

DE
TRANSFORMATIONE
SERIEI DIVERGENTIS

$$1 - mx + m(m+n)x^2 - m(m+n)(m+2n)x^3 + m(m+n)(m+2n)(m+3n)x^4 \text{ etc.}$$

IN FRACTIONEM CONTINUAM.

Auctore
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§. I.

Cum olim indolem huiusmodi serierum diuergentium effem perscrutatus, et veram summam seriei hypergeometricae

$$1 - 1 + 2 - 6 + 24 - 120 + 720 - \text{etc.}$$

assignauiffem ope transformationis in fractionem continuam, mentionem quoque feci istius seriei multo latius patentis :

$$1 - mx + m(m+n)x^2 - m(m+n)(m+2n)x^3 + m(m+n)(m+2n)(m+3n)x^4 - \text{etc.}$$

cuius summam inueneram aequari huic fractioni continuae:

$$\frac{1}{1 + \frac{mx}{1 + \frac{nx}{1 + \frac{(m+n)x}{1 + \frac{2nx}{1 + \frac{(m+2n)x}{1 + \text{etc.}}}}}}}$$

cuius

cuius rei veritatem ex conuersione aequationis Riccatianae in fractionem continuam deduxeram. Cum autem haec demonstratio nimis longe petita videri queat, eandem reductionem hic ex principiis simplicioribus sum traditurus.

§. 2. Primo autem istam seriem generalem in formam concinniore contrahi conueniet ponendo $mx = a$ et $nx = b$, vt proposita fit ista series infinita:

$$1 - a + a(a+b) - a(a+b)(a+2b) + a(a+b)(a+2b)(a+3b) - \text{etc.}$$

Praeterea vero vt sequentes resolutiones commodius peragi queant, neque tot clausulis sit opus, statuam vt sequitur:

$$a = A, a + b = B, a + 2b = C, a + 3b = D, \text{ etc.}$$

ficque habebitur ista series :

$$1 - A + AB - ABC + ABCD - \text{etc.}$$

cuius summam quaesitam designemus littera S, ita vt fit

$$S = 1 - A + AB - ABC + ABCD - \text{etc.}$$

hinc porro

$$\frac{1}{S} = \frac{1}{1 - A + AB - ABC + ABCD - \text{etc.}}$$

§. 3. Cum igitur fit $\frac{1}{S} > 1$, postrema aequatio reducatur ad hanc formam :

$$\frac{1}{S} = 1 + \frac{A - AB + ABC - ABCD + \text{etc.}}{1 - A + AB - ABC + ABCD \text{ etc.}}$$

Nunc autem ponamus $\frac{1}{S} = 1 + \frac{A}{P}$, eritque

$$P = \frac{1 - A + AB - ABC + ABCD \text{ etc.}}{1 - B + BC - BCD + BCDE \text{ etc.}}$$

quae expressio cum iterum vnitatem superet, ob $B - A = b$, $C - A = 2b$, $D - A = 3b$, etc. ea dabit

$$P = 1 + \frac{b - 2bB + 3bBC - 4bBCD + \text{etc.}}{1 - B + BC - BCD + BCDE - \text{etc.}}$$

Ponatur ergo $P = 1 + \frac{b}{Q}$ eritque

E 3

Q =

$$Q = \frac{1 - B + BC - BCD + BCDE - \text{etc.}}{1 - 2B + 3BC - 4BCD + \text{etc.}},$$

vnde deducimus

$$Q = 1 + \frac{B - 2BC + 3BCD - 4BCDE \text{ etc.}}{1 - 2B + 3BC - 4BCD + \text{etc.}}.$$

Hanc ob rem ponamus nunc $Q = 1 + \frac{B}{R}$, ac prodibit

$$R = \frac{1 - 2B + 3BC - 4BCD + \text{etc.}}{1 - 2C + 3CD - 4CDE + \text{etc.}}.$$

§. 4. Hic ergo tam in numeratore quam in denominatore iidem coefficientes occurrunt, at litterae maiusculae in denominatore vno gradu sunt promotae. Cum igitur sit $C - B = b$, $D - B = 2b$, $E - B = 3b$, etc. fiet

$$R = 1 + \frac{2b - 2.3bc + 3.4bcd + 4.5bcde - \text{etc.}}{1 - 2C + 3CD - 4CDE + 5CDEF - \text{etc.}}.$$

Quod si ergo ponamus $R = 1 + \frac{2b}{S}$, erit

$$S = \frac{1 - 2C + 3CD - 4CDE + \text{etc.}}{1 - 3C + 6CD - 10CDE + \text{etc.}}$$

vbi in denominatore manifesto occurrunt numeri trigonales, quae expressio reducitur ad hanc:

$$S = 1 + \frac{C - 3CD + 6CDE - 10CDEF + \text{etc.}}{1 - 3C + 6CD - 10CDE + \text{etc.}}$$

Quod si ergo statuamus $S = 1 + \frac{C}{T}$, erit

$$T = \frac{1 - 3C + 6CD - 10CDE + 15CDEF - \text{etc.}}{1 - 3D + 6DE - 10DEF + 15DEFG - \text{etc.}}$$

§. 5. Ista forma ob $D - C = b$, $E - C = 2b$, $F - C = 3b$, etc. abit in hanc:

$$T = 1 + \frac{3b - 2.6bd + 3.10bde - 4.15bdef + \text{etc.}}{1 - 3D + 6DE - 10DEF + 15DEFG - \text{etc.}}$$

Ponamus $T = 1 + \frac{3b}{U}$, vt fiat

$$U = \frac{1 - 3D + 6DE - 10DEF + 15DEFG - \text{etc.}}{1 - 4D + 10DE - 20DEF + 35DEFG - \text{etc.}}$$

vbi in denominatore reperiuntur numeri pyramidales primi siue summae trigonalium, hincque nanciscimur:

U =

$$U = 1 + \frac{D - 4DE + 10DEF - 20DEFG + \text{etc.}}{1 - 4D + 10DE - 20DEF + 35DEFG - \text{etc.}}$$

vbi iam supra et infra occurrunt numeri pyramidales. Statua-
tur porro $U = 1 + \frac{D}{V}$ fietque

$$V = \frac{1 - 4D + 10DE - 20DEF + 35DEFG - \text{etc.}}{1 - 4E + 10EF - 20EFG + 35EFGH - \text{etc.}}$$

§. 6. Hinc calculum vt supra profequendo, cum fit
 $E - D = b$, $F - D = 2b$, $G - D = 3b$, erit

$$V = 1 + \frac{4b - 2 \cdot 10bE + 3 \cdot 20bEF - 4 \cdot 35bEFG + \text{etc.}}{1 - 4E + 10EF - 20EFG + 35EFGH + \text{etc.}}$$

Sit $V = 1 + \frac{4b}{X}$, vt fiat

$$X = \frac{1 - 4E + 10EF - 20EFG + 35EFGH - \text{etc.}}{1 - 5E + 15EF - 35EFG + 70EFGH - \text{etc.}}$$

quae expressio reducitur ad hanc:

$$X = 1 + \frac{E - 5EF + 15EFG - 35EFGH + \text{etc.}}{1 - 5E + 15EF - 35EFG + \text{etc.}}$$

Sit $X = 1 + \frac{E}{Y}$ eritque

$$Y = \frac{1 - 5E + 15EF - 35EFG + 70EFGH - \text{etc.}}{1 - 5F + 15FG - 35FGH + 70FGHI - \text{etc.}}$$

§. 7. Cum igitur fit $F - E = b$, $G - E = 2b$, $H - E = 3b$, etc. erit

$$Y = 1 + \frac{5b - 2 \cdot 15b \cdot F + 3 \cdot 35bFG - 4 \cdot 70bFGH + \text{etc.}}{1 - 5F + 15FG - 35FGH + 70FGHI - \text{etc.}}$$

Sit nunc $Y = 1 + \frac{5b}{Z}$, vt fiat

$$Z = \frac{1 - 5F + 15FG - 35FGH + 70FGHI - \text{etc.}}{1 - 10F + 21FG - 56FGH + 126FGHI - \text{etc.}}$$

Cum igitur initio posuerimus $\frac{1}{S} = 1 + \frac{A}{P}$, erit summa quaesita

$S = \frac{1}{1 + \frac{A}{P}}$; tum vero factae sunt sequentes positiones:

$$P = 1 + \frac{b}{Q}, \quad Q = 1 + \frac{b}{R}, \quad R = 1 + \frac{2b}{S}, \quad S = 1 + \frac{c}{T}, \quad T = 1 + \frac{3b}{U},$$

$$U = 1 + \frac{D}{V}, \quad V = 1 + \frac{4b}{X}, \quad X = 1 + \frac{E}{Y}, \quad Y = 1 + \frac{5b}{Z}, \text{ etc.}$$

quibus

quibus valoribus ordine substitutis oritur ista fractio continua:

$$S = \frac{1}{1 + \frac{A}{1 + \frac{b}{1 + \frac{B}{1 + \frac{2b}{1 + \frac{C}{1 + \frac{3b}{1 + \frac{D}{1 + \frac{4b}{1 + \text{etc.}}}}}}}}}}$$

Quod si ergo loco litterarum A, B, C, D, etc. valores assumptos restituamus, ut nobis fit ista series diuergens:

$$1 - a + a(a+b) - a(a+b)(a+2b) + a(a+b)(a+2b)(a+3b) - \text{etc.}$$

eius summa exprimetur per sequentem fractionem continuam:

$$S = \frac{1}{1 + \frac{a}{1 + \frac{b}{1 + \frac{a+b}{1 + \frac{2b}{1 + \frac{a+2b}{1 + \frac{3b}{1 + \frac{a+3b}{1 + \frac{4b}{1 + \text{etc.}}}}}}}}}}$$

quae est eadem forma quam olim dederam.

§. 8. Haec transformatio eo magis est notatu digna, quod tutissimam ac fortasse vnicam nobis viam aperit, valorem seriei diuergentis vero proxime saltem determinandi. Si enim fractio continua more solito in fractiones simplices resoluatur $1, \frac{1}{1+a}, \frac{1+b}{1+a+b},$ etc. eae alternatim sunt maiores et minores quam valor seriei diuergentis, et continuo propius ad istum valorem accedunt. Tum vero etiam singularia olim exposui artificia, quae multo promptius ad verum valorem deducunt.

§. 9. Praeterea vero etiam notasse iuuabit, talem fractionem continuam:

$$1 + \frac{a}{1 + \frac{\beta}{1 + \frac{\gamma}{1 + \frac{\delta}{1 + \text{etc.}}}}}$$

in genere satis commode ad dimidium partium numerum redigi posse. Posito enim eius valore = S, eum ita repraesentare licebit:

$$S = 1 + \frac{a}{1 + \frac{\beta}{P}}, \quad P = 1 + \frac{\gamma}{1 + \frac{\delta}{Q}}, \quad Q = 1 + \frac{e}{1 + \frac{\zeta}{R}}, \text{ etc.}$$

Iam prima harum formularum erit

$$S = 1 + \frac{aP}{P + \beta} = 1 + a - \frac{a\beta}{\beta + P},$$

secunda deinde formula dat

$$P = 1 + \frac{\gamma Q}{Q + \delta} = 1 + \gamma - \frac{\gamma\delta}{\delta + Q},$$

eodem modo tertia praebet

$$Q = 1 + \frac{\varepsilon R}{R + \zeta} = 1 + \frac{\varepsilon - \varepsilon \zeta}{\zeta + R}, \text{ etc.}$$

Hi igitur valores successiue substituti, producent hanc nouam fractionem continuam:

$$S = 1 + \frac{a - a\beta}{1 + \beta + \frac{\gamma - \gamma\delta}{1 + \delta + \frac{\varepsilon - \varepsilon\zeta}{1 + \zeta + \frac{\eta - \eta\theta}{1 + \theta + \dots \text{etc.}}}}$$

§. 10. Cum igitur nostro casu series diuergens

$$S = 1 - a + a(a+b) - a(a+b)(a+2b) + a(a+b)(a+2b)(a+3b) - \text{etc.}$$

perducta fit ad istam fractionem continuam:

$$S = \frac{1}{1 + \frac{a}{1 + \frac{b}{1 + \frac{a+b}{1 + \frac{2b}{1 + \frac{a+2b}{1 + \frac{3b}{1 + \frac{a+3b}{1 + \dots \text{etc.}}}}}}}}}}$$

sumamus hic

$$a = a, \beta = b, \gamma = a+b, \delta = 2b, \varepsilon = a+2b, \text{ etc.}$$

eritque

$$S =$$

$$S = \frac{1+a-ab}{1+a+2b-2b(a+b)} = \frac{1+a-ab}{1+a+4b-3b(a+2b)} = \frac{1+a-ab}{1+a+6b-4b(a+3b)} = \frac{1+a-ab}{1+a+8b-5b(a+4b)} \text{ etc.}$$

Appendix.

De fractione continua Brouncheriana.

§. 11. Cum olim multum fuisset occupatus in Analyfi indaganda, quae *Brouncherum* ad istam singularem fractionem perduxerit, quandoquidem mihi haud probabile est visum, eum per tot ambages, quales a *Wallisio* commemorantur, eo fuisse perductum, tandem mihi quidem satis dilucide ostendisse sum visus, *Brouncherum* hanc formam ex serie *Leibniziana* $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \text{etc.}$ quam magnus *Gregorius*, iam ante inuenerat, deduxisse potius quam ex interpolatione seriei $1, \frac{1}{2}, \frac{1 \cdot 3}{2 \cdot 4}, \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}, \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8}, \text{etc.}$ quemadmodum *Wallisius* suspicabatur, si quidem consideratio illius seriei per ratiocinium satis planum ad formam *Brouncherianam* manuducit.

§. 12. Haec observatio autem nunc quidem eo maiore attentione digna videtur, postquam *Cel. Dan. Bernoullius* memoriam formae *Brouncherianae* renouare haud sit dedignatus. Quoniam igitur non ita pridem facilem methodum exposui istam formam ex serie $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \text{etc.}$ derivandi, Geometris haud ingratum fore arbitror, si methodum inuersam in medium protulero, cuius ope formulam *Brouncherianam* vicissim ad seriem *Leibnizianam* reducere licet.

§. 13. Considerabo igitur fractionem istam continuam quasi eius valor nondum esset cognitus, statuendo:

$$S = \frac{1}{1 + \frac{1}{2 + \frac{9}{2 + \frac{25}{2 + \frac{49}{2 + \frac{81}{2 + \text{etc.}}}}}}}}$$

quam per partes sequenti modo repraesento:

$$S = \frac{1}{1 + \frac{1}{-1 + P}}, \quad P = \frac{3 + 9}{-3 + Q}, \quad Q = \frac{5 + 25}{-5 + R}, \quad R = \frac{7 + 49}{-7 + S}, \quad \text{etc.}$$

Ex his enim partibus debite coniunctis ipsa forma proposita manifesto enascitur.

§. 14. Singulas igitur has partes seorsim evoluamus, ac prima quidem reducta ad fractionem simplicem praebet $S = \frac{P-1}{P}$, ideoque $S = 1 - \frac{1}{P}$, secunda vero erit $\frac{3Q}{Q-3}$, vnde fit $\frac{1}{P} = \frac{Q-3}{3Q}$, sive $\frac{1}{P} = \frac{1}{3} - \frac{1}{Q}$, simili modo pars tertia dat $Q = \frac{5R}{R-5}$, ideoque $\frac{1}{Q} = \frac{1}{5} - \frac{1}{R}$; eodem modo ex sequentibus partibus nanciscemur $\frac{1}{R} = \frac{1}{7} - \frac{1}{S}$, $\frac{1}{S} = \frac{1}{9} - \frac{1}{T}$, etc. Quare si isti valores successive substituantur, obtinebimus hanc expressionem:

$$S = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \text{etc.}$$

ita ita ut nunc certi simus esse $S = \frac{\pi}{4}$.

§. 15. Simili modo etiam aliarum huiusmodi fractionum continuarum valorem inuestigare licebit. Veluti si proposita fuerit

fuerit haec forma:

$$S = \frac{1}{1+1} + \frac{1}{1+4} + \frac{1}{1+9} + \frac{1}{1+16} + \dots + \frac{1}{1+n^2} + \dots$$

ea sequenti modo in membra distribuatur:

$$S = \frac{1}{1+1} - \frac{1}{-1+P} + \frac{1}{-2+Q} - \frac{1}{-3+R} + \frac{1}{-4+S} - \dots$$

his enim singulis partibus euolutis reperietur:

$$S = 1 - \frac{1}{P} + \frac{1}{Q} - \frac{1}{R} + \frac{1}{S} - \frac{1}{T} + \dots$$

unde fequitur fore

$$S = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots = \frac{1}{2}$$

Haec igitur methodus haud parum in recessu habere videtur.