METHODVS FACILIS
INVENIENDI INTEGRALE HVIVS FORMVLAE

$$\int \frac{\partial x}{x} \cdot \frac{x^{n+p} - 2 x^n \cos. \zeta + x^{n-p}}{x^{2n} - 2 x^n \cos. \theta + 1},$$

CASV QVO POST INTEGRATIONEM PONITVR VEL $x = 1$ VEL $x = \infty$.

Auctore
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Conuent. exhib. d. 18 *Mart.* 1776.

§. 1.

Denotet S integrale huius formulae generaliter sumtum, ita vt quaeri debeat valor ipsius S, casu quo statuitur $x = 1$; vbi primum obseruo, formam propositam multo concinniorem reddi, si fractionis numerator et denominator per x^n diuidantur; tum enim habebimus

$$S = \int \frac{\partial x}{x} \cdot \frac{x^p - 2 \cos. \zeta + x^{-p}}{x^n - 2 \cos. \theta + x^{-n}}.$$

Hic statim patet, denominatorem $x^n - 2 \cos. \theta + x^{-n}$ semper in n factores resolui posse, qui singuli sint formae $x^1 - 2\cos.\omega + x^{-1}$, vbi angulum ω ita capi oportet, vt, dum iste euanescit, simul etiam ipse denominator ad nihilum redigatur.

§. 2.

§. 2. Poſito autem iſto factore $x^1 - 2\cos.\omega + x^{-1} = 0$, inde fit $x = \cos.\omega + \sqrt{-1}\,\sin.\omega$, vnde in genere colligitur

$x^\lambda = \cos.\lambda\omega + \sqrt{-1}\,\sin.\lambda\omega$ et

$x^{-\lambda} = \cos.\lambda\omega - \sqrt{-1}\,\sin.\lambda\omega.$

Hinc ergo denominator iſtum accipiet valorem: $2\cos.n\omega - 2\cos.\theta$, qui igitur euaneſcet, ſi pro $n\omega$ ſumatur aliquis ex his valoribus:

$$\theta,\ \theta + 2\pi,\ \theta + 4\pi,\ \theta + 6\pi,\ \theta + 8\pi,\ \text{etc.}$$

quare cum numerus horum valorum debeat eſſe $= n$, omnes valores anguli ω erunt ſequentes:

$$\frac{\theta}{n},\ \frac{\theta + 2\pi}{n},\ \frac{\theta + 4\pi}{n},\ \frac{\theta + 6\pi}{n},\ \ldots\ldots\ \frac{\theta + 2(n-1)\pi}{n}.$$

Praeterea vero cum ſit $\cos.n\omega = \cos.\theta$, erit etiam $\sin.n\omega = \sin.\theta$.

§. 3. Cum igitur denominator habeat n factores huius formae: $x^1 - 2\cos.\omega + x^{-1}$, noſtra fractio, quicunque fuerit eius numerator, in n fractiones ſimplices reſolui poterit, quarum denominatores erunt illi n factores denominatoris. Scribamus igitur breuitatis gratia Π loco numeratoris $x^p - 2\cos.\zeta + x^{-p}$, atque haec fractio: $\frac{\Pi}{x^n - 2\cos.\theta + x^{-n}}$ reſoluetur in n fractiones ſimplices, quarum ſingulae hanc habebunt formam: $\frac{P}{x^1 - 2\cos.\omega + x^{-1}}$, quocirca ſtatuamus:

$$\frac{\Pi}{x^n - 2\cos.\theta + x^{-n}} = \frac{P}{x^1 - 2\cos.\omega + x^{-1}} + R;$$

vbi littera R omnes reliquas complectatur fractiones, vnde ſtatim habebimus

$$\frac{\Pi(x^1 - 2\cos.\omega + x^{-1})}{x^n - 2\cos.\theta + x^{-n}} = P + R(x^1 - 2\cos.\omega + x^{-1}).$$

Quodſi

Quodſi iam faciamus $x^1 - 2\,\text{coſ.}\,\omega + x^{-1} = 0$, hinc colligemus numeratorem P: erit enim

$$P = \frac{\Pi\,(x^1 - 2\,\text{coſ.}\,\omega + x^{-1})}{x^n - 2\,\text{coſ.}\,\theta + x^{-n}},$$

ſiquidem in hac aequatione ponatur $x = \text{coſ.}\,\omega + \sqrt{-1}\,\text{ſin.}\,\omega$.

§. 4. At vero iam vidimus, ſi ipſi x hunc valorem tribuamus, illius fractionis tam numeratorem quam denominatorem euaneſcere, quamobrem ſecundum regulam notiſſimam loco numeratoris et denominatoris ſua ſcribamus differentialia, ac prodibit $P = \frac{\Pi\,(x^1 - x^{-1})}{n\,x^n - n\,x^{-n}}$. Nunc igitur ſi loco x valor aſſignatus ſcribatur, primo pro Π nanciſcemur hunc valorem:

$$\Pi = 2\,\text{coſ.}\,p\,\omega - 2\,\text{coſ.}\,\zeta;$$

ex fractione autem oritur iſte valor: $\frac{\text{ſin.}\,\omega}{n\,\text{ſin.}\,n\,\omega}$, quae ergo expreſſio cum ſit realis, numerator quaeſitus erit

$$P = \frac{2\,\text{ſin.}\,\omega\,(\text{coſ.}\,p\,\omega - \text{coſ.}\,\zeta)}{n\,\text{ſin.}\,n\,\omega}.$$

Iam autem vidimus eſſe $\text{ſin.}\,n\,\omega = \text{ſin.}\,\theta$, vnde iſte numerator erit $P = \frac{2\,\text{ſin.}\,\omega\,(\text{coſ.}\,p\,\omega - \text{coſ.}\,\zeta)}{n\,\text{ſin.}\,\theta}$.

§. 5. Quaelibet igitur fractio partialis ex reſolutione fractionis propoſitae oriunda erit huiusmodi:

$$\frac{2\,(\text{coſ.}\,p\,\omega - \text{coſ.}\,\zeta)}{n\,\text{ſin.}\,\theta} \cdot \frac{\text{ſin.}\,\omega}{x^1 - 2\,\text{coſ.}\,\omega + x^{-1}},$$

in qua forma ſi angulo ω ſucceſſiue tribuantur omnes valores ſupra aſſignati, qui erant

$$\frac{\theta}{n}, \frac{\theta + 2\pi}{n}, \frac{\theta + 4\pi}{n}, \ldots \ldots \frac{\theta + 2(n-1)\pi}{n},$$

orientur omnes fractiones partiales, quae in vnam ſummam col-

collectae ipsam formam propositam $\frac{x^p - 2\cos.\zeta + x^{-p}}{x^n - 2\cos.\theta + x^{-n}}$ producere debebunt, vnde etiam singulae in $\frac{\partial x}{x}$ ductae et integratae, tum vero in vnam summam collectae, exhibebunt integrale quaesitum S.

§. 6. Consideremus igitur fractionem: $\frac{\sin.\omega}{x^1 - 2\cos.\omega + x^{-1}}$, quae ducta in $\frac{\partial x}{x}$ praebet $\frac{\partial x \sin.\omega}{x^2 - 2\cos.\omega + 1}$, cuius integrale, ita sumtum vt euanescat posito $x = 0$, constat esse $=$ A tang. $\frac{x \sin.\omega}{1 - x\cos.\omega}$. Hinc igitur ex hac fractione partiali oritur ista pars integralis: $\frac{2(\cos.p\omega - \cos.\zeta)}{n\sin.\theta}$. A tang. $\frac{x\sin.\omega}{1 - x\cos.\omega}$, vnde ergo facile deducuntur omnes n partes integralis quaesiti, si loco ω ordine omnes eius valores assignati substituantur atque in vnam summam colligantur.

§. 7. Quoniam autem hoc loco eum tantum integralis valorem postulamus, qui oritur posito $x = 1$, hoc casu fiet

$$\text{A tang.} \frac{x\sin.\omega}{1 - x\cos.\omega} = \text{A tang.} \frac{\sin.\omega}{1 - \cos.\omega}.$$

At vero ista formula $\frac{\sin.\omega}{1-\cos.\omega}$ exprimit cotangentem anguli $\frac{1}{2}\omega$, ideoque tangentem anguli $\frac{\pi - \omega}{2}$, ita vt hoc casu pars integralis futura sit $\frac{\cos.p\omega - \cos.\zeta}{n\sin.\theta}(\pi - \omega)$. Hic autem in transitu notasse iuuabit, si desideretur integrale pro casu $x = \infty$, tum proditurum esse A tang. $-\frac{\sin\omega}{\cos.\omega}$; quia igitur est $-\frac{\sin.\omega}{\cos.\omega}$ tangens anguli $\pi - \omega$, cum ante habuissemus $\frac{\pi-\omega}{2}$, hinc patet, casu $x = \infty$ etiam totum integrale duplo maius fore quam casu $x = 1$.

§. 8.

§. 8. Tribuamus igitur angulo ω ſucceſſiue omnes eius valores, eosque ordine hic ſiſtamus, eritque

$$S = \frac{\operatorname{cof.}\frac{p\theta}{n} - \operatorname{cof.}\zeta}{n \operatorname{fin.}\theta} \cdot \left(\pi - \frac{\theta}{n}\right)$$
$$+ \frac{\operatorname{cof.}\frac{p}{n}(\theta + 2\pi) - \operatorname{cof.}\zeta}{n \operatorname{fin.}\theta} \cdot \frac{((n-2)\pi - \theta)}{n}$$
$$+ \frac{\operatorname{cof.}\frac{p}{n}(\theta + 4\pi) - \operatorname{cof.}\zeta}{n \operatorname{fin.}\theta} \cdot \frac{((n-4)\pi - \theta)}{n}$$
$$+ \frac{\operatorname{cof.}\frac{p}{n}(\theta + 6\pi) - \operatorname{cof.}\zeta}{n \operatorname{fin.}\theta} \cdot \frac{((n-6)\pi - \theta)}{n}$$
etc.

quarum formularum numerus debet eſſe $= n$. Haec autem expreſſio ſtatim in duas partes diſtinguatur hoc modo indicandas: $S = \frac{Q}{n \operatorname{fin.}\theta} - \frac{R \operatorname{cof.}\zeta}{n \operatorname{fin.}\theta}$, ita vt ſit

$$Q = \frac{n\pi - \theta}{n} \operatorname{cof.}\frac{p\theta}{n} + \frac{((n-2)\pi - \theta)}{n} \operatorname{cof.}\frac{p}{n}(\theta + 2\pi)$$
$$+ \frac{((n-4)\pi - \theta)}{n} \operatorname{cof.}\frac{p}{n}(\theta + 4\pi) \text{ etc.},$$

$$R = \frac{n\pi - \theta}{n} + \frac{(n-2)\pi - \theta}{n} + \frac{(n-4)\pi - \theta}{n} + \frac{(n-6)\pi - \theta}{n} + \text{etc.}$$

ita vt iam nobis incumbatur in valores litterarum Q et R inquirere.

§. 9. Primo autem ſtatim patet, valorem ipſius R eſſe progreſſionem arithmeticam decreſcentem differentia $\frac{2\pi}{n}$, vnde ſumma n terminorum erit $= \pi - \theta$, ita vt ſit $R = \pi - \theta$. At vero inuentio progreſſionis Q maiorem requirit apparatum, quem in finem ſequentes inueſtigationes generaliores praemittamus.

§. 10. Consideretur primo progressio ista cosinuum, quorum anguli in progressione arithmetica progrediantur et quorum numerus sit n:

$$t = \text{cos.}(\alpha + 2\beta) + \text{cos.}(\alpha + 4\beta) + \text{cos.}(\alpha + 6\beta) + \ldots\ldots + \text{cos.}(\alpha + 2n\beta).$$

Iam multiplicemus vtrinque per $2\,\text{sin.}\,\beta$, et cum sit

$$2\,\text{sin.}\,\beta\,\text{cos.}\,\gamma = \text{sin.}(\beta + \gamma) - \text{sin.}(\gamma - \beta),$$

proueniet sequens forma:

$$2t\,\text{sin.}\,\beta = -\text{sin.}(\alpha + \beta) + \text{sin.}(\alpha + 3\beta) + \text{sin.}(\alpha + 5\beta) + \ldots\ldots \text{sin.}(\alpha + (2n+1)\beta)$$
$$- \text{sin.}(\alpha + 3\beta) - \text{sin.}(\alpha + 5\beta) - \ldots$$

vbi omnes termini intermedii manifesto se destruunt, ita vt soli extremi remaneant, hincque ergo fiet

$$t = \frac{\text{sin.}[\alpha + (2n+1)\beta] - \text{sin.}(\alpha + \beta)}{2\,\text{sin.}\,\beta}.$$

§. 11. Deinde vero iidem cosinus combinentur cum numeris naturalibus $1, 2, 3, 4, \ldots\ldots n$, ac statuatur

$$u = 1\,\text{cos.}(\alpha + 2\beta) + 2\,\text{cos.}(\alpha + 4\beta) + 3\,\text{cos.}(\alpha + 6\beta) + \ldots\ldots + n\,\text{cos.}(\alpha + 2n\beta)$$

qua expressione ducta in $2\,\text{sin.}\,\beta$, adhibita resolutione qua modo sumus vsi, consequemur

$$2u\,\text{sin.}\,\beta = -\text{sin.}(\alpha + \beta) + \text{sin.}(\alpha + 3\beta) + 2\,\text{sin.}(\alpha + 5\beta) + \ldots\ldots + \text{sin.}(\alpha + (2n+1)\beta)$$
$$- 2\,\text{sin.}(\alpha + 3\beta) - 3\,\text{sin.}(\alpha + 5\beta) - \ldots$$

quae forma reducitur ad istam:

$$n\,\text{sin.}(\alpha + (2n+1)\beta) - 2u\,\text{sin.}\,\beta = \text{sin.}(\alpha + \beta) + \text{sin.}(\alpha + 3\beta) + \ldots\ldots + \text{sin.}(\alpha + (2n-1)\beta)$$

quae vocetur $= v$.

§. 12.

§. 12. Nunc iſta feries denuo ducatur in $2\,\text{fin.}\,\beta$, et cum in genere ſit

$$2\,\text{fin.}\,\beta\,\text{fin.}\,\gamma = \text{cof.}\,(\gamma - \beta) - \text{cof.}\,(\gamma + \beta),$$

nanciſcemur

$$2\,v\,\text{fin.}\,\beta = \text{cof.}\,\alpha - \text{cof.}\,(\alpha + 2\beta) - \text{cof.}\,(\alpha + 4\beta) - \ldots - \text{cof.}\,(\alpha + 2n\beta)$$
$$+ \text{cof.}\,(\alpha + 2\beta) + \text{cof.}\,(\alpha + 4\beta) + \ldots$$

vnde ob terminos medios omnes ſe deſtruentes colligitur

$v = \frac{\text{cof.}\,\alpha - \text{cof.}\,(\alpha + 2n\beta)}{2\,\text{fin.}\,\beta}$; quare cum ſit

$u = \frac{n\,\text{fin.}\,(\alpha + (2n+1)\beta) - v}{2\,\text{fin.}\,\beta}$, hinc obtinemus

$$u = \frac{n\,\text{fin.}\,(\alpha + (2n+1)\beta)}{2\,\text{fin.}\,\beta} - \frac{[\text{cof.}\,\alpha - \text{cof.}\,(\alpha + 2n\beta)]}{4\,\text{fin.}\,\beta^2}.$$

§. 13. Combinemus nunc ambas ſummationes modo traditas in genere, ac ſtatuamus

$$V = (a+b)\,\text{cof.}\,(\alpha + 2\beta) + (a + 2b)\,\text{cof.}\,(\alpha + 4\beta) + (a + 3b)$$
$$\text{cof.}\,(\alpha + 6\beta) + \ldots\ldots (a + nb)\,\text{cof.}\,(\alpha + 2n\beta),$$

atque euidens eſt fore $V = a\,t + b\,u$, vnde loco t et u valoribus ſubſtitutis erit

$$V = \frac{a\,\text{fin.}\,(\alpha + (2n+1)\beta) - a\,\text{fin.}\,(\alpha + \beta)}{2\,\text{fin.}\,\beta} + \frac{b\,n\,\text{fin.}\,(\alpha + (2n+1)\beta)}{2\,\text{fin.}\,\beta}$$
$$- \frac{b\,\text{cof.}\,\alpha + b\,\text{cof.}\,(\alpha + 2n\beta)}{4\,\text{fin.}\,\beta^2}.$$

§. 14. Iam ſatis perſpicuum eſt progreſſionem, quam ſupra littera Q exhibuimus, in iſta forma generali pro V inventa contineri, quandoquidem vtrinque idem terminorum numerus n occurrit, atque coëfficientes coſinuum feriei Q etiam progreſſionem arithmeticam conſtituunt. Quamobrem pro coëfficientibus primo faciamus

$$a + b = \frac{n\pi - \theta}{n} \text{ et } a + 2\,b = \frac{(n-2)\pi - \theta}{n},$$

vnde deducimus

 $b =$

$$b = -\frac{2\pi}{n} \text{ et } a = \frac{(n+2)\pi - \theta}{n}.$$

Nunc etiam angulos inter ſe coaequemus faciamusque

$$\alpha + 2\beta = \frac{p\theta}{n} \text{ et } \alpha + 4\beta = \frac{p}{n}(\theta + 2\pi),$$

vnde colligimus $\beta = \frac{\pi p}{n}$, hincque porro

$$\alpha = \frac{p}{n}(\theta - 2\pi) = -\frac{p}{n}(2\pi - \theta),$$

hocque modo fiet $V = Q$. At anguli in expreſſione ipſius V occurrentes erunt: primo

$$\alpha + (2n + 1)\beta = -\frac{p}{n}(\pi - \theta)$$

vbi cum p ſit numerus integer, poſtrema pars $2\pi p$, totam circumferentiam exprimens, omiſſa eſt, ex quo habebimus

$$\text{ſin.}\,(\alpha + (2n+1)\beta = -\text{ſin.}\,\frac{p}{n}(\pi - \theta).$$

Deinde occurrit angulus

$$\alpha + \beta = -\frac{p}{n}(\pi - \theta), \text{ cuius ſinus eſt}$$

$$\text{ſin.}\,(\alpha + \beta) = -\text{ſin.}\,\frac{p}{n}(\pi - \theta).$$

Denique erit

$$\alpha + 2n\beta = -\frac{p}{n}(2\pi - \theta) \text{ et}$$

$$\text{coſ.}\,(\alpha + 2n\beta) = \text{coſ.}\,\frac{p}{n}(2\pi - \theta).$$

His igitur valoribus ſubſtitutis prodibit

$$Q = V = -\frac{\frac{(n+2)\pi - \theta}{n}\,\text{ſin.}\,\frac{p}{n}(\pi - \theta) + \frac{(n+2)\pi - \theta}{n}\,\text{ſin.}\,\frac{p}{n}(\pi - \theta)}{2\,\text{ſin.}\,\frac{\pi p}{n}}$$

$$+ \frac{2\pi\,\text{ſin.}\,\frac{p}{n}(\pi - \theta)}{2\,\text{ſin.}\,\frac{\pi p}{n}}$$

$$+ \frac{\frac{2\pi}{n}\,\text{coſ.}\,\frac{p}{n}(2\pi - \theta) - \frac{2\pi}{n}\,\text{coſ.}\,\frac{p}{n}(2\pi - \theta)}{4\,\text{ſin.}\,\frac{\pi p^2}{n}}$$

quae

quae expressio manifesto reducitur ad hanc:

$$Q = V = \frac{\pi \sin. \frac{p}{n}(\pi - \theta)}{\sin. \frac{\pi p}{n}}.$$

§. 15. Inuentis igitur valoribus litterarum Q et R, valor integralis quem quaerimus pro casu $x = 1$ erit

$$S = \frac{\pi \sin. \frac{p}{n}(\pi - \theta)}{n \sin. \theta \sin. \frac{\pi p}{n}} - \frac{(\pi - \theta)\cos. \zeta}{n \sin. \theta}.$$

Sin autem integrale quaeratur a termino $x = 0$ vsque ad $x = \infty$, eius valor duplo maior euadet.

§. 16. His iam in genere expeditis, consideremus casum iam saepius tractatum, quo est $\zeta = 90°$ et $\theta = 90°$, haecque formula integranda proponitur: $\int \frac{\partial x}{x} \cdot \frac{x^p + x^{-p}}{x^n + x^{-n}}$, atque pro eius valore casu $x = 1$ habebimus $S = \frac{\pi \sin. \frac{p\pi}{2n}}{n \sin. \frac{p\pi}{n}}$, quae ob $\sin. \frac{p\pi}{n} = 2 \sin. \frac{p\pi}{2n} \cos. \frac{p\pi}{2n}$, abit in formulam illam notissimam $\frac{\pi}{2n \cos. \frac{p\pi}{2n}} = \frac{\pi}{2n}$ sec. $\frac{p\pi}{2n}$. Sin autem tantum sumamus $\zeta = 90$, vt formula integranda sit $\int \frac{\partial x}{x} \cdot \frac{x^p + x^{-p}}{x^n - 2\cos. \theta + x^{-n}}$, eius valor ab $x = 0$ vsque ad $x = 1$ extensus erit $\frac{\pi \sin. \frac{p}{n}(\pi - \theta)}{n \sin. \theta \sin. \frac{p\pi}{n}}$, quae expressio reducitur ad hanc:

$$\frac{\pi}{n \sin. \theta}\left(\cos. \frac{p\theta}{n} - \sin. \frac{p\theta}{n} \cot. \frac{p\pi}{n}\right).$$

OBSERVATIONES
in integrationem traditam.

I. Primum hic obſeruo terminum medium in numeratore exhibitum nullo modo integrationem turbare, quoniam, ſi ſolus adeſſet, integratio nulla laboraret difficultate; tum enim formula $\int \frac{\partial x}{x} \cdot \frac{1}{x^n - 2 \operatorname{cof.} \theta + x^{-n}}$ reducitur ad hanc formam: $\int \frac{x^{n-1} \partial x}{x^{2n} - 2 x^n \operatorname{cof.} \theta + 1}$, quae poſito $x^n = y$ abit in hanc: $\frac{1}{n} \int \frac{\partial y}{y - 2 y \operatorname{cof.} \theta + 1}$, cuius integrale eſt $\frac{1}{n \operatorname{fin.} \theta}$ A tang. $\frac{y \operatorname{fin.} \theta}{1 - y \operatorname{cof.} \theta}$, cuius valor, caſu $x = 1$, fit $\frac{1}{n \operatorname{fin.} \theta}$ A tang. $\frac{\operatorname{fin.} \theta}{1 - \operatorname{cof.} \theta} = \frac{(\pi - \theta)}{2 n \operatorname{fin.} \theta}$, qui ductus in $- 2 \operatorname{cof.} \zeta$ praebet illam ipſam partem hinc oriundam in forma ſupra inuenta, quamobrem ſuperfluum foret hunc terminum in calculo retinere, vnde hanc formam integralem ſumus contemplaturi: $\int \frac{\partial x}{x} \cdot \frac{x^p + x^{-p}}{x^n - 2 \operatorname{cof.} \theta + x^{-n}}$, cuius valorem caſu $x = 1$ inuenimus $= \frac{\pi \operatorname{fin.} \frac{p}{n} (\pi - \theta)}{n \operatorname{fin.} \theta \operatorname{fin.} \frac{p \pi}{n}}$, quem breuitatis gratia littera P deſignemus, ita vt ſit $P = \frac{\pi \operatorname{fin.} \frac{p}{n} (\pi - \theta)}{n \operatorname{fin.} \theta \operatorname{fin.} \frac{p \pi}{n}}$. Tum vero etiam inuenimus caſu $x = \infty$ valorem huius formulae eſſe $= 2 P$.

II. Secundo loco probe notari oportet, exponentem p neceſſario minorem eſſe debere quam exponentem n, quia alioquin fractio foret ſpuria, et variabilis x in numeratore tot vel plures dimenſiones eſſet habitura quam in denominatore. Quoties autem hoc euenit, integrali praeter partes, quas per reſolutiolutionem in fractiones partiales sumus nacti, quantitas quaedam integra adiici debet, id quod in nostra solutione non est factum, quamobrem tales casus hinc prorsus excludi conuenit. Ceterum quolibet casu has partes integras facile erit adiicere ad partes quas nobis nostra methodus suppeditabit.

III. Ex ipsa solutione, quam dedimus, perspicuum est exponentem p necessario integrum statui debere, quia alias operationes ibi exhibitae locum habere non possent; vnde eo magis mirum videbitur, quod conclusiones inuentae subsistere queant, etiamsi iste exponens p fuerit numerus fractus quicunque, dummodo minor quam n, propterea quod hos casus semper ad exponentes integros reducere licet. Ad hoc ostendendum ponamus esse $p = \frac{q}{\lambda}$, atque forma nostra, posito $x = z^{\lambda}$, reducetur ad hanc formam: $\lambda \int \frac{\partial z}{z} \cdot \frac{z^q + z^{-q}}{z^{\lambda n} - 2 \operatorname{cos.} \theta + z^{-\lambda n}}$, vbi cum omnes exponentes sint integri, ac pro terminis integrationis, qui erant $x = 0$, $x = 1$ et $x = \infty$, etiam fiat $z = 0$, $z = 1$ et $z = \infty$, pro $z = 1$ valor integralis erit $\frac{\lambda \pi \operatorname{sin.} \frac{q}{\lambda n} (\pi - \theta)}{\lambda n \operatorname{sin.} \theta \operatorname{sin.} \frac{q \pi}{\lambda n}}$, qui, restituto loco $\frac{q}{\lambda}$ valore p, abit in hunc: $\frac{\pi \operatorname{sin.} \frac{p}{n} (\pi - \theta)}{n \operatorname{sin.} \theta \operatorname{sin.} \frac{p \pi}{n}}$, quae expressio cum superiore prorsus congruit. Atque hinc intelligitur, quominus etiam exponenti p valores irrationales tribuantur, dumne superent exponentem n, semper hoc euenire debere.

IV. Hic iam quaestio oritur maximi momenti, vtrum etiam exponenti p dare liceat valores imaginarios nec ne? Hoc autem affirmandum videtur, quandoquidem imaginaria certe non sint maiora quam n; vnde concludimus, dummodo valor ipsius p ita capiatur imaginarius, vt ipsa formula differentialis

 ma-

maneat realis, tum etiam conclusiones nostras veritati consentaneas esse mansuras. Hoc autem euenit, si statuamus $p = q\sqrt{-1}$; tum enim, cum in genere sit $e^{\phi\sqrt{-1}} + e^{-\phi\sqrt{-1}} = 2\cos.\phi$, quia nostro casu est $\phi = q\,l\,x$, ipsa formula integralis erit

$$\int \frac{\partial x}{x} \cdot \frac{2\cos.\, q\,l\,x}{x^n - 2\cos.\theta + x^{-n}}.$$

Nunc igitur videamus quamnam formam nostrum integrale casu $x = 1$ sit recepturum, et quoniam sinus angulorum imaginariorum sunt etiam imaginarii, quandoquidem

$$e^{\phi\sqrt{-1}} - e^{-\phi\sqrt{-1}} = \frac{2}{\sqrt{-1}}\,\text{sin.}\,\phi,$$

loco ϕ scribamus $\psi\sqrt{-1}$, eritque

$$\text{sin.}\,\psi\sqrt{-1} = \frac{\sqrt{-1}}{2}\,(e^{-\psi} - e^{+\psi}),$$

vnde in forma integrali erit $\frac{p}{n} = \frac{q\sqrt{-1}}{n}$, ideoque loco ψ scribamus $\frac{q}{n}(\pi - \theta)$ pro numeratore, et $\frac{q\pi}{n}$ pro denominatore, ex quo valor integralis ab $x = 0$ ad $x = 1$ extensus erit

$$\frac{\pi}{n\,\text{sin.}\,\theta} \cdot \frac{e^{-\frac{q}{n}(\pi-\theta)} - e^{+\frac{q}{n}(\pi-\theta)}}{e^{-\frac{q\pi}{n}} - e^{+\frac{q\pi}{n}}}.$$

Hinc igitur formemus sequens Theorema notatu dignissimum:

Quodsi ista formula integralis:

$$\int \frac{\partial x}{x} \cdot \frac{\cos.\, q\,l\,x}{x^n - 2\cos.\theta + x^{-n}},$$

a termino $x = 0$ *vsque ad* $x = 1$ *extendatur, eius volor semper erit* $= \frac{\pi}{2\,n\,\text{sin.}\,\theta} \cdot \frac{e^{-\frac{q}{n}(\pi-\theta)} - e^{+\frac{q}{n}(\pi-\theta)}}{e^{-\frac{q\pi}{n}} - e^{+\frac{q\pi}{n}}}$. *Sin autem integrale extendatur ab* $x = 0$ $x = \infty$, *valor prodibit duplo maior.*

Hoc

Hoc theorema vtique eo maiorem attentionem meretur, quod nulla via patet, eius veritatem directe demonstrandi.

V. Reuertamur autem ad formam integralem primo expositam, et quoniam numerator duabus constat partibus x^p et x^{-p}, vnde summa integralium pro $x = 1$ inuenta est $= P$, at pro casu $x = \infty$ duplo maior $= 2P$, hic maxime notatu dignum occurrit, quod pro termino $x = \infty$ vtraque pars numeratoris eundem producat valorem $= P$. Semper enim erit, integrale ab $x = 0$ ad $x = \infty$ extendendo,

$$\int \frac{\partial x}{x} \frac{x^p}{x^n - 2\cos.\theta + x^{-n}} = \int \frac{\partial x}{x} \cdot \frac{x^{-p}}{x^n - 2\cos.\theta + x^{-n}} = P.$$

Ad hoc ostendendum ponamus pro posteriore formula $x = \frac{1}{z}$, eaque induet hanc formam:

$$-\int \frac{\partial z}{z} \cdot \frac{z^{-p} + z^{+p}}{z^{-n} - 2\cos.\theta + z^n},$$

quae cum sit priori formae prorsus similis, solo signo — excepto, eius valor a termino $z = 0$ vsque ad $z = \infty$, negatiue sumtus, primae formulae erit aequalis. Cum autem sit $z = \frac{1}{x}$, isti termini integralis erunt ab $x = \infty$ vsque ad $x = 0$, qui ergo si inuertantur, etiam signum integralis erit mutandum, sicque ipsi priori formulae aequale euadet; quare cum ambae formulae coniunctae summam habeant $= 2P$, vtriusque seorsim sumtae valor erit $= P$, vnde deducitur sequens theorema notatu pariter dignissimum:

Istius formulae integralis: $\int \frac{\partial x}{x} \cdot \frac{x^{\pm p}}{x^n - 2\cos.\theta + x^{-n}}$ *valor a termino* $x = 0$ *vsque ad* $x = \infty$ *extensus semper est*

$$= P = \frac{\pi \sin.\frac{p}{n}(\pi - \theta)}{n \sin.\theta \sin.\frac{p\pi}{n}},$$

Euidens

Euidens autem eſt hanc aequalitatem pro caſu $x = 1$, neutiquam locum habere poſſe.

VI. Quoniam in noſtra formula differentiali tantum occurrit terminus $2 \operatorname{cof.} \theta$, cuius valor idem manet, etiamſi pro θ ſumeremus $\theta \pm 2 i \pi$, maxime hic mirum videri debet, quod tum valor integralis maxime diuerſus ſit proditurus, ſcilicet $= \frac{\pi \operatorname{fin.} \frac{p}{n} (\pi - \theta \pm 2 i \pi)}{n \operatorname{fin.} \theta \operatorname{fin.} \frac{p \pi}{n}}$, vnde merito quaeritur, quisnam horum valorum veritati ſit conformis, ad quod certe nihil aliud reſponderi poteſt, niſi quod omnes veritati aeque conſentanei ſint cenſendi, id quod eo minus mirum videri debet, quod omnes huiusmodi formulae integrales reuera ſunt functiones multiformes, atque adeo infinitiformes, id quod ex hoc exemplo ſimpliciſſimo: $\int \frac{\partial x}{1 + x x}$ intelligi poteſt. Cum enim eius integrale exhibeat arcum circuli cuius tangens eſt x, tales autem arcus innumerabiles dentur, quorum eadem ſit tangens $= x$, neceſſe eſt, vt omnes aeque in hac forma integrali contineantur. Quin etiam in noſtro valore inuento P loco π quoque ſcribere licet $\pi + 2 i \pi$, eiusque valor nihilominus cum veritate conſiſtere poterit. Verum in huiusmodi integrationibus perpetuo valores minimi deſiderari ſolent, hocque modo omnis difficultas e medio eſt ſublata.

VII. Deinde in Analyſi ſupra adhibita ſuppoſuimus omnes factores denominatoris inter ſe eſſe inaequales, id quod vtique ſemper euenit, niſi ſit $\operatorname{cof.} \theta = \pm 1$, quippe quibus caſibus denominator quadratum inuoluit: ſit enim is

$$= x^{-n} (x^n \pm 1)^2;$$

ex quo patet omnes factores $x^n \pm 1$ bis occurrere debere. Hoc incommodum etiam innuitur per ipſam noſtram formulam P, quae

quae casu $\theta = 0$ valorem indicat infinitum. Verum posito $\theta = \pi$, singulare phaenomenon se offert, dum formulae pro P inuentae tam numerator quam denominator euanescunt, atque adeo fractio determinatum nanciscitur valorem. Ponamus enim $\theta = \pi - \omega$, existente ω infinite paruo, eritque $\sin.\theta = \sin.\omega = \omega$; at ob $\pi - \theta = \omega$, in numeratore habebimus $\sin.\frac{p\omega}{n} = \frac{p\omega}{n}$, vnde valor ipsius P resultat $\frac{\pi p}{n n \sin.\frac{p\pi}{n}}$, qui cum penitus sit determinatus, nullum plane dubium superesse potest, quin cum veritate conspiret, vnde sequens enascitur Theorema maxime memorabile:

Theorema.

Proposita formula differentiali

$$\frac{\partial x}{x} \cdot \frac{x^p + x^{-p}}{x^n - 1 + x^{-n}} = \frac{x^{n-1}\partial x\,(x^p + x^{-p})}{(x^n + 1)^2},$$

si eius integrale a termino $x = 0$ *vsque ad* $x = 1$ *extendatur, eius valor semper erit* $\frac{\pi p}{n n \sin.\frac{p\pi}{n}}$; *sin autem vsque ad terminum* $x = \infty$ *extendatur, eius valor erit duplo maior* $= \frac{2\pi p}{n n \sin.\frac{p\pi}{n}}$.

Demonstratio huius Theorematis directa.

Formula ista integralis resoluatur sequenti modo:

$$\int \frac{\partial x}{x} \cdot \frac{x^{n+p} + x^{n-p}}{(1 + x^n)^2} = \frac{Q}{1 + x^n} + \int \frac{\partial x}{x}\,\frac{R}{1 + x^n}.$$

Sumantur igitur differentialia simulque ducantur in $\frac{x}{\partial x}$, positoque $\partial Q' = Q\,\partial x$, orietur ista aequatio:

$$\frac{x^{n+p} + x^{n-p}}{(1 + x^n)^2} = \frac{Q' x}{1 + x^n} - \frac{n Q x^n}{(1 + x^n)^2} + \frac{R}{1 + x^n},$$

quae

quae per $1 + x^n$ multiplicata hoc modo repraesentetur:

$$\frac{x^{n+p} + x^{n-p} + n\,Q\,x^n}{1 + x^n} = Q'x + R,$$

vbi iam Q ita accipi debet, vt illa fractio ad integrum reuocetur. Facile autem patet, hoc fieri statuendo

$$n\,Q = -x^p + x^{n-p},$$

tum enim illa fractio fiet

$$\frac{x^{n-p} + x^{2n-p}}{1 + x^n} = x^{n-p},$$

ita vt nunc habeamus $x^{n-p} = Q'x + R$. Cum igitur sit $Q = \frac{x^{n-p} - x^p}{n}$, erit

$$Q'x = \frac{(n-p)\,x^{n-p} - p\,x^p}{n},$$

hincque colligitur $R = \frac{p}{n}(x^{n-p} + x^p)$, quocirca formula integralis proposita reducta est ad hanc formam:

$$\frac{x^{n-p} - x^p}{n(1 + x^n)} + \frac{p}{n}\int \frac{\partial x}{x} \cdot \frac{x^{n-p} + x^p}{1 + x^n},$$

quod integrale ita est sumendum, vt euanescat posito $x = 0$. Nunc igitur statuamus $x = 1$, ac prior pars absoluta euanescit, formulae autem integralis valor, per ea quae dudum sunt inuenta, prodit $\frac{p}{n} \cdot \frac{\pi}{n \text{ sin. } \frac{p\pi}{n}} = \frac{\pi p}{n\,n \text{ sin. } \frac{p\pi}{n}}$, qui ergo cum ante inuento perfecte congruit.

VIII. Tribuatur nunc etiam in hac postrema forma exponenti p valor imaginarius, ponendo $p = q\sqrt{-1}$, et cum, vt ante vidimus, hinc fiat $x^p + x^{-p} = 2 \text{ cos. } q\,l\,x$, formula integralis proposita erit $= 2\int \frac{\partial x}{x} \cdot \frac{x^n \text{ cos. } q\,l\,x}{(1 + x^n)^2}$. Pro eius

valore

valore autem iam ante vidimus fore

$$\text{fin.}\ \frac{\pi q \sqrt{-1}}{n} = \frac{e^{-\frac{\pi q}{n}} - e^{+\frac{\pi q}{n}}}{2\sqrt{-1}},$$

quamobrem valor noftrae formulae, ab $x = 0$ ad $x = 1$ extenfus, erit $\frac{2\pi q}{nn(e^{\frac{\pi q}{n}} - e^{-\frac{\pi q}{n}})}$, vnde deducimus fequens theorema omni attentione dignum.

Theorema.

Si valor iftius formulae integralis: $\int \frac{x^{n-1}\partial x(x^p + x^{-p})}{(1+x^n)^2}$, *a termino* $x = 0$ *vsque* $x = 1$ *extendatur, is femper aequabitur huic formulae:* $\frac{\pi q}{nn(e^{\frac{\pi q}{n}} - e^{-\frac{\pi q}{n}})}$. Cuius autem Theorematis demonftratio ex principiis iam cognitis vix elici poffe videtur.

IX. Praeterea etiam perfpicuum eft, methodum, qua vfi fumus ad noftram formulam integrandam, fubfiftere non poffe, nifi terminus medius denominatoris binario fit minor, quam ob cauffam eum hac forma $2\,\text{cof.}\,\theta$ expreffimus. Quamobrem hinc oritur quaeftio maximi momenti: vtrum noftrae conclufiones etiamnunc valeant, fi terminus ille medius binario maior acciperetur, fiue fi angulus θ foret imaginarius, nec ne? Verum etiam hoc cafu nullum dubium fupereffe poteft, quin formula noftra finalis etiamnunc veritati confentanea fit futura. Ante omnia autem hic eft obferuandum, illi termino medio $2\,\text{cof.}\,\theta$ valorem negatiuum tribui conuenire, quia alioquin ipfe denominator in nihilum abiret, dum quantitas noftra variabilis x a termino 0 vsque ad 1 augetur. Hanc ob rem ftatuamus angulum $\theta = \pi - \eta$, et valor nofter integralis erit

 $\int \partial x$

$$\int \frac{\partial x}{x} \cdot \frac{x^p + x^{-p}}{x^n + 2\,\mathrm{cos}.\,\eta + x^{-n}} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = \frac{\pi\,\mathrm{fin}.\,\frac{p\eta}{n}}{n\,\mathrm{fin}.\,\eta\,\mathrm{fin}.\,\frac{\pi p}{n}}.$$

In hac igitur forma faciamus angulum η imaginarium, ponendo $\eta = \Phi\sqrt{-1}$, eritque, per ea quae iam fupra obferuauimus, $2\,\mathrm{cof}.\,\Phi\sqrt{-1} = e^{\Phi} + e^{-\Phi}$, ita vt iam nofter denominator fit

$$x^n + e^{\Phi} + e^{-\Phi} + x^{-n} = \frac{1}{x^n}(x^n + e^{\Phi})(x^n + e^{-\Phi}),$$

quem idcirco ftatim in duos factores reales formae $x + k$ refoluere licet; tum vero fiet

$$\mathrm{fin}.\,\theta = \mathrm{fin}.\,\Phi\sqrt{-1} = \frac{e^{-\Phi} - e^{+\Phi}}{2\sqrt{-1}},$$

fimilique modo erit

$$\mathrm{fin}.\,\frac{p}{n}\eta = \mathrm{fin}.\,\frac{p}{n}\Phi\sqrt{-1} = \frac{e^{-\frac{p\Phi}{n}} - e^{+\frac{p\Phi}{n}}}{2\sqrt{-1}},$$

vnde formula noftra integralis emergit realis

$$= \frac{\pi\left(e^{-\frac{p\Phi}{n}} - e^{+\frac{p\Phi}{n}}\right)}{n\left(e^{-\Phi} - e^{+\Phi}\right)\mathrm{fin}.\,\frac{p\pi}{n}}.$$

Statuamus autem hic breuitatis gratia $e^{\Phi} = f$, vt fit $e^{-\Phi} = \frac{1}{f}$, atque noftra formula integralis fequentem induet formam:

$$\int \frac{\partial x}{x} \cdot \frac{x^p + x^{-p}}{x^n + (f + \frac{1}{f}) + x^{-n}} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = \frac{\pi\left(f^{\frac{p}{n}} - f^{-\frac{p}{n}}\right)}{n\left(f - f^{-1}\right)\mathrm{fin}.\,\frac{p\pi}{n}},$$

id quod tanquam Theorema omni attentione dignum fpectari poteft; vbi per fe intelligitur, valorem eiusdem integralis, vsque ad $x = \infty$ extenfum, fore duplo maiorem.

X. Quodfi iam in hac forma etiam exponenti p valorem imaginarium tribuamus, pariter nullo modo dubitari poterit, sit, quin conclusio nostra vera sit mansura. Ponamus igitur $p = q\sqrt{-1}$, eritque vt ante $x^p + x^{-p} = 2 \operatorname{cos.} q\,l\,x$; tum vero erit $\operatorname{sin.}\frac{p\pi}{n} = \frac{e^{-\frac{q\pi}{n}} - e^{+\frac{q\pi}{n}}}{2\sqrt{-1}}$, pro integralis autem numeratore erit $f^{\frac{p}{n}} - f^{-\frac{p}{n}} = 2\sqrt{-1}\operatorname{sin.}\frac{q}{n}lf$. His igitur valoribus substitutis sequens nanciscimur

Theorema.

Valor istius formulae integralis: $\int \frac{\partial x}{x} \cdot \frac{\operatorname{cos.} q\,l\,x}{x^n + (f + \frac{1}{f}) + x^{-n}}$, *a termino* $x = 0$ *vsque ad* $x = 1$ *extensus, semper aequabitur formulae*

$$\frac{2\pi \operatorname{sin.}\frac{q}{n}lf}{n(f - f^{-1})\left(e^{\frac{q\pi}{n}} - e^{-\frac{q\pi}{n}}\right)}.$$

XI. Deinde iam pridem obseruaui, omnia huiusmodi integralia satis commode per series infinitas exprimi posse. Cum enim ista fractio:

$$\frac{x^p}{x^n - 2\operatorname{cos.}\theta + x^{-n}} = \frac{x^{n+p}}{x^{2u} - 2x^n \operatorname{cos.}\theta + 1},$$

resoluatur in hanc seriem:

$$\frac{1}{\operatorname{sin.}\theta}\left(x^{n+p}\operatorname{sin.}\theta + x^{2n+p}\operatorname{sin.}2\theta + x^{3n+p}\operatorname{sin.}3\theta \text{ etc.}\right)$$

integrale istud: $\int \frac{\partial x}{x} \cdot \frac{x^p}{x^{2n} - 2x^n \operatorname{cos.}\theta + 1}$, a termino $x = 0$ vsque ad $x = 1$ extensum, aequabitur huic seriei infinitae:

$$\frac{1}{\operatorname{sin.}\theta}\left(\frac{\operatorname{sin.}\theta}{n+p} + \frac{\operatorname{sin.}2\theta}{2u+p} + \frac{\operatorname{sin.}3\theta}{3n+p} + \frac{\operatorname{sin.}4\theta}{4n+p} + \text{etc.}\right)$$

Hinc ergo, si p negatiue caperemus, tum formula nostra principalis $\int \frac{\partial x}{x} \cdot \frac{x^p + x^{-p}}{x^n - 2\operatorname{cos.}\theta + x^{-n}}$, ab $x = 0$ ad $x = 1$ ex-

tensa

tenſa, ſemper aequabitur huic ſeriei infinitae geminatae:

$$\frac{1}{\text{ſin.}\,\theta}\left\{\begin{array}{l}\frac{\text{ſin.}\,\theta}{n+p}+\frac{\text{ſin.}\,2\theta}{2n+p}+\frac{\text{ſin.}\,3\theta}{3n+p}+\frac{\text{ſin.}\,4\theta}{4n+p}\ \text{etc.}\\ +\frac{\text{ſin.}\,\theta}{n-p}+\frac{\text{ſin.}\,2\theta}{2n-p}+\frac{\text{ſin.}\,3\theta}{3n-p}+\frac{\text{ſin.}\,4\theta}{4n-p}\ \text{etc.}\end{array}\right\},$$

quae binis homologis coniungendis contrahitur in hanc ſeriem:

$$\frac{2n}{\text{ſin.}\,\theta}\left(\frac{\text{ſin.}\,\theta}{nn-pp}+\frac{2\,\text{ſin.}\,2\theta}{4nn-pp}+\frac{3\,\text{ſin.}\,3\theta}{9nn-pp}+\frac{4\,\text{ſin.}\,4\theta}{16nn-pp}+\text{etc.}\right)$$

XII. Hinc iam manifeſto pro caſu, quo ponitur $p = q\sqrt{-1}$, iſta ſeries infinita exoritur:

$$\frac{2n}{\text{ſin.}\,\theta}\left(\frac{\text{ſin.}\,\theta}{nn+qq}+\frac{2\,\text{ſin.}\,2\theta}{4nn+qq}+\frac{3\,\text{ſin.}\,3\theta}{9nn+qq}+\frac{4\,\text{ſin.}\,4\theta}{16nn+qq}+\text{etc.}\right)$$

quae ergo exprimit valorem huius formulae integralis:

$$\int\frac{\partial x}{x}\cdot\frac{\text{coſ.}\,q\,l\,x}{x^n-2\,\text{coſ.}\,\theta+x^{-n}},$$

ſcilicet ab $x = 0$ ad $x = 1$ extenſae, ita vt iſtius ſeriei ſumma finito modo expreſſa ſit etiam

$$\frac{\pi}{n\,\text{ſin.}\,\theta}\cdot\left(\frac{e^{-\frac{q}{n}(\pi-\theta)}-e^{+\frac{q}{n}(\pi-\theta)}}{e^{-\frac{q\pi}{n}}-e^{+\frac{q\pi}{n}}}\right).$$

Quin etiam facile intelligitur hic quoque angulum θ imaginarium accipi poſſe. Vidimus enim poſito $\theta = \Phi\sqrt{-1}$ fore

$$\text{ſin.}\,\theta = \frac{e^{-\Phi}-e^{+\Phi}}{2\sqrt{-1}},\ \text{hincque in genere}$$

$$\text{ſin.}\,\lambda\,\theta = \frac{e^{-\lambda\Phi}-e^{+\lambda\Phi}}{2\sqrt{-1}}.$$

Quare ſi ſtatuamus $e^{\Phi} = f$, erit $\frac{\text{ſin.}\,\lambda\,\theta}{\text{ſin.}\,\theta} = \frac{f^{\lambda}-f^{-\lambda}}{f-\frac{1}{f}}$, vnde ſeries illa ſatis concinnam formam accipiet.

XII. Denique operationes, quibus in integratione nostrae formulae fumus vfi, confiftere nequeunt, nifi exponens n fuerit numerus integer. Interim tamen valor integralis, quem inuenimus pro cafu vel $x = 1$ vel $x = \infty$, veritati conformis deprehenditur, non folum quando pro n numerus fractus quicunque fed etiam adeo imaginarius accipitur, quorum prius facile oftenditur. Sit enim $n = \frac{m}{\lambda}$, ac ponatur $x = y^{\lambda}$, atque ob $\frac{\partial x}{x} = \frac{\lambda \partial y}{y}$, orietur haec forma integralis exponentibus integris contenta: $\int \frac{\lambda \partial y}{y} \cdot \frac{y^{\lambda p} + y^{-\lambda p}}{y^m - 2 \operatorname{cof.} \theta + y^{-m}}$, cuius ergo valor cafu $x = 1$ debet effe fecundum formulam inuentam

$$\frac{\lambda \pi}{m} \cdot \frac{\operatorname{fin.} \frac{\lambda p}{m} (\pi - \theta)}{\operatorname{fin.} \theta \operatorname{fin.} \frac{\lambda p \pi}{m}},$$

qui, fi iam loco m valor λn reftituatur, manifefto abit in ipfam noftram formulam fupra inuentam: $\frac{\pi}{n} \frac{\operatorname{fin.} \frac{p}{n} (\pi - \theta)}{\operatorname{fin.} \theta \operatorname{fin.} \frac{p\pi}{n}}$. Hinc autem nulli amplius dubio relinquitur, quin veritas haec fubfiftat, etiamfi n fuerit numerus imaginarius. Ponamus igitur $n = m \sqrt{-1}$; et formula integralis reducetur ad hanc formam: $\int \frac{\partial x}{x} \cdot \frac{x^p + x^{-p}}{2 \operatorname{cof.} m\, l\, x - 2 \operatorname{cof.} \theta}$, cuius ergo valor cafu $x = 1$ certe erit $\frac{p}{m \sqrt{-1}} \cdot \frac{e^{\frac{p}{m}(\pi - \theta)} - e^{-\frac{p}{m}(\pi - \theta)}}{\operatorname{fin.} \theta \left(e^{\frac{\pi p}{m}} - e^{-\frac{\pi p}{m}}\right)}$, vbi mirum videbitur iftum valorem femper effe imaginarium, licet ipfa formula differentialis, dum variabilis x, a termino o vsque ad terminum $x = 1$, maneat realis, id quod merito maxime videtur paradoxum. Interim tamen non defunt cafus, quibus valor integralis formulae differentialis realis manifefto euadit imaginarius, id quod in ifta formula fimpliciori $\int \frac{\partial x}{x \operatorname{cof.} m\, l\, x}$ oftendiffe

fuffiſufficiet, quae vtique, dum x a o ad 1 augetur, conſtanter manet realis. Ad hanc ergo formulam integrandam ſtatuamus $l\,x = -z$, vbi notetur, dum x a o vsque ad 1 progreditur, tum quantitatem z ab ∞ vsque ad o decreſcere. Nunc igitur formula noſtra integralis erit $\int\frac{-\partial z}{\text{coſ.}\,m\,z}$, cum vero conſtet eſſe $\int\frac{\partial\Phi}{\text{ſin.}\,\Phi} = l\,\text{tang.}\,\frac{1}{2}\,\Phi$, ſumamus $\Phi = 90^\circ - m\,z$, eritque $\partial\Phi = -m\,\partial z$, hincque

$$\int\frac{-m\,\partial z}{\text{coſ.}\,m\,z} = +\,l\,\text{tang.}\,(45^\circ - \tfrac{1}{2}\,m\,z),$$

quod integrale manifeſto euaneſcit pro termino $z = 0$, dum autem ab hoc termino quantitas z in infinitum vsque augetur, infinities tangens huius anguli fiet negatiua, eiusque logarithmus propterea imaginarius, vnde non amplius mirabimur, quod formulae differentialis realis integrale euadere poſſit certis caſibus imaginarium.

XIII. Hoc igitur modo euictum eſt formulae noſtrae differentialis propoſitae $\int\frac{\partial x}{x}\cdot\frac{x^p + x^{-p}}{x^n - 2\,\text{coſ.}\,\theta + x^{-n}}$, integrale aſſignatum a termino $x = 0$ vsque ad $x = 1$ ſemper cum veritate conſiſtere, quicunque valores ternis litteris n, p et θ, tribuantur, ſiue integri, ſiue fracti, ſiue etiam imaginarii. Interim tamen dantur caſus iam initio indicati, quibus iſti valores integrales a veritate aberrabunt, quippe quod ſemper vſu venire debet, quoties exponens p maior eſt exponente n, quam ob cauſſam ſedulo excludere debemus omnes caſus, quibus formula $p - n$ euadit realis et poſitiua. His autem exceptis variae formulae, ad quas hic ſumus perducti, ita ſunt comparatae, vt maxima attentione dignae videantur, ſimulque non contemnenda incrementa Scientiae analyticae promittant.

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