

DE MIRIS PROPRIETATIBVS
CVRVAE ELASTICAE

sub aequatione $y = \int \frac{x \cdot x^{\frac{d}{2}}}{\sqrt{1-x^2}} dx$ contentae.

Auctore

L. E V L E R O.

§. I.

Tab. II. Fig. 1. Sit EGF lamina elastica, quae ope funiculi terminis E et F alligati incuruetur in curuam elasticam EGF, tum vero, si funiculus eo vsque constringatur, donec anguli in E et F fiant recti, ea curua elastica oritur, quae vocari solet rectangularia et in aequatione contenta $y = \int \frac{x \cdot x^{\frac{d}{2}}}{\sqrt{1-x^2}} dx$, cuius nonnullas proprietates prorsus singulares et admirandas hic sum commemoraturus.

Fig. 2. §. 2. Sit igitur CAC' talis curua elastica, rectae CC', quae funiculum refert, vtrinque normaliter insistens, et euidens est rectam AD, ad punctum medium D inter vtrumque terminum C et C' perpendiculariter ductam, fore curuae diametrum, et punctum A eius quasi verticem referre. Tum vero si ex C ad rectam CC' erigatur perpendicular CB, quod tanquam axem hic spectabimus, in eoque capiamus abscissam CP = x et vocemus applicatam PM = y, posita altitudine AD = AB = 1, erit,

vti constat, $dy = \frac{xx dx}{\sqrt{1-x^2}}$; vnde si arcus curuae CM pos-
natur $= s$, fiet $ds = \frac{dx}{\sqrt{1-x^2}}$; atque tam ex natura rei
quam ex hac aequatione intelligere licet totam hanc cur-
vam constare ex infinitis portionibus CA C', C'A'C'',
C''A''C''', etc. inter se similibus et aequalibus super rec-
ta CC''' vtrinque in infinitum producta constitutus, vnde
etiam tota haec curva infinitos habebit diametros AD,
A'D', A''D'', etc. totidemque vertices A, A', A'', A''',
etc. tam dextrorum quam sinistrorum. Puncta autem
C, C', C'', C''', etc., quoniam circa eorum singula cur-
va similiter alternatim protenditur, centra vocari poterunt.
Quemadmodum autem singularum harum portionum alti-
tudines AD, A'D', A''D'', etc. unitate designamus, ponam-
mus semilatinitudinem cuiusque portionis CD = AB = a;
ipso vero arcus CA = C'A = C'A' = etc. = c, et quo-
modo haec binæ quantitates a et c se ad unitatem seu al-
titudinem AD habent deinceps accuratius inuestigabimus.

§. 3. His de quantitatibus ad hanc curuam perti-
nentibus notatis quantitates variables PM = y et CM = s,
ad abscissam CP = x referamus, vnde statim patet, tam
y quam s fore functiones infinitiformes eiusdem abscissae
CP = x. Cum enim applicata PM vtrinque in infinitum
producta curuam fecet in infinitis punctis M, M', M'',
M''', etc. applicata y infinitos recipiet valores, scilicet PM,
PM', PM'', PM''', PM''''', etc qui ex principali PM = y
et quantitate constante AB = CD = a erunt PM = y;
PM' = 2a - y; PM'' = 4a + y; PM''' = 6a - y;
PM'''' = 8a + y; PM''''' = 10a - y;

qui omnes valores in his generalibus formis continentur:

$$4ia + y \text{ et } (4i+2)a - y,$$

vbi littera i omnes numeros integros tam positivos quam negatiuos denotare potest. Simili modo eidem abscissae $CP = x$ respondebunt infiniti arcus curuae, qui erunt

$$CM = s; CAM' = 2c - s; CAA'M'' = 4c + s;$$

$$CAA'A''M''' = 6c - s;$$

qui omnes etiam in his geminis formulis continentur:

$$4ic + s, (4i+2)c - s$$

sumendo pro i successiue omnes numeros tam positiuos quam negatiuos.

Tab. II.

Fig. 3.

§. 4. Sufficiet igitur solam huius curuae portionem CMA considerasse, quoniam reliquæ omnes ei sunt aequales, pro qua posuimus $CB = AD = 1$, $AB = CD = a$, et arcum $CMA = c$. Tum vero pro puncto indefinito M si vocentur coordinatae $CP = x$, $PM = y$ et arcus $CM = s$ erit

$$dy = \frac{x x d x}{\sqrt{(1-x^2)}} \text{ et } ds = \frac{d x}{\sqrt{(1-x^2)}}.$$

His positis ad curuam in M ducamus normalem MN basi CD productæ occurrentem in N . Hinc si ducatur ad basin perpendicularum $MQ = x$, ob $CQ = y$ erit interuallum $QN = \frac{x d x}{dy} = \frac{\sqrt{(1-x^2)}}{x}$ et ipsa normalis $MN = \frac{x d s}{dy} = \frac{s}{x}$, ita ut rectangulum $MQ \cdot MN$ sit $= 1 = AD^2$. Hinc si vocetur angulus $CNM = \Phi$, qui metitur amplitudinem arcus CM , erit sin. $\Phi = x x$,

$$\cos. \Phi = \sqrt{(1-x^2)} \text{ et } \tan. \Phi = \frac{x x}{\sqrt{(1-x^2)}}.$$

§. 5.

§. 5. Quaeramus nunc etiam radium osculi curvae in puncto M, qui sit MO, hunc in finem faciamus $\frac{dy}{dx} = p = \frac{xx}{\sqrt{1-x^4}}$, unde fit $\sqrt{1+p^2} = \frac{1}{\sqrt{1-x^4}}$; hinc porro fiat $\frac{p}{\sqrt{1+p^2}} = xx = q$, eritque uti constat radius osculi $= \frac{dx}{dq} = \frac{1}{2x}$; sicque erit $MO = \frac{1}{2x}$, ideoque $MO = \frac{1}{2} MN$, ita ut centrum curvaturae cadat in punctum medium normalis MN; ex quo patet radium osculi MO reciproce esse proportionalem intervallo MQ = x, quae est proprietas, quam natura elasticæ postulat. Cum enim vis laminam in puncto C tendens directionem habeat MN, eius momentum respectu puncti M erit vi multiplicatae per QM = x aequale, cui per naturam elasticitatis radius osculi in M reciproce debet esse proportionalis. Manifestum igitur est radium osculi in ipso puncto C esse infinitum; in altero autem termino A = $\frac{1}{2} = \frac{1}{2} AD$; sicque in hoc puncto A curvatura erit maxima.

§. 6. Nunc etiam videamini, quomodo ex data abscissa CP = x tam applicata PM = y, quam ipse arcus CM = s proxime per series infinitas exprimi queat, id quod dupli modo praestari potest. Prior maxime obvius in eo consistit ut formula $\frac{1}{\sqrt{1-x^4}} = (1-x^4)^{-\frac{1}{2}}$ in seriem resoluatur, quae erit:

$$1 + \frac{1}{2}x^4 + \frac{1}{2} \cdot \frac{3}{4}x^8 + \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6}x^{12} + \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \frac{7}{8}x^{16} + \text{etc.}$$

vnde per integrationem colligitur:

$$PM = y = \frac{1}{3}x^3 + \frac{1}{4} \cdot \frac{1}{2}x^7 + \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{1}{12}x^{11} + \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \frac{1}{16}x^{15} \text{ etc.}$$

tum vero etiam arcus:

$$CM = s = x + \frac{1}{2} \cdot \frac{1}{3}x^5 + \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{1}{5}x^9 + \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \frac{1}{9}x^{13} + \text{etc.}$$

E 3

Hinc

Hinc igitur patet, si abscissa x fuerit valde parua, tum fore proxime $y = \frac{1}{3}x^3$ et $s = x$. Verum si capiamus $x = 1$, per series ambae quantitates a et c ita exprimentur, vt sit

$$a = \frac{1}{3} + \frac{1}{2} \cdot \frac{1}{7} + \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{1}{11} + \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \frac{1}{15} + \text{etc.}$$

$$c = 1 + \frac{1}{2} \cdot \frac{1}{5} + \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{1}{9} + \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \frac{1}{13} + \text{etc.}$$

Hae autem series nimis lente conuergunt, quam vt inde valores litterarum a et c satis exacte definiri queant.

§. 7. Alter modus non adeo obuius in eo consistit, vt statuatur

$$y = \int \frac{x \, dx \, d \, x}{\sqrt{(1 - x^4)}} = u \sqrt{(1 - x^4)},$$

sumtis igitur differentialibus erit

$$x \, x \, d \, x = d \, u \, (1 - x^4) - 2 \, u \, x^3 \, d \, x, \text{ siue}$$

$$\frac{d \, u}{d \, x} (1 - x^4) - 2 \, u \, x^3 - x \, x = 0.$$

Fingatur nunc ista series:

$$u = \alpha x^5 + \beta x^7 + \gamma x^{11} + \delta x^{15} + \varepsilon x^{19} + \text{etc.}$$

quandoquidem iam nouimus, si x fuerit valde paruum, fieri debere $y = \frac{1}{3}x^3$, ideoque etiam $u = \frac{1}{3}x^3$; deinde ex forma aequationis manifestum est, in serie exponentes ipsius x continuo quaternario crescere debere. Hac igitur serie substituta fiat sequens euolutio:

$$\frac{d \, u}{d \, x} = 3 \alpha x^5 + 7 \beta x^7 + 11 \gamma x^{11} + 15 \delta x^{15} + 19 \varepsilon x^{19} + \text{etc.}$$

$$-\frac{x^4 \, du}{dx} = -3 \alpha x^6 - 7 \beta x^8 - 11 \gamma x^{12} - 15 \delta x^{16} - \text{etc.}$$

$$-2 \, u \, x^3 = -2 \alpha x^6 - 2 \beta x^8 - 2 \gamma x^{12} - 2 \delta x^{16} - \text{etc.}$$

$$-x \, x = -x \, x$$

Singulis igitur membris ad nihilum redactis fiet

$\alpha =$

39 (३९)

$\alpha = \frac{1}{3}; \beta = \frac{1.5}{3.7}; \gamma = \frac{1.5.9}{3.7.11}; \delta = \frac{1.5.9.13}{3.7.11.13}; \text{ etc.}$
quamobrem habebimus:

$$y = \left(\frac{1}{3}x^3 + \frac{1.5}{3.7}x^7 + \frac{1.5.9}{3.7.11}x^{11} + \frac{1.5.9.13}{3.7.11.13}x^{15} + \text{etc.} \right) \sqrt[3]{(1-x^4)}.$$

§. 8. Simili modo si statuamus

$$s = \int \frac{dx}{\sqrt[3]{(1-x^4)}} = v \sqrt[3]{(1-x^4)},$$

peruenietur ad hanc aequationem:

$$\frac{dv}{dx}(1-x^4) - 2v x^3 - 1 = 0,$$

vbi iam statuamus

$$v = \alpha x + \beta x^5 + \gamma x^9 + \delta x^{13} + \epsilon x^{17} + \zeta x^{21} + \text{etc.}$$

cuius evolutio ita reprezentetur:

$$\frac{dv}{dx} = \alpha + 5\beta x^4 + 9\gamma x^8 + 13\delta x^{12} + 17\epsilon x^{16} + \text{etc.}$$

$$-\frac{x^4 dv}{dx} = -\alpha - 5\beta - 9\gamma - 13\delta - \text{etc.}$$

$$-2v x^3 = -2\alpha - 2\beta - 2\gamma - 2\delta - \text{etc.}$$

$$-1 = -1$$

Hinc reperiuntur coefficientes

$$\alpha = 1; \beta = \frac{1}{3}; \gamma = \frac{5.7}{5.9}; \delta = \frac{5.7.11}{5.9.13}; \epsilon = \frac{5.7.11.13}{5.9.13.17} \text{ etc.}$$

vnde colligitur fore

$$s = (x + \frac{1}{3}x^5 + \frac{5.7}{5.9}x^9 + \frac{5.7.11}{5.9.13}x^{13} + \text{etc.}) \sqrt[3]{(1-x^4)}.$$

His autem seriebus plane ad valores litterarum α et ϵ e-
ruendos vti non licet: facto enim $x=1$ formula $\sqrt[3]{(1-x^4)}$
euanscit; tum autem ipsae series in infinitum ex crescunt.

§. 9. Pro litteris autem α et ϵ cognoscendis alias
adhiberi conueniet methodos inde petendas, quod integra-
lia harum formularum: $\int \frac{dx}{\sqrt[3]{(1-x^4)}}$ et $\int \frac{dx}{\sqrt[3]{(1-x^4)}}$, pro eo tan-

tum

tum casu quaeruntur, quo post integrationem fit $x = r$.
Hunc in finem formula $\frac{1}{\sqrt{1-x^2}}$ ita repreſentetur:

$\frac{(1+x^2)^{-\frac{1}{2}}}{\sqrt{1-x^2}}$ et numerat̄or $(1+x^2)^{-\frac{1}{2}}$ in ſeriem convergat̄ur, quae erit

$$1 - \frac{1}{2}x^2 + \frac{1}{2} \cdot \frac{5}{4}x^4 - \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6}x^6 + \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \frac{7}{8}x^8 - \text{etc.}$$

ita vt loco $\frac{1}{\sqrt{1-x^2}}$ ſcripti ſimus hanc ſeriem:

$$\frac{1}{\sqrt{1-x^2}} (1 - \frac{1}{2}x^2 + \frac{1}{2} \cdot \frac{5}{4}x^4 - \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6}x^6 + \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \frac{7}{8}x^8 - \text{etc.})$$

quo facto tam pro y quam pro s ſequentes formulae integrandae occurſent:

$$\int \frac{dx}{\sqrt{1-x^2}}, \int \frac{x dx}{\sqrt{1-x^2}}, \int \frac{x^2 dx}{\sqrt{1-x^2}}, \text{etc.}$$

§. 10. Harum autem formularum integralia hic non in genere requiruntur, sed tantum pro caſu quo post integrationem ponitur $x = r$. Hoc autem caſu no- vimus, si $r : \pi$ denotet rationem diametri ad peripheriam, eſſe

$$\int \frac{dx}{\sqrt{1-x^2}} = \frac{\pi}{2}, \quad \int \frac{x dx}{\sqrt{1-x^2}} = \frac{r}{2} \cdot \frac{\pi}{2},$$

$$\int \frac{x^2 dx}{\sqrt{1-x^2}} = \frac{r}{2} \cdot \frac{3}{4} \cdot \frac{\pi}{2}, \quad \int \frac{x^6 dx}{\sqrt{1-x^2}} = \frac{r}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \frac{\pi}{2},$$

et ita porro, quibus valoribus substitutis primo ex formula

$$s = \int \frac{x^2 dx}{\sqrt{1-x^2}} = \int \frac{x^2 dx}{\sqrt{1-x^2}} (1+x^2)^{-\frac{1}{2}},$$

colligimus fore

$$a = \frac{\pi}{2} \left(1 - \frac{1}{2^2} \cdot \frac{3}{4} + \frac{1}{2^2} \cdot \frac{5}{4^2} \cdot \frac{5}{6} - \frac{1}{2^2} \cdot \frac{3}{4^2} \cdot \frac{5}{6^2} \cdot \frac{7}{8} + \text{etc.} \right)$$

ex altera autem formula $s = \int \frac{dx}{\sqrt{1-x^2}}$, colligitur longitu- do totius arcus

CA =

$$CA = c - \frac{\pi}{2} \left(1 - \frac{1^2}{2^2} + \frac{1^2}{2^2} \cdot \frac{3^2}{4^2} - \frac{1^2}{2^2} \cdot \frac{3^2}{4^2} \cdot \frac{5^2}{6^2} + \frac{1^2}{2^2} \cdot \frac{3^2}{4^2} \cdot \frac{5^2}{6^2} \cdot \frac{7^2}{8^2} - \dots \right)$$

Verum etiam hae series non satis sunt aptae pro veris valoribus quantitatum a et c cognoscendis.

§. 11. Superest autem adhuc alia methodus eosdem valores per producta ex infinitis factoribus exprimendi, cuius rationem, quamquam a me iam dudum fusius est explicata, hic sequenti modo succincte exponam. Consideretur haec formula: $z = x^n V(1-x^4)$, et cum sit

$$dz = nx^{n-1} V(1-x^4) - \frac{2x^{n+3} dx}{V(1-x^4)} = \frac{nx^{n-1} dx - (n+2)x^{n+3} dx}{V(1-x^4)},$$

hinc vicissim integrando erit

$$x^n V(1-x^4) = n \int \frac{x^{n-1} dx}{V(1-x^4)} - (n+2) \int \frac{x^{n+3} dx}{V(1-x^4)},$$

quare si haec integralia tantum desiderentur pro casu $x=1$, sicut

$$\int \frac{x^{n-1} dx}{V(1-x^4)} = \frac{n+2}{n} \int \frac{x^{n+3} dx}{V(1-x^4)}.$$

Simili modo erit

$$\int \frac{x^{n+3} dx}{V(1-x^4)} = \frac{n+6}{n+4} \int \frac{x^{n+7} dx}{V(1-x^4)} \text{ et}$$

$$\int \frac{x^{n+7} dx}{V(1-x^4)} = \frac{n+10}{n+8} \int \frac{x^{n+11} dx}{V(1-x^4)} \text{ etc.}$$

Quod si ergo hoc modo in infinitum ascendamus, erit

$$\int \frac{x^{n-1} dx}{V(1-x^4)} = \frac{n+2}{n} \cdot \frac{n+6}{n+4} \cdot \frac{n+10}{n+8} \cdot \frac{n+14}{n+12} \cdots \int \frac{x^{n+\infty} dx}{V(1-x^4)}.$$

§. 12. Substituamus nunc successiue pro n numeros 1, 2, 3, 4, ac prodibunt sequentes quatuor reductiones ad producta infinita, casu scilicet $x = 1$.

$$\text{I. } \int \frac{dx}{\sqrt{1-x^4}} = \frac{\pi}{2} \cdot \frac{7}{5} \cdot \frac{11}{9} \cdot \frac{15}{13} \cdot \frac{19}{17} \cdots \cdots \int \frac{x^{1+\infty} dx}{\sqrt{1-x^4}} = c.$$

$$\text{II. } \int \frac{xdx}{\sqrt{1-x^4}} = \frac{\pi}{2} \cdot \frac{8}{6} \cdot \frac{12}{10} \cdot \frac{16}{14} \cdot \frac{20}{18} \cdots \cdots \int \frac{x^{2+\infty} dx}{\sqrt{1-x^4}} = \frac{\pi}{4}.$$

$$\text{III. } \int \frac{x^2 dx}{\sqrt{1-x^4}} = \frac{5}{3} \cdot \frac{9}{7} \cdot \frac{13}{11} \cdot \frac{17}{15} \cdot \frac{21}{19} \cdots \cdots \int \frac{x^{3+\infty} dx}{\sqrt{1-x^4}} = a.$$

$$\text{IV. } \int \frac{x^3 dx}{\sqrt{1-x^4}} = \frac{6}{4} \cdot \frac{10}{8} \cdot \frac{14}{12} \cdot \frac{18}{16} \cdot \frac{22}{20} \cdots \cdots \int \frac{x^{4+\infty} dx}{\sqrt{1-x^4}} = \frac{\pi}{8}.$$

§. 13. Hic iam probe notandum est postremas formulas integrales inter se omnes esse aequales. Cum enim in genere sit

$$\int \frac{x^{n-1} dx}{\sqrt{1-x^4}} = \frac{n+2}{n} \int \frac{x^{n+3} dx}{\sqrt{1-x^4}}$$

sumto $n = \infty$ erit

$$\int \frac{x^{\infty-1} dx}{\sqrt{1-x^4}} = \int \frac{x^{\infty+3} dx}{\sqrt{1-x^4}}.$$

Quod si ergo harum quatuor formularum quamlibet per aliam diuidamus, postremi factores integrales se mutuo tollunt eritque

$$\frac{\text{I.}}{\text{II.}} = \frac{4c}{\pi} = \frac{2 \cdot 3}{1 \cdot 4} \cdot \frac{6 \cdot 7}{5 \cdot 8} \cdot \frac{10 \cdot 11}{9 \cdot 12} \cdot \frac{14 \cdot 15}{13 \cdot 16} \cdot \frac{18 \cdot 19}{17 \cdot 20} \cdot \text{etc.}$$

$$\frac{\text{I.}}{\text{III.}} = \frac{c}{a} = \frac{3 \cdot 3}{1 \cdot 5} \cdot \frac{7 \cdot 7}{5 \cdot 9} \cdot \frac{11 \cdot 11}{9 \cdot 13} \cdot \frac{15 \cdot 15}{13 \cdot 17} \cdot \frac{19 \cdot 19}{17 \cdot 21} \cdot \text{etc.}$$

$$\frac{\text{I.}}{\text{IV.}} = 2c = \frac{3 \cdot 4}{1 \cdot 6} \cdot \frac{7 \cdot 8}{5 \cdot 10} \cdot \frac{11 \cdot 12}{9 \cdot 14} \cdot \frac{15 \cdot 16}{13 \cdot 18} \cdot \frac{19 \cdot 20}{17 \cdot 22} \cdot \text{etc.}$$

$$\begin{aligned} \text{I. } &= \frac{\pi}{4a} = \frac{5.4}{2.5} \cdot \frac{7.9}{6.9} \cdot \frac{11.12}{10.13} \cdot \frac{15.16}{14.17} \cdot \frac{19.20}{18.21} \cdot \text{etc.} \\ \text{II. } &= \frac{\pi}{2} = \frac{4.4}{2.6} \cdot \frac{8.8}{6.10} \cdot \frac{12.12}{10.14} \cdot \frac{16.16}{14.18} \cdot \frac{20.20}{18.22} \cdot \text{etc.} \\ \text{III. } &= 2a = \frac{4.5}{3.6} \cdot \frac{8.9}{7.10} \cdot \frac{12.13}{11.14} \cdot \frac{16.17}{15.18} \cdot \frac{20.21}{19.22} \cdot \text{etc.} \\ \text{IV. } &= \end{aligned}$$

§. 14. Hae iam expressiones multo sunt aptiores ad veros valores litterarum a et c proxime definiendos. Pro valore autem ipsius c inueniendo formula $\frac{I}{II}$ maxime videtur idonea, unde fit

$$\begin{aligned} \frac{4c}{\pi} &= \frac{7.3}{2.1} \cdot \frac{3.7}{4.5} \cdot \frac{5.11}{6.9} \cdot \frac{7.15}{8.13} \cdot \frac{9.19}{10.17} \cdot \text{etc. siue} \\ \frac{4c}{\pi} &= \frac{3}{2} \cdot \frac{21}{20} \cdot \frac{55}{54} \cdot \frac{105}{104} \cdot \frac{171}{170} \cdot \text{etc.} \end{aligned}$$

quae pro facilitiori calculo ita potest exhiberi:

$$\frac{4c}{\pi} = (1 + \frac{1}{2})(1 + \frac{1}{20})(1 + \frac{1}{54})(1 + \frac{1}{104})(1 + \frac{1}{170}) \text{ etc.}$$

At vero quantitas a commodissime definitur siue ex hac forma:

$$\begin{aligned} \frac{\pi}{4a} &= \frac{2.3}{1.5} \cdot \frac{4.7}{3.9} \cdot \frac{6.11}{5.13} \cdot \frac{8.15}{7.17} \cdot \frac{10.19}{9.21} \cdot \text{etc. siue} \\ \frac{\pi}{4a} &= \frac{6}{5} \cdot \frac{28}{27} \cdot \frac{66}{65} \cdot \frac{120}{119} \cdot \frac{190}{189} \cdot \text{etc.} \end{aligned}$$

quae commodi ergo ita repraesentetur:

$$\frac{\pi}{4a} = (1 + \frac{1}{1.5})(1 + \frac{1}{2.9})(1 + \frac{1}{5.13})(1 + \frac{1}{7.17})(1 + \frac{1}{9.21}) \text{ etc.}$$

vel etiam pari successu definietur quantitas a ex formula $\frac{III}{IV}$, quae dat

$$2a = \frac{2.5}{3.2} \cdot \frac{4.9}{5.7} \cdot \frac{6.13}{7.11} \cdot \frac{8.17}{9.15} \cdot \frac{10.21}{11.19} \cdot \text{etc. siue}$$

$$2a = \frac{10}{9} \cdot \frac{36}{35} \cdot \frac{78}{77} \cdot \frac{136}{135} \cdot \frac{210}{209} \cdot \text{etc. siue}$$

$$2a = (1 + \frac{1}{5.5})(1 + \frac{1}{6.7})(1 + \frac{1}{7.11})(1 + \frac{1}{9.16})(1 + \frac{1}{11.19}) \text{ etc.}$$

Interim ramen satis taedioso calculo opus foret, si valores harum litterarum usque ad partem millionesimam

vnitatis iustas exquirere vellemus: verum infra, cum propriales magis absconditas huius curvae detexerimus, satis prompte hos valores exhibere licebit.

§. 15. At vero pro eodem scopo series pro a et ϵ supra §. 10. inventae optimo cum successu usurpari possunt, quanquam ipsi termini parum decrescent, propterea quod in istis seriebus signa + et - alternantur. Hinc enim insigne subsidium nascitur ad summas harum serierum proxime inueniendas. Si enim habeatur huiusmodi series:

$$A - A' + A'' - A''' + A'''' - A''''' \text{ etc.}$$

cuius termini A, A', A'', A'''' continuo fiant minores, tum inde formetur series differentiarum

$$A - A' = B, A' - A'' = B', A'' - A''' = B'' \text{ etc.}$$

hincque porro series differentiarum secundarum

$$B - B' = C, B' - B'' = C', B'' - B''' = C'' \text{ etc.}$$

sicque hoc modo continuo differentiae capiantur, tum summa seriei propositae semper erit

$$\frac{A}{2} + \frac{B}{4} + \frac{C}{8} + \frac{D}{16} + \frac{E}{32} + \text{ etc.}$$

§. 16. Quo nunc hanc regulam ad series §. 10. applicemus, euoluamus in fractionibus decimalibus singulos terminos qui ibi occurront.

$\frac{1}{2}$	$= 0, 500000$	$\frac{1^2}{2^2} \cdot \frac{3^2}{4^2} \cdot \frac{5^2}{6^2} \cdot \frac{7}{8}$	$= 0, 085450$
$\frac{3^2}{2^2}$	$= 0, 250000$	$\frac{1^2}{2^2} \cdot \frac{3^2}{4^2} \cdot \frac{5^2}{6^2} \cdot \frac{7^2}{8^2}$	$= 0, 074769$
$\frac{1^2}{2^2} \cdot \frac{3}{4}$	$= 0, 187500$	$\frac{1^2}{2^2} \cdot \frac{3^2}{4^2} \cdot \frac{5^2}{6^2} \cdot \frac{7^2}{8^2} \cdot \frac{9}{10}$	$= 0, 067292$
$\frac{1^2}{2^2} \cdot \frac{3^2}{4^2}$	$= 0, 140625$	$\frac{1^2}{2^2} \cdot \frac{3^2}{4^2} \cdot \frac{5^2}{6^2} \cdot \frac{7^2}{8^2} \cdot \frac{9^2}{10^2}$	$= 0, 060563$
$\frac{1^2}{2^2} \cdot \frac{3^2}{4^2} \cdot \frac{5}{6}$	$= 0, 117188$	$\frac{1^2}{2^2} \cdot \frac{3^2}{4^2} \cdot \frac{5^2}{6^2} \cdot \frac{7^2}{8^2} \cdot \frac{9^2}{10^2} \cdot \frac{11}{12}$	$= 0, 055516$
$\frac{1^2}{2^2} \cdot \frac{3^2}{4^2} \cdot \frac{5^2}{6^2}$	$= 0, 097657$	$\frac{1^2}{2^2} \cdot \frac{3^2}{4^2} \cdot \frac{5^2}{6^2} \cdot \frac{7^2}{8^2} \cdot \frac{9^2}{10^2} \cdot \frac{11^2}{12^2}$	$= 0, 050890$

45 (5000)

$\frac{1^2}{2^2} \cdot \frac{3^2}{4^2} \cdot \frac{5^2}{6^2} \cdot \frac{7^2}{8^2} \cdot \frac{9^2}{10^2} \cdot \frac{11^2}{12^2} \cdot \frac{13}{14}$								$= 0,047255$
$\frac{1^2}{2^2} \cdot \frac{13^2}{14^2}$								$= 0,043880$
$\frac{1^2}{2^2} \cdot \frac{13^2}{14^2} \cdot \frac{15}{16}$								$= 0,041138$
$\frac{1^2}{2^2} \cdot \frac{15^2}{16^2}$								$= 0,038567$
$\frac{1^2}{2^2} \cdot \frac{15^2}{16^2} \cdot \frac{17}{18}$								$= 0,036424$
$\frac{1^2}{2^2} \cdot \frac{17^2}{18^2}$								$= 0,034400$
$\frac{1^2}{2^2} \cdot \frac{17^2}{18^2} \cdot \frac{19}{20}$								$= 0,032700$
$\frac{1^2}{2^2} \cdot \frac{19^2}{20^2}$								$= 0,031065$

§. 17. His praeparatis calculum instituamus pro valore litterae c inueniendo, et cum effe-

$$\frac{z^c}{\pi} = 1 - \frac{1^2}{2^2} + \frac{1^2}{2^2} \cdot \frac{3^2}{4^2} - \frac{1^2}{2^2} \cdot \frac{3^2}{4^2} \cdot \frac{5^2}{6^2} + \text{etc.}$$

binis primis terminis ad finitram translatis erit

$$\frac{z^c}{\pi} = \frac{3}{4} - \frac{1^2}{2^2} \cdot \frac{3^2}{4^2} - \frac{1^2}{2^2} \cdot \frac{3^2}{4^2} \cdot \frac{5^2}{6^2} + \frac{1^2}{2^2} \cdot \frac{3^2}{4^2} \cdot \frac{5^2}{6^2} \cdot \frac{7^2}{8^2} \text{ etc.}$$

Nunc singuli huius seriei termini sibi inuicem subscriban-
tur iisque subiungantur series differentiarum litteris B, C,
D etc. insignitarum hoc modo:

A	B	C	D
0, 140025			
0, 097657	0, 042968		
0, 074769	0, 022888	0, 020080	
0, 060563	0, 014206	0, 008682	0, 011398
0, 050890	0, 009673	0, 004533	0, 004149
0, 043880	0, 007010	0, 002663	0, 001870
0, 038567	0, 005313	0, 001697	0, 000966
0, 034400	0, 004107	0, 001146	0, 000551
0, 031065	0, 003335	0, 000832	0, 000314

F 3

E

E	F	G	H
०, ००७२४९	०, ००४९७०	०, ००३५९५	
०, ००२२७९	०, ००१३७५	०, ०००८८६	०, ००२७०९
०, ०००९०४	०, ०००४८९	०, ०००३११	०, ०००५७५
०, ०००४१५	०, ०००१७८		
०, ०००२३७			

§. 18. Hinc igitur summa nostrae seriei sequenti modo colligetur:

$$\begin{array}{ll}
 \frac{1}{2} A = 0, 070312 & 0, 084503 \\
 \frac{1}{4} B = 0, 010742 & \frac{1}{54} F = 0, 000078 \\
 \frac{1}{8} C = 0, 002510 & \frac{1}{128} G = 0, 000028 \\
 \frac{1}{16} D = 0, 000712 & \frac{1}{256} H = 0, 000011 \\
 \frac{1}{32} E = 0, 000227 & \text{pro reliquis } 000007 \\
 \hline
 & 0, 084503 \\
 & 0, 084627 \\
 & \text{adde } \frac{1}{4} = 0, 750000 \\
 & \hline
 & \text{erit } \frac{\pi}{4} = 0, 834627
 \end{array}$$

Hinc ergo erit $c = \pi \cdot 0, 417314 = 1, 311031$.

§. 19. Simili modo computabitur interuallum AB
 $= CD = a$. Erat autem

$$\frac{2a}{\pi} = \frac{1}{2} - \frac{1^2}{2^2} \cdot \frac{5}{4} + \frac{1^2}{2^2} \cdot \frac{3^2}{4^2} \cdot \frac{5}{6} - \frac{1^2}{2^2} \cdot \frac{3^2}{4^2} \cdot \frac{5^2}{6^2} \cdot \frac{7}{8} + \text{etc.}$$

vbi bini primi termini

$$\frac{1}{2} - \frac{5}{16} = \frac{5}{16} = 0, 312500$$

dant ad alteram partem translati

$$\frac{2a}{\pi} = 0, 312500 = \frac{1^2}{2^2} \cdot \frac{3^2}{4^2} \cdot \frac{5}{6} - \frac{1^2}{2^2} \cdot \frac{3^2}{4^2} \cdot \frac{5^2}{6^2} \cdot \frac{7}{8} + \text{etc.}$$

vnde calculus sequenti modo expediatur:

A

47 (8:34)

A	B	C	D
0, 117188	0, 031738	0, 013580	
0, 085450	0, 018158	0, 006382	0, 007198
0, 067292	0, 011776	0, 003515	0, 002867
0, 055516	0, 008261	0, 002144	0, 001371
0, 047255	0, 00611-	0, 001403	0, 000741
0, 041138	0, 004714	0, 000990	0, 000413
0, 036424	0, 003724		
0, 032700			

E	F	G	H
0, 004331	0, 002835	0, 001969	
0, 001496	0, 000866	0, 000564	0, 001405
0, 000630	0, 000302		
0, 000328			

§. 20. Hinc igitur seriei summa colligitur

$$\begin{array}{ll}
 \frac{1}{2} A = 0, 058594 & 0, 068810 \\
 \frac{1}{2} B = 0, 007934 & \frac{1}{2} F = 0, 000044 \\
 \frac{1}{2} C = 0, 001697 & \frac{1}{2} G = 0, 000015 \\
 \frac{1}{2} D = 0, 000450 & \frac{1}{2} H = 0, 000005 \\
 \frac{1}{2} E = 0, 000135 & \\
 \\
 0, 068810 & 0, 068874 \\
 \text{adde } \frac{1}{2} = 0, 312500 & \\
 \end{array}$$

$$\text{et prodit } \frac{2a}{\pi} = 0, 381374$$

hinc ergo $a = \pi \cdot 0, 190687 = 0, 599061$.

§. 20. His valoribus quantitatum a et c proxime
veris inuentis, quos autem deinceps adhuc accuratius defi-
nire docebo, progredior ad illas proprietates huius curvae
magis

magis abstrusas, quas sum pollicitus demonstrandas, quippe quas per solitas calculi operationes vix ac ne vix quidem eruere licet, et quae propterea profundioris indaginis merito sunt censendae. Ac primo quidem hic eam insignem relationem, quae inter ternas principales dimensiones huius curvae, scilicet altitudinem $B C = A D$, et inter latitudinem $A B = C D$ atque ipsam curvae longitudinem $A M C$ intercedit, et quam iam pridem detexi, hic accuratius exponam, et sequenti Theoremate complectar.

Theorema I.

§. 21. In curva elastica rectangula $A M C$, cuius vertex est A et centrum alternationis C , ternae dimensiones principales, quae sunt: 1) altitudo $B C = A D$; 2) latitudo $A B = C D$; ac 3) longitudo arcus $A M C$, ita a te inuicem pendent, vt rectangulum ex latitudine $A B$ in longitudinem arcus $A M C$ aequale sit areae circuli circa diametrum altitudinis $B C$ descripti, sive positis vt fecimus $B C = A D = r$, $A B = C D = a$ et arcu $A M C = \sigma$ erit $A C = \frac{\pi}{4}$.

Demonstratio.

§. 22. Insignis ista proprietas deducitur ex formulae quas supra per producta in infinitum excurrentia expressimus (§. 13.) quarum prima dabat

$$\frac{a}{\pi} = \frac{2 \cdot 3}{1 \cdot 4} \cdot \frac{6 \cdot 7}{5 \cdot 8} \cdot \frac{10 \cdot 11}{9 \cdot 12} \cdot \frac{14 \cdot 15}{13 \cdot 16} \text{ etc } \text{ ultima vero}$$

$$2a = \frac{4 \cdot 5}{3 \cdot 6} \cdot \frac{8 \cdot 9}{7 \cdot 10} \cdot \frac{12 \cdot 13}{11 \cdot 14} \cdot \frac{16 \cdot 17}{15 \cdot 18} \text{ etc.}$$

Quod si iam in priore expressione primus factor simplex $\frac{2}{\pi}$ seorsim exhibeat, ex reliquis autem sequentibus binis inter

inter se combinentur habebitur:

$$\frac{4c}{\pi} = \frac{2}{1} \cdot \frac{3 \cdot 5}{4 \cdot 3} \cdot \frac{7 \cdot 10}{8 \cdot 9} \cdot \frac{11 \cdot 14}{12 \cdot 13} \cdot \frac{15 \cdot 18}{16 \cdot 17}, \text{ etc.}$$

Quod si ergo haec expressio per alteram multiplicetur, omnes factores praeter primum manifesto se mutuo tollunt, ita ut proditum sit $\frac{8ac}{\pi} = 2$, vnde fit $ac = \frac{\pi}{4}$, quae est ipsa illa proprietas quam demonstrare oportebat.

§. 23. Etsi haec veritas modo prorsus singulari ex contemplatione infiniti est conclusa: tamen deinceps obseruauit, eandem quoque per operationes calculi magis consuetas elici posse. Quaeramus enim in genere pro quoquis curvae puncto indefinito M productum ex applicata PM = y et arcu CM = s, sitque hoc productum P = ys, erit $dP = y dx + s dy$, hincque iterum integrando $P = \int y ds + \int s dy$, quas ambas formulas seorsim euoluamus. Pro priori initio ostendimus esse

$$y = \frac{1}{3}x^3 + \frac{1 \cdot 3}{2 \cdot 7}x^7 + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 11}x^{11} + \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 15}x^{15} + \text{etc.}$$

quae series ducta in $ds = \frac{dx}{\sqrt{1-x^4}}$, per singulos terminos ita integretur, ut post integrationem statuatur $x=1$, quippe in quo versatur casus nostri theorematis.

§. 24 Pro hac autem investigatione habebimus

$$\int \frac{x^n dx}{\sqrt{1-x^4}} = \frac{1}{2} - \frac{1}{2} \sqrt{(1-x^4)} = \frac{1}{2},$$

posito $x=1$; tum vero in genere vidimus esse (§. 11.)

$$\int \frac{x^n + s dx}{\sqrt{1-x^4}} = \frac{n}{n+2} \int \frac{x^{n-1} dx}{\sqrt{1-x^4}},$$

vnde deducimus

$$\int \frac{x^7 dx}{\sqrt{1-x^4}} = \frac{2}{3} \cdot \frac{1}{2}$$

$$\int \frac{x^{11} dx}{\sqrt{1-x^4}} = \frac{8}{10} \cdot \frac{4}{5} \cdot \frac{1}{2} = \frac{2}{3} \cdot \frac{4}{5} \cdot \frac{1}{2}$$

$$\int \frac{x^{15} dx}{\sqrt{1-x^4}} = \frac{12}{14} \cdot \frac{8}{10} \cdot \frac{4}{5} \cdot \frac{1}{2} = \frac{2}{3} \cdot \frac{4}{5} \cdot \frac{6}{7} \cdot \frac{1}{2}$$

Hinc igitur pro nostro casu, quo $x=1$, erit

$$\begin{aligned} \int y ds &= \frac{1}{3} \cdot \frac{1}{2} + \frac{1}{5} \cdot \frac{2}{3} \cdot \frac{1}{2} + \frac{1}{3} \cdot \frac{2}{4} \cdot \frac{1}{5} \cdot \frac{2}{3} \cdot \frac{4}{5} \cdot \frac{1}{2} \\ &+ \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{6}{5} \cdot \frac{1}{6} \cdot \frac{2}{3} \cdot \frac{4}{7} \cdot \frac{6}{5} \cdot \frac{1}{8} + \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \frac{7}{8} \cdot \frac{1}{9} \cdot \frac{2}{3} \cdot \frac{4}{5} \cdot \frac{6}{7} \cdot \frac{8}{9} \cdot \frac{1}{2}, \text{ etc.} \end{aligned}$$

quae series reducitur ad sequentem formam:

$$\int y ds = \frac{1}{2} \left(\frac{1}{3} + \frac{1}{3 \cdot 7} + \frac{1}{5 \cdot 11} + \frac{1}{7 \cdot 15} + \frac{1}{9 \cdot 19} + \text{etc.} \right).$$

Eodem modo euoluatur altera formula $\int s dy$, et cum per seriem priorem effet

$$s = x + \frac{1}{2} \cdot \frac{1}{3} x^5 + \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{1}{5} x^9 + \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \frac{1}{7} x^{13} + \text{etc.}$$

at vero $dy = \frac{x x dx}{\sqrt{1-x^4}}$, singulis terminis integrandis operarum ante datarum pro casu $x=1$ reperietur

$$\int s dy = \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{2}{5} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{1}{5} \cdot \frac{2}{7} \cdot \frac{4}{9} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \frac{1}{7} \cdot \frac{2}{9} \cdot \frac{4}{11} \cdot \frac{6}{13} \cdot \frac{1}{2},$$

quae series contrahitur in sequentiam formam:

$$\int s dy = \frac{1}{2} \left(1 + \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 9} + \frac{1}{7 \cdot 13} + \frac{1}{9 \cdot 17} + \frac{1}{11 \cdot 21} + \text{etc.} \right).$$

His igitur duabus seriebus coniunctis fiet:

$$P = \frac{1}{2} \left(1 + \frac{1}{3} + \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} + \frac{1}{5 \cdot 9} + \frac{1}{5 \cdot 11} + \frac{1}{7 \cdot 13} + \text{etc.} \right).$$

§. 25. Quod si in hac serie bini termini sequentes in unum contrahantur, obtinebitur sequens series:

$$P = y s = \frac{2}{1 \cdot 3} + \frac{2}{5 \cdot 7} + \frac{2}{9 \cdot 11} + \frac{2}{13 \cdot 15} + \frac{2}{17 \cdot 19} + \text{etc.}$$

Quoniam autem porro est $\frac{2}{3} = 1 - \frac{1}{3}$ et $\frac{2}{5} = \frac{1}{5} - \frac{1}{7}$, $\frac{2}{9} = \frac{1}{9} - \frac{1}{11}$, etc. ista series resoluitur in hanc formam:

$$P = 1$$

$$P = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \text{etc.}$$

Quae cum sit notissima series *Leibniziana* cuius summa $= \frac{\pi}{4}$, erit $P = y s = \frac{\pi}{4}$, casu scilicet quo $x = 1$. Verum hoc casu assumsi fieri $y = a$ et $s = c$, sicque etiam hinc apparet esse productum $a c = \frac{\pi}{4}$.

Praeparatio
ad sequentes huius curvae proprietates magis
abstrusas.

§. 26. In dissertatione, cui titulus: *Plenior explicatio circa comparationem quantiarum in formula integrali* $\int \frac{z dz}{\sqrt{(1+mzz+nz^4)}}$ *contentarum*, quaeque Parti posteriori Actorum pro anno 1781 inserta fuit, ostendi: si $\Pi:z$ denotet valorem huius formulae integralis: $\int \frac{dz}{\sqrt{(1+mzz+nz^4)}}$, ita sumtum ut evanescat posito $z = 0$, tum plures huius generis quantitates transcendentes modo prorsus singulare inter se comparari posse. Scilicet si propositae fuerint duae huiusmodi formulae: $\Pi:x$ et $\Pi:y$, atque ex litteris x et y ita determinetur tertia z , ut sit

$$z = \frac{x \sqrt{(1+mz^2+nz^4)}}{1-nx^2y^2} + \frac{y \sqrt{(1+mz^2+nz^4)}}{1-nx^2y^2},$$

vnde fit

$$\sqrt{(1+mzz+nz^4)} = \frac{(mxy + \sqrt{(1+mzx+nzx^2)(1+myy+ny^2)(1+nx^2yy)+2nxy(xx+yy)})}{(1-nx^2y^2)^2},$$

tum semper erit

$\Pi:z = \Pi:x + \Pi:y + \beta xy z$,
ita ut quantitas transcendentis $\Pi:z$ superet summam datarum $\Pi:x$ et $\Pi:y$ quantitate algebraica $\beta xy z$.

§. 27. Evidens iam est has formulas generales duplii modo ad institutum nostrum accommodari posse,

scilicet tam ad arcus huius curvae inter se comparandos, quam ad applicatas cuique abscissae z respondentes. Pro utroque casu autem erit $m = 0$ et $n = -1$, tum vero in numeratore pro arcibus sumi debebit $\alpha = 1$ et $\beta = 0$, at pro applicatis $\alpha = 0$ et $\beta = 1$.

§. 28. Quod si iam littera z deuotet abscissam quamcunque in axe CB assumtam, applicatam ei respondentem designemus charactere $\Pi : z$, arcum vero respondentem hoc charactere $\Theta : z$, eritque ex natura nostrae elasticae

$$\Pi : z = \int \frac{z \, dz}{\sqrt{(1 - z^2)}} \text{ et } \Theta : z = \int \frac{dz}{\sqrt{(1 - z^2)}},$$

quibus characteribus in sequentibus vtemur. Tum igitur sumto $z = 0$ erit $\Pi : 0 = 0$ et $\Theta : 0 = 0$. Sumto autem $z = 1$ erit $\Pi : 1 = AB = a$ et $\Theta : 1 = CA = c$. Praeterea vero notari oportet, sumta abscissa z negatiua, tam applicatam quam arcus longitudinem etiam fore negatiuas; sicque erit $\Pi : (-z) = -\Pi : z$, similius modo $\Theta : (-z) = -\Theta : z$. His igitur praemissis duplcam istam comparationem in sequentibus Problematis ad nostrum institutum accommodabimus.

Problema I.

Tab II. *Propositis in nostra curva elastica binis arcibus CX et CY, absindere arcum CZ, qui aequalis sit summae arcuum CX + CY.*
Fig 4.

Solutio.

§. 29. Vocentur abscissae his arcibus respondentes $Cx = x$; $Cy = y$ et $Cz = z$, eruntque applicatae stabi-

stabilito signandi modo $x X = \Pi : x$, $y Y = \Pi : y$ et $z Z = \Pi : z$,
ipso vero arcus $CX = \Theta : x$, $CY = \Theta : y$ et $CZ = \Theta : z$,
et quoniam requiritur ut sit $\Theta : z = \Theta : x + \Theta : y$, re-
gula generalis supra allata, quoniam hoc casu littera $\beta = 0$,
pro datis litteris x et y ita definire iubet z , ut sit.

$$z = \frac{x\sqrt{1-x^4} + y\sqrt{1-y^4}}{1+x^2y^2},$$

tum autem erit

$$\sqrt{1-z^4} = \frac{(1-xxyy)\sqrt{(1-x^4)(1-y^4)} - zxy(xx+yy)}{(1+x^2y^2)^2},$$

vnde patet, quomodo ex binis abscissis datis $Cx = x$ et
 $Cy = y$ quaesitam z construi oporteat, ut arcus CZ aequalis fiat summae arcuum $CX + CY$.

§. 30. Quemadmodum hic ex datis abscissis x et y determinauimus abscissam z , ita viceversa, si dentur abscissae x et z , tertia y ex iis simili modo determinabitur. Cum enim hic esse debeat $\Theta : y = \Theta : z - \Theta : x$, euidens est hic y eodem modo per z et $-x$ definiri, quo ante z per $+x$ et $+y$ expressimus. Hinc igitur erit

$$y = \frac{z\sqrt{1-x^4} - x\sqrt{1-z^4}}{1+x^2z^2} \text{ et}$$

$$\sqrt{1-y^4} = \frac{(1-xxzz)\sqrt{(1-x^4)(1-z^4)} + xxz(xz+zz)}{(1+x^2z^2)^2}.$$

Parique modo ex datis y et z abscissa x ita determinabitur, ut sit

$$x = \frac{z\sqrt{1-y^4} - y\sqrt{1-z^4}}{1+y^2z^2} \text{ et}$$

$$\sqrt{1-x^4} = \frac{(1-yyzz)\sqrt{(1-y^4)(1-z^4)} + yyz(yz+zz)}{(1+y^2z^2)^2}.$$

§. 31. Hinc igitur patet ternas quantitates x , y et z ita inter se referri, ut quaelibet per binas reliquas simili fere modo determinetur; quamobrem istam relatio-

nem accuratis euoluamus, quo clarus pateat, quomodo a fe inuicem pendeant. Ex primis autem valoribus, sumitis quadratis erit

$$zz = \frac{(xx+yy)(1-xxyy) + 2xy\sqrt{(1-x^4)(1-y^4)}}{(1+xxyy)^2};$$

ex valore autem formulae $\sqrt{(1-z^4)}$ colligitur:

$$\sqrt{(1-x^4)(1-y^4)} = \frac{(1+xxyy)^2 \sqrt{(1-z^4)} + 2xy(xx+yy)}{1+xxyy},$$

qui valor si ibi substituatur, orietur haec aequatio:

$$zz(1-xxyy) = xx+yy + 2xy\sqrt{(1-z^4)}.$$

Similique modo ex binis reliquis determinationibus fiet

$$yy(1-xxzz) = zz + xx - 2xz\sqrt{(1-y^4)} \text{ et}$$

$$xx(1-yyzz) = yy + zz - 2yz\sqrt{(1-x^4)}.$$

§. 32. Quod si has aequationes ab omni irrationalitate liberemus, ex singulis eadem resultabit aequatio rationalis, quae erit

$$\left. \begin{array}{l} +x^4 - 2xxyy + 2x^4yyzz + x^4y^4z^4 \\ +y^4 - 2xxzz + 2xxy^4zz \\ +z^4 - 2yyzz + 2xxyyz^4 \end{array} \right\} = 0,$$

quaeque etiam ita exhiberi potest:

$$0 = \left. \begin{array}{l} x^4 + y^4 + z^4 - 2xxyy - 2xxzz - 2yyzz \\ + 2xxyyzz(xx+yy+zz) + x^4y^4z^4 \end{array} \right\},$$

vbi iam manifesto ternae litterae x , y et z aequaliter ingrediuntur; quoniam enim hic litterarum x , y , z tantum quadrata insunt, perinde est siue eae negatiue capiantur, siue positivae.

§. 33. Quoties ergo ternae abscissae $Cx = x$, $Cy = y$ et $Cz = z$, eam inter se tenent rationem, quam assignauimus, tum arcus CZ semper aequabitur summae bivorum reliquorum CX et CY . Cum igitur hinc sit $CZ - CY = CX$, erit arcus $YZ = CX$, vnde si puncta Y et Z pro labitu accipientur, a punto C semper arcus CX abscondi poterit, qui arcui YZ erit aequalis. Ac vicissim proposito arcu CX , a punto quovis dato Y abscondi poterit arcus YZ , illi arcui CX aequalis. Sin autem terminus Z vt datus spectetur, ab eo retro abscondi poterit arcus ZY ipsi CX aequalis, quae cum sint sat obvia, superfluum foret pro iis peculiaria problemata constituere.

Theorema II.

§. 34. Si ternae abscissae $Cx = x$, $Cy = y$, $Cz = z$, ita fuerint assumtae, vt arcus CZ aequetur summae CX et CY , tum ternae applicatae $xX = \Pi : x$, $yY = \Pi : y$, $zZ = \Pi : z$ ita inter se erunt relatae, vt sit

$$\Pi : z = \Pi : x + \Pi : y + xyz,$$

sive erit

$$zZ = xX + yY + \frac{c_x \cdot c_y \cdot c_z}{c_B}.$$

Demonstratio.

§. 35. Cum relatio inter formulas $\Pi : x$, $\Pi : y$ et $\Pi : z$ eandem relationem inter abscissas x , y et z praebeat quam pro formulis: $\Theta : x$, $\Theta : y$ et $\Theta : z$ assignauimus, quoniam pro hoc casu littera β in forma generali adhibita unitati aequatur, vi relationis generalis erit

$$\Pi : z$$

$$\Pi : z = \Pi : x + \Pi : y + xyz,$$

vnde, ad homogeneitatem obseruandam, quia altitudo CB
vnitate est definita, solidum xyz per eius quadratum di-
vidi oportet, vnde fiet

$$zZ = xX + yY + \frac{cx \cdot cy \cdot cz}{cB^2}.$$

§. 36. Cum igitur characteres $\Theta : z$ et $\Pi : z$ cer-
tas functiones transcendentes abscissae z denotent, quas
constat neque per logarithmos neque per arcus circularis ex-
primi posse, quandoquidem per formulas integrales $\int \frac{dz}{\sqrt{1-z^4}}$
et $\int \frac{z^2 dz}{\sqrt{1-z^4}}$ definiuntur, earum valores saltem per series
infinitas exhibuisse iuuabit: erit autem per modum priorem

$$\Theta : z = z + \frac{1}{2} \cdot \frac{1}{5} z^5 + \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{1}{5} z^9 + \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \frac{1}{15} z^{13} + \text{etc.}$$

$$\Pi : z = \frac{1}{3} z^3 + \frac{1}{2} \cdot \frac{3}{7} z^7 + \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{1}{11} z^{11} + \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \frac{1}{15} z^{15} + \text{etc.}$$

Ex altera autem resolutione erit ex §. 8.

$$\Theta : z = (z + \frac{2}{5} z^5 + \frac{2 \cdot 7}{5 \cdot 9} z^9 + \frac{2 \cdot 7 \cdot 11}{5 \cdot 9 \cdot 13} z^{13} + \text{etc.}) \sqrt{1-z^4} \text{ et}$$

$$\Pi : z = (\frac{1}{3} z^3 + \frac{1 \cdot 5}{3 \cdot 7} z^7 + \frac{1 \cdot 5 \cdot 9}{3 \cdot 7 \cdot 11} z^{11} + \frac{1 \cdot 5 \cdot 9 \cdot 13}{3 \cdot 7 \cdot 11 \cdot 15} z^{15} + \text{etc.}) \sqrt{1-z^4}.$$

Problema

Elementa principalia nostrae curvae elasticae, scilicet
latitudinem AB = a et totum arcum CA = c, respectu al-
titudinis CB = 1, accuratius determinare quam supra fieri
licuit

Solutio

§. 37. Hunc in finem accipiatur punctum Z in
ipso vertice curvae A, vt fiat $z = 1$, eritque

$$\Pi : z = AB = a \text{ et } \Theta : z = CA = c,$$

tum

tum igitur erit $V(x - z^4) = 0$. Nunc querantur bini arcus CX et CY, quorum summa sit aequalis arcui CA = c. Positis ergo eorum abscissis $Cx = x$ et $Cy = y$ ex §. 31. erit

$$x - x^2 - y^2 - xy^2 = 0,$$

vnde fit $y^2 = \frac{x^2}{1+x^2}$. Quod si igitur y hoc modo per x determinetur, tum erit $\Theta : x + \Theta : y = c$; tum vero ob $\Pi : z = a$ erit $a = \Pi : x + \Pi : y + xy$.

§. 38. Quo nunc series pro $\Theta : x$ et $\Theta : y$, item pro $\Pi : x$ et $\Pi : y$, maxime conuergentes reddantur, abscissas x et y proxime inter se aequales accipiamus. Si enim vellemus statuere $y = x$, prodiret

$$x = y = V(-1 + \sqrt{2}),$$

qui valor irrationalis minime idoneus foret ad nostras series euoluendas. Hanc ob rem sumamus $x^2 = \frac{1}{2}$, erit $y^2 = \frac{1}{3}$, ideoque $x = \sqrt[4]{2}$ et $y = \sqrt[4]{3}$, vnde per priores series fiet

$$\Theta : x = \frac{1}{\sqrt[4]{2}} \left(1 + \frac{1}{2 \cdot 5} \cdot \frac{1}{2^2} + \frac{1 \cdot 3}{2 \cdot 4 \cdot 9} \cdot \frac{1}{2^4} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 13} \cdot \frac{1}{2^6} + \text{etc.} \right)$$

$$\Pi : x = \frac{1}{2\sqrt[4]{2}} \left(\frac{1}{3} + \frac{1}{2 \cdot 7} \cdot \frac{1}{3^2} + \frac{1 \cdot 3}{2 \cdot 4 \cdot 11} \cdot \frac{1}{3^4} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 15} \cdot \frac{1}{3^6} + \text{etc.} \right)$$

Simili vero modo erunt:

$$\Theta : y = \sqrt[4]{3} \left(1 + \frac{1}{2 \cdot 5} \cdot \frac{1}{3^2} + \frac{1 \cdot 3}{2 \cdot 4 \cdot 9} \cdot \frac{1}{3^4} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 13} \cdot \frac{1}{3^6} + \text{etc.} \right)$$

$$\Pi : y = \frac{1}{2\sqrt[4]{3}} \left(\frac{1}{3} + \frac{1}{2 \cdot 7} \cdot \frac{1}{3^2} + \frac{1 \cdot 3}{2 \cdot 4 \cdot 11} \cdot \frac{1}{3^4} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 15} \cdot \frac{1}{3^6} + \text{etc.} \right)$$

§. 39. Hae series manifesto tantopere conuergunt, vt, qui laborem calculi suscipere voluerit, veros litterarum Acta Acad. Imp. Sc. Tom. VI. P. II.

H a et

a et c valores tam exacte definire queat quam libuerit; valo-
res autem quas supra assignauimus, iam tam parum a veritate
discrepant, vt pro nostro instituto abunde sufficere possint;
quandoquidem hic de eo tantum agitur, vt valores inuenti
calculum subducendo comprobari queant, quamobrem ad
alias insignes proprietates huius curuae progrediamur.

Problema III.

Tab. II. *Proposito in curua elatica arcu quocunque P Q, a
Fig. 5. punto dato R abscindere arcum R S, qui illi arcui P Q sit
aequalis.*

Solutio.

§. 40. Quoniam igitur in curua quatuor puncta
 P, Q, R, S consideranda veniunt, sint abscissae illis res-
pondentes $Cp = p, Cq = q, Cr = r, Cs = s$, pro qui-
bus ponamus breuitatis gratia formulas irrationales

$$\sqrt{r-p^4} = P, \sqrt{r-q^4} = Q, \sqrt{r-r^4} = R \text{ et } \sqrt{r-s^4} = S.$$

His positis, quoniam arcus RS aequalis esse debet arcui
 PQ , requiritur vt sit $CS - CR = CQ - CP$, hoc est
 $\Theta : s - \Theta : r = \Theta : q - \Theta : p$, cui aequationi vt per regulam
supra datam satisfaciamus, quaeramus arcum $\Theta : v$, vt sit
 $\Theta : v = \Theta : q - \Theta : p$, et secundum praecpta superiora esse
debet $v = \frac{qP - pQ}{1 + ppqq}$, vnde fit

$$\sqrt{r-v^4} = V = \frac{(r-ppq)PQ + pqr(pq + qr)}{(1 + ppqq)^2}.$$

Hoc iam arcu inuento esse debet $\Pi : s = \Pi : r + \Pi : v$;
quare per eadem praecpta fiet $s = \frac{rv + vr}{1 + rrvv}$, hincque
porro

$$S = \frac{(r - rrvv)Rv - rv(rv + vv)}{(1 + rrvv)^2}.$$

Sub-

Substituamus nunc in his formulis valores pro v et V inventos; ac primo erit

$$1 + rrvv = \frac{(1+ppqq)^2 + rrppQQ + rrqqPP - rrpqPQ}{(1+ppqq)^2},$$

quae aequatio, si loco PP et QQ valores substituantur, ad hanc reducitur:

$$1 + rrvv = \frac{(1+ppqq)^2 + rr(pp+qq)(1-ppqq) - rppqrPQ}{(1+ppqq)^2}.$$

At vero pro numeratore erit

$$rV + vR = \frac{r(1-ppqq)PQ - rppqr(pp+qq) + (qPR - pQR)(1+ppqq)}{(1+ppqq)^2},$$

consequenter abscissa quaesita CS = s ita erit expressa:

$$s = \frac{r(1-ppqq)PQ + rppqr(pp+qq) + (qPR - pQR)(1+ppqq)}{(1+ppqq)^2 + rr(pp+qq)(r-ppqq) - rppqrPQ}.$$

Quod autem ad valorem litterae S attinet, quia eo in nostro calculo non indigemus, eius euolutione supersedemus.

§. 41. Hinc igitur videmus, quomodo abscissa s per ternas abscissas datas p , q et r exprimatur; ubi quidem plurimum abest, ut litterae p , q , r in eam aequaliter ingrediantur: cum tamen ex aequatione proposita

$$\Theta : s = \Theta : r + \Theta : q - \Theta : p$$

intelligatur, istas litteras P, Q et R simili modo in valorem ipsius s ingredi debere, si modo littera p negatiue acciperetur. Neque igitur ullum est dubium, quin forma inuenta ita transformari possit, ut ista paritas litterarum p , q et r elucescat, id quod tamen neutiquam liquet.

§. 42. Cum autem esse debeat

$$\Theta : s = \Theta : r + \Theta : q - \Theta : p,$$

euidens est, manente littera p binas reliquias q et r inter se commutari posse, vnde etiam vera esse debet ista expressio:

$$s = \frac{q(1-prr)PR + 2pqr(pp+rr) + (rPQ - pQR)(1+prr)}{(1+prr)^2 + qr(pp+rr)(1-prr) - 2prqrPR}.$$

Deinde manente r litterae p et q ita permutari poterunt, si loco q scribatur $-p$ et $-q$ loco p , tum autem erit

$$s = \frac{-p(1-qrr)QR + 2pqr(qq+rr) + (qPR + rPQ)(1+qrr)}{(1+qrr)^2 + pp(qq+rr)(1-qrr) + 2irppQR}.$$

Atque haec tres expressiones, quantumvis diuersae videantur, tamen certe eundem valorem exprimunt.

§. 43. Insignis igitur hic occurrit quaestio analytica, quomodo istae tres expressiones tractari debeant, ut perfecta permutabilitas inter ternas litteras p , q , r perspiciatur. Facile quidem intelligitur, si tres istae expressiones in se inuicem multiplicentur, ita ut productum aequetur cubo s^3 , tum tam in numeratore quam in denominatore ternas litteras p , q , r , pari modo esse ingressuras; verum tale productum nimis foret perplexum, quam ut ullum usum habere posset.

Solutio.

§. 44. Quae hactenus de curua elatica rectangula sunt tradita etiam ad omnes curuas elasticas in genere accommodari poterunt. Cum enim pro data abscissa z sit applicata $= \int \frac{dz}{\sqrt{1-(\alpha+\beta z z)^2}}$ et ipse arcus $= \int \frac{dz}{\sqrt{1-(\alpha+\beta z z)^2}}$, praecepta generalia supra tradita pro comparatione harum quantitatum transcendentium simili modo applicari poterunt. Interim tamen hic conditio maxime necessaria probet

be notari debet, qua postulatur ut denominator, qui evolutus est $\sqrt{(1 - \alpha\alpha - 2\alpha\beta z^2 - \beta\beta z^4)}$, ad hanc formam: $\sqrt{(1 + mz^2 + nz^4)}$, reduci queat, quod manifesto fieri nequit nisi $1 - \alpha\alpha$ fuerit quantitas positiva. His igitur casibus $\alpha\alpha > 1$ omnes comparationes, quas tam inter arcus quam inter applicatas docuimus, simili modo ad curvas elasticas obliquangulas traduci poterunt.
