+1} + \Phi\right) \times \\ &\times \text{cof. } \left(\frac{2\rho}{2i+1} - \Phi\right) \text{cof. } \left(\frac{4\rho}{2i+1} + \Phi\right) \text{cof. } \left(\frac{4\rho}{2i+1} - \Phi\right) \times \\ &\times \text{cof. } \left(\frac{6\rho}{2i+1} + \Phi\right) \text{cof. } \left(\frac{6\rho}{2i+1} - \Phi\right) \dots \times \\ &\times \text{cof. } \left(\frac{2i\rho}{2i+1} + \Phi\right) \text{cof. } \left(\frac{2i\rho}{2i+1} - \Phi\right). \end{aligned}$$

## II. Resolutio sinus anguli multipli $n\Phi$ in factores.

§. 16. Manentibus denominationibus initio stabilitis iam notauimus esse  $\sin. n\Phi = \frac{1}{2\sqrt{-1}}(u^n - u^{-n})$ ; vnde nobis incumbit in factores huius formae  $u^n - u^{-n}$  inquirere. Hic autem statim elucet, cunctos eius factores non habere posse formam supra arctam  $u - 2 \text{cof. } \omega + u^{-1}$ , quandoquidem ex  $n$  huiusmodi factoribus maxima potestas negatiua proditura esset  $+u^{-n}$ , cum tamen hic sit  $-u^{-n}$ ; facile autem patet formae nostrae propositae  $u^n - u^{-n}$  vnum factorem certe esse  $u - u^{-1}$ , quandoquidem  $uu - 1$  semper est factor formae  $u^{2n} - 1$ . Hoc igitur primo factore constituto, reliquorum, quorum ergo numerus erit  $= 2n - 1$ , forma tuto assumi potest supra vsurpata  $u - 2 \text{cof. } \omega + u^{-1}$ , vbi ergo angulum  $\omega$  ita comparatum esse oportet, vt si poneretur  $u - 2 \text{cof. } \omega + u^{-1} = 0$ , id quod fit capiendo  $u = \text{cof. } \omega + \sqrt{-1} \sin. \omega$ , ipsa forma proposita  $u^n - u^{-n}$  ad nihilum redigeretur.

§. 17. Supra autem iam obseruauimus, si fuerit  $u = \text{cof. } \omega + \sqrt{-1} \sin. \omega$ , tum prodire  
 $u^n - u^{-n} = 2 \sqrt{-1} \sin. \omega,$

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quem

quem ergo valorem nihilo aequalem esse oportet, unde manente  $\varrho$  caractere anguli recti, quia omnium horum angulorum  $2\varrho, 4\varrho, 6\varrho, 8\varrho$ , et in genere  $2i\varrho$ , sinus evanescent, idonei anguli pro  $\omega$  accipiendi hanc progressionem constituent:  $\frac{2\varrho}{n}; \frac{4\varrho}{n}; \frac{6\varrho}{n}; \frac{8\varrho}{n}$ ; etc. quorum quidem numerus est infinitus; verum ob rationes iam ante allegatas hic tantum priores, numero  $n-1$ , accipi conveniet, ita ut ultimus valorum idoneorum futurus sit  $\frac{2(n-1)\varrho}{n}$ .

§. 18. Hic quidem videtur, primo loco statui debuiffe angulum  $\frac{2\varrho}{n}$ ; verum hinc nasceretur factor  $u - 2 + u^{-1}$ , qui involuit quadratum, dum est  $\frac{1}{u}(u-1)^2$ , eiusque radix  $u-1$ , quae his casibus tantum accipi debet, iam in nostro factore primo  $u - u^{-1}$  continetur. Praeterea vero in isto factore primo etiam continetur  $u+1$ , qui autem in valore nostro ordinem proxime sequente, qui esset  $\frac{2n\varrho}{n} = 2\varrho = \omega$ , contineretur, siquidem hinc resultaret factor

$$u + 2 + u^{-1} = \frac{1}{u}(u+1)^2.$$

Caeterum vero patet, si angulos illos pro  $\omega$  datos ultra terminum praescriptum continuare vellemus, eorum cosinus iam in antecedentibus contineri. Veluti si sumeremus  $\omega = \frac{2(n+1)\varrho}{n}$ , foret  $4\varrho - \omega = \frac{2(n-1)\varrho}{n}$ , qui in assignatis valoribus iam continetur.

§. 19. Restituamus nunc loco  $u$  valorem assumtum  $\text{cof. } \Phi - \sqrt{-1} \text{ sin. } \Phi$ , ac pro primo factore habebimus  $u - u^{-1} = 2\sqrt{-1} \text{ sin. } \Phi$ . Deinde vero pro quolibet factore reliquorum fiet

$$u - 2 \text{ cof. } \omega + u^{-1} = 2(\text{cof. } \Phi - \text{cof. } \omega),$$

quamobrem si loco  $\omega$  successive valores debitos scribamus, forma

forma proposita  $u^n - u^{-n}$  sequenti producto acquabitur;

$$2^n \sqrt{-1} \sin. \Phi (\cos. \Phi - \cos. \frac{2\varrho}{n}) (\cos. \Phi - \cos. \frac{4\varrho}{n}) \times \\ \times (\cos. \Phi - \cos. \frac{6\varrho}{n}) (\cos. \Phi - \cos. \frac{8\varrho}{n}) ;$$

quorum factorum ultimus erit  $\cos. \Phi - \cos. (\frac{2(n-1)\varrho}{n})$ .

§. 20. Cum igitur sit  $\sin. n \Phi = \frac{1}{2^n \sqrt{-1}} (u^n - u^{-n})$ , hoc valore, quem modo inuenimus, substituto, reperiemus:

$$\sin. n \Phi = 2^{n-1} \sin. \Phi (\cos. \Phi - \cos. \frac{2\varrho}{n}) (\cos. \Phi - \cos. \frac{4\varrho}{n}) \times \\ \times (\cos. \Phi - \cos. \frac{6\varrho}{n}) \dots (\cos. \Phi - \cos. \frac{2(n-1)\varrho}{n}),$$

vnde sequentes casus speciales deriuasse iuuabit:

$$\sin. \Phi = \sin. \Phi.$$

$$\sin. 2 \Phi = 2 \sin. \Phi \cos. \Phi.$$

$$\sin. 3 \Phi = 4 \sin. \Phi (\cos. \Phi - \cos. \frac{2}{3}\varrho) (\cos. \Phi - \cos. \frac{4}{3}\varrho).$$

$$\sin. 4 \Phi = 8 \sin. \Phi (\cos. \Phi - \cos. \frac{2}{4}\varrho) (\cos. \Phi - \cos. \frac{4}{4}\varrho) (\cos. \Phi - \cos. \frac{6}{4}\varrho).$$

$$\sin. 5 \Phi = 16 \sin. \Phi (\cos. \Phi - \cos. \frac{2}{5}\varrho) (\cos. \Phi - \cos. \frac{4}{5}\varrho) (\cos. \Phi - \cos. \frac{6}{5}\varrho) \times \\ \times (\cos. \Phi - \cos. \frac{8}{5}\varrho).$$

$$\sin. 6 \Phi = 32 \sin. \Phi (\cos. \Phi - \cos. \frac{2}{6}\varrho) (\cos. \Phi - \cos. \frac{4}{6}\varrho) (\cos. \Phi - \cos. \frac{6}{6}\varrho) \times \\ \times (\cos. \Phi - \cos. \frac{8}{6}\varrho) (\cos. \Phi - \cos. \frac{10}{6}\varrho).$$

$$\sin. 7 \Phi = 64 \sin. \Phi (\cos. \Phi - \cos. \frac{2}{7}\varrho) (\cos. \Phi - \cos. \frac{4}{7}\varrho) (\cos. \Phi - \cos. \frac{6}{7}\varrho) \times \\ \times (\cos. \Phi - \cos. \frac{8}{7}\varrho) (\cos. \Phi - \cos. \frac{10}{7}\varrho) \times \\ \times (\cos. \Phi - \cos. \frac{12}{7}\varrho).$$

$$\sin. 8 \Phi = 128 \sin. \Phi (\cos. \Phi - \cos. \frac{2}{8}\varrho) (\cos. \Phi - \cos. \frac{4}{8}\varrho) (\cos. \Phi - \cos. \frac{6}{8}\varrho) \times \\ \times (\cos. \Phi - \cos. \frac{8}{8}\varrho) (\cos. \Phi - \cos. \frac{10}{8}\varrho) \times \\ \times (\cos. \Phi - \cos. \frac{12}{8}\varrho) (\cos. \Phi - \cos. \frac{14}{8}\varrho).$$

etc.

etc.

§. 21. Seposito iam primo factore  $\sin. \Phi$ , in reliquis, quorum numerus est  $n - 1$ , idem vsu venit, quod ante, ut scilicet illi factores ab extremis aequidistantes in vnum contrahi queant. Cum enim vltimus factor sit  $\cos. \frac{2(n-1)\rho}{n}$ , ob angulum  $\frac{2(n-1)\rho}{n} = 2\rho - \frac{2\rho}{n}$ , eius cosinus erit  $= -\cos. \frac{2\rho}{n}$ , ita vt iste factor sit  $\cos. \Phi + \cos. \frac{2\rho}{n}$ , qui ergo per primum multiplicatus dat productum  $\cos. \Phi^2 - (\cos. \frac{2\rho}{n})^2$ , Simili modo factores secundus et penultimus contrahentur in hoc productum:  $\cos. \Phi^2 - (\cos. \frac{4\rho}{n})^2$ , hocque modo factorum numerus ad dimidium reuocabitur, siquidem  $n - 1$  fuerit numerus par, ideoque  $n$  impar; casibus autem, quibus  $n - 1$  est impar, ideoque  $n$  par, insuper accedet factor medius, qui semper est  $\cos. \Phi$ ; vnde hoc productum ita se habebit:

$$\sin. n \Phi = 2^{n-1} \sin. \Phi [\cos. \Phi^2 - (\cos. \frac{2\rho}{n})^2] [\cos. \Phi^2 - (\cos. \frac{4\rho}{n})^2] \times \\ \times [\cos. \Phi^2 - \cos. \frac{6\rho}{n})^2] [\cos. \Phi^2 - \cos. \frac{8\rho}{n})^2] \text{ etc.}$$

§. 22. Supra autem vidimus esse

$$\cos. \Phi^2 - \cos. \Psi^2 = \frac{1}{2} \cos. 2\Phi - \frac{1}{2} \cos. 2\Psi,$$

quae formula denuo resoluitur in hos duos factores:

$$\sin. (\Phi + \Psi) \sin. (\Psi - \Phi),$$

sicque hoc modo omnium factorum numerus crit  $n - 1$ . Hac igitur resolutione adhibita reperietur:

$$\sin. n \Phi = 2^{n-1} \sin. \Phi \sin. (\frac{2\rho}{n} + \Phi) \sin. (\frac{2\rho}{n} - \Phi) \sin. (\frac{4\rho}{n} + \Phi) \times \\ \times \sin. (\frac{4\rho}{n} - \Phi) \sin. (\frac{6\rho}{n} + \Phi) \sin. (\frac{6\rho}{n} - \Phi) \text{ etc.}$$

si modo obseruetur casibus, quibus  $n$  est numerus par, adiungendum esse factorem  $\cos. \Phi$ .

§. 23. Quo indoles huius expressionis clarius perspiciatur, sequentes casus speciales euoluamus :

$$\begin{aligned} \text{fin. } \Phi &= \text{fin. } \Phi. \\ \text{fin. } 2\Phi &= 2 \text{ fin. } \Phi \text{ cof. } \Phi. \\ \text{fin. } 3\Phi &= 4 \text{ fin. } \Phi \text{ fin. } \left(\frac{2}{3}\varrho + \Phi\right) \text{ fin. } \left(\frac{2}{3}\varrho - \Phi\right). \\ \text{fin. } 4\Phi &= 8 \text{ fin. } \Phi \text{ fin. } \left(\frac{2}{4}\varrho + \Phi\right) \text{ fin. } \left(\frac{2}{4}\varrho - \Phi\right) \text{ cof. } \Phi. \\ \text{fin. } 5\Phi &= 16 \text{ fin. } \Phi \text{ fin. } \left(\frac{2}{5}\varrho + \Phi\right) \text{ fin. } \left(\frac{2}{5}\varrho - \Phi\right) \text{ fin. } \left(\frac{4}{5}\varrho + \Phi\right) \times \\ &\quad \times \text{ fin. } \left(\frac{4}{5}\varrho - \Phi\right). \\ \text{fin. } 6\Phi &= 32 \text{ fin. } \Phi \text{ fin. } \left(\frac{2}{6}\varrho + \Phi\right) \text{ fin. } \left(\frac{2}{6}\varrho - \Phi\right) \text{ fin. } \left(\frac{4}{6}\varrho + \Phi\right) \text{ fin. } \left(\frac{4}{6}\varrho - \Phi\right) \times \\ &\quad \times \text{ cof. } \Phi. \\ \text{fin. } 7\Phi &= 64 \text{ fin. } \Phi \text{ fin. } \left(\frac{2}{7}\varrho + \Phi\right) \text{ fin. } \left(\frac{2}{7}\varrho - \Phi\right) \text{ fin. } \left(\frac{4}{7}\varrho + \Phi\right) \text{ fin. } \left(\frac{4}{7}\varrho - \Phi\right) \times \\ &\quad \times \text{ fin. } \left(\frac{6}{7}\varrho + \Phi\right) \text{ fin. } \left(\frac{6}{7}\varrho - \Phi\right). \\ \text{fin. } 8\Phi &= 128 \text{ fin. } \Phi \text{ fin. } \left(\frac{2}{8}\varrho + \Phi\right) \text{ fin. } \left(\frac{2}{8}\varrho - \Phi\right) \text{ fin. } \left(\frac{4}{8}\varrho + \Phi\right) \text{ fin. } \left(\frac{4}{8}\varrho - \Phi\right) \times \\ &\quad \times \text{ fin. } \left(\frac{6}{8}\varrho + \Phi\right) \text{ fin. } \left(\frac{6}{8}\varrho - \Phi\right) \text{ cof. } \Phi. \\ &\quad \text{etc.} \qquad \qquad \qquad \text{etc.} \end{aligned}$$

§. 24. Ut nunc has formulas rursus generales reddamus, duos casus constitui conuenit, prouti numerus  $n$  fuerit vel par vel impar.

I. Sit  $n = 2i$ , erit

$$\begin{aligned} \text{fin. } 2i\Phi &= 2^{2i-1} \text{ fin. } \Phi \text{ cof. } \Phi \text{ fin. } \left(\frac{\varrho}{i} + \Phi\right) \text{ fin. } \left(\frac{\varrho}{i} - \Phi\right) \times \\ &\quad \times \text{ fin. } \left(\frac{2\varrho}{i} + \Phi\right) \text{ fin. } \left(\frac{2\varrho}{i} - \Phi\right) \text{ fin. } \left(\frac{3\varrho}{i} + \Phi\right) \text{ fin. } \left(\frac{3\varrho}{i} - \Phi\right) \times \\ &\quad \times \dots \text{ fin. } \left(\frac{(i-1)\varrho}{i} + \Phi\right) \text{ fin. } \left(\frac{(i-1)\varrho}{i} - \Phi\right). \end{aligned}$$

II. Sit

II. Sit  $n = 2i + 1$ , erit

$$\begin{aligned} \text{fin. } (2i + 1) \Phi &= 2^{2i} \text{fin. } \Phi \text{fin. } \left(\frac{2\rho}{2i+1} + \Phi\right) \text{fin. } \left(\frac{2\rho}{2i+1} - \Phi\right) \times \\ &\times \text{fin. } \left(\frac{4\rho}{2i+1} + \Phi\right) \text{fin. } \left(\frac{4\rho}{2i+1} - \Phi\right) \text{fin. } \left(\frac{6\rho}{2i+1} + \Phi\right) \times \\ &\times \text{fin. } \left(\frac{6\rho}{2i+1} - \Phi\right) \dots \text{fin. } \left(\frac{2i\rho}{2i+1} + \Phi\right) \text{fin. } \left(\frac{2i\rho}{2i+1} - \Phi\right). \end{aligned}$$

§. 25. Quoniam hic omnes factores praeter cof.  $\Phi$  sunt sinus, eos, si lubet, in cofinus conuertere possumus, ope formulae  $\text{fin. } \Psi = \text{cof. } (\rho - \Psi)$ , ac pro binis casibus principalibus sequentes expressiones reperiemus:

Si  $n = 2i$ , erit

$$\begin{aligned} \text{fin. } 2i \Phi &= 2^{2i-1} \text{fin. } \Phi \text{cof. } \Phi \text{cof. } \left(\frac{\rho}{i} + \Phi\right) \text{cof. } \left(\frac{\rho}{i} - \Phi\right) \times \\ &\times \text{cof. } \left(\frac{2\rho}{i} + \Phi\right) \text{cof. } \left(\frac{2\rho}{i} - \Phi\right) \text{cof. } \left(\frac{3\rho}{i} + \Phi\right) \times \\ &\times \text{cof. } \left(\frac{3\rho}{i} - \Phi\right) \text{cof. } \left(\frac{4\rho}{i} + \Phi\right) \text{cof. } \left(\frac{4\rho}{i} - \Phi\right) \times \\ &\times \dots \text{cof. } \left(\frac{(i-1)\rho}{i} + \Phi\right) \text{cof. } \left(\frac{(i-1)\rho}{i} - \Phi\right). \end{aligned}$$

Si  $n = 2i + 1$ , erit

$$\begin{aligned} \text{fin. } (2i + 1) \Phi &= 2^{2i} \text{fin. } \Phi \text{cof. } \left(\frac{\rho}{2i+1} + \Phi\right) \text{cof. } \left(\frac{\rho}{2i+1} - \Phi\right) \times \\ &\times \text{cof. } \left(\frac{3\rho}{2i+1} + \Phi\right) \text{cof. } \left(\frac{3\rho}{2i+1} - \Phi\right) \text{cof. } \left(\frac{5\rho}{2i+1} + \Phi\right) \times \\ &\times \text{cof. } \left(\frac{5\rho}{2i+1} - \Phi\right) \dots \text{cof. } \left(\frac{(2i-1)\rho}{2i+1} + \Phi\right) \times \\ &\times \text{cof. } \left(\frac{(2i-1)\rho}{2i+1} - \Phi\right). \end{aligned}$$

§. 26. Cum igitur duplici modo anguli multipli  $n\Phi$  tam cofinum quam sinum per factores secundum sinus siue cofinus procedentes exhibuerimus, hic istas expressiones inuentas coniunctim ante oculos exponamus:

I. Si

==== (41) =====

I. Si fuerit  $n = 2i$ , erit

Pro cofinu :

$$1^{\circ}. \text{ cof. } 2i\phi = 2^{2i-1} \text{ fin. } \left(\frac{\rho}{2i} + \phi\right) \text{ fin. } \left(\frac{\rho}{2i} - \phi\right) \text{ fin. } \left(\frac{3\rho}{2i} + \phi\right) \times \\ \times \text{ fin. } \left(\frac{3\rho}{2i} - \phi\right) \text{ fin. } \left(\frac{5\rho}{2i} + \phi\right) \text{ fin. } \left(\frac{5\rho}{2i} - \phi\right) \times \\ \times \dots \text{ fin. } \left(\frac{(2i-1)\rho}{2i} + \phi\right) \text{ fin. } \left(\frac{(2i-1)\rho}{2i} - \phi\right).$$

$$2^{\circ}. \text{ cof. } 2i\phi = 2^{2i-1} \text{ cof. } \left(\frac{\rho}{2i} + \phi\right) \text{ cof. } \left(\frac{\rho}{2i} - \phi\right) \text{ cof. } \left(\frac{3\rho}{2i} + \phi\right) \times \\ \times \text{ cof. } \left(\frac{3\rho}{2i} - \phi\right) \text{ cof. } \left(\frac{5\rho}{2i} + \phi\right) \text{ cof. } \left(\frac{5\rho}{2i} - \phi\right) \times \\ \times \dots \text{ cof. } \left(\frac{(2i-1)\rho}{2i} + \phi\right) \text{ cof. } \left(\frac{(2i-1)\rho}{2i} - \phi\right).$$

Pro finu :

$$1^{\circ}. \text{ fin. } 2i\phi = 2^{2i-1} \text{ fin. } \phi \text{ cof. } \phi \text{ fin. } \left(\frac{\rho}{2i} + \phi\right) \text{ fin. } \left(\frac{\rho}{2i} - \phi\right) \text{ fin. } \left(\frac{2\rho}{2i} + \phi\right) \times \\ \times \text{ fin. } \left(\frac{2\rho}{2i} - \phi\right) \text{ fin. } \left(\frac{3\rho}{2i} + \phi\right) \text{ fin. } \left(\frac{3\rho}{2i} - \phi\right) \times \\ \times \dots \text{ fin. } \left(\frac{(i-1)\rho}{2i} + \phi\right) \text{ fin. } \left(\frac{(i-1)\rho}{2i} - \phi\right).$$

$$2^{\circ}. \text{ fin. } 2i\phi = 2^{2i-1} \text{ fin. } \phi \text{ cof. } \phi \text{ cof. } \left(\frac{\rho}{2i} + \phi\right) \text{ cof. } \left(\frac{\rho}{2i} - \phi\right) \times \\ \times \text{ cof. } \left(\frac{2\rho}{2i} + \phi\right) \text{ cof. } \left(\frac{2\rho}{2i} - \phi\right) \text{ cof. } \left(\frac{3\rho}{2i} + \phi\right) \text{ cof. } \left(\frac{3\rho}{2i} - \phi\right) \times \\ \times \dots \text{ cof. } \left(\frac{(i-1)\rho}{2i} + \phi\right) \text{ cof. } \left(\frac{(i-1)\rho}{2i} - \phi\right).$$

II. Si fuerit  $n = 2i + 1$ , erit

Pro cofinu :

$$1^{\circ}. \text{ cof. } (2i + 1)\phi = 2^{2i} \text{ cof. } \phi \text{ fin. } \left(\frac{\rho}{2i+1} + \phi\right) \text{ fin. } \left(\frac{\rho}{2i+1} - \phi\right) \times \\ \times \text{ fin. } \left(\frac{3\rho}{2i+1} + \phi\right) \text{ fin. } \left(\frac{3\rho}{2i+1} - \phi\right) \text{ fin. } \left(\frac{5\rho}{2i+1} + \phi\right) \times \\ \times \text{ fin. } \left(\frac{5\rho}{2i+1} - \phi\right) \dots \text{ fin. } \left(\frac{(2i-1)\rho}{2i+1} + \phi\right) \text{ fin. } \left(\frac{(2i-1)\rho}{2i+1} - \phi\right).$$

$$\begin{aligned}
 2^{\circ}. \operatorname{cof.} (2i + 1) \Phi &= 2^{2i} \operatorname{cof.} \Phi \operatorname{cof.} \left( \frac{2\rho}{2i+1} + \Phi \right) \operatorname{cof.} \left( \frac{2\rho}{2i+1} - \Phi \right) \times \\
 &\times \operatorname{cof.} \left( \frac{4\rho}{2i+1} + \Phi \right) \operatorname{cof.} \left( \frac{4\rho}{2i+1} - \Phi \right) \operatorname{cof.} \left( \frac{6\rho}{2i+1} + \Phi \right) \times \\
 &\times \operatorname{cof.} \left( \frac{6\rho}{2i+1} - \Phi \right) \dots \operatorname{cof.} \left( \frac{2i\rho}{2i+1} + \Phi \right) \operatorname{cof.} \left( \frac{2i\rho}{2i+1} - \Phi \right) \Phi.
 \end{aligned}$$

Pro finu :

$$\begin{aligned}
 1^{\circ}. \operatorname{fin.} (2i + 1) \Phi &= 2^{2i} \operatorname{fin.} \Phi \operatorname{fin.} \left( \frac{2\rho}{2i+1} + \Phi \right) \operatorname{fin.} \left( \frac{2\rho}{2i+1} - \Phi \right) \times \\
 &\times \operatorname{fin.} \left( \frac{4\rho}{2i+1} + \Phi \right) \operatorname{fin.} \left( \frac{4\rho}{2i+1} - \Phi \right) \operatorname{fin.} \left( \frac{6\rho}{2i+1} + \Phi \right) \times \\
 &\times \operatorname{fin.} \left( \frac{6\rho}{2i+1} - \Phi \right) \dots \operatorname{fin.} \left( \frac{2i\rho}{2i+1} + \Phi \right) \operatorname{fin.} \left( \frac{2i\rho}{2i+1} - \Phi \right).
 \end{aligned}$$

$$\begin{aligned}
 2^{\circ}. \operatorname{fin.} (2i + 1) \Phi &= 2^{2i} \operatorname{fin.} \Phi \operatorname{cof.} \left( \frac{\rho}{2i+1} + \Phi \right) \operatorname{cof.} \left( \frac{\rho}{2i+1} - \Phi \right) \times \\
 &\times \operatorname{cof.} \left( \frac{3\rho}{2i+1} + \Phi \right) \operatorname{cof.} \left( \frac{3\rho}{2i+1} - \Phi \right) \operatorname{cof.} \left( \frac{5\rho}{2i+1} + \Phi \right) \times \\
 &\times \operatorname{cof.} \left( \frac{5\rho}{2i+1} - \Phi \right) \dots \operatorname{cof.} \left( \frac{(2i-1)\rho}{2i+1} + \Phi \right) \operatorname{cof.} \left( \frac{(2i-1)\rho}{2i+1} - \Phi \right).
 \end{aligned}$$

§. 27. Hinc patet, casu quo  $n = 2i$ , tam cofinum quam finum per aequales angulos exprimi, unde si binae formae pro cofinu  $2i\Phi$  inuentae per se inuicem diuidantur, nascetur ista aequatio :

$$\begin{aligned}
 1 &= \operatorname{tang.} \left( \frac{\rho}{2i} + \Phi \right) \operatorname{tang.} \left( \frac{\rho}{2i} - \Phi \right) \operatorname{tang.} \left( \frac{3\rho}{2i} + \Phi \right) \times \\
 &\times \operatorname{tang.} \left( \frac{3\rho}{2i} - \Phi \right) \operatorname{tang.} \left( \frac{5\rho}{2i} + \Phi \right) \operatorname{tang.} \left( \frac{5\rho}{2i} - \Phi \right) \times \\
 &\times \dots \operatorname{tang.} \left( \frac{(2i-1)\rho}{2i} + \Phi \right) \operatorname{tang.} \left( \frac{(2i-1)\rho}{2i} - \Phi \right).
 \end{aligned}$$

Hinc igitur sequentes casus speciales deducuntur :

Si  $i = 1$ , erit

$$1 = \operatorname{tang.} \left( \frac{1}{2} \rho + \Phi \right) \operatorname{tang.} \left( \frac{1}{2} \rho - \Phi \right).$$

Si  $i = 2$ , erit

$$1 = \operatorname{tang.} \left( \frac{1}{4} \rho + \Phi \right) \operatorname{tang.} \left( \frac{1}{4} \rho - \Phi \right) \operatorname{tang.} \left( \frac{3}{4} \rho + \Phi \right) \operatorname{tang.} \left( \frac{3}{4} \rho - \Phi \right).$$

Si  $i = 3$ , erit



$$1 = \text{tang.} \left(\frac{1}{2} \varrho + \Phi\right) \text{tang.} \left(\frac{1}{2} \varrho - \Phi\right) \text{tang.} \left(\frac{3}{2} \varrho + \Phi\right) \times \\ \times \text{tang.} \left(\frac{3}{2} \varrho - \Phi\right) \text{tang.} \left(\frac{5}{2} \varrho + \Phi\right) \text{tang.} \left(\frac{5}{2} \varrho - \Phi\right). \\ \text{etc.} \qquad \qquad \qquad \text{etc.}$$

Quae aequalitas quidem sponte in oculos incurrit, quoniam productum ex primo factore in ultimum est = 1, similique modo productum ex secundo in penultimum = 1, et ita porro, siquidem bini tales anguli iunctim sumpti conficiunt angulum rectum  $\varrho$ .

§. 28. Diuidamus simili modo binos valores pro sinu  $2i\Phi$  inuentos, priorem per posteriorem, ac resultabit ista aequalitas:

$$1 = \text{tang.} \left(\frac{\varrho}{i} + \Phi\right) \text{tang.} \left(\frac{\varrho}{i} - \Phi\right) \text{tang.} \left(\frac{2\varrho}{i} + \Phi\right) \text{tang.} \left(\frac{2\varrho}{i} - \Phi\right) \times \\ \times \text{tang.} \left(\frac{3\varrho}{i} + \Phi\right) \text{tang.} \left(\frac{3\varrho}{i} - \Phi\right) \dots \times \\ \times \text{tang.} \left(\frac{(i-1)\varrho}{i} + \Phi\right) \text{tang.} \left(\frac{(i-1)\varrho}{i} - \Phi\right),$$

vnde sequuntur hi casus speciales:

Si  $i = 1$ , erit

$$1 = \text{tang.} (\varrho + \Phi) \text{tang.} (\varrho - \Phi).$$

Si  $i = 2$ , erit

$$1 = \text{tang.} \left(\frac{1}{2} \varrho + \Phi\right) \text{tang.} \left(\frac{1}{2} \varrho - \Phi\right).$$

Si  $i = 3$ , erit

$$1 = \text{tang.} \left(\frac{1}{3} \varrho + \Phi\right) \text{tang.} \left(\frac{1}{3} \varrho - \Phi\right) \text{tang.} \left(\frac{2}{3} \varrho + \Phi\right) \text{tang.} \left(\frac{2}{3} \varrho - \Phi\right). \\ \text{etc.} \qquad \qquad \qquad \text{etc.}$$

Vbi iterum ratio per se est manifesta, cum in genere fit  $\text{tang.} \psi \text{tang.} (\varrho - \psi) = 1$ , ob  $\text{tang.} (\varrho - \psi) = \text{cot.} \psi$ .

### III. Resolutio tangentis anguli $n\phi$ in factores.

§. 29. Ista resolutio peculiari Analyfi non eget, propterea quod  $\text{tang. } n\phi = \frac{\text{fin. } n\phi}{\text{cof. } n\phi}$ , quamobrem tantum opus est, ut singuli factores ipsius  $\text{fin. } n\phi$  diuidantur per singulos factores ipsius  $\text{cof. } n\phi$ . Cum igitur tam pro sinu quam cosinu duplicem dederimus solutionem, ea combinatione uti conueniet, quae simplicissimam solutionem suppeditat. Hic autem iterum duos casus a se inuicem distinxisse iuuabit, prouti numerus  $n$  fuerit vel par vel impar.

#### I. Casus quo $n = 2i$ .

§. 30. Hic primo casus speciales euoluamus, tribuendo numero  $n$  valores 1, 2, 3 etc. eritque

$$\text{tang. } 2\phi = \frac{\text{fin. } \phi \text{ cof. } \phi}{\text{cof. } (\frac{1}{2}\phi - \phi) \text{ cof. } (\frac{1}{2}\phi + \phi)},$$

$$\text{tang. } 4\phi = \frac{\text{fin. } \phi \text{ cof. } \phi \text{ cof. } (\frac{3}{4}\phi + \phi) \text{ cof. } (\frac{3}{4}\phi - \phi)}{\text{cof. } (\frac{1}{4}\phi + \phi) \text{ cof. } (\frac{1}{4}\phi - \phi) \text{ cof. } (\frac{3}{4}\phi + \phi) \text{ cof. } (\frac{3}{4}\phi - \phi)}.$$

Cum igitur hic nulla contractio locum inueniat, has expressiones ulterius continuare superfluum foret; at quando  $n$  est numerus impar, ob contractionem formulae prodeunt notatu maxime dignae

#### II. Casus quo $n = 2i + 1$ .

§. 31. Loco  $i$  ergo sumamus successiue numeros 0, 1, 2, 3, etc., ac primo quidem:

Si  $i = 0$  et  $n = 1$ , erit

$$\text{tang. } \phi = \frac{\text{fin. } \phi}{\text{cof. } \phi}.$$

Si

Si  $i = 1$  et  $n = 3$ , erit

$$\text{tang. } 3 \Phi = \text{tang. } \Phi \text{ tang. } \left(\frac{2}{3} \rho + \Phi\right) \text{ tang. } \left(\frac{2}{3} \rho - \Phi\right).$$

Si  $i = 2$  et  $n = 5$ , erit

$$\begin{aligned} \text{tang. } 5 \Phi = & \text{tang. } \Phi \text{ tang. } \left(\frac{2}{5} \rho + \Phi\right) \text{ tang. } \left(\frac{2}{5} \rho - \Phi\right) \times \\ & \times \text{tang. } \left(\frac{4}{5} \rho + \Phi\right) \text{ tang. } \left(\frac{4}{5} \rho - \Phi\right). \end{aligned}$$

et generaliter, pro quocunque numero impari  $2n + 1$ , erit

$$\begin{aligned} \text{tang. } (2i + 1) \Phi = & \text{tang. } \Phi \text{ tang. } \left(\frac{n\rho}{2i+1} + \Phi\right) \text{ tang. } \left(\frac{n\rho}{2i+1} - \Phi\right) \times \\ & \times \text{tang. } \left(\frac{4\rho}{2i+1} + \Phi\right) \text{ tang. } \left(\frac{4\rho}{2i+1} - \Phi\right) \times \\ & \times \text{tang. } \left(\frac{6\rho}{2i+1} + \Phi\right) \text{ tang. } \left(\frac{6\rho}{2i+1} - \Phi\right) \times \\ & \times \dots \text{tang. } \left(\frac{n\rho}{2i+1} + \Phi\right) \text{ tang. } \left(\frac{n\rho}{2i+1} - \Phi\right). \end{aligned}$$

§. 32. Hic igitur egregia theoremata Trigonometrica deducimus. Scilicet ex angulo  $3 \Phi$  habemus:

$$\text{tang. } 3 \Phi = \text{tang. } \Phi \text{ tang. } (60^\circ + \Phi) \text{ tang. } (60^\circ - \Phi),$$

feu quia  $\text{tang. } 60 + \omega = \text{cot. } (30 - \omega)$ , erit

$$\text{tang. } 3 \Phi = \text{tang. } \Phi \text{ cot. } (30^\circ - \Phi) \text{ cot. } (30^\circ + \Phi),$$

vnde colligitur

$$\text{tang. } 3 \Phi \text{ tang. } (30^\circ - \Phi) \text{ tang. } (30^\circ + \Phi) = \text{tang. } \Phi.$$

Veluti si fuerit  $\Phi = 20^\circ$ , habebimus sequentes relationes:

$$\text{tang. } 60^\circ = \text{tang. } 20^\circ \times \text{tang. } 80^\circ \times \text{tang. } 40^\circ, \text{ ideoque}$$

$$\text{tang. } 60^\circ \times \text{tang. } 10^\circ = \text{tang. } 20^\circ \times \text{tang. } 40^\circ,$$

hinc sequentem proportionem deducimus

$$\text{tang. } 10^\circ : \text{tang. } 20^\circ = \text{tang. } 40^\circ : \text{tang. } 60^\circ,$$

ideoque logarithmos fumendo erit

$$l \text{ tang. } 10^\circ + l \text{ tang. } 60^\circ = l \text{ tang. } 20^\circ + l \text{ tang. } 40^\circ.$$

Ex tabulis autem est

$$\begin{array}{r|l}
 l \text{ tang. } 10^\circ = 9,2463188 & l \text{ tang. } 20^\circ = 9,5610659 \\
 l \text{ tang. } 60 = 10,2385606 & l \text{ tang. } 40 = 9,9238135 \\
 \hline
 l \text{ tg. } 10^\circ + l \text{ tg. } 60^\circ = 9,4848794 & l \text{ tg. } 20^\circ + l \text{ tg. } 40^\circ = 9,4848794.
 \end{array}$$

§. 33. Simili modo formula pro tang.  $5\Phi$  inuenta praebet

$$\text{tang. } 5\Phi = \text{tang. } \Phi \text{ tang. } (36^\circ + \Phi) \text{ tang. } (36^\circ - \Phi) \times \\
 \times \text{tang. } (72^\circ + \Phi) \text{ tang. } (72^\circ - \Phi), \text{ siue}$$

$$\text{tang. } 5\Phi \text{ tang. } (18^\circ + \Phi) \text{ tang. } (18^\circ - \Phi) = \text{tang. } \Phi \times \\
 \times \text{tang. } (36^\circ + \Phi) \text{ tang. } (36^\circ - \Phi),$$

hincque in logarithmis :

$$\begin{aligned}
 & l \text{ tang. } 5\Phi + l \text{ tang. } (18^\circ + \Phi) + l \text{ tang. } (18^\circ - \Phi) \\
 & = l \text{ tang. } \Phi + l \text{ tang. } (36^\circ + \Phi) + l \text{ tang. } (36^\circ - \Phi).
 \end{aligned}$$

Sit exempli gratia  $\Phi = 10^\circ$ , eritque

$$\begin{aligned}
 & l \text{ tang. } 50 + l \text{ tang. } 28^\circ + l \text{ tang. } 8^\circ \\
 & = l \text{ tang. } 10^\circ + l \text{ tang. } 46^\circ + l \text{ tang. } 26^\circ,
 \end{aligned}$$

veluti in sequente schematismo videre licet :

$$\begin{array}{r|l}
 l \text{ tang. } 50 = 10,0761865 & l \text{ tang. } 10^\circ = 9,2463188 \\
 l \text{ tang. } 28 = 9,7256744 & l \text{ tang. } 46 = 10,0151628 \\
 l \text{ tang. } 8 = 9,1478025 & l \text{ tang. } 26 = 9,6881818 \\
 \hline
 l \text{ tg. } 50^\circ + l \text{ tg. } 28^\circ + l \text{ tg. } 8^\circ = 8,9496634 & \text{Summa} = 8,9496634.
 \end{array}$$

§. 34. Quoniam igitur haftenus tam sinus et cosinus quam tangentes angulorum multiplorum per producta expressimus, istas formulas commode per logarithmos euoluere licebit, qui deinceps per differentiationem tolli poterunt, dum scilicet angulus  $\Phi$  tanquam variabilis spectatur.

Euolu-

## Euolutio

formularum pro  $\text{cof. } n \Phi$  inuentarum, per logarithmos  
et differentiationem.

§. 35. Cum supra inuenerimus

$$\text{cof. } 2 \Phi = 2 \text{ cof. } \left(\frac{1}{2} \xi + \Phi\right) \text{ cof. } \left(\frac{1}{2} \xi - \Phi\right),$$

erit sumtis logarithmis

$$l \text{ cof. } 2 \Phi = l 2 + l \text{ cof. } \left(\frac{1}{2} \xi + \Phi\right) + l \text{ cof. } \left(\frac{1}{2} \xi - \Phi\right),$$

quae aequatio quia vera est pro angulo quocunque  $\Phi$ , spectetur  $\Phi$  ut quantitas variabilis, ac differentiatio nobis praebebit hanc aequationem:

$$-\frac{2 \partial \Phi \text{ fin. } 2 \Phi}{\text{cof. } 2 \Phi} = -\frac{\partial \Phi \text{ fin. } \left(\frac{1}{2} \xi + \Phi\right)}{\text{cof. } \left(\frac{1}{2} \xi + \Phi\right)} + \frac{\partial \Phi \text{ fin. } \left(\frac{1}{2} \xi - \Phi\right)}{\text{cof. } \left(\frac{1}{2} \xi - \Phi\right)},$$

quae per  $-\partial \Phi$  diuisa per tangentes dabit

$$2 \text{ tang. } 2 \Phi = \text{tang. } (45^\circ + \Phi) - \text{tang. } (45^\circ - \Phi),$$

id quod exemplo illustrasse iuuabit. Sit igitur  $\Phi = 17^\circ, 30'$ , eritque

$$2 \text{ tang. } 35^\circ = \text{tang. } (62^\circ, 30') - \text{tang. } (27^\circ, 30').$$

Est vero ex tabulis

$2 \text{ tang. } 35^\circ = 1,4004150$	$\text{tang. } 62^\circ, 30' = 1,9209821$
	$\text{tang. } 27^\circ, 30' = 0,5205671$
	$\text{Differentia} = 1,4004150$

§. 36. Deinde quia supra inuenimus

$$\text{cof. } 3 \Phi = 4 \text{ cof. } (60^\circ + \Phi) \text{ cof. } (60^\circ - \Phi) \text{ cof. } \Phi,$$

erit per logarithmos

$$l \text{ cof. } 3 \Phi = l 4 + l \text{ cof. } \Phi + l \text{ cof. } (60^\circ + \Phi) + l \text{ cof. } (60^\circ - \Phi).$$

Hinc

Hinc differentiando erit

$$3 \operatorname{tang.} 3\Phi = \operatorname{tang.} \Phi + \operatorname{tang.} (60^\circ + \Phi) - \operatorname{tang.} (60 - \Phi).$$

*Exemplum.* Sumatur  $\Phi = 25^\circ$ , eritque

$$3 \operatorname{tang.} 75^\circ = \operatorname{tang.} 25^\circ + \operatorname{tang.} 85^\circ - \operatorname{tang.} 35^\circ,$$

cuius veritas ex subiecto calculo videre licet:

3 tang. 75° = 11, 1961524	<table style="border-collapse: collapse; margin: 0 auto;"> <tr> <td style="border-right: 1px solid black; padding-right: 5px;">tang. 25° = 0, 4663077</td> <td style="padding-left: 5px;">0, 4663077</td> </tr> <tr> <td style="border-right: 1px solid black; padding-right: 5px;">tang. 85. = 11, 4300520</td> <td style="padding-left: 5px;">11, 4300520</td> </tr> <tr> <td style="border-right: 1px solid black; padding-right: 5px;">Summa = 11, 8963597</td> <td style="padding-left: 5px;">11, 8963597</td> </tr> <tr> <td style="border-right: 1px solid black; padding-right: 5px;">tang. 35° = 0, 7002075</td> <td style="padding-left: 5px;">0, 7002075</td> </tr> <tr> <td style="border-right: 1px solid black; padding-right: 5px;"></td> <td style="padding-left: 5px; border-top: 1px solid black;">11, 1961522.</td> </tr> </table>	tang. 25° = 0, 4663077	0, 4663077	tang. 85. = 11, 4300520	11, 4300520	Summa = 11, 8963597	11, 8963597	tang. 35° = 0, 7002075	0, 7002075		11, 1961522.
tang. 25° = 0, 4663077	0, 4663077										
tang. 85. = 11, 4300520	11, 4300520										
Summa = 11, 8963597	11, 8963597										
tang. 35° = 0, 7002075	0, 7002075										
	11, 1961522.										

*Aliud exemplum.* Sumamus  $\Phi = 29^\circ$ , eritque

$$3 \operatorname{tang.} 87^\circ = \operatorname{tang.} 29^\circ + \operatorname{tang.} 89^\circ - \operatorname{tang.} 31^\circ,$$

veluti ex subiecto calculo videre licet

3 tang. 87° = 57, 243411	<table style="border-collapse: collapse; margin: 0 auto;"> <tr> <td style="border-right: 1px solid black; padding-right: 5px;">tang. 29° = 0, 554309</td> <td style="padding-left: 5px;">0, 554309</td> </tr> <tr> <td style="border-right: 1px solid black; padding-right: 5px;">tang. 89 = 57, 289962</td> <td style="padding-left: 5px;">57, 289962</td> </tr> <tr> <td style="border-right: 1px solid black; padding-right: 5px;">Summa = 57, 844271</td> <td style="padding-left: 5px;">57, 844271</td> </tr> <tr> <td style="border-right: 1px solid black; padding-right: 5px;">tang. 31° = 0, 600860</td> <td style="padding-left: 5px;">0, 600860</td> </tr> <tr> <td style="border-right: 1px solid black; padding-right: 5px;"></td> <td style="padding-left: 5px; border-top: 1px solid black;">57, 243411.</td> </tr> </table>	tang. 29° = 0, 554309	0, 554309	tang. 89 = 57, 289962	57, 289962	Summa = 57, 844271	57, 844271	tang. 31° = 0, 600860	0, 600860		57, 243411.
tang. 29° = 0, 554309	0, 554309										
tang. 89 = 57, 289962	57, 289962										
Summa = 57, 844271	57, 844271										
tang. 31° = 0, 600860	0, 600860										
	57, 243411.										

§. 37. Sumamus etiam exemplum quo anguli occurrunt maiores recto, fitque  $\Phi = 58^\circ$ , effeque oportet:

$$3 \operatorname{tang.} 174^\circ = \operatorname{tang.} 58^\circ + \operatorname{tang.} 118^\circ - \operatorname{tang.} 2^\circ,$$

quae aequatio reducitur ad sequentem:

$$3 \operatorname{tang.} 6^\circ = \operatorname{tang.} 62^\circ + \operatorname{tang.} 2^\circ - \operatorname{tang.} 58^\circ,$$

et calculus ita se habebit:

tang.

$\text{tang. } 6^\circ = 0,1051042$ <hr style="border: 0; border-top: 1px solid black; margin: 5px 0;"/> $3 \text{ tang. } 6^\circ = 0,3153126$	$\text{tang. } 62^\circ = 1,8807265$ <hr style="border: 0; border-top: 1px solid black; margin: 5px 0;"/> $\text{tang. } 2^\circ = 0,0349208$ <hr style="border: 0; border-top: 1px solid black; margin: 5px 0;"/> $\text{Summa} = 1,9156473$ $\text{tang. } 38^\circ = 1,6003345$ <hr style="border: 0; border-top: 1px solid black; margin: 5px 0;"/> $\text{different.} = 0,3153128.$
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§. 38. Sit nunc in genere  $n = 2i$ , et quia inuenimus  
 $\text{cof. } 2i\Phi = 2^{2i-1} \text{cof.} \left(\frac{\rho}{2i} + \Phi\right) \text{cof.} \left(\frac{\rho}{2i} - \Phi\right) \text{cof.} \left(\frac{3\rho}{2i} + \Phi\right) \times$   
 $\times \text{cof.} \left(\frac{3\rho}{2i} - \Phi\right) \text{cof.} \left(\frac{5\rho}{2i} + \Phi\right) \dots \text{cof.} \left(\frac{(2i-1)\rho}{2i} + \Phi\right) \times$   
 $\times \text{cof.} \left(\frac{(2i-1)\rho}{2i} - \Phi\right),$

hinc per logarithmos erit

$$l \text{ cof. } 2i\Phi = l 2^{2i-1} + l \text{ cof.} \left(\frac{\rho}{2i} + \Phi\right) + l \text{ cof.} \left(\frac{\rho}{2i} - \Phi\right) \\
+ l \text{ cof.} \left(\frac{3\rho}{2i} + \Phi\right) + \text{cof.} l \left(\frac{3\rho}{2i} - \Phi\right) \dots \\
+ l \text{ cof.} \left(\frac{(2i-1)\rho}{2i} + \Phi\right) + l \text{ cof.} \left(\frac{(2i-1)\rho}{2i} - \Phi\right),$$

vnde ergo per differentiationem nanciscimur

$$2i \text{ tang. } 2i\Phi = \text{tang.} \left(\frac{\rho}{2i} + \Phi\right) - \text{tang.} \left(\frac{\rho}{2i} - \Phi\right) \\
+ \text{tang.} \left(\frac{3\rho}{2i} + \Phi\right) - \text{tang.} \left(\frac{3\rho}{2i} - \Phi\right) \text{ etc. } \dots \\
+ \text{tang.} \left(\frac{(2i-1)\rho}{2i} + \Phi\right) - \text{tang.} \left(\frac{(2i-1)\rho}{2i} - \Phi\right).$$

§. 39. Hinc ergo pro  $i$  fumendis numeris minoribus habebimus :

$$4 \text{ tang. } 4\Phi = \text{tang.} \left(22\frac{1}{2}^\circ + \Phi\right) \text{tang.} \left(22\frac{1}{2}^\circ - \Phi\right) \\
+ \text{tang.} \left(67\frac{1}{2}^\circ + \Phi\right) - \text{tang.} \left(67\frac{1}{2}^\circ - \Phi\right); \\
6 \text{ tang. } 6\Phi = \text{tang.} \left(15^\circ + \Phi\right) - \text{tang.} \left(15^\circ - \Phi\right) \\
+ \text{tang.} \left(45^\circ + \Phi\right) - \text{tang.} \left(45^\circ - \Phi\right) \\
+ \text{tang.} \left(75^\circ + \Phi\right) - \text{tang.} \left(75^\circ - \Phi\right);$$

$$\begin{aligned}
 8 \operatorname{tang.} 8 \Phi &= \operatorname{tang.} (11 \frac{1}{4}^\circ + \Phi) - \operatorname{tang.} (11 \frac{1}{4}^\circ - \Phi) \\
 &+ \operatorname{tang.} (33 \frac{3}{4}^\circ + \Phi) - \operatorname{tang.} (33 \frac{3}{4}^\circ - \Phi) \\
 &+ \operatorname{tang.} (56 \frac{1}{4}^\circ + \Phi) - \operatorname{tang.} (56 \frac{1}{4}^\circ - \Phi) \\
 &+ \operatorname{tang.} (78 \frac{3}{4}^\circ + \Phi) - \operatorname{tang.} (78 \frac{3}{4}^\circ - \Phi).
 \end{aligned}$$

§. 40. Sumamus nunc etiam  $n = 2i + 1$ , quo casu vidimus esse

$$\begin{aligned}
 \operatorname{cof.} (2i + 1) \Phi &= 2^{2i} \operatorname{cof.} \Phi \operatorname{cof.} \left( \frac{2\varrho}{2i+1} + \Phi \right) \operatorname{cof.} \left( \frac{2\varrho}{2i+1} - \Phi \right) \times \\
 &\times \operatorname{cof.} \left( \frac{4\varrho}{2i+1} + \Phi \right) \operatorname{cof.} \left( \frac{4\varrho}{2i+1} - \Phi \right) \operatorname{cof.} \left( \frac{6\varrho}{2i+1} + \Phi \right) \times \\
 &\times \dots \operatorname{cof.} \left( \frac{2i\varrho}{2i+1} + \Phi \right) \operatorname{cof.} \left( \frac{2i\varrho}{2i+1} - \Phi \right),
 \end{aligned}$$

hinc sumtis logarithmis erit

$$\begin{aligned}
 l \operatorname{cof.} (2i + 1) \Phi &= l 2^{2i} + l \operatorname{cof.} \Phi + l \operatorname{cof.} \left( \frac{2\varrho}{2i+1} + \Phi \right) \\
 &+ l \operatorname{cof.} \left( \frac{2\varrho}{2i+1} - \Phi \right) \dots + l \operatorname{cof.} \left( \frac{2i\varrho}{2i+1} + \Phi \right) \\
 &+ l \operatorname{cof.} \left( \frac{2i\varrho}{2i+1} - \Phi \right),
 \end{aligned}$$

vnde differentiando nanciscimur

$$\begin{aligned}
 (2i + 1) \operatorname{tang.} (2i + 1) \Phi &= \operatorname{tang.} \Phi + \operatorname{tang.} \left( \frac{2\varrho}{2i+1} + \Phi \right) \\
 &- \operatorname{tang.} \left( \frac{2\varrho}{2i+1} - \Phi \right) + \operatorname{tang.} \left( \frac{4\varrho}{2i+1} + \Phi \right) \\
 &- \operatorname{tang.} \left( \frac{4\varrho}{2i+1} - \Phi \right) \dots \\
 &+ \operatorname{tang.} \left( \frac{2i\varrho}{2i+1} + \Phi \right) - \operatorname{tang.} \left( \frac{2i\varrho}{2i+1} - \Phi \right).
 \end{aligned}$$

§. 41. Casum quo  $2i + 1 = 3$  iam evoluimus, fit igitur  $2i + 1 = 5$ , eritque

$$\begin{aligned}
 5 \operatorname{tang.} 5 \Phi &= \operatorname{tang.} \Phi + \operatorname{tang.} (36^\circ + \Phi) - \operatorname{tang.} (36^\circ - \Phi) \\
 &+ \operatorname{tang.} (72^\circ + \Phi) - \operatorname{tang.} (72^\circ - \Phi).
 \end{aligned}$$

Sir



Sit nunc  $2i + 1 = 7$ , eritque

$$\begin{aligned}
 7 \text{ tang. } 7 \Phi &= \text{tang. } \Phi + \text{tang. } (25\frac{2}{7}^\circ + \Phi) \\
 &\quad - \text{tang. } (25\frac{2}{7}^\circ - \Phi) + \text{tang. } (51\frac{1}{7}^\circ + \Phi) \\
 &\quad - \text{tang. } (51\frac{1}{7}^\circ - \Phi) + \text{tang. } (76\frac{3}{7}^\circ + \Phi) \\
 &\quad - \text{tang. } (76\frac{3}{7}^\circ - \Phi).
 \end{aligned}$$

Posito  $2i + 1 = 9$  habebimus

$$\begin{aligned}
 9 \text{ tang. } 9 \Phi &= \text{tang. } \Phi + \text{tang. } (20^\circ + \Phi) - \text{tang. } (20^\circ - \Phi) \\
 &\quad + \text{tang. } (40^\circ + \Phi) - \text{tang. } (40^\circ - \Phi) \\
 &\quad + \text{tang. } (60^\circ + \Phi) - \text{tang. } (60^\circ - \Phi) \\
 &\quad + \text{tang. } (80^\circ + \Phi) - \text{tang. } (80^\circ - \Phi).
 \end{aligned}$$

