

EVOLVTIO  
FORMVLAE INTEGRALIS

$$\int dx \left( \frac{1}{1-x} + \frac{1}{1x} \right)$$

A TERMINO  $x = 0$  VSQVE AD  $x = 1$   
EXTENSAE.

Auctore  
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*Conuent. exhib. die 29 Febr. 1776.*

§. I.

Ista formula integralis eo magis est notatu digna, quod eius valorem ostendi conuenire cum eo, quem praebet ista expressio:  $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots + \frac{1}{n} - \ln n$ , si numerus  $n$  sumatur infinite magnus, et quem per approximationem olim inveni esse  $= 0,5772156649015325$ , cuius valorem nullo adhuc modo ad mensuras transcendentis iam cognitae redigere potui; vnde haud inutile erit resolutionem huius formulae propositae pluribus modis tentare. Ac primo quidem, quoniam duabus constat partibus  $\int \frac{dx}{1-x}$  et  $\int \frac{dx}{1x}$ , manifestum est prioris partis valorem  $-\ln(1-x)$ , posito  $x = 1$ , fore  $-\ln 0$  ideoque  $= \infty$ ; tum vero etiam facile perspicitur, posterioris partis valorem quoque esse infinitum, sed signo contrario affectum, ita vt haud difficulter intelligatur aggregatum earum partium finitum habere valorem.

## Euolutio prima geometrica.

§. 2. Primo igitur hanc formulam per quadraturas exhibeamus, considerando lineam curuam, cuius abscissae  $x$  respondeat applicata  $y = \frac{1}{1-x} + \frac{1}{1+x}$ , tum vero eius area  $\int y \, dx$  abscissae  $x$  insitens ipsum valorem quaesitum repraesentabit, quamobrem formam huius curuae accuratius perpendamus. Ac primo quidem euidens est, hanc curuam neuiquam in regionem abscissarum negatiuarum porrigi, sed a termino  $x = 0$  incipere. Posito autem  $x = 0$  manifesto fit  $y = 1$ , ob  $\int x = \infty$ ; at existente  $x$  infinite paruo fiet  $y = 1 + x + \frac{1}{1+x}$ , vbi facile perspicitur postremum membrum  $\frac{1}{1+x}$  esse negatiuum et quasi infinities maius quam  $x$ , ita vt fiat  $y = 1 - ix$ , existente  $i$  numero maximo; vnde patet, si curuam ad axem  $A O$  referamus in eoque abscissas  $x$  a puncto  $A$  capiamus, in ipso puncto  $A$  applicatam fore  $AC = 1$ , et curuam in  $C$  hanc applicatam tangere, propterea quod decrementum applicatae infinities superat incrementum abscissae. Curua igitur originem ducet ab ipso puncto  $C$ , hincque continuo propius ad axem inflectetur, quem tandem in distantia infinita attinget. Posito enim  $x = \infty$  fit  $y = -\frac{1}{\infty} + \frac{1}{\infty}$ ; vbi notetur prius membrum  $\frac{1}{\infty}$  prae altero euanescere, ita vt iste valor sit positius, vnde patet, hanc curuam a puncto  $C$  ad axem continuo propius esse accessuram.

Tab. I.  
Fig. 1.

§. 3. Consideremus nunc abscissam  $AB = 1$ , vbi sumto  $x = 1$  fit  $y = \frac{1}{0} + \frac{1}{2}$ , vnde nihil plane concludere liceret, hanc ob causam statuamus  $x = 1 - \omega$ , vt fiat  $y = \frac{1}{\omega} + \frac{1}{1-\omega}$ . Iam  $\int (1 - \omega)$  in seriem euoluendo fiet

$$y = \frac{1}{\omega} - \frac{1}{\omega + \frac{1}{2}\omega^2 + \frac{1}{3}\omega^3 + \text{etc.}} = \frac{\frac{1}{2} + \frac{1}{3}\omega}{1 + \frac{1}{2}\omega + \frac{1}{3}\omega^2}.$$

Fiat

numeratore loco  $l x$  scribamus, prodibitque

$$y = \frac{-\frac{1}{2}(1-x)^2 - \frac{1}{3}(1-x)^3 - \frac{1}{4}(1-x)^4 - \text{etc.}}{(1-x)lx}$$

atque hinc

$$y = \frac{-\frac{1}{2}(1-x) - \frac{1}{3}(1-x)^2 - \frac{1}{4}(1-x)^3 - \text{etc.}}{lx}$$

hinc igitur per partes integrando valor quaesitus erit

$$\int y \partial x = -\frac{1}{2} \int \frac{(1-x) \partial x}{lx} - \frac{1}{3} \int \frac{(1-x)^2 \partial x}{lx} - \frac{1}{4} \int \frac{(1-x)^3 \partial x}{lx} - \text{etc.}$$

quae formulae singulae facile ad formulam illam generalem reducuntur, qua ostendi esse

$$\int \frac{x^m - x^n}{lx} \partial x = l \frac{m+1}{n+1}. (*)$$

Hinc enim statim erit  $\int \frac{(1-x) \partial x}{lx} = l \frac{1}{2}$ , et quia est

$$(1-x)^2 = 1-x-(x-xx), \text{ erit}$$

$$\int \frac{(1-x)^2 \partial x}{lx} = l \frac{1}{2} - l \frac{2}{3} = l \frac{1.3}{2^2}$$

Simili modo facile patebit fore

$$\int \frac{(1-x)^3 \partial x}{lx} = l \frac{1.3^3}{2^3.4};$$

$$\int \frac{(1-x)^4 \partial x}{lx} = l \frac{1.3^6.5}{2^4.4^2};$$

$$\int \frac{(1-x)^5 \partial x}{lx} = l \frac{1.3^{10}.5^3}{2^5.4^{10}.6};$$

$$\int \frac{(1-x)^6 \partial x}{lx} = l \frac{1.3^{15}.5^{15}.7}{2^6.4^{20}.6^6}; \text{ etc.}$$

§. 6. Ex his igitur valor nostrae formulae  $\int y \partial x$  per seriem logarithmicam prorsus singularem sequenti modo exprimitur:

$$\begin{aligned} \int y \partial x = & \frac{1}{2} l 2 + \frac{1}{3} l \frac{2^2}{1.3} + \frac{1}{4} l \frac{2^3.4}{1.3^3} + \frac{1}{5} l \frac{2^4.4^4}{1.3^6.5} \\ & + \frac{1}{6} l \frac{2^5.4^{10}.6}{1.3^{10}.5^3} + \frac{1}{7} l \frac{2^6.4^{20}.6^6}{1.3^{15}.5^{15}.7} + \text{etc.} \end{aligned}$$

Vbi

(\*) Hoc integrale duplici modo ab Ill. huius dissertationis Auctore fuit inventum in Tomo XIX. Nouorum Commentariorum pag. 70 et 79. F.

Vbi probe notandum est omnes logarithmos capi debere hyperbolicos; facile autem intelligitur, terminos huius serici continuo prodire minores, neque tamen hanc seriem tantopere conuergere, vt ex ea valor quaesitus commode computari possit.

### Euolutio tertia.

§. 7. Vtatur eadem resolutione logarithmi  $x$  in seriem infinitam, ac ponamus breuitatis gratia  $1 - x = t$ , vt fit

$$\log x = -t - \frac{1}{2} t t - \frac{1}{3} t^3 - \frac{1}{4} t^4 - \text{etc. eritque}$$

$$\frac{1}{\log x} = \frac{1}{t(1 + \frac{1}{2} t + \frac{1}{3} t t + \frac{1}{4} t^3 + \frac{1}{5} t^4 + \text{etc.})}$$

Iam fractionem  $\frac{1}{1 + \frac{1}{2} t + \frac{1}{3} t t + \text{etc.}}$  conuertamus more solito

in seriem recurrentem, quae fit

$$1 + \alpha t + \beta t t + \gamma t^3 + \delta t^4 + \epsilon t^5 + \zeta t^6 + \text{etc.}$$

vbi coëfficientes  $\alpha, \beta, \gamma, \delta, \text{etc.}$  ita erunt comparati, vt fit

$\alpha + \frac{1}{2} = 0,$	hincque	$\alpha = -\frac{1}{2},$
$\beta + \frac{1}{2}\alpha + \frac{1}{3} = 0,$		$\beta = -\frac{1}{12},$
$\gamma + \frac{1}{2}\beta + \frac{1}{3}\alpha + \frac{1}{4} = 0,$		$\gamma = -\frac{1}{24},$
$\delta + \frac{1}{2}\gamma + \frac{1}{3}\beta + \frac{1}{4}\alpha + \frac{1}{5} = 0,$		$\delta = -\frac{19}{720},$
etc.		etc.

vnde hanc seriem tanquam cognitam spectare licet.

§. 8. Hoc igitur valore substituto erit

$$\frac{1}{\log x} = -\frac{1}{t} - \alpha - \beta t - \gamma t t - \delta t^3 - \epsilon t^4 + \text{etc.},$$

quare cum fit  $\frac{1}{1-x} = \frac{1}{t}$ , erit

$$y = -\alpha - \beta t - \gamma t t - \delta t^3 - \epsilon t^4 - \text{etc. siue}$$

$$y = -\alpha - \beta(1-x) - \gamma(1-x)^2 - \delta(1-x)^3 - \text{etc.}$$

Cum

Cum nunc in genere fit

$$\int \partial x (1-x)^n = C \frac{(1-x)^{n+1}}{n+1} = \frac{1}{n+1} \frac{(1-x)^n}{1-x}$$

posito  $x = 1$ , quemadmodum assumimus, erit

$$\int \partial x (1-x)^n = \frac{1}{n+1}$$

Hinc igitur singulis integralibus collectis reperietur

$$\int y \partial x = -\frac{\alpha}{2} - \frac{\beta}{3} - \frac{\gamma}{4} - \frac{\delta}{5} - \frac{\epsilon}{6} - \text{etc.},$$

unde per valores ante euolutos fiet

$$\int y \partial x = \frac{1}{4} + \frac{1}{36} + \frac{1}{96} + \frac{19}{3600} + \text{etc.}$$

quae series utique parum est conuergens.

### Euolutio quarta.

§. 9. Cum habeamus  $y = \frac{1-x+1-x}{(1-x)l x}$ , quemadmodum ante partem  $l x$  in seriem infinitam resolimus, ita nunc vicissim ipsam quantitatem in seriem per logarithmos ipsius  $x$  procedentem euoluamus. Quia enim est  $x = e^{l x}$ , erit

$$x = 1 + l x + \frac{1}{2} (l x)^2 + \frac{1}{6} (l x)^3 + \frac{1}{24} (l x)^4 + \text{etc.}$$

vbi loco  $l x$  breuitatis ergo scribamus  $u$ , atque hanc seriem tantum in numeratorem introducamus, vt fiat

$$y = \frac{-\frac{1}{2} u u - \frac{1}{6} u^3 - \frac{1}{24} u^4 - \frac{1}{120} u^5 - \text{etc.}}{u(1-x)}$$

$$y = \frac{-\frac{1}{2} u - \frac{1}{6} u u - \frac{1}{24} u^3 - \frac{1}{120} u^4 - \text{etc.}}{1-x}$$

ideoque

$$\int y \partial x = -\frac{1}{2} \int \frac{\partial x l x}{1-x} - \frac{1}{6} \int \frac{\partial x (l x)^2}{1-x} - \frac{1}{24} \int \frac{\partial x (l x)^3}{1-x} - \frac{1}{120} \int \frac{\partial x (l x)^4}{1-x} - \text{etc.}$$

(9)

§. 10. Cum nunc in genere, sumto scilicet integrali ab  $x=0$  ad  $x=1$ , fit  $\int \partial x (lx)^n = \pm 1.2.3.4.5. \dots n$ , vbi signum  $+$  valet quando  $n$  est numerus par, contra vero signum  $-$ , erit porro

$$\int x^{n-1} \partial x (lx)^\lambda = \pm \frac{1.2.3.4.5. \dots \lambda}{n^{\lambda+1}},$$

vbi signum  $+$  valet si  $\lambda$  fuerit numerus par, inferius vero si impar. Hinc igitur singulas nostras formulas per series integremus, dum loco  $\frac{1}{1-x}$  seriem scribimus

$$1 + x + xx + x^3 + x^4 + x^5 + \text{etc.}$$

atque hinc primo nanciscemur

$$\int \frac{\partial x lx}{1-x} = -1 \left( 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \text{etc.} \right)$$

$$\int \frac{\partial x (lx)^2}{1-x} = 1.2 \left( 1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \frac{1}{5^3} + \frac{1}{6^3} + \text{etc.} \right)$$

$$\int \frac{\partial x (lx)^3}{1-x} = -1.2.3 \left( 1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \frac{1}{6^4} + \text{etc.} \right)$$

$$\int \frac{\partial x (lx)^4}{1-x} = 1.2.3.4 \left( 1 + \frac{1}{2^5} + \frac{1}{3^5} + \frac{1}{4^5} + \frac{1}{5^5} + \frac{1}{6^5} + \text{etc.} \right)$$

etc.

etc.

His igitur seriebus substitutis reperiemus

$$\int y \partial x = \frac{1}{2} \left( 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \text{etc.} \right)$$

$$- \frac{1}{3} \left( 1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \frac{1}{5^3} + \text{etc.} \right)$$

$$+ \frac{1}{4} \left( 1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \text{etc.} \right)$$

$$- \frac{1}{5} \left( 1 + \frac{1}{2^5} + \frac{1}{3^5} + \frac{1}{4^5} + \frac{1}{5^5} + \text{etc.} \right)$$

cuius expressionis iam nuper ostendi valorem esse numerum illum memorabilem 0,5772156649015325. (\*)

Euo-

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(\*) V. Differtatio: *De numero memorabili in summatione progressionis harmonicæ naturalis occurrente.* Acta Acad. pro Anno 1781. Pars posterior, pag. 49. seqq.

F.

### Evolutio quinta.

§. 11. Utamur hic eadem resolutione in seriem ipsius numeri  $x$ , sed eam alio modo adhibeamus. Scilicet cum posito  $lx = u$  fit

$x = 1 + u + \frac{1}{2}uu + \frac{1}{6}u^3 + \frac{1}{24}u^4 + \frac{1}{120}u^5 + \text{etc.}$

erit formulae nostrae ipsa pars prior

$$\frac{1}{1-x} = \frac{1}{u + \frac{1}{2}uu + \frac{1}{6}u^3 + \frac{1}{24}u^4 + \frac{1}{120}u^5 + \text{etc.}}$$

$$\frac{1}{u} \cdot \frac{1}{1 + \frac{1}{2}u + uu + \frac{1}{24}u^3 + \text{etc.}}$$

Hanc fractionem:

$$\frac{1}{1 + \frac{1}{2}u + \frac{1}{2}uu + \frac{1}{24}u^3 + \text{etc.}}$$

more solito in seriem recurrentem conuertamus, quae fit

$$1 - Au + Buu - Cu^3 + Du^4 - Eu^5 + \text{etc.}$$

eritque facta comparatione:

$A = \frac{1}{2}$	ergo $A = \frac{1}{2}$
$B = \frac{1}{2}A - \frac{1}{6}$	$B = \frac{1}{12}$
$C = \frac{1}{2}B - \frac{1}{6}A + \frac{1}{24}$	$C = 0$
$D = \frac{1}{2}C - \frac{1}{6}B + \frac{1}{24}A - \frac{1}{120}$	$D = -\frac{1}{720}$
$E = \frac{1}{2}D - \frac{1}{6}C + \frac{1}{24}B - \frac{1}{120}A + \frac{1}{720}$	$E = 0$

§. 12. Hac igitur serie introducta et loco  $u$  restituto valore  $lx$ , formula nostra  $\frac{1}{1-x} + \frac{1}{lx} = y$  sequentem induet formam:

$$y = -\frac{1}{u} + A - Bu + Cuu - Du^3 + \text{etc.} + \frac{1}{u}, \text{ siue}$$

$$y = A - Blx + C(lx)^2 - D(lx)^3 + E(lx)^4 - \text{etc.}$$

vnde cum in genere fit

$\int dx$

$$\int \partial x (l x)^n = \pm 1. 2. 3. 4. \dots n,$$

postquam scilicet absoluta integratione positum fuerit  $x = 1$ , ubi signum superius valet, quando  $n$  est numerus par, inferius vero si  $n$  impar: hoc obseruato nanciscimur valorem quaesitum  $\int y \partial x$  sequenti modo expressum:

$$\int y \partial x = A + 1B + 1. 2 C + 1. 2. 3 D + 1. 2. 3. 4 E + 1. \dots 5 F + \text{etc.}$$

quae series vtique parum conuergit, ob coëfficientes litterarum A, B, C, D; verum perpendendum est, ipsos valores harum litterarum continuo magis decrefcere, quandoquidem certum est, seriei valorem esse debere 0, 5772156649015325, quare operae pretium erit harum litterarum seriem accuratius euoluere.

### TRANSFORMATIO FRACTIONIS

$$\frac{1}{1 + \frac{1}{2}u + uu + \frac{1}{24}u^3 + u^4 + \text{etc.}}$$

#### IN SERIEM

$$1 - Au + Buu - Cu^3 + Du^4 - Eu^5 + \text{etc.}$$

§. 13. Designet littera  $s$  summam huius seriei, eritque

$$s = \frac{u}{e^u - 1}, \text{ vnde fit } e^u = \frac{u + s}{s}, \text{ hincque } u = l(u + s) - ls,$$

ergo differentiando erit

$$\partial u = \frac{\partial u + \partial s}{u + s} - \frac{\partial s}{s} = \frac{s \partial u - u \partial s}{s(u + s)},$$

sive statim ponatur  $s = pu$ , vt fit  $u = l \frac{1+p}{p}$ , vnde fit  $\partial u = -\frac{\partial p}{p(p+1)}$ , quae expressio quo seriem praebeat concinniore,

statuamus  $p = q - \frac{1}{2}$ , vt iam fit  $s = (q - \frac{1}{2})u$ , tum vero erit

$$\partial u = -\frac{\partial q}{qq - \frac{1}{4}}, \text{ vnde colligitur haec aequatio:}$$

$$qq - \frac{1}{4} + \frac{\partial q}{\partial u} = 0.$$



§. 14. Ex hac igitur aequatione inuestigari debet series valorem ipsius  $q$  exhibens, vbi ante omnia principium huius seriei inde constitui oportet, quod posito  $u = 0$  fieri debeat  $s = 1$  et  $q = \frac{1}{2} + \frac{1}{2}$ ; vnde patet seriei pro  $q$  fingendae primum terminum esse debere  $\frac{1}{2}$ ; tum vero facile perspicitur in hac serie potestates ipsius  $u$  tantum impares assumi debere. Quamobrem fingatur ista series:

$$q = \frac{1}{2} + au + bu^3 + cu^5 + du^7 + eu^9 + \text{etc. eritque}$$

$$qq = \frac{1}{uu} + 2a + 2bu^2 + 2cu^4 + 2du^6 + 2eu^8 + 2fu^{10} + \text{etc.}$$

$$+ aa + 2ab + 2ac + 2ad + 2ae + \text{etc.}$$

$$+ bb + 2bc + 2bd + \text{etc.}$$

$$+ cc$$

$\frac{\partial q}{\partial u} = -\frac{1}{uu} + a + 3b + 5c + 7d + 9e + 11f + \text{etc.}$   
 harum ergo serierum summa debet esse  $\frac{1}{4}$ , vnde deducuntur sequentes determinaciones:

$3a = \frac{1}{4}$	ergo $a = \frac{1}{12}$
$5b + aa = 0$	$b = -\frac{aa}{5}$
$7c + 2ab = 0$	$c = -\frac{2ab}{7}$
$9d + 2ac + bb = 0$	$d = -\frac{2ac - bb}{9}$
$11e + 2ad + 2bc = 0$	$e = -\frac{2ad - 2bc}{11}$
$13f + 2ae + 2bd + cc = 0$	$f = -\frac{2ae - 2bd - cc}{13}$
etc.	etc.

ex quibus formulis valores numerici litterarum  $a, b, c, d$ , computari poterunt.

§. 15. His autem litteris  $a, b, c, d$ , etc. definitis ipsa series pro  $s$  quaesita erit

$s = 1$

$$s = 1 - \frac{1}{2}u + auu + bu^4 + cu^6 + du^8 + eu^{10} + fu^{12} + gu^{14} + \text{etc.}$$

quare cum supra posuerimus

$$s = 1 - Au + Buu - Cu^3 + Du^4 - Eu^5 + Fu^6 - Gu^7 + Hu^8 - \text{etc.}$$

valores harum litterarum maiuscularum per minusculas sequenti modo definientur:

$$A = \frac{1}{2}, B = a, C = 0, D = b, E = 0, F = c, G = 0, H = d, \text{ etc.}$$

ficque potestatum imparium coëfficientes per se euanescent. Evidens autem est ope formularum hic inuentarum valores litterarum  $a, b, c, d$ , etc. multo facilius et promptius assignari posse quam per relationes supra allatas, scilicet erit  $A = \frac{1}{2}$ ,  $B = a = \frac{1}{12}$ ,  $C = 0$ ,  $D = -\frac{1}{720}$ ,  $E = 0$ ,  $F = \frac{1}{30240}$ , etc.

§. 16. Quoniam hoc modo calculus istarum litterarum mox ad numeros vehementer magnos deduceret, loco litterarum  $a, b, c, d$ , etc. quaeramus alias  $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}$ , etc. quarum signa iam alternentur et quarum valores ad illos sequenti modo referantur:

$$a = \frac{\mathfrak{A}}{12}, b = -\frac{\mathfrak{B}}{12^2}, c = +\frac{\mathfrak{C}}{12^3}, d = -\frac{\mathfrak{D}}{12^4}, e = +\frac{\mathfrak{E}}{12^5}, \text{ etc.}$$

ita vt iam fit nostra series

$$s = 1 - \frac{1}{2}u + \frac{\mathfrak{A}u^2}{12} - \frac{\mathfrak{B}u^4}{12^2} + \frac{\mathfrak{C}u^6}{12^3} - \frac{\mathfrak{D}u^8}{12^4} + \text{etc.}$$

atque istae nouae litterae  $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}$  sequenti modo determinabuntur:

$$\begin{array}{l|l} \mathfrak{A} = 1 & \mathfrak{D} = \frac{2\mathfrak{A}\mathfrak{C} + \mathfrak{B}\mathfrak{B}}{9}, \\ \mathfrak{B} = \frac{\mathfrak{A}\mathfrak{A}}{5} & \mathfrak{E} = \frac{2\mathfrak{A}\mathfrak{D} + 2\mathfrak{B}\mathfrak{C}}{11}, \\ \mathfrak{C} = \frac{2\mathfrak{A}\mathfrak{B}}{7} & \mathfrak{F} = \frac{2\mathfrak{A}\mathfrak{E} + 2\mathfrak{B}\mathfrak{D} + \mathfrak{C}\mathfrak{C}}{13}, \text{ etc.} \end{array}$$

qui valores nunc haud difficulter in numeris euoluentur ac reperientur:

$$\mathfrak{A} = 1, \mathfrak{B} = \frac{1}{5}, \mathfrak{C} = \frac{2}{35}, \mathfrak{D} = \frac{3}{175}, \mathfrak{E} = \frac{2}{385}, \mathfrak{F} = \frac{1322}{875875}, \text{ etc.}$$

§. 17. Introducamus igitur istas novas litteras  $\mathfrak{A}$ ,  $\mathfrak{B}$ ,  $\mathfrak{C}$ ,  $\mathfrak{D}$ , etc. in seriem §. 12. pro  $\int y \partial x$  inuentam eritque

$$\int y \partial x = \frac{1}{2} + \frac{1 \cdot \mathfrak{A}}{12} - \frac{1 \cdot 2 \cdot 3 \mathfrak{B}}{12^2} + \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \mathfrak{C}}{12^3} - \frac{1 \cdot 1 \cdot \dots \cdot 7 \mathfrak{D}}{12^4} + \text{etc.}$$

hanc autem seriem non satis conuergere iam supra obseruauimus.

### TRANSFORMATIO FRACTIONIS

$$\frac{1}{1 + \frac{1}{2}t + \frac{1}{3}t^2 + \frac{1}{4}t^3 + \frac{1}{5}t^4 + \text{etc}}$$

#### IN SERIEM

$$1 + \alpha t + \beta t^2 + \gamma t^3 + \delta t^4 + \varepsilon t^5 + \zeta t^6 + \text{etc.}$$

§. 17. Ad hanc transformationem perducti fumus supra in §. 7. ubi euolutio litterarum  $\alpha$ ,  $\beta$ ,  $\gamma$ ,  $\delta$ , etc. mox fiebat nimis molesta. Nunc igitur simili modo utamur quo ante, positaque hac serie quam quaerimus  $= s$ , erit  $s = \frac{-t}{1(1-t)}$ , ideoque  $1(1-t) = -\frac{t}{s}$ , ergo differentiando  $-\frac{\partial t}{1-t} = -\frac{\partial v}{v}$ , seu  $\frac{\partial v}{v} + \frac{\partial t}{1-t} = 0$ , cui hanc formam tribuamus:

$$v v + \frac{\partial v}{\partial t} (1-t) = 0,$$

ex qua aequatione series idonea pro  $v$  elici debet.

§. 18. Cum igitur posito  $t = 0$  fiat  $s = 1$ , hoc casu esse debeat  $v = \frac{1}{t}$ , quamobrem fingamus istam seriem:

$$v = \frac{1}{t} + a + b t + c t^2 + d t^3 + e t^4 + \text{etc.},$$

qui valor sequenti modo substituatur

$$\begin{aligned}
 \frac{\partial v}{\partial t} &= -\frac{1}{tt} + * + b + 2ct + 3dtt + 4et^3 + 5ft^4 + 6gt^5 + \text{etc.} \\
 -\frac{\partial v}{\partial t} &= +\frac{1}{t} - * - b - 2c - 3d - 4e - 5f - \text{etc.} \\
 +\partial v &= +\frac{1}{tt} + \frac{2a}{t} + 2b + 2c + 2d + 2e + 2f + 2g + \text{etc.} \\
 &\quad + aa + 2ab + 2ac + 2ad + 2ae + 2af + \text{etc.} \\
 &\quad + bb + 2bc + 2bd + 2be + \text{etc.} \\
 &\quad + cc + 2cd + \text{etc.}
 \end{aligned}$$

Hinc igitur prodeunt sequentes determinaciones:

$$\begin{aligned}
 1 + 2a &= 0, \\
 3b + aa &= 0, \\
 4c + 2ab - b &= 0, \\
 5d + 2ac - 2c + bb &= 0, \\
 6e + 2ad + 2bc - 3d &= 0, \\
 7f + 2ae + 2bd - 4e + cc &= 0,
 \end{aligned}$$

quae formulae ob  $a = -\frac{1}{2}$  contrahuntur in sequentes:

$$\begin{array}{ll}
 3b = -\frac{1}{2} & \text{ergo } a = -\frac{1}{2} \\
 4c = 2b & b = -\frac{1}{12} \\
 5d = 3c - bb & c = -\frac{1}{24} \\
 6e = 4d - 2bc & d = -\frac{19}{720} \\
 7f = 5e - 2bd - cc & e = -\frac{3}{160} \\
 8g = 6f - 2be - 2cd & f = -\frac{827}{32 \cdot 2 \cdot 0 \cdot 7}
 \end{array}$$

§. 19. Hinc igitur erit series quaesita

$$s = 1 + at + btt + ct^3 + dt^4 + \text{etc.},$$

quae supra posita fuerat

$$s = 1 + \alpha t + \beta tt + \gamma t^3 + \delta t^4 + \text{etc.}$$

litterae igitur latinae et graecae prorsus conueniunt, eritque ergo

$$\int y \partial x = -\frac{a}{2} - \frac{b}{3} - \frac{c}{4} - \frac{d}{5} - \frac{e}{6} - \text{etc.}$$

et

et valoribus modo inuentis substitutis

$$\int y \partial x = \frac{1}{2 \cdot 2} + \frac{1}{3 \cdot 12} + \frac{1}{4 \cdot 24} + \frac{19}{5 \cdot 720} + \frac{3}{6 \cdot 160} + \text{etc.}$$

§. 20. Quo autem calculus harum litterarum expeditior reddatur, ponamus  $a = -\frac{A}{2}$ ,  $b = -\frac{B}{4}$ ,  $c = -\frac{C}{8}$ ,  $d = -\frac{D}{12}$ ,  $e = -\frac{E}{32}$ , etc. vt fit

$$\int y \partial x = \frac{A}{2 \cdot 2} + \frac{B}{3 \cdot 4} + \frac{C}{4 \cdot 8} + \frac{D}{5 \cdot 16} + \frac{E}{6 \cdot 32} + \text{etc.}$$

pro his autem litteris habebimus sequentes determinationes:

$A = 1,$	$E = \frac{8D + 2BC}{6},$
$B = \frac{1}{3},$	$F = \frac{10E + 2BD + CC}{7},$
$C = \frac{4B}{4},$	$G = \frac{12F + 2BE + 2CD}{8},$
$D = \frac{6C + BB}{5},$	etc.

vnde colligitur

$$A = 1, B = \frac{1}{3}, C = \frac{1}{3}, D = \frac{19}{45}, E = \frac{39}{50}, \text{etc.}$$

Haec igitur ad calculos superiores subleuandos sufficere poterunt.

