INNVMERA THEOREMATA CIRCA FORMVLAS INTEGRALES QVORVM DEMONSTRATIO VIRES ANALYSEOS SVPERARE VIDEATVR.

Auctore
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Conuent. exhib. die 18 *Mart.* 1776.

Fundamentum horum Theorematum in eiusmodi formulis integralibus $\int V \partial x$ est constitutum, quarum valor a termino $x = 0$ vsque ad certum terminum definitum $x = k$ per expressionem finitam assignari queat. Quod si enim istum valorem littera P designemus, ita vt sit $\int V \partial x \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = k \end{bmatrix} = P$, quoniam ipsa variabilis x in P non amplius inest, ea tanquam functio alius cuiuspiam quantitatis p, quae simul in functione V contineatur, spectari poterit; tum autem sub iisdem integrationis terminis innumerabiles aliae formulae integrales tam per differentiationem quam per integrationem, quemadmodum iam aliquoties fusius exposui, deriuari possunt, quae sunt:

 Per

III.

$$\int \frac{x^p - 1 - p\,lx - \frac{1}{2}pp(lx)^2}{\Delta} \cdot \frac{\partial x}{x(lx)^3} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = \infty \end{bmatrix} = \int \partial p \int \partial p \int P\, \partial p.$$

IV.

$$\int \frac{x^p - 1 - p\,lx - \frac{1}{2}pp(lx)^2 - \frac{1}{6}p^3(lx)^3}{\Delta} \cdot \frac{\partial x}{x(lx)^4} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = \infty \end{bmatrix} = \int \partial p \int \partial p \int \partial p \int P\, \partial p.$$

etc.

Haecque Theoremata aeque ſubſiſtunt, ſiue p ſit numerus poſitiuus, ſiue negatiuus, ſiue etiam integer, ſiue fractus, dum ne ſit $p - n > 0$, et integralia $\int P \partial p$, $\int \partial p \int P \partial p$, $\int \partial p \int \partial p \int P \partial p$, omniaque hinc deducta ita capiantur, vt euaneſcant poſito $p = 0$.

ORDO SECVNDVS
Theorematum ex hac forma principali deductorum:

$$\int \frac{x^p}{x^{-n}(1 + x^n)^2} \frac{\partial x}{x} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = \infty \end{bmatrix} = \frac{\pi p}{nn \sin. \frac{p}{n}\pi}.$$

Ponamus hic iterum denominatorem $x^{-n}(1 + x^n)^2 = \Delta$, ſitque $P = \frac{\pi p}{nn \sin. \frac{p}{n}\pi}$, ita vt P iterum ſit functio ipſius p, ac primo per differentiationem hinc deducentur ſequentia Theoremata:

I.

$$\int \frac{x^p}{\Delta} \cdot \frac{\partial x\, lx}{x} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = \infty \end{bmatrix} = \frac{\partial P}{\partial p}.$$

II.

$$\int \frac{x^p}{\Delta} \cdot \frac{\partial x\,(lx)^2}{x} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = \infty \end{bmatrix} = \frac{\partial \partial P}{\partial p^2}.$$

III.

III.

$$\int \frac{x^p}{\Delta} \cdot \frac{\partial x\, (l\, x)^3}{x} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = \infty \end{bmatrix} = \frac{\partial^3 P}{\partial p^3}.$$

IV.

$$\int \frac{x^p}{\Delta} \cdot \frac{\partial x\, (l\, x)^4}{x} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = \infty \end{bmatrix} = \frac{\partial^4 P}{\partial p^4}.$$

etc.

Per integrationem autem inde fequentia Theoremata oriuntur:

I.

$$\int \frac{x^p - 1}{\Delta} \cdot \frac{\partial x}{x\, l\, x} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = \infty \end{bmatrix} = \int P\, \partial p.$$

II.

$$\int \frac{x^p - 1 - p\, l\, x}{\Delta} \cdot \frac{\partial x}{x\, (l\, x)^2} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = \infty \end{bmatrix} = \int \partial p \int P\, \partial p.$$

III.

$$\int \frac{x^p - 1 - p\, l\, x - \frac{1}{2} p\, p\, (l\, x)^2}{\Delta} \cdot \frac{\partial x}{x\, (l\, x)^3} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = \infty \end{bmatrix}$$
$$= \int \partial p \int \partial p \int P\, \partial p.$$

IV.

$$\int \frac{x^p - 1 - p\, l\, x - \frac{1}{2} p\, p\, (l\, x)^2 - \frac{1}{6} p^3 (l\, x)^3}{\Delta} \cdot \frac{\partial x}{x\, (l\, x)^4} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = \infty \end{bmatrix}$$

etc.
$$= \int \partial p \int \partial p \int \partial p \int P\, \partial p.$$

vbi circa integrationes eadem funt obferuanda, quae ante fuerant praecepta.

ORDO

ORDO TERTIVS

Theorematum ex hac forma principali deductorum:

$$\int \frac{\partial x}{x} \cdot \frac{x^p}{x^n + (f + \frac{1}{f}) + x^{-n}} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = \infty \end{bmatrix} = \frac{\pi (f^{\frac{p}{n}} - f^{-\frac{p}{n}})}{n (f^1 - f^{-1}) \sin. \frac{p}{n} \pi}.$$

Ponamus hic iterum pro denominatore

$$\Delta = x^n + (f + \tfrac{1}{f}) = x^{-n},$$

tum vero fit

$$P = \frac{\pi (f^{\frac{p}{n}} - f^{-\frac{p}{n}})}{n (f^1 - f^{-1}) \sin. \frac{p}{n} \pi} = \frac{\pi (f^{1 + \frac{p}{n}} - f^{1 - \frac{p}{n}})}{n (ff - 1) \sin. \frac{p}{n} \pi}.$$

His positis vt ante per differentiationem sequentia Theoremata deducuntur:

I.

$$\int \frac{x^p}{\Delta} \cdot \frac{\partial x \, l x}{x} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = \infty \end{bmatrix} = \frac{\partial P}{\partial p}.$$

II.

$$\int \frac{x^p}{\Delta} \cdot \frac{\partial x (l x)^2}{x} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = \infty \end{bmatrix} = \frac{\partial \partial P}{\partial p^2}.$$

III.

$$\int \frac{x^p}{\Delta} \cdot \frac{\partial x (l x)^3}{x} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = \infty \end{bmatrix} = \frac{\partial^3 P}{\partial p^3}.$$

IV.

$$\int \frac{x^p}{\Delta} \cdot \frac{\partial x (l x)^4}{x} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = \infty \end{bmatrix} = \frac{\partial^4 P}{\partial p^4}.$$

etc.

Per integrationem autem eliciuntur sequentia:

I.

I.

$$\int \frac{x^p - 1}{\Delta} \cdot \frac{\partial x}{x\, l\, x} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = \infty \end{bmatrix} = \int P\, \partial p.$$

II.

$$\int \frac{x^p - 1 - p\, l\, x}{\Delta} \cdot \frac{\partial x}{x\, (l\, x)^2} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = \infty \end{bmatrix} = \int \partial p \int P\, \partial p.$$

III.

$$\int \frac{x^p - 1 - p\, l\, x - \frac{1}{2}\, p\, p\, (l\, x)^2}{\Delta} \cdot \frac{\partial x}{x\, (l\, x)^3} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = \infty \end{bmatrix}$$
$$= \int \partial p \int \partial p \int P\, \partial p.$$

IV.

$$\int \frac{x^p - 1 - p\, l\, x - \frac{1}{2}\, p\, p\, (l\, x)^2 - \frac{1}{6}\, p^3\, (l\, x)^3}{\Delta} \cdot \frac{\partial x}{x\, (l\, x)^3} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = \infty \end{bmatrix}$$

etc.
$$= \int \partial p \int \partial p \int \partial p \int P\, \partial p.$$

Vbi denuo eadem ſunt obſeruanda, quae ſupra ſunt praecepta.

ORDO QVARTVS

Theorematum ex hac forma principali deductorum:

$$\int \frac{\partial x}{x} \cdot \frac{x^p + x^{-p}}{x^n + 2 \operatorname{coſ.} \theta + x^{-n}} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = \frac{\pi \operatorname{ſin.} \frac{p}{n} \theta}{n \operatorname{ſin.} \theta \operatorname{ſin.} \frac{p}{n} \pi}.$$

Statuamus hic iterum $\Delta = x^n + 2 \operatorname{coſ.} \theta + x^{-n}$, ſitque

$$P = \frac{\pi \operatorname{ſin.} \frac{p}{n} \theta}{n \operatorname{ſin.} \theta \operatorname{ſin.} \frac{p}{n} \pi},$$

ita vt P tanquam functio ipſius p ſpectari poſſit; vbi probe notandum est, hunc valorem integralem subsistere non posse, nisi sit $p < n$, ideoque fractio $\frac{p}{n}$ vnitate minor, atque sub iisdem conditionibus per differentiationem sequentia hinc deducuntur Theoremata:

I.

$$\int \frac{x^p - x^{-p}}{\Delta} \cdot \frac{\partial x \, l x}{x} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = \frac{\partial P}{\partial p}.$$

II.

$$\int \frac{x^p + x^{-p}}{\Delta} \cdot \frac{\partial x \, (l x)^2}{x} \begin{bmatrix} \text{ab } = 0 \\ \text{ad } = 1 \end{bmatrix} = \frac{\partial \partial P}{\partial p^2}.$$

III.

$$\int \frac{x^p - x^{-p}}{\Delta} \cdot \frac{\partial x \, (l x)^3}{x} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = \frac{\partial^3 P}{\partial p^3}.$$

IV.

$$\int \frac{x^p + x^{-p}}{\Delta} \cdot \frac{\partial x \, (l x)^4}{x} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = \frac{\partial^4 P}{\partial p^4}.$$

etc.

Per integrationem autem colliguntur sequentia:

I.

$$\int \frac{x^p - x^{-p}}{\Delta} \cdot \frac{\partial x}{x \, l x} \begin{bmatrix} \text{ad } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = \int P \, \partial p.$$

II.

$$\int \frac{x^p + x^{-p} - 2}{\Delta} \cdot \frac{\partial x}{x \, (l x)^2} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = \int \partial p \int P \, \partial p.$$

III.

III.

$$\int \frac{x^p - x^{-p} - 2\,p\,l\,x}{\Delta} \cdot \frac{\partial x}{x\,(l\,x)^3} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = \int \partial p \int \partial p \int \mathrm{P}\,\partial p.$$

IV.

$$\int \frac{x^p + x^{-p} - 2 - \frac{2}{2}\,p\,p\,(l\,x)^2}{\Delta} \cdot \frac{\partial x}{x\,(l\,x)^4} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix}$$

etc.

$$= \int \partial p \int \partial p \int \partial p \int \mathrm{P}\,\partial p.$$

Quod ſi eadem integralia extendantur ab $x = 0$ ad $x = \infty$, eorum valores duplo euadent maiores.

ORDO QVINTVS
Theorematum ex hac forma principali deductorum:

$$\int \frac{\partial x}{x} \cdot \frac{x^p + x^{-p}}{x^{-n}\,(1 + x^n)^2} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = \frac{\pi\,p}{n\,n\,\text{ſin.}\,\frac{p}{n}\,\pi}.$$

Statuamus igitur hic pro denominatore $\Delta = x^{-n}\,(1 + x^n)^2$, ſitque $\mathrm{P} = \frac{\pi\,p}{n\,n\,\text{ſin.}\,\frac{p}{n}\,\pi}$, ita vt P ſpectari poſſit tanquam functio ipſius p, vbi perpetuo fractio $\frac{p}{n}$ vnitate minor ſupponitur, quibus poſitis per differentiationem ſequentia naſcuntur Theoremata:

I.

$$\int \frac{x^p - x^{-p}}{\Delta} \cdot \frac{\partial x\,l\,x}{x} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = \frac{\partial \mathrm{P}}{\partial p}.$$

II.

$$\int \frac{x^p + x^{-p}}{\Delta} \cdot \frac{\partial x\,(l\,x)^2}{x} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = \frac{\partial \partial \mathrm{P}}{\partial p^2}.$$

III.

III.

$$\int \frac{x^p - x^{-p}}{\Delta} \cdot \frac{\partial x\,(l\,x)^3}{x} \begin{bmatrix}\text{ab } x = 0\\ \text{ad } x = 1\end{bmatrix} = \frac{\partial^3 P}{\partial p^3}.$$

IV.

$$\int \frac{x^p + x^{-p}}{\Delta} \cdot \frac{\partial x\,(l\,x)^4}{x} \begin{bmatrix}\text{ab } x = 0\\ \text{ad } x = 1\end{bmatrix} = \frac{\partial^4 P}{\partial p^4}.$$

Per integrationem vero ſequentia deducuntur:

I.

$$\int \frac{x^p - x^{-p}}{\Delta} \cdot \frac{\partial x}{x\,l\,x} \begin{bmatrix}\text{ab } x = 0\\ \text{ad } x = 1\end{bmatrix} = \int P\,\partial p.$$

II.

$$\int \frac{x^p + x^{-p} - 2}{\Delta} \cdot \frac{\partial x}{x(l\,x)^2} \begin{bmatrix}\text{ab } x = 0\\ \text{ad } x = 1\end{bmatrix} = \int \partial p \int P\,\partial p.$$

III.

$$\int \frac{x^p - x^{-p} - 2\,p\,(l\,x)}{\Delta} \cdot \frac{\partial x}{x(l\,x)^3} \begin{bmatrix}\text{ab } x = 0\\ \text{ad } x = 1\end{bmatrix}$$
$$= \int \partial p \int \partial p \int P\,\partial p.$$

IV.

$$\int \frac{x^p + x^{-p} - 2 - p\,p\,(l\,x)^2}{\Delta} \cdot \frac{\partial x}{x\,(l\,x)^4} \begin{bmatrix}\text{ab } x = 0\\ \text{ad } x = 1\end{bmatrix}$$
$$= \int \partial p \int \partial p \int \partial p \int P\,\partial p$$

etc.

At ſi haec integralia ab $x = 0$ ad $x = \infty$ capiantur, eorum valores euadent duplo maiores.

ORDO

ORDO SEXTVS

Theorematum ex forma principali deductorum:

$$\int \frac{\partial x}{x} \cdot \frac{x^p + x^{-p}}{x^n + (f + \frac{1}{f}) + x^{-n}} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = \frac{\pi (f^{\frac{p}{n}} - f^{-\frac{p}{n}})}{n (f^1 - f^{-1}) \text{ fin. } \frac{p}{n} \pi}.$$

Statuamus

$$\Delta = x^n + (f + \tfrac{1}{f}) + x^{-n} = \frac{1}{x^n} (x^n + f) (x^n + \tfrac{1}{f}), \text{ et fit}$$

$$P = \frac{\pi (f^{\frac{p}{n}} - f^{-\frac{p}{n}})}{n (f^1 - f^{-1}) \text{ fin. } \frac{p}{n} \pi},$$

vbi iterum fractio $\frac{p}{n}$ vnitate minor fupponitur. His obferuatis per differentiationem colligimus:

I.

$$\int \frac{x^p - x^{-p}}{\Delta} \cdot \frac{\partial x \, l x}{x} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = \frac{\partial P}{\partial p}.$$

II.

$$\int \frac{x^p + x^{-p}}{\Delta} \cdot \frac{\partial x \, (l x)^2}{x} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = \frac{\partial \partial P}{\partial p^2}.$$

III.

$$\int \frac{x^p - x^{-p}}{\Delta} \cdot \frac{\partial x \, (l x)^3}{x} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = \frac{\partial^3 P}{\partial p^3}.$$

IV.

$$\int \frac{x^p + x^{-p}}{\Delta} \cdot \frac{\partial x \, (l x)^4}{x} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = \frac{\partial^4 P}{\partial p^4}.$$

etc.

Per integrationem autem fequentia Theoremata nafcuntur:

I.

$$\int \frac{x^p - x^{-p}}{\Delta} \cdot \frac{\partial x}{x\, l\, x} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = \int P\, \partial p.$$

II.

$$\int \frac{x^p + x^{-p} - 2}{\Delta} \cdot \frac{\partial x}{x\, (l\, x)^2} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = \int \partial p \int P\, \partial p.$$

III.

$$\int \frac{x^p - x^{-p} - 2\, p\, l\, x}{\Delta} \cdot \frac{\partial x}{x\, (l\, x)^3} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix}$$
$$= \int \partial p \int \partial p \int P\, \partial p.$$

IV.

$$\int \frac{x^p + x^{-p} - 2 - p\, p\, (l\, x)^2}{\Delta} \cdot \frac{\partial x}{x\, (l\, x)^4} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix}$$
$$= \int \partial p \int \partial p \int \partial p \int P\, \partial p.$$

Quod ſi haec integralia ab $x = 0$ ad $x = \infty$ extendantur, eorum valores erunt duplo maiores. Ceterum hic perſpicuum eſt, quantitatem f eſſe debere poſitiuam, quia alias poteſtates $f^{\pm \frac{p}{n}}$ fieri poſſent imaginariae.

ORDO SEPTIMVS

Theorematum ex hac forma principali deductorum:

$$\int \frac{\partial x}{x} \cdot \frac{\operatorname{cof.} p\, l\, x}{x^n + 2 \operatorname{cof.} \theta + x^{-n}} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = \frac{\pi}{2\, n \operatorname{fin.} \theta} \left(\frac{e^{\frac{p}{n}\theta} - e^{-\frac{p}{n}\theta}}{e^{\frac{p}{n}\pi} - e^{-\frac{p}{n}\pi}} \right).$$

Statua-

Statuamus hic iterum pro denominatore

$\Delta = x^n + 2\,\text{cos}.\,\theta + x^{-n}$,

ſitque

$$P = \frac{\pi}{2\,n\,\text{ſin}.\,\theta} \cdot \frac{e^{\frac{p}{n}\theta} - e^{-\frac{p}{n}\theta}}{e^{\frac{p}{n}\pi} - e^{-\frac{p}{n}\pi}},$$

quae ergo quantitas iterum vt functio ipſius p ſpectari poteſt; vbi autem non amplius neceſſe eſt vt fractio $\frac{p}{n}$ ſit vnitate minor. Hinc igitur per differentiationem ſequentia deriuantur Theoremata:

I.

$$\int \frac{\text{ſin}.\,p\,l\,x}{\Delta} \cdot \frac{\partial\,x\,l\,x}{x} \begin{bmatrix} \text{ab} & x = 0 \\ \text{ad} & x = 1 \end{bmatrix} = - \frac{\partial\,P}{\partial\,p}.$$

II.

$$\int \frac{\text{coſ}.\,p\,l\,x}{\Delta} \cdot \frac{\partial x\,(l\,x)^2}{x} \begin{bmatrix} \text{ab} & x = 0 \\ \text{ad} & x = 1 \end{bmatrix} = - \frac{\partial\,\partial\,P}{\partial\,p^2}.$$

III.

$$\int \frac{\text{ſin}.\,p\,l\,x}{\Delta} \cdot \frac{\partial x\,(l\,x)^3}{x} \begin{bmatrix} \text{ab} & x = 0 \\ \text{ad} & x = 1 \end{bmatrix} = + \frac{\partial^3\,P}{\partial\,p^3}.$$

IV.

$$\int \frac{\text{coſ}.\,p\,l\,x}{\Delta} \cdot \frac{\partial x\,(l\,x)^4}{x} \begin{bmatrix} \text{ab} & x = 0 \\ \text{ad} & x = 1 \end{bmatrix} = + \frac{\partial^4\,P}{\partial\,p^4}.$$

etc.

Per integrationem vero

I.

$$\int \frac{\text{ſin}.\,p\,l\,x}{\Delta} \cdot \frac{\partial\,x}{x\,l\,x} \begin{bmatrix} \text{ab} & x = 0 \\ \text{ad} & x = 1 \end{bmatrix} = \int P\,\partial\,p.$$

II.

$$\int \frac{1 - \text{cos}.\, p\, l\, x}{\Delta} \cdot \frac{\partial x}{x\,(l\,x)^2} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = \int \partial p \int P\, \partial p.$$

III.

$$\int \frac{p\, l\, x - \text{sin}.\, p\, l\, x}{\Delta} \cdot \frac{\partial x}{x\,(l\,x)^3} \begin{bmatrix} \text{ab } x = 0 \\ \text{ab } x = 1 \end{bmatrix}$$
$$= \int \partial p \int \partial p \int P\, \partial p.$$

IV.

$$\int \frac{\frac{1}{2}\, p\, p\,(l\,x)^2 - 1 + \text{cos}.\, p\, l\, x}{\Delta} \cdot \frac{\partial x}{x\,(l\,x)^4} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix}$$
$$= \int \partial p \int \partial p \int \partial p \int P\, \partial p.$$

V.

$$\int \frac{\frac{1}{6}\, p^3\,(l\,x)^3 - p\, l\, x + \text{sin}.\, p\, l\, x}{\Delta} \cdot \frac{\partial x}{x\,(l\,x)^5} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix}$$
$$= \int \partial p \int \partial p \int \partial p \int \partial p \int P\, \partial p.$$

etc.

Haec igitur integralia, si ab $x = 0$ ad $x = \infty$ extendantur, iterum duplo fiunt maiora.

ORDO OCTAVUS

Theorematum ex hac forma principali deductorum:

$$\int \frac{\partial x}{x} \cdot \frac{\text{cos}.\, p\, l\, x}{x^{-n}\,(x^n + 1)^2} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = \frac{\pi}{n\,n} \cdot \frac{p}{e^{\frac{p}{n}\pi} - e^{-\frac{p}{n}\pi}}.$$

Statuamus hic pro denominatore $\Delta = x^{-n}\,(x^n + 1)^2$, sitque

P =

$$P = \frac{\pi}{n n} \cdot \frac{p}{e^{\frac{p}{n}\pi} - e^{-\frac{p}{n}\pi}},$$

atque per differentiationem hinc deducentur ſequentia Theoremata:

I.

$$\int \frac{\text{ſin.}\, p\, l\, x}{\Delta} \cdot \frac{\partial x\, l\, x}{x} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = - \frac{\partial P}{\partial p}.$$

II.

$$\int \frac{\text{coſ.}\, p\, l\, x}{\Delta} \cdot \frac{\partial x\, (l\, x)^2}{x} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = - \frac{\partial \partial P}{\partial p^2}.$$

III.

$$\int \frac{\text{ſin.}\, p\, l\, x}{\Delta} \cdot \frac{\partial x\, (l\, x)^3}{x} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = + \frac{\partial^3 P}{\partial p^3}.$$

IV.

$$\int \frac{\text{coſ.}\, p\, l\, x}{\Delta} \cdot \frac{\partial x\, (l\, x)^4}{x} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = + \frac{\partial^4 P}{\partial p^4}.$$

Per integrationem vero elicitur

I.

$$\int \frac{\text{ſin.}\, p\, l\, x}{\Delta} \cdot \frac{\partial x}{x\, l\, x} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = \int P\, \partial p.$$

II.

$$\int \frac{1 - \text{coſ.}\, p\, l\, x}{\Delta} \cdot \frac{\partial x}{x\, (l\, x)^2} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = \int \partial p \int P\, \partial p.$$

III.

$$\int \frac{p\, l\, x - \text{ſin.}\, p\, l\, x}{\Delta} \cdot \frac{\partial x}{x\, (l\, x)^3} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = \int \partial p \int \partial p \int P\, \partial p.$$

IV.

$$\int \frac{\frac{1}{2}pp(lx)^2 - 1 + \text{cos}.\,p\,l\,x}{\Delta} \cdot \frac{\partial x}{x(lx)^4} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix}$$
$$= \int \partial p \int \partial p \int \partial p \int P\,\partial p.$$

V.

$$\int \frac{\frac{1}{6}p^3(lx)^3 - p\,l\,x + \text{sin}.\,p\,l\,x}{\Delta} \cdot \frac{\partial x}{x(lx)^5} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix}$$
$$= \int \partial p \int \partial p \int \partial p \int \partial p \int P\,\partial p.$$

etc.

ORDO NONUS
Theorematum ex hac forma principali deductorum:

$$\int \frac{\partial x}{x} \cdot \frac{\text{cos}.\,p\,l\,x}{x^n + (f + \frac{1}{f}) + x^{-n}} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = \frac{2\pi\,\text{sin}.\,\frac{p}{n}.\,lf}{n(f - \frac{1}{f})(e^{\frac{p}{n}\pi} - e^{-\frac{p}{n}\pi})}.$$

Statuatur $\Delta = x^n + (f + \frac{1}{f}) + x^{-n}$ sitque

$$P = \frac{2\pi\,\text{sin}.\,\frac{p}{n}.\,lf}{n(f - \frac{1}{f})(e^{\frac{p}{n}\pi} - e^{-\frac{p}{n}\pi})}$$

atque hinc per differentiationem sequentia prodeunt Theoremata:

I.

$$\int \frac{\text{sin}.\,p\,l\,x}{\Delta} \cdot \frac{\partial x\,l\,x}{x} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = -\frac{\partial P}{\partial p}.$$

II.

$$\int \frac{\text{cos}.\,p\,l\,x}{\Delta} \cdot \frac{\partial x\,(lx)^2}{x} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = -\frac{\partial\partial P}{\partial p^2}.$$

III.

$$\int \frac{\text{fin.}\, p\, l\, x}{\Delta} \cdot \frac{\partial x\, (l\, x)^3}{x} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = + \frac{\partial^3 P}{\partial x^3}.$$

IV.

$$\int \frac{\text{cof.}\, p\, l\, x}{\Delta} \cdot \frac{\partial x\, (l\, x)^4}{x} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = + \frac{\partial^4 P}{\partial x^4}.$$

etc.

Per integrationem vero

I.

$$\int \frac{\text{fin.}\, p\, l\, x}{\Delta} \cdot \frac{\partial x}{x\, l\, x} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = \int P\, \partial p.$$

II.

$$\int \frac{1 - \text{cof.}\, p\, l\, x}{\Delta} \cdot \frac{\partial x}{x\, (l\, x)^2} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = \int \partial p \int P\, \partial p.$$

III.

$$\int \frac{p\, l\, x - \text{fin.}\, p\, l\, x}{\Delta} \cdot \frac{\partial x}{x\, (l\, x)^3} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = \int \partial p \int \partial p \int P \partial p.$$

IV.

$$\int \frac{\frac{1}{2} pp (l\, x)^2 - 1 + \text{cof.}\, p\, l\, x}{\Delta} \cdot \frac{\partial x}{x\, (l\, x)^4} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix}$$
$$= \int \partial p \int \partial p \int \partial p \int P\, \partial p.$$

etc.

Hic manifeſtum eſt quantitatem f negatiuam accipi non poſſe, quia alias iam ipſa functio P fieret imaginaria.

Adiungamus his theoremata ſimpliciora, quae ex hactenus allatis naſcuntur, dum angulus θ ſumitur rectus, vt ſit coſ. $\theta = 0$ et ſin. $\theta = 1$. Hinc ergo ſequentes ordines adiiciamus

ORDO DECIMVS

Theorematum ex hac forma principali deductorum:

$$\int \frac{\partial x}{x} \cdot \frac{x^p}{x^n + x^{-n}} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = \infty \end{bmatrix} = \frac{\pi}{2\, n \text{ coſ. } \frac{\pi p}{2n}}.$$

Haec forma ſcilicet nata eſt ex prima, ſumendo $\theta = \frac{\pi}{2}$, vnde poſito $\Delta = x^n + x^{-n}$ et $P = \frac{\pi}{2\, n \text{ coſ. } \frac{\pi p}{2n}}$ naſcuntur eaedem formulae, quae in ordine primo ſunt allatae. Hic autem imprimis notari meretur, quod integrale $\int P\, \partial p$ per logarithmos exhiberi poteſt: erit enim

$$\int P\, \partial p = \int \frac{\pi\, \partial p}{2\, n \text{ coſ. } \frac{\pi p}{2n}} = l \text{ tang. } (45^\circ + \tfrac{\pi p}{4n})$$

quod integrale ita eſt ſumtum, vt euaneſcat facto $p = 0$.

ORDO VNDECIMVS

Theorematum ex hac forma principali deductorum:

$$\int \frac{\partial x}{x} \cdot \frac{x^p + x^{-p}}{x^n + x^{-n}} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = \frac{\pi}{2\, n \text{ coſ. } \frac{\pi p}{2n}}.$$

Hic ſcilicet ordo natus eſt ex quarto, ponendo $\theta = \frac{\pi}{2}$; quamobrem ſtatuamus $\Delta = x^n + x^{-n}$ et $P = \frac{\pi}{2\, n \text{ coſ. } \frac{\pi p}{2n}}$, eademque theoremata inde naſcuntur, quae ſupra pro ordine quar-

quarto ſunt allata, vbi ergo iterum commode vſu venit vt ſit
$\int P\,\partial p = l\,\text{tang.}\,(45^\circ + \frac{\pi p}{4n})$.

ORDO XII.

Theorematum ex hac forma principali deductorum:

$$\int \frac{\partial x}{x} \cdot \frac{\text{cof.}\, p\, l\, x}{x^n + x^{-n}} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = -\frac{\pi}{2n} \cdot \frac{1}{e^{\frac{p\pi}{2n}} + e^{-\frac{p\pi}{2n}}}.$$

Quod ſi ergo ſtatuamus

$$\Delta = x^n + x^{-n} \text{ et } P = \frac{\pi}{2n\,(e^{\frac{p\pi}{2n}} + e^{-\frac{p\pi}{2n}})},$$

eadem plane Theoremata hinc oriuntur, quae ſupra pro caſu ſeptimo ſunt allata. Hic autem iterum notaſſe iuuabit integrale $\int P\,\partial p$ reuera exhiberi poſſe. Cum enim ſit

$$\int P\,\partial p = \int \frac{\pi\,\partial p}{2n\,(e^{\frac{p\pi}{2n}} + e^{-\frac{p\pi}{2n}})},$$

ponatur $\frac{\pi p}{2n} = z$, eritque

$$\int P\,\partial p = \int \frac{\partial z}{e^z + e^{-z}} = \int \frac{e^z\,\partial z}{e^{2z} + 1}.$$

Sit porro $e^z = v$, erit $\partial v = e^z\,\partial z$, hincque fiet

$$\int P\,\partial p = \int \frac{\partial v}{1 + v v} = \text{A tang. } v;$$

quare retro ſubſtituendo habebimus

$$\int P\,\partial p = \text{A tang. } e^{\frac{\pi p}{2n}}.$$

Denique adhuc referamus formulas illas integrales, in quarum denominatore erat $1 - x^{2n}$, quas quidem iam olim breuiter tetigi, nunc autem vberius euoluam.

ORDO XIII.

Theorematum ex hac forma principali deductorum:

$$\int \frac{\partial x}{x} \cdot \frac{x^p - x^{-p}}{x^n - x^{-n}} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = \frac{\pi}{2n} \text{ tang. } \frac{\pi p}{2n}.$$

Hic igitur iterum ſtatuamus

$\Delta = x^n - x^{-n}$ et $P = \frac{\pi}{2n}$ tang. $\frac{\pi p}{2n}$,

atque per differentiationem nanciſcemur ſequentia Theoremata.

I.

$$\int \frac{x^p + x^{-p}}{\Delta} \cdot \frac{\partial x \, l x}{x} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = \frac{\partial P}{\partial p}.$$

II.

$$\int \frac{x^p - x^{-p}}{\Delta} \cdot \frac{\partial x (l x)^2}{x} \begin{bmatrix} \text{ab } = 0 \\ \text{ad } = 1 \end{bmatrix} = \frac{\partial \partial P}{\partial p^2}.$$

III.

$$\int \frac{x^p + x^{-p}}{\Delta} \cdot \frac{\partial x (l x)^3}{x} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = \frac{\partial^3 P}{\partial p^3}.$$

IV.

$$\int \frac{x^p - x^{-p}}{\Delta} \cdot \frac{\partial x (l x)^4}{x} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = \frac{\partial^4 P}{\partial p^4}.$$

etc.

Integratio autem ſequentia ſuppeditat:

I.

$$\int \frac{x^p + x^{-p} - 2}{\Delta} \cdot \frac{\partial x}{x \, l x} \begin{bmatrix} \text{ad } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = \int P \, \partial p.$$

II.

II.

$$\int \frac{x^p - x^{-p} - 2p\,lx}{\Delta} \cdot \frac{\partial x}{x(lx)^2} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = \int \partial p \int P \, \partial p.$$

III.

$$\int \frac{x^p + x^{-p} - 2 - \frac{2}{2} pp (lx)^2}{\Delta} \cdot \frac{\partial x}{x(lx)^3} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = \int \partial p \int \partial p \int P \, \partial p.$$

IV.

$$\int \frac{x^p - x^{-p} - 2p\,lx - \frac{2}{6} p^3 (lx)^3}{\Delta} \cdot \frac{\partial x}{x(lx)^4} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = \int \partial p \int \partial p \int \partial p \int P \, \partial p.$$

V.

$$\int \frac{x^p + x^{-p} - 2 - \frac{2}{2} pp (lx)^2 - \frac{2}{24} p^4 (lx)^4}{\Delta} \cdot \frac{\partial x}{x(lx)^5} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = \int \partial p \int \partial p \int \partial p \int \partial p \int P \, \partial p.$$

etc.

vbi iterum notetur formulam integralem $\int P \, \partial p$ actu exhiberi posse; erit enim

$$\int P \, \partial p = \int \frac{\pi \, \partial p}{2n} \text{ tang. } \frac{\pi p}{2n} = -l \text{ cos. } \frac{\pi p}{2n} = +l \text{ sec. } \frac{\pi p}{2n}.$$

Hic probe notandum est, fractionem $\frac{p}{n}$ semper esse debere vnitate minorem.

ORDO

ORDO XIV.

Theorematum ex hac forma generali deductorum:

$$\int \frac{\partial x}{x} \cdot \frac{\text{ſin.}\, p\, l\, x}{x^{-n} - x^{+n}} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = \frac{\pi}{4n} \cdot \frac{e^{-\frac{p\pi}{2n}} - e^{+\frac{p\pi}{2n}}}{e^{+\frac{p\pi}{2n}} + e^{-\frac{p\pi}{2n}}}.$$

Statuatur igitur vt hactenus $\Delta = x^{-n} - x^{n}$ et

$$P = \frac{\pi}{4n} \cdot \frac{e^{-\frac{p\pi}{2n}} - e^{+\frac{p\pi}{2n}}}{e^{\frac{p\pi}{2n}} + e^{-\frac{p\pi}{2n}}},$$

atque differentiatio nobis praebebit ſequentia Theoremata:

I.

$$\int \frac{\text{coſ.}\, p\, l\, x}{\Delta} \cdot \frac{\partial x\, l\, x}{x} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = \frac{\partial P}{\partial p}.$$

II.

$$\int \frac{\text{ſin.}\, p\, l\, x}{\Delta} \cdot \frac{\partial x\, (l\, x)^2}{x} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = - \frac{\partial \partial P}{\partial p^2}.$$

III.

$$\int \frac{\text{coſ.}\, p\, l\, x}{\Delta} \cdot \frac{\partial x\, (l\, x)^3}{x} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = - \frac{\partial^3 P}{\partial p^3}.$$

IV.

$$\int \frac{\text{ſin.}\, p\, l\, x}{\Delta} \cdot \frac{\partial x\, (l\, x)^4}{x} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = + \frac{\partial^4 P}{\partial p^4}.$$

etc.

Per

Per integrationem autem impetramus sequentia:

I.

$$\int \frac{1 - \cos. p\, l\, x}{\Delta} \cdot \frac{\partial x}{x\, l\, x} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = \int P\, \partial p.$$

II.

$$\int \frac{p\, l\, x - \sin. p\, l\, x}{\Delta} \cdot \frac{\partial x}{x\, (l\, x)^2} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = \int \partial p \int P\, \partial p.$$

III.

$$\int \frac{\frac{1}{2} p\, p\, (l\, x)^2 - 1 + \cos. p\, l\, x}{\Delta} \cdot \frac{\partial x}{x\, (l\, x)^3} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix}$$
$$= \int \partial p \int \partial p \int P\, \partial p.$$

IV.

$$\int \frac{\frac{1}{6} p^3\, (l\, x)^3 - p\, l\, x + \sin. p\, l\, x}{\Delta} \cdot \frac{\partial x}{x\, (l\, x)^4} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix}$$
$$= \int \partial p \int \partial p \int \partial p \int P\, \partial p.$$

etc.

Vbi iterum commode euenit vt $\int P\, \partial p$ exhiberi possit, siquidem habemus

$$\int P\, \partial p = \int \frac{\pi\, \partial p}{4\, n} \cdot \frac{e^{-\frac{p\pi}{2n}} - e^{+\frac{p\pi}{2n}}}{e^{\frac{p\pi}{2n}} + e^{-\frac{p\pi}{2n}}}.$$

Ponatur enim $\frac{p\pi}{2n} = \Phi$, eritque

$$\int P \,\partial p = \int \tfrac{1}{2} \partial \Phi . \frac{e^{-\Phi} - e^{+\Phi}}{e^{\Phi} + e^{-\Phi}},$$

vbi denominatoris differentiale eſt $e^{\Phi}\,\partial\Phi - e^{-\Phi}\,\partial\Phi$, vnde concluditur

$$\int P \,\partial p = - l \sqrt{(e^{\Phi} + e^{-\Phi})} + C$$

quae conſtans **C** ita aſſumi debet, vt integrale euaneſcat poſito $\Phi = 0$, vnde fit

$$\int P \,\partial p = \tfrac{1}{2} l \frac{2}{e^{\frac{p\pi}{2n}} + e^{-\frac{p\pi}{2n}}}.$$

Hic autem perinde eſt, vtrum fractio $\frac{p}{n}$ maior ſit minorue vnitate.

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