CONSIDERATIO MOTVS PLANE SINGVLARIS, QVI IN FILO PERFECTE FLEXILI LOCVM HABERE POTEST.

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§. 1.

Quanquam theoria non ſolum aequilibrii ſed etiam motus pro omnibus filis tam perfecte flexibilibus quam etiam elaſticis ita perfecte ſit explorata, vt nihil amplius deſiderari poſſe videatur: tamen formulae pro motu determinando traditae etiamnunc omni vſu caruerunt; cum pro nullo adhuc caſu motus huiusmodi filorum definiri potuerit exceptis ſolis illis caſibus, quibus talia fila motum reciprocum ſeu oſcillatorium eumque adeo infinite paruum recipere valent. Huius autem defectus cauſa neutiquam theoriae mechanicae eſt tribuenda ſed vnica imperfectioni analyſeos adſcribi debet: ita vt ante vix quicquam in hoc genere ſperari poſſit, quam ſcientia analyſeos inſignia incrementa acceperit.

§. 2. Quin etiam caſus ſimpliciſſimus, quo motus fili perfecte flexilis a nullis plane viribus ſollicitati in eodem plano concitari poteſt, nullis adhuc artificiis a me quidem adhibitis

bitis expediri potuit. Quod quidem eo minus eſt mirandum, cum ſi loco fili conſiderentur plures virgae ita inuicem iunctae, vt circa iuncturas liberrime commoueri queant, motus nullo adhuc modo perfecte aſſignari potuerit, ſtatim ac plures duabus virgis hoc modo fuerint coniunctae.

Tab. IV. Fig. 1. §. 3. Quo igitur ſummas has difficultates penitius perſpiciamus, conſideremus filum quodcunque flexile E Y F quod a viribus quibuscunque ſollicitatum in ipſo plano tabulae vtcunque promoueatur, et ſumta in hoc plano recta fixa O A, pro axe habenda, elapſo tempore t teneat filum ſitum in figura exhibitum E Y F, a cuius puncto quocunque indefinito Y ad axem ducatur normalis Y X, vocenturque coordinatae $OX = x$ et $XY = y$, ipſa autem portio fili $EY = s$, vt ſit $\partial x^2 + \partial y^2 = \partial s^2$. Tum vero hoc tempore fili elementum $Y y = \partial s$ ſollicitetur a duabus viribus $YP = P\partial s$ et $YQ = Q\partial s$, quarum directiones ſint coordinatis parallelae. Quibus poſitis manifeſtum eſt, ambas coordinatas x et y ſpectari debere tanquam functiones duarum variabilium, arcus ſcilicet $EY = s$ ac temporis t. Vnde ſumto tempore t conſtante, vt fili figura quam ipſo tempore tenet exploretur, erit per ea quae de functionibus duarum variabilium iam ſatis ſunt explicata, $\partial x = \partial s \left(\frac{\partial x}{\partial s}\right)$ et $\partial y = \partial s \left(\frac{\partial y}{\partial s}\right)$, hincque ergo $\left(\frac{\partial x}{\partial s}\right)^2 + \left(\frac{\partial y}{\partial s}\right)^2 = 1$. At vero ſumto ſolo tempore t variabili, manente arcu $EY = s$ inuariato, coordinatae x et y pro eodem fili puncto Y ita variabunt, vt ſit $\partial x = \partial t \left(\frac{\partial x}{\partial t}\right)$ et $\partial y = \partial t \left(\frac{\partial y}{\partial t}\right)$, vbi notetur formulam $\left(\frac{\partial x}{\partial t}\right)$ exprimere celeritatem puncti y ſecundum directionem YP, et $\left(\frac{\partial y}{\partial t}\right)$ celeritatem ſecundum directionem YQ, vnde porro acceleratio motus pro puncto Y ſecundum directionem YP erit $= \left(\frac{\partial \partial x}{\partial t^2}\right)$ et ſecundum directionem $YQ = \left(\frac{\partial \partial y}{\partial t^2}\right)$. Praeterea

terea vero hic erit monendum, etiam ipsas vires sollicitantes P et Q vtcunque a tempore t pendere posse.

§. 4. His expositis secundum praecepta pro motu huius fili tradita ex viribus sollicitantibus deriuentur isti valores:

$$P' = P - \frac{1}{2g}\left(\frac{\partial\partial x}{\partial t^2}\right) \text{ et } Q' = Q - \frac{1}{2g}\left(\frac{\partial\partial y}{\partial t^2}\right),$$

vbi g denotat altitudinem lapsus grauium pro vno minuto secundo, siquidem tempus t in minutis secundis exprimere lubuerit. Tum vero hic littera s non solum nobis longitudinem arcus E Y sed etiam eius pondus denotare assumitur, quandoquidem filo per totam longitudinem eandem crassitiem tribuimus.

§. 5. Per has autem quantitates deriuatas P' et Q' totus fili motus ex hac aequatione satis simplici inuestigari debet

$$\left(\frac{\partial y}{\partial s}\right)\int P'\,\partial s - \left(\frac{\partial x}{\partial s}\right)\int Q'\,\partial s = 0.$$

In quibus formulis integralibus sola quantitas s pro variabili est habenda, tempore t manente constante. Hinc igitur si loco P' et Q' substituamus eorum valores, aequatio nostra pro motu determinando erit

$$\left(\frac{\partial y}{\partial s}\right)\int P\,\partial s - \left(\frac{\partial x}{\partial s}\right)\int Q\,\partial s = \frac{1}{2g}\left(\frac{\partial y}{\partial s}\right)\int \partial s\left(\frac{\partial\partial x}{\partial t^2}\right) - \frac{1}{2g}\left(\frac{\partial x}{\partial s}\right)\int \partial s\left(\frac{\partial\partial y}{\partial t^2}\right).$$

Praeterea vero si tensio fili hoc tempore in puncto Y ponatur $= T$, erit

$$T = -\left(\frac{\partial x}{\partial s}\right)\int P'\,\partial s - \left(\frac{\partial y}{\partial s}\right)\int Q'\,\partial s, \text{ siue}$$

$$T = -\left(\frac{\partial x}{\partial s}\right)\int P\,\partial s - \left(\frac{\partial y}{\partial s}\right)\int Q\,\partial s + \frac{1}{2g}\left(\frac{\partial x}{\partial s}\right)\int \partial s\left(\frac{\partial\partial x}{\partial t^2}\right)$$
$$+ \frac{1}{2g}\left(\frac{\partial y}{\partial s}\right)\int \partial s\left(\frac{\partial\partial y}{\partial t^2}\right).$$

§. 6. Quod si ergo filum a nullis plane viribus sollicitari ponamus, ita vt motus fili flexilis super plano horizontali vtcunque proiecti determinari debeat, ob vires $P = 0$ et

$Q = 0$, tota motus determinatio pendebit a resolutione huius aequationis satis simplicis:

$$0 = \left(\frac{\partial y}{\partial s}\right)\int \partial s \left(\frac{\partial \partial x}{\partial t^2}\right) - \left(\frac{\partial x}{\partial s}\right)\int \partial s \left(\frac{\partial \partial y}{\partial t^2}\right),$$

quae autem quomodo tractari debeat nullo plane modo perspicitur. Tum vero tensio euadet:

$$T = \frac{1}{2g}\left(\frac{\partial x}{\partial s}\right)\int \partial s \left(\frac{\partial \partial x}{\partial t^2}\right) + \frac{1}{2g}\left(\frac{\partial y}{\partial s}\right)\int \partial s \left(\frac{\partial \partial y}{\partial t^2}\right).$$

Quamobrem Geometrae erunt hortandi, vt omnes vires intendere velint ad resolutionem huius aequationis expediendam.

§. 7. Equidem meos conatus etiam irritos hic communicare non dubito dum forte aliis occasionem praebere poterunt feliciori successu hunc laborem exsequendi. Primo igitur mihi erat propositum, hanc aequationem:

$$\left(\frac{\partial y}{\partial s}\right)\int \partial s \left(\frac{\partial \partial x}{\partial t^2}\right) = \left(\frac{\partial x}{\partial s}\right)\int \partial s \left(\frac{\partial \partial y}{\partial t^2}\right),$$

a formulis integralibus liberare, quem in finem loco functionum x et y alias u et v in calculum introduxi, ponendo

$$\int \partial s \left(\frac{\partial \partial x}{\partial t^2}\right) = \left(\frac{\partial \partial u}{\partial t^2}\right) \text{ et } \int \partial s \left(\frac{\partial \partial y}{\partial t^2}\right) = \left(\frac{\partial \partial v}{\partial t^2}\right),$$

hinc autem differentiando sola variabili adhibita s, prodibit

$$\frac{\partial \partial x}{\partial t^2} = \frac{\partial^3 u}{\partial s \partial t^2} \text{ et } \frac{\partial \partial y}{\partial t^2} = \frac{\partial^3 v}{\partial s \partial t^2}.$$

Hinc autem porro colligemus, dum nunc solam t vt variabilem spectamus, cum sit $\partial t \left(\frac{\partial \partial x}{\partial t^2}\right) = \partial t \left(\frac{\partial^3 u}{\partial s \partial t^2}\right)$, erit integrando $\left(\frac{\partial x}{\partial t}\right) = \left(\frac{\partial \partial u}{\partial s \partial t}\right) + E$, quae constans E etiam arcum s vtcunque in se complecti potest, eodemque modo erit $\left(\frac{\partial y}{\partial t}\right) = \left(\frac{\partial \partial v}{\partial s \partial t}\right) + F$. Hae aequationes porro ducantur in ∂t ac denuo integrentur manente s constante, prodibit

$$x = \left(\frac{\partial u}{\partial s}\right) + E\,t + G \text{ et } y = \left(\frac{\partial v}{\partial s}\right) + F\,t + H,$$

vbi E, F, G, H possunt esse functiones ipsius s tantum.

§. 8.

§. 8. Hos valores denuo differentiemus sumta sola s pro variabili ac positis breuitatis gratia $\partial E = E' \partial s$, $\partial F = F' \partial s$, $\partial G = G' \partial s$ et $\partial H = H' \partial s$, obtinebimus

$$\left(\frac{\partial x}{\partial s}\right) = \left(\frac{\partial \partial u}{\partial s^2}\right) + E' t + G' \text{ et } \left(\frac{\partial y}{\partial s}\right) = \frac{\partial \partial v}{\partial s^2} + F' t + H'.$$

Quare cum esse oporteat $\left(\frac{\partial x}{\partial s}\right)^2 + \left(\frac{\partial y}{\partial s}\right)^2 = 1$, omissis functionibus adiectis E, F, G, H, requiritur vt fiat $\left(\frac{\partial \partial u}{\partial s^2}\right)^2 + \left(\frac{\partial \partial v}{\partial s^2}\right)^2 = 1$. Tum vero ipsa aequatio pro motu induet hanc formam:

$$\left(\frac{\partial \partial v}{\partial s^2}\right) \cdot \left(\frac{\partial \partial u}{\partial t^2}\right) = \left(\frac{\partial \partial u}{\partial s^2}\right) \cdot \left(\frac{\partial \partial v}{\partial t^2}\right),$$

vbi quidem breuitati consulentes functiones illas arbitrarias ipsius s praetermisimus. Simili modo pro tensione habebimus:

$$T = \frac{1}{2g}\left(\frac{\partial \partial u}{\partial s^2}\right) \cdot \left(\frac{\partial \partial u}{\partial t^2}\right) + \frac{1}{2g}\left(\frac{\partial \partial v}{\partial s^2}\right) \cdot \left(\frac{\partial \partial v}{\partial t^2}\right).$$

§. 9. Totum ergo negotium iam huc est reductum, quemadmodum ambas functiones ipsarum s et t, quas posuimus u et v, comparatas esse oporteat, vt fiat

$$\left(\frac{\partial \partial v}{\partial s^2}\right) \cdot \left(\frac{\partial \partial u}{\partial t^2}\right) = \left(\frac{\partial \partial u}{\partial s^2}\right) \cdot \left(\frac{\partial \partial v}{\partial t^2}\right),$$

siue vt haec proportio non parum elegans locum habeat:

$$\frac{\partial \partial u}{\partial s^2} : \frac{\partial \partial u}{\partial t^2} = \frac{\partial \partial v}{\partial s^2} : \frac{\partial \partial v}{\partial t^2},$$

cui quidem conditioni haud difficulter infinitis modis satisfieri potest. At vero altera conditio adimplenda nunc maximae difficultati videtur obnoxia, vt scilicet euadat $\left(\frac{\partial \partial u}{\partial s^2}\right)^2 + \left(\frac{\partial \partial v}{\partial s^2}\right)^2 = 1$. Hinc igitur manifesto perspicitur, hunc casum, qui sine dubio in hoc genere tanquam simplicissimus est spectandus, tantis difficultatibus ac tenebris etiamnunc esse inuolutum, vt nulla plane via pateat ad scopum optatum perueniendi.

§. 10. Talis reductio etiam in genere fieri potest in aequatione latissime patente:

$$2g\left(\frac{\partial y}{\partial s}\right)\int P\,\partial s - 2g\left(\frac{\partial x}{\partial s}\right)\int Q\,\partial s = \left(\frac{\partial y}{\partial s}\right)\int \partial s\left(\frac{\partial\partial x}{\partial t^2}\right) - \left(\frac{\partial x}{\partial s}\right)\int \partial s\left(\frac{\partial\partial y}{\partial t^2}\right),$$

atque adeo facilius ita instituetur. Ponatur statim $x = \left(\frac{\partial u}{\partial s}\right)$ et $y = \left(\frac{\partial v}{\partial s}\right)$. Hinc igitur erit $\left(\frac{\partial x}{\partial s}\right) = \left(\frac{\partial\partial u}{\partial s^2}\right)$ et $\left(\frac{\partial y}{\partial s}\right) = \left(\frac{\partial\partial v}{\partial s^2}\right)$, ita vt nunc esse debeat $\left(\frac{\partial\partial u}{\partial s^2}\right)^2 + \left(\frac{\partial\partial v}{\partial s^2}\right) = 1$. Porro vero erit $\left(\frac{\partial x}{\partial t}\right) = \left(\frac{\partial\partial u}{\partial s\,\partial t}\right)$ et $\left(\frac{\partial y}{\partial t}\right) = \left(\frac{\partial\partial v}{\partial s\,\partial t}\right)$, quae formulae exprimunt celeritates puncti Y secundum directiones Y P et Y Q. Tum vero habebimus insuper $\left(\frac{\partial\partial x}{\partial t^2}\right) = \left(\frac{\partial^3 u}{\partial s\,\partial t^2}\right)$ et $\left(\frac{\partial\partial y}{\partial t^2}\right) = \left(\frac{\partial^3 v}{\partial s\,\partial t^2}\right)$, atque nunc integratio succedit: erit enim

$$\int \partial s\left(\frac{\partial\partial x}{\partial t^2}\right) = \left(\frac{\partial\partial u}{\partial t^2}\right) + \Gamma : t \text{ et}$$

$$\int \partial s\left(\frac{\partial\partial y}{\partial t^2}\right) = \left(\frac{\partial\partial v}{\partial t^2}\right) + \Delta : t,$$

vbi functiones quascunque temporis loco constantium sunt adiectae, propterea quod in istis integrationibus tempus t vt constans est spectatum. Quamobrem si vires P et Q etiam x vel y inuoluunt, hoc modo tota aequatio inter binas functiones u et v subsistet.

§. 11. Nihilo vero minus nullum adhuc fructum mihi quidem hinc percipere licuit, ac praecipua huius difficultatis caussa in hoc sita esse videtur: quod innumeras figuras diversas quas filum successiue induit, vix vllo modo ita per calculum exprimere licet, vt ad quoduis tempus definiri queat quales functiones ipsarum s et t binae coordinatae x et y sint futurae. Hanc ob rem istud argumentum ordine inuerso tractare institui, dum scilicet ad quoduis tempus figuram fili tanquam datam spectabo atque in vires P et Q inquiram, quae filo talem motum imprimere valeant.

Status quaestionis.

Tab. IV. Fig. 2. §. 12. Sumamus igitur initio, vbi erat $t = 0$, filum super plano horizontali in directum fuisse extensum, ita vt situm

tum tenuerit E F, eiusque longitudinem E F ſtatuamus $= a$. Hinc vero elapſo tempore $= t$ acceperit figuram E Y F′, quae ſit arcus circularis rectam E F pro axe aſſumtam tangens in ipſo puncto E, ita vt fili terminus E perpetuo maneat immotus. Radius autem huius circuli ſit E O $= r$, functio quaecunque data temporis t, vnde neceſſe eſt vt poſito $t = 0$ iſta functio r euadat infinita. Sit nunc E Y portio quaecunque indefinita fili $= s$, ductoque radio O Y erit angulus $EOY = \frac{s}{r}$, cuius ſinus erit $\frac{EX}{EO} = \frac{x}{r}$, coſinus vero $1 - \frac{y}{r}$, vnde coordinatae $EX = x$ et $XY = y$ ita per binas variabiles s et t exprimentur, vt ſit $x = r \sin. \frac{s}{r}$ et $y = r(1 - \cos. \frac{s}{r})$. Quibus poſitis quaeſtio ſoluenda huc redit: vt inueſtigentur vires P et Q, quae filo talem motum qualem hic deſcripſimus inducere valeant. Quae quidem quaeſtio maxime adhuc erit indeterminata, propterea quod pro motu determinando vnicam tantum habemus aequationem: Tab. II. Fig. 2.

$$2g\left(\frac{\partial y}{\partial s}\right)\int P\,\partial s - 2g\left(\frac{\partial x}{\partial s}\right)\int Q\,\partial s = \left(\frac{\partial y}{\partial s}\right)\int \partial s\left(\frac{\partial\partial x}{\partial t^2}\right) - \left(\frac{\partial x}{\partial s}\right)\int \partial s\left(\frac{\partial\partial y}{\partial t^2}\right),$$

vnde alterutra quantitatum P et Q arbitrio noſtro relinquetur.

Euolutio formularum
in hanc aequationem ingredientium.

§. 13. Cum littera r ſit functio temporis t tantum, ſumta ſola s variabili impetrabimus has formulas $\left(\frac{\partial x}{\partial s}\right) = \cos. \frac{s}{r}$ et $\left(\frac{\partial y}{\partial s}\right) = \sin. \frac{s}{r}$, vnde ſponte fit $\left(\frac{\partial x}{\partial s}\right)^2 + \left(\frac{\partial y}{\partial s}\right)^2 = 1$, vti rei natura poſtulat. Sumto autem ſolo tempore t variabili ponamus breuitatis gratia $\partial r = r'\,\partial t$, ac differentiando reperiemus

$$\left(\frac{\partial x}{\partial t}\right) = r' \sin. \frac{s}{r} - \frac{r's}{r} \cos. \frac{s}{r} \text{ et}$$

$$\left(\frac{\partial y}{\partial t}\right) = r'\left(1 - \cos. \frac{s}{r}\right) - \frac{sr'}{r} \sin. \frac{s}{r}.$$

§. 14. Hae formulae cum ambas celeritates puncti Y exprimant, hinc istas celeritates pro statu fili initiali, vbi erat $t = 0$ filumque in directum extensum, cognoscere licebit, id quod patebit si statuamus $r = \infty$. Tum igitur erit sin. $\frac{s}{r} = \frac{s}{r}$ et cos. $\frac{s}{r} = 1 - \frac{ss}{2rr}$, ex quo pro hoc casu erit

$$\left(\frac{\partial x}{\partial t}\right) = \frac{r's^3}{2r^3} \text{ et } \left(\frac{\partial y}{\partial t}\right) = -\frac{r'ss}{2rr}.$$

Videndum igitur est, num istae formulae casu $r = \infty$ seu $t = 0$ valores finitos recipere queant nec ne, id quod ab indole functionis r pendet. Veluti si sit $r = \frac{1}{t^n}$ ita vt exponens n sit positiuus, quoniam posito $t = 0$ fieri debet $r = \infty$, eritque $r' = -\frac{n}{t^{n+1}}$, hoc casu habebitur

$$\left(\frac{\partial x}{\partial t}\right) = -\tfrac{1}{2} n s^3 t^{2n-1} \text{ et } \left(\frac{\partial y}{\partial t}\right) = +\tfrac{1}{2} n s s t^{n-1}.$$

Hinc ergo intelligitur si n sit 1 fore

$$\left(\frac{\partial x}{\partial t}\right) = -\tfrac{1}{2} s^3 t = 0 \text{ et } \left(\frac{\partial y}{\partial t}\right) = \tfrac{1}{2} n s s.$$

Quo igitur casu sola celeritas $\left(\frac{\partial x}{\partial t}\right)$ euanescit. At si fuerit $n = \frac{1}{2}$, fiet $\left(\frac{\partial x}{\partial t}\right) = -\frac{1}{4}s^3$. Altera vero $\left(\frac{\partial y}{\partial t}\right) = \frac{ss}{4\sqrt{t}} = \infty$. Hinc igitur patet, pro indole functionis r euenire posse vt celeritates initiales modo fiant $= 0$, modo determinatum obtineant valorem, modo etiam in infinitum excrescant, solo termino E ipso excepto vbi $s = 0$, ille enim certe quieuisse necesse est.

§. 15. Progrediamur nunc etiam ad differentialia secunda sumendo solum t variabile, quem in finem statuamus $\partial r' = r'' \partial t$, et subducto calculo reperiemus:

$$\left(\frac{\partial \partial x}{\partial t^2}\right) = r'' \text{ sin. } \frac{s}{r} - \frac{r''s}{r} \text{ cos. } \frac{s}{r} - \frac{r'r's s}{r^3} \text{ sin. } \frac{s}{r} \text{ et}$$

$$\left(\frac{\partial \partial y}{\partial t^2}\right) = r''\left(1 - \text{cos. } \frac{s}{r}\right) - \frac{r''}{r} s \text{ sin. } \frac{s}{r} + \frac{r'r's s}{r^3} \text{ cos. } \frac{s}{r}.$$

§. 16.

§. 16. Nunc igitur has formulas ducamus in ∂s easque ita integremus vt ſola quantitas s pro variabili habeatur, ac reperiemus:

$$\int \partial s \left(\frac{\partial \partial x}{\partial t^2}\right) = r'' \int \partial s \text{ ſin.} \frac{s}{r} - \frac{r''}{r} \int s\, \partial s \text{ coſ.} \frac{s}{r}$$
$$- \frac{r' r'}{r s} \int s\, s\, \partial s \text{ ſin.} \frac{s}{r} + \Gamma : t,$$

eodemque modo

$$\int \partial s \left(\frac{\partial \partial y}{\partial t^2}\right) = r'' s - r'' \int \partial s \text{ coſ.} \frac{s}{r} - \frac{r''}{r} \int s\, \partial s \text{ ſin.} \frac{s}{r}$$
$$+ \frac{r' r'}{r^3} \int s\, s\, \partial s \text{ coſ.} \frac{s}{r} + \Delta : t,$$

vbi loco conſtantium adiecimus functiones quascunque ipſius t, propterea quod tempus ſpectatum eſt vt conſtans.

§. 17. Supereſt igitur tantum vt formulas integrales euoluamus, hoc modo:

$$\int \partial s \text{ ſin.} \frac{s}{r} = - r \text{ coſ.} \frac{s}{r}; \quad \int \partial s \text{ coſ.} \frac{s}{r} = r \text{ ſin.} \frac{s}{r};$$
$$\int s\, \partial s \text{ coſ.} \frac{s}{r} = r s \text{ ſin.} \frac{s}{r} + r r \text{ coſ.} \frac{s}{r};$$
$$\int s\, \partial s \text{ ſin.} \frac{s}{r} = - r s \text{ coſ.} \frac{s}{r} + r r \text{ ſin.} \frac{s}{r};$$
$$\int s s \partial s \text{ ſin.} \frac{s}{r} = - r s s \text{ coſ.} \frac{s}{r} + 2 r r s \text{ ſin.} \frac{s}{r} + 2 r^3 \text{ coſ.} \frac{s}{r} \text{ et}$$
$$\int s s \partial s \text{ coſ.} \frac{s}{r} = r s s \text{ ſin.} \frac{s}{r} + 2 r r s \text{ coſ.} \frac{s}{r} - 2 r^3 \text{ ſin.} \frac{s}{r}:$$

hinc igitur erit

$$\int \partial s \left(\frac{\partial \partial x}{\partial t^2}\right) = - 2 \text{ coſ.} \frac{s}{r} (r r'' + r' r')$$
$$- s \text{ ſin.} \frac{s}{r} \left(r'' + \frac{2 r' r'}{r}\right) + \frac{r' r' s s}{r r} \text{ coſ.} \frac{s}{r} \text{ et}$$
$$\int \partial s \left(\frac{\partial \partial y}{\partial t^2}\right) = - 2 \text{ ſin.} \frac{s}{r} (r r'' + r' r')$$
$$+ s \left(r'' + \left(r'' + \frac{2 r' r'}{r}\right) \text{ coſ.} \frac{s}{r}\right) + \frac{r' r' s s}{r r} \text{ ſin.} \frac{s}{r} + \Delta : t.$$

§. 18. Nunc igitur ad aequationem noſtram conſtituendam prior formula ducatur in $\left(\frac{\partial y}{\partial s}\right) = \text{ ſin.} \frac{s}{r}$ altera vero in $-(\partial x)$

$-\left(\frac{\partial x}{\partial s}\right) = -\cos.\frac{s}{r}$, et membrum dextrum aequationis noſtrae euadet

$$-r''s\cos.\frac{s}{r} - \left(r'' + \frac{2r'r'}{r}\right)s + \sin.\frac{s}{r}\,\Gamma : t - \cos.\frac{s}{r}\,\Delta : t,$$

quoniam igitur membrum ſiniſtrum eſt

$$2g\sin.\frac{s}{r}\int P\,\partial s - 2g\cos.\frac{s}{r}\int Q\,\partial s,$$

aequatio, ex qua tota motus natura eſt definienda, erit

$$2g\sin.\frac{s}{r}\int P\,\partial s - 2g\cos.\frac{s}{r}\int Q\,\partial s = -r''s\cos.\frac{s}{r} - \left(r'' + \frac{2r'r'}{r}\right)s + \sin.\frac{s}{r}\,\Gamma : t - \cos.\frac{s}{r}\,\Delta : t,$$

vnde cum duae adhuc inſint incognitae P et Q, alteram pro lubitu accipere licebit.

§. 19. Conſideremus etiam tenſionem T, quam filum in ſingulis punctis ſuſtinebit, quae cum in genere fuerit

$$T = -\left(\frac{\partial x}{\partial s}\right)\int P\,\partial s - \left(\frac{\partial y}{\partial s}\right)\int Q\,\partial s + \frac{1}{2g}\left(\frac{\partial x}{\partial s}\right)\int \partial s\left(\frac{\partial\partial x}{\partial t^2}\right) + \frac{1}{2g}\left(\frac{\partial y}{\partial s}\right)\int \partial s\left(\frac{\partial\partial y}{\partial t^2}\right),$$

ſubſtitutis valoribus modo inuentis fiet

$$T = -\cos.\frac{s}{r}\int P\,\partial s - \sin.\frac{s}{r}\int Q\,\partial s - \frac{1}{g}(rr'' + r'r') + \frac{1}{2g}r''s\sin.\frac{s}{r} + \frac{1}{2g}\cdot\frac{r'r'ss}{rr} + \frac{1}{2g}\,\Gamma : t\cos.\frac{s}{r} + \frac{1}{2g}\,\Delta : t\sin.\frac{s}{r}.$$

§. 20. Cum igitur ex priore aequatione ſit

$$\int Q\,\partial s = \tan.\frac{s}{r}\int P\,\partial s + \frac{1}{2g}r''s + \frac{1}{2g}\left(r'' + \frac{2r'r'}{r}\right)\frac{s}{\cos.\frac{s}{r}} - \frac{1}{2g}\tan.\frac{s}{r}\,\Gamma : t + \frac{1}{2g}\,\Delta : t,$$

ſi hic valor in expreſſione tenſionis ſubſtituatur, prodibit

T =

$$T = -\frac{\int P\,\partial s}{\cos.\frac{s}{r}} - \frac{1}{2g}\left(r'' + \frac{2r'r'}{r}\right) s \tang.\frac{s}{r} - \frac{1}{g}(rr'' + r'r')$$

$$+ \frac{1}{2g} \cdot \frac{r'r's\,s}{rr} + \frac{1}{2g\cos.\frac{s}{r}}\Gamma : t$$

ficque per tenfionem formula $\int P\,\partial s$ ita exprimitur, vt fit

$$\int P\,\partial s = -T\cos.\frac{s}{r} - \frac{1}{2g}\left(r'' + \frac{2r'r'}{r}\right) s \sin.\frac{s}{r} - \frac{1}{g}(rr'' + r'r')\cos.\frac{s}{r}$$

$$+ \frac{1}{2g}\cos.\frac{s}{r} \cdot \frac{r'r's\,s}{rr} + \frac{1}{2g}\Gamma : t$$

vnde differentiando, fi ponamus $\partial T = T'\,\partial s$ quandoquidem hic fola quantitas s variabilis affumitur, fiet

$$P = -T'\cos.\frac{s}{r} + \frac{T}{r}\sin.\frac{s}{r} - \frac{1}{2g}\left(r'' + \frac{2r'r'}{r}\right)\sin.\frac{s}{r} - \frac{s}{2gr}\left(r'' + \frac{2r'r'}{r}\right)\cos.\frac{s}{r}$$

$$+ \frac{1}{gr}(rr'' + r'r')\sin.\frac{s}{r} - \frac{1}{2gr}\sin.\frac{s}{r} \cdot \frac{r'r's\,s}{rr} + \frac{r'r's}{grr}\cos.\frac{s}{r}$$

quae manifefto reducitur ad hanc

$$P = -T'\cos.\frac{s}{r} + \frac{T}{r}\sin.\frac{s}{r} + \frac{r''}{2g}\sin.\frac{s}{r} - \frac{s}{2gr}\left(r'' + \frac{2r'r'}{r}\right)\cos.\frac{s}{r}$$

$$- \frac{1}{2gr}\sin.\frac{s}{r} \cdot \frac{r'r's\,s}{rr} + \frac{r'r's}{grr}\cos.\frac{s}{r}.$$

Simili modo, quia ex prima aequatione eft

$$\int P\,\partial s = \cot.\frac{s}{r}\int Q\,\partial s - \frac{1}{2g\sin.\frac{s}{r}} r''\,s\cot.\frac{s}{r}$$

$$- \frac{1}{2g}\left(r'' + \frac{2r'r'}{r}\right)\frac{s}{\sin.\frac{s}{r}} + \frac{1}{2g}\Gamma : t - \frac{1}{2g}\cot.\frac{s}{r}\Delta : t,$$

qui valor in expreffione tenfionis fubftitutus praebet

$$T = -\frac{\int Q\,\partial s}{\sin.\frac{s}{r}} + \frac{1}{2g\sin.\frac{s}{r}} r''s + \frac{1}{2g}\left(r'' + \frac{2r'r'}{r}\right) s\cot.\frac{s}{r}$$

$$- \frac{1}{g}(rr'' + r'r') + \frac{1}{2g} \cdot \frac{r'r's\,s}{rr} + \frac{1}{2g\sin.\frac{s}{r}}\Delta : t,$$

inde porro colligitur

$$\int Q\partial s = -T\,\text{sin.}\,\frac{s}{r} + \frac{r''s}{2g} + \frac{1}{2g}\left(r'' + \frac{2r'r'}{r}\right)\text{cos.}\,\frac{s}{r}$$
$$-\frac{1}{g}(rr'' + r'r')\,\text{sin.}\,\frac{s}{r} + \frac{\text{sin.}\,\frac{s}{r}}{2g}\cdot\frac{r'r'ss}{rr} + \frac{1}{2g}\Delta t$$

vnde tandem differentiando elicitur Q

$$Q = -T'\,\text{sin.}\,\frac{s}{r} - \frac{T}{r}\,\text{cos.}\,\frac{s}{r} + \frac{r''}{2g} + \frac{1}{2g}\left(r'' + \frac{2r'r'}{r}\right)\text{cos.}\,\frac{s}{r}$$
$$-\frac{s}{2gr}\left(r'' + \frac{2r'r'}{r}\right)\text{sin.}\,\frac{s}{r} - \frac{1}{gr}(rr'' + r'r')\,\text{cos.}\,\frac{s}{r}$$
$$+\frac{1}{2gr}\,\text{cos.}\,\frac{s}{r}\cdot\frac{r'r's.s}{rr} + \frac{1}{g}\,\frac{r'r's}{rr}.\text{sin.}\,\frac{s}{r};\ \text{siue}$$
$$Q = -T'\,\text{sin.}\,\frac{s}{r} - \frac{T}{r}\,\text{cos.}\,\frac{s}{r} + \frac{r''}{2g} - \frac{rr''}{2gr}\,\text{cos.}\,\frac{s}{r}$$
$$-\frac{sr''}{2gr}\,\text{sin.}\,\frac{s}{r} + \frac{r'r's.s}{2gr^3}\,\text{cos.}\,\frac{s}{r}.$$

§. 22. Hoc igitur modo ambas litteras incognitas P et Q per tenſionem definiuimus, vbi notari meretur has litteras deſignare vires acceleratrices filo in puncto y applicatas. Quoniam enim elementi $Y y = \partial s$ maſſa quoque exprimitur per ∂s, vires motrices vtique erunt $P\partial s$ et $Q\partial s$, prouti ſupra asſumſimus. Non ſolum autem ipſas has vires P et Q per tenſionem expreſſimus, ſed etiam formulas integrales $\int P\partial s$ et $\int Q\partial s$.

§. 23. Cum autem in formulis pro P et Q inuentis non ſolum tenſio ipſa T inſit ſed etiam eius differentiale $\partial T = T'\partial s$, operae pretium erit per combinationem harum formularum ſiue T ſiue T′ eliminare. Hoc modo reperiemus

$$P\,\text{sin.}\,\frac{s}{r} - Q\,\text{cos.}\,\frac{s}{r} = \frac{T}{r} - \frac{r''}{2g}\,\text{cos.}\,\frac{s}{r} - \frac{1}{2g}\left(r'' + \frac{2r'r'}{r}\right)$$
$$+\frac{1}{gr}(rr'' + r'r') - \frac{r'r'ss}{2gr^3} = \frac{T}{r} - \frac{r''}{2g}\,\text{cos.}\,\frac{s}{r} + \frac{r''}{g} - \frac{r'r's.s}{2gr^3};$$
$$P\,\text{cos.}\,\frac{s}{r} + Q\,\text{sin.}\,\frac{s}{r} = -T' + \frac{r''}{2g}\,\text{sin.}\,\frac{s}{r} - \frac{s}{2gr}\left(r'' + \frac{2r'r'}{r}\right)$$
$$+\frac{r'r's}{grr} = -T' + \frac{r''}{2g}\,\text{sin.}\,\frac{s}{r} - \frac{r''s}{2gr};$$

vbi

vbi notasse iuuabit, exprimere formulam posteriorem $P \cos. \frac{s}{r} + Q \sin. \frac{s}{r}$ vim tangentialem qua filum in puncto Y sollicitatur, alteram vero formulam $P \sin. \frac{s}{r} - Q \cos. \frac{s}{r}$ vim normalem eidem puncto applicatam, quarum ergo vtraque ex tensione T, quam quidem pro lubitu fingere licet, perfecte determinabitur. Atque hinc pro ipso fili initio E vbi $s = 0$ fiet

$$P \sin. \frac{s}{r} - Q \cos. \frac{s}{r} = \frac{T}{r} = -Q$$

ideoque $Q = -\frac{T}{r}$. Similique modo

$$P \cos. \frac{s}{r} + Q \sin. \frac{s}{r} = -T' = P.$$

§. 24. His formulis euolutis ponamus vim tangentialem acceleratricem secundum directionem $Y y$ agentem $= \Theta$, at vim normalem secundum directionem Y O versus centrum circuli tendentem $= \Pi$, ita vt sit

$$\Theta = P \cos. \frac{s}{r} + Q \sin. \frac{s}{r} \text{ et}$$

$$\Pi = Q \cos. \frac{s}{r} - P \sin. \frac{s}{r}$$

atque valores harum duarum virium erunt

$$\Theta = -T' + \frac{r''}{2g} \sin. \frac{s}{r} - \frac{r''s}{2gr} \text{ et}$$

$$\Pi = -\frac{T}{r} + \frac{r''}{2g} \cos. \frac{s}{r} - \frac{r''}{2g} + \frac{r'r's\,s}{2gr^3}.$$

Nunc igitur cum quaestio in se sit indeterminata, sequentia Problemata specialia percurramus, in quibus ratio virium sollicitantium praescribitur, vt filo motus supra assignatus inducatur.

Problema I.

§. 25. *Definire vires tangentiales ad motum supra descriptum in filo producendum requisitas.*

Solutio.

Cum igitur hic solae vires tangentiales requirantur, vires normales Π euanescent ita vt sit $\Pi = 0$, vnde ex postre-

ma aequatione colligitur tensio:

$$T = \frac{r''r}{2g} \text{cos.} \frac{s}{r} - \frac{rr''}{2g} + \frac{r'r'}{2grr} ss,$$

cuius differentiale sumto solo s variabili praebet

$$T' = -\frac{r''}{2g} \text{sin.} \frac{s}{r} - \frac{r'r's}{grr},$$

quo valore substituto reperimus vim tangentialem:

$$\Theta = \frac{r''}{g} \text{sin.} \frac{s}{r} - \left(\frac{rr''-2r'r'}{2grr}\right)s + \frac{r''}{g} \text{sin.} \frac{s}{r} - \frac{r''s}{2gr},$$

quae ergo in ipso termino E vbi $s = 0$ euadit $\Theta = 0$, in fine autem fili seu puncto T vbi $s = a$ erit

$$\Theta = \frac{r''}{g} \text{sin.} \frac{a}{r} - \frac{(rr''-2rr)}{2grr} a.$$

Corollarium.

§. 26. Quia hic r denotat radium circuli secundum quem filum elapso tempore t incuruatur, iam supra monuimus r talem esse debere functionem ipsius T, quae fiat infinita posito $P = 0$: consideremus vnicum casum.

Exemplum.

§. 27. Sumamus $r = \frac{1}{t}$, erit $r' = -\frac{1}{tt}$ et $r'' = \frac{2}{t^3}$; hinc igitur fiet vis tangentialis quaesita $\Theta = \frac{2}{gt^3} \text{sin.} st$; tensio autem erit $T = \frac{1}{gt^4}(1 - \text{cos.} st) + \frac{ss}{2gtt}$: hinc igitur sequentia notari merentur: 1) In ipso igitur initio vbi $t = 0$ vires tangentiales vbique infinitae requiruntur, vnde etiam tensio euadet infinita. 2) Elapso autem quouis tempore pro singulis fili punctis vires tangentiales erunt reciproce vt cubus temporis. 3) Pro ipso autem fili termino E, vbi $s = 0$, tam vis tangentialis Θ quam tensio euanescit, id quod natura rei postulat, cum punctum E maneat immotum. 4) Supra vidimus, celeritates puncti Y secundum directiones YP et YQ esse, prio-

rem

rem $\left(\frac{\partial x}{\partial t}\right) = r' \sin. \frac{s}{r} - \frac{r's}{r} \cos. \frac{s}{r}$. Alteram vero

$$\left(\frac{\partial y}{\partial t}\right) = r'\left(1 - \cos. \frac{s}{r}\right) - \frac{r'}{r} s \sin. \frac{s}{r},$$

quae ergo hoc casu euadent

$$\left(\frac{\partial y}{\partial t}\right) = \frac{(1 - \cos. st)}{tt} + \frac{1}{t} s \sin. st.$$

$$\left(\frac{\partial x}{\partial t}\right) = -\frac{1}{tt} \sin. st + \frac{1}{t} s \cos. st,$$

quae casu $t = 0$, quo fit $\sin. st = st$ et $\cos. st = 1 - \frac{sstt}{2}$, erunt $\left(\frac{\partial x}{\partial t}\right) = -\frac{1}{2} s^3 t = 0$ et $\left(\frac{\partial y}{\partial t}\right) = \frac{1}{2} ss$, vnde patet, quo hic casus locum habere queat, initio singulis fili punctis Y in directione Y Q eiusmodi celeritates imprimi debere, quae sint quadrato arcus E Y $= s$ proportionales. Tum vero ipso initio viribus opus esse infinitis, quae deinceps in ratione triplicata temporis decrescent.

Problema II.

§. 28. *Definire vires normales* Π, *ad motum supra descriptum in filo producendum requisitas.*

Solutio.

Hic igitur esse debet $\Theta = 0$, vnde colligimus:

$$T' = \frac{r''}{2g} \sin. \frac{s}{r} - \frac{r''s}{2gr},$$

vnde deducimus integrando:

$$T = -\frac{rr''}{2g} \cos. \frac{s}{r} - \frac{r''ss}{4gr} + f : t,$$

quo valore substituto reperitur vis normalis quaesita

$$\Pi = -\frac{r''}{2g}\left(1 - 2\cos. \frac{s}{r}\right) + \left(\frac{rr'' + 2r'r'}{4gr^3}\right) ss - \frac{1}{r} f : t,$$

vnde pro termino fili E fiet $\Pi = +\frac{r''}{2g} - \frac{1}{r} f : t$ et tensio $T = -\frac{rr''}{2g} + f : t$.

Exem-

Exemplum.

§. 29. Consideremus hic iterum casum quo $r = \frac{1}{t}$, ideoque $r' = -\frac{1}{tt}$ et $r'' = \frac{2}{t^3}$, eritque vis normalis:

$$\Pi = -\frac{1}{gt^3}(1 - 2\cos. st) + \frac{1}{gt}ss - tf:t,$$

et tensio

$$T = -\frac{1}{gt^4}\cos. st - \frac{ss}{2gtt} + f:t.$$

Hinc igitur pro termino fili E vbi $s = 0$ fiet

$$\Pi = +\frac{1}{gt^3} - t:f:t \text{ et } T = -\frac{1}{gt^4} + f:t,$$

motus autem filo in ipso initio imprimendus erit vt ante

$$\left(\frac{\partial x}{\partial t}\right) = 0 \text{ et } \left(\frac{\partial y}{\partial t}\right) = \frac{1}{2}ss.$$

Corollarium.

§. 30. Hoc igitur problema etiam nunc est indeterminatum, quoniam functio arbitrio nostro relinquitur. Eam igitur ita assumere licebit, vt tensio in ipso fili termino E euanescat, quod ergo fiet si functio $f:t = \frac{1}{gt^4}$, vnde fiet vis normalis:

$$\Pi = \frac{-2}{gt^3}(1 - \cos. st) + \frac{1}{gt}ss,$$

quae ergo in ipso puncto E euanescit. Hinc igitur patet quo maius euadat tempus t, has vires normales continuo fieri minores.

Problema III.

§. 31. *Inuenire tam vires tangentiales quam normales ad motum propositum fili requisitas, ita vt durante motu tensio fili in singulis punctis perpetuo sit nulla.*

Solutio.

Solutio.

Cum igitur sit $T = 0$ ideoque etiam $T' = 0$, vires quaesitae sequenti modo exprimentur:

$$\Theta = \frac{r''}{2g} \text{ sin.} \frac{s}{r} - \frac{r''s}{2gr} \text{ et}$$

$$\Pi = \frac{-r''}{2g}(1 - \text{cos.} \frac{s}{r}) + \frac{r'r'}{2gr^3} ss,$$

quae ambae euanescunt pro termino fili E vbi fit $s = 0$. Ex his duabus viribus etiam vires initio consideratae P et Q assignari poterunt. Cum enim sit

$$P = \Theta \text{ cos.} \frac{s}{r} - \Pi \text{ sin.} \frac{s}{r} \text{ et}$$

$$Q = \Theta \text{ sin.} \frac{s}{r} + \Pi \text{ cos.} \frac{s}{r},$$

hinc colligitur fore

$$P = \frac{r''}{2g} \text{ sin.} \frac{s}{r} - \frac{r''}{2gr} s \text{ cos.} \frac{s}{r} - \frac{r'r'}{2gr^3} ss \text{ sin.} \frac{s}{r} \text{ et}$$

$$Q = \frac{r''}{2g}(1 - \text{cos.} \frac{s}{r}) - \frac{r''}{2gr} s \text{ sin.} \frac{s}{r} + \frac{r'r'}{2gr^3} ss \text{ cos.} \frac{s}{r}.$$

Exemplum.

§. 32. Sit iterum $r = \frac{1}{t}$, vt sit $r' = \frac{-1}{tt}$ et $r'' = \frac{2}{t^3}$, fietque $\Theta = \frac{1}{gt^3} \text{ sin.} st - \frac{s}{gtt}$ et

$$\Pi = -\frac{1}{gt^3}(1 - \text{cos.} st) + \frac{ss}{2gt},$$

vel loco harum duarum virium applicatae concipi possunt sequentes:

$$P = \frac{1}{gt^3} \text{ sin.} st - \frac{1}{gtt} s \text{ cos.} st - \frac{1}{2gt} ss \text{ sin.} st.$$

$$Q = \frac{1}{gt^3}(1 - \text{cos.} st) - \frac{1}{gtt} s \text{ sin.} \frac{s}{r} + \frac{1}{2gt} ss \text{ cos.} \frac{s}{r},$$

ab his scilicet viribus filum, quod initio erat in directum extensum, tandem post tempus infinitum quasi in vnicum punctum conglomerabitur.

Scho-

Scholion.

§. 33. Hinc igitur infinitos casus deducere licet, quibus motus fili, dum a certis viribus continuo sollicitatur, perfecte determinari potest. Atque hi casus maxime sunt memorabiles, cum hactenus nullo plane casu talem motum inuestigare licuerit, ne eo quidem excepto, quo filo nullae plane vires applicatae concipiuntur. Simili autem modo infinitos alios huiusmodi casus euoluere licebit, quibus filum successiue secundum alios atque alios arcus circulares quacunque lege incuruatur; semper enim per theoriam generalem eiusmodi vires assignare licebit, quibus tales motus producentur.