DE SVMMA SERIEI EX NVMERIS PRIMIS FORMATAE

$$\frac{1}{3}-\frac{1}{5}+\frac{1}{7}+\frac{1}{11}-\frac{1}{13}-\frac{1}{17}+\frac{1}{19}+\frac{1}{23}-\frac{1}{29}+\frac{1}{31} \text{ etc.}$$

VBI NVMERI PRIMI FORMAE $4n-1$ HABENT SIGNVM POSITIVVM, FORMAE AVTEM $4n+1$ SIGNVM NEGATIVVM.

§. 1.

Cum iam *Euclides* demonstrasset, multitudinem numerorum primorum reuera esse infinitam, ego iam pridem ostendi etiam summam seriei reciprocae numerorum primorum: scilicet

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{11}+\frac{1}{13}+\frac{1}{17}+\frac{1}{19}+ \text{ etc.}$$

esse infinite magnam, atque adeo referre logarithmum summae seriei harmonicae

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+ \text{ etc.}$$

id quod non parum mirum videbatur, cum vulgo summa seriei harmonicae ad genus quasi infimum infinitorum referri soleat. Cum autem non solum logarithmus numeri infiniti sit etiam infinitus, sed etiam logarithmi horum ipsorum logarithmorum etiamnunc sint infiniti; manifestum est dari insuper infinitos gradus inferiores infinitorum. Ita si A denotet summam seriei reciprocae numerorum primorum, etiam lA

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lA adhuc erit infinite magnus, sed ad ordinem infinitorum infinities inferiorem pertinere censendus est; tum vero etiamnunc hae formulae: llA, $lllA$, $llllA$, etc. erunt infinitae, quanquam quaelibet earum infinities sit minor quam praecedens.

§. 2. Quoniam porro numeri primi praeter binarium quasi a natura in duas classes distinguuntur, prouti fuerint vel formae $4n+1$, vel formae $4n-1$, dum priores omnes sunt summae duorum quadratorum, posteriores vero ab hac proprietate penitus excluduntur: series reciprocae ex vtraque classe formatae, scilicet:

$$\frac{1}{5}+\frac{1}{13}+\frac{1}{17}+\frac{1}{29}+ \text{ etc. et}$$

$$\frac{1}{3}+\frac{1}{7}+\frac{1}{11}+\frac{1}{19}+\frac{1}{23}+ \text{ etc.}$$

ambae erunt pariter infinitae, id quod etiam de omnibus speciebus numerorum primorum est tenendum. Ita si ex numeris primis ii tantum excerpantur, qui sunt formae $100n+1$, cuiusmodi sunt 101, 401, 601, 701 etc., non solum multitudo eorum est infinita, sed etiam summa huius seriei ex illis formatae, scilicet:

$$\frac{1}{101}+\frac{1}{401}+\frac{1}{601}+\frac{1}{701}+\frac{1}{1201}+\frac{1}{1301}+\frac{1}{1601}+\frac{1}{1801}+\frac{1}{1901}+ \text{ etc.}$$

etiam est infinita.

§. 3. Consideremus hic autem imprimis discrimen inter numeros primos formae $4n+1$ et $4n-1$, et quia ambae series ex vtroque ordine formatae sunt infinitae et quasi eiusdem ordinis; nullum est dubium, quin earum differentia habeat valorem determinatum. Hanc ob rem terminis ex forma $4n-1$ formatis tribuamus signum $+$, reliquis vero signum $-$, vt orietur ista series:

$$\tfrac{1}{3} - \tfrac{1}{5} + \tfrac{1}{7} + \tfrac{1}{11} - \tfrac{1}{13} - \tfrac{1}{17} + \tfrac{1}{19} + \tfrac{1}{23} - \tfrac{1}{29} + \tfrac{1}{31} \text{ etc},$$

in qua nullus plane ordo in ſignis cernitur. Hoc tamen non obſtante videamus, quomodo eius ſumma vero ſaltem proxime aſſignari queat. Non parum enim probabile videtur, iſtam ſummam, ſi non fuerit rationalis vel irrationalis, ſaltem ad quodpiam genus notabile quantitatum tranſcendentium pertinere debere. Interim tamen, quoniam non ſolum non in ſignis, verum etiam multo minus in ipſis fractionibus nullus plane ordo perſpicitur, primo intuitu nulla patere videtur via, qua ad eius ſummam peruenire liceat.

§. 4. Quando autem contemplamur ſeriem notiſſimam Leibnitzianam pro quadratura circuli datam

$$\tfrac{\pi}{4} = 1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \tfrac{1}{9} - \tfrac{1}{11} \text{ etc.}$$

videmus, in ea omnes numeros impares formae $4n+1$ habere ſignum $+$, reliquos vero formae $4n-1$ ſignum $-$, vnde mutatis ſignis omnes termini noſtrae ſeriei cum ſuis ſignis in hac ſerie Leibnitziana occurrunt. Quod ſi ergo ex hac ſerie omnes numeros compoſitos expungamus, tandem praeter vnitatem ipſa noſtra ſeries propoſita, contrariis ſignis, remanebit. Quamobrem ſummam noſtrae ſeriei obtinebimus, ſi ex ſerie Leibnitziana ſucceſſiue omnes numeros compoſitos excludamus, quandoquidem omnium terminorum, qui per quamuis operationem excluduntur, ſumma facile aſſignare licebit.

§. 5. Incipiamus igitur ab ipſa ſerie Leibnitziana, ſtatuendo

$$A = 1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \tfrac{1}{9} - \tfrac{1}{11} + \tfrac{1}{13} - \tfrac{1}{15} + \tfrac{1}{17} - \tfrac{1}{19} + \text{ etc.}$$

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ita vt ſit $A = \tfrac{\pi}{4}$, ex eaque primo omnes numeros compoſitos per 3 diuiſibiles excludamus. Quem in finem hinc formemus iſtam ſeriem:

$$(A-1)\tfrac{1}{3} = -\tfrac{1}{9} + \tfrac{1}{15} - \tfrac{1}{21} + \tfrac{1}{27} - \tfrac{1}{33} + \tfrac{1}{39} - \text{ etc.}$$

quae vtique continet omnes numeros compoſitos per 3 diviſibiles, vnde ſi haec ſeries ad illam addatur, omnes iſti numeri compoſiti excludentur fietque

$$\tfrac{4}{3}A - \tfrac{1}{3} = 1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} - \tfrac{1}{11} + \tfrac{1}{13} + \tfrac{1}{17} - \tfrac{1}{19} - \tfrac{1}{23} + \tfrac{1}{25} \text{ etc.} = B,$$

ita vt ſit $B = \tfrac{4}{3}A - \tfrac{1}{3}$. In hac igitur ſerie, cuius ſumma nouimus, nulli amplius numeri compoſiti per 3 diuiſibiles occurrunt, ſed primus terminus compoſitus hic occurrens eſt $+\tfrac{1}{25}$.

§. 6. Nunc igitur ex poſtrema hac ſerie omnes terminos per 5 diuiſibiles excludamus, quem in finem hinc formemus iſtam ſeriem:

$$(B - 1 + \tfrac{1}{3})\tfrac{1}{5} = \tfrac{1}{25} - \tfrac{1}{35} - \tfrac{1}{55} + \tfrac{1}{65} + \tfrac{1}{85} - \tfrac{1}{95} - \text{ etc.},$$

quae omnes complectitur terminos per quinque diuiſibiles. qui adhuc in ſeriem B ingrediebantur, er quidem iisdem ſignis affectos, quare hac ſerie ab illa ablata remanebit haec:

$$\tfrac{4}{5}B + \tfrac{1}{5}(1 - \tfrac{1}{3}) = 1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} - \tfrac{1}{11} + \tfrac{1}{13} + \tfrac{1}{17} - \tfrac{1}{19} - \tfrac{1}{23} + \text{ etc.}$$

vbi iam ab initio omnes termini ſunt numeri primi, et non primus qui hic occurret erit $\tfrac{1}{49}$, ſequentes vero $\tfrac{1}{77}, -\tfrac{1}{91}$. Ponamus autem ſummam huius ſeriei $= C$, vt ſit

$$C = \tfrac{4}{5}B + \tfrac{1}{5}(1 - \tfrac{1}{3}).$$

§. 7. Nunc igitur hinc excludamus omnes terminos qui adhuc per ſeptem ſunt diuiſibiles, quos complectetur ſequens forma:

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$$(C - 1 + \tfrac{1}{3} - \tfrac{1}{5})\tfrac{1}{7} = -\tfrac{1}{49} - \tfrac{1}{77} + \tfrac{1}{91} + \text{etc.}$$

quorum terminorum ſigna ſunt contraria, quare haec ſeries ad ſeriem C addita dabit

$$\tfrac{6}{7}C - \tfrac{1}{7}(1 - \tfrac{1}{3} + \tfrac{1}{5}) = 1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} - \tfrac{1}{11} + \tfrac{1}{13} + \tfrac{1}{17} + \text{etc.}$$

in qua primus terminus compoſitus erit $+\frac{1}{13\cdot 7}$; ſequentes vero $-\frac{1}{11\cdot 13}$, $-\frac{1}{11\cdot 17}$ etc. Hanc autem ſeriem vocemus D vt ſit

$$D = \tfrac{6}{7}C - \tfrac{1}{7}(1 - \tfrac{1}{3} + \tfrac{1}{5}).$$

§. 8. Iam ex ſerie modo inuenta D expungamus terminos, qui adhuc ſunt per 11 diuiſibiles, quos complectetur iſta forma:

$$D - (1 + \tfrac{1}{3} - \tfrac{1}{5} + \tfrac{1}{7})\tfrac{1}{11} = -\tfrac{1}{121} + \tfrac{1}{143} + \tfrac{1}{187} \text{ etc.}$$

qui termini in ſerie D contraria habent ſigna; quamobrem ſi haec ſeries ad illam addatur, iſti termini excludentur, prodibitque

$$\tfrac{12}{11}D - \tfrac{1}{11}(1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7}) = 1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} - \tfrac{1}{11} + \text{etc.}$$

in qua primus terminus non primus eſt $\frac{1}{169}$; iſtam autem ſeriem deſignemus littera E, ita vt ſit

$$E = \tfrac{12}{11}D - \tfrac{1}{11}(1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7}).$$

§. 9. Ex hac igitur ſerie excludamus omnes terminos, qui adhuc inſunt per 13 diuiſibiles, quos ergo complectetur haec forma:

$$(E - 1 + \tfrac{1}{3} - \tfrac{1}{5} + \tfrac{1}{7} + \tfrac{1}{11})\tfrac{1}{13} = \tfrac{1}{169} + \tfrac{1}{221} \text{ etc.}$$

hique termini eadem habent ſigna ac in ipſa ſerie E. Haec igitur ſeries ab illa debet ſubtrahi, vnde prodit

$$\tfrac{12}{13}E + \tfrac{1}{13}(1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} - \tfrac{1}{11}) = 1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} - \tfrac{1}{11} + \text{etc.}$$

vbi

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vbi primus terminus non primus eſt $\frac{1}{289}$. Totam autem hanc ſeriem deſignemus littera F, vt ſit

$$F = \tfrac{12}{13}E + \tfrac{1}{13}(1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} - \tfrac{1}{11}).$$

§. 10. Quod ſi nunc iſtas operationes vlterius continuemus, dum ſucceſſiue hinc excludimus terminos adhuc per 17 diuiſibiles, tum vero per 19, per 23, etc. tandem relinquetur tantum ſeries numerorum primorum poſt vnitatem ſequentium, quae ſi deſignetur littera Z, quam vt infiniteſimam ſpectari oportet, erit vtique

$$Z = 1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} - \tfrac{1}{11} + \tfrac{1}{13} + \tfrac{1}{17} - \tfrac{1}{19} \text{ etc.}$$

conſequenter ſumma ſeriei in titulo propoſitae ſumma erit $1 - Z$. Ac manifeſtum eſt, ad hunc valorem continuo propius accedere iſtas formulas:

$$1 - A,\ 1 - B,\ 1 - C,\ 1 - D,\ 1 - E,\ 1 - F, \text{ etc.}$$

§. 11. Quemadmodum autem valores omnium harum litterarum ſucceſſiue ex antecedentibus colligi debeant, ex ſequentibus formulis fiet manifeſtum:

$B = \frac{4}{3}A - \frac{1}{3}\cdot 1$

$C = \frac{4}{5}B + \frac{1}{5}(1 - \frac{1}{3})$

$D = \frac{6}{7}C - \frac{1}{7}(1 - \frac{1}{3} + \frac{1}{5})$

$E = \frac{12}{11}D - \frac{1}{11}(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7})$

$F = \frac{12}{13}E + \frac{1}{13}(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} - \frac{1}{11})$

$G = \frac{16}{17}F + \frac{1}{17}(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} - \frac{1}{11} + \frac{1}{13})$

$H = \frac{20}{19}G - \frac{1}{19}(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} - \frac{1}{11} + \frac{1}{13} + \frac{1}{17})$

$I = \frac{24}{23}H - \frac{1}{23}(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} - \frac{1}{11} + \frac{1}{13} + \frac{1}{17} - \frac{1}{19})$

etc. etc.

Vbi

Vbi notandum, fi denominator primus fuerit formae $4n+1$, tum numeratorem primae partis fore vnitate minorem, fiue $4n$, alteram vero partem addi debere. Sin autem denominator primus fuerit $4n-1$, tum numeratorem primae partis fore vnitate maiorem, fiue $4n$, alteram partem vero hoc cafu fubtrahi debere.

§. 12. Quo nunc omnes hos valores in numeris per fractiones decimales exprimamus, ante omnia notetur effe

$$A = \tfrac{\pi}{4} = 0,7853981634.$$

Pro reliquis autem litteris computentur fequentes valores:

$1-\frac{1}{3}=$ - - - - - - - - $b=0,6666666666$

$1-\frac{1}{3}+\frac{1}{5}=$ - - - - - - - $c=0,8666666666$

$1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}=$ - - - - - - $d=0,7238095238$

$1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}-\frac{1}{11}=$ - - - - - $e=0,6329004329$

$1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}-\frac{1}{11}+\frac{1}{13}=$ - - - - $f=0,7098235098$

$1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}-\frac{1}{11}+\frac{1}{13}+\frac{1}{17}=$ - - - $g=0,7686470392$

$1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}-\frac{1}{11}+\frac{1}{13}+\frac{1}{17}-\frac{1}{19}=$ - - $h=0,7160154603$

$1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}-\frac{1}{11}+\frac{1}{13}+\frac{1}{17}-\frac{1}{19}-\frac{1}{23}=$ - $i=0,6725371994$

in quo ordine primus terminus a vnitate aequatur.

§. 13. Pro computo autem ipfarum litterarum A, B, C, D, E, etc. praeftabit fequentibus vti formulis, quibus fimul valores numericos harum litterarum adfcribamus

$$B = A+\tfrac{1}{3}(A-a) = 0,713864$$
$$C = B-\tfrac{1}{5}(B-b) = 0,704424$$
$$D = C+\tfrac{1}{7}(C-c) = 0,681247$$

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$$E = D+\tfrac{1}{11}(D-d) = 0,677377$$
$$F = E-\tfrac{1}{13}(E-e) = 0,673956$$
$$G = F-\tfrac{1}{17}(F-f) = 0,676066$$
$$H = G+\tfrac{1}{19}(G-g) = 0,671193$$
$$I = H+\tfrac{1}{23}(H-h) = 0,669245$$
$$K = I-\tfrac{1}{29}(I-i) = 0,669358.$$

§. 14. Quanquam autem calculum huc vsque produximus, tamen vltra tertiam figuram decimalem de fumma noftrae feriei certi effe non poffumus, atque adeo in dubio relinquere cogimur, vtrum ifta fumma aliquanto maior vel minor fit quam 0,669. Sin autem hunc valorem pro vero affumamus, ipfa feries propofita

$$\tfrac{1}{3}-\tfrac{1}{5}+\tfrac{1}{7}+\tfrac{1}{11}-\tfrac{1}{13}-\tfrac{1}{17}+\tfrac{1}{19} \text{ etc.}$$

fummam habebit 0,331, ideoque hic valor tantillo foret minor quam $\frac{1}{3}$. Quoniam vero ablato $\frac{1}{5}$, iterum addi debeat $\frac{1}{7}+\frac{1}{11}$, quarum fractionum fumma maior eft quam $\frac{1}{5}$, vtique fieri poffet, vt verus valor fuperaret $\frac{1}{3}$, id quod hoc loco in dubio eft relinquendum. Datur vero alia methodus, multo accuratius in fummam huius feriei inquirendi, quam hic euoluemus, quandoquidem operae pretium videtur, veram huius feriei fummam propius cognouiffe.

§. 15. Eadem methodo qua hic ex prima ferie Leibnitziana fucceffiue terminos compofitos expulimus, fi omnes plane terminos praeter vnitatem remoueamus, reperiemus

$$\tfrac{\pi}{4} = \tfrac{3}{4}\cdot\tfrac{5}{4}\cdot\tfrac{7}{8}\cdot\tfrac{11}{12}\cdot\tfrac{13}{12}\cdot\tfrac{17}{16}\cdot\tfrac{19}{20} \text{ etc.}$$

vbi

vbi in numeratoribus omnes numeri primi occurrunt praeter 2, denominatores vero ſunt numeri pariter pares vnitate vel maiores vel minores. Deinde vero ſi iſta ſeries reciproca quadratorum imparium:

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \frac{1}{11^2} + \frac{1}{13^2} + \text{etc.}$$

cuius ſummam oſtendi eſſe $= \frac{\pi\pi}{8}$, ſimili modo tractetur, reperietur

$$\frac{\pi\pi}{8} = \frac{3.3}{2.4} \cdot \frac{5.5}{4.6} \cdot \frac{7.7}{6.8} \cdot \frac{11.11}{10.12} \cdot \frac{13.13}{12.14} \cdot \text{etc.}$$

vbi iterum in numeratoribus omnes numeri primi bis occurrunt, in denominatoribus vero iidem tam vnitate aucti quam minuti. Quare ſi hanc expreſſionem per quadratum illius, quod eſt

$$\frac{\pi\pi}{15} = \frac{3.3}{4.4} \cdot \frac{5.5}{4.4} \cdot \frac{7.7}{8.8} \cdot \frac{11.11}{12.12} \cdot \frac{13.13}{12.12} \text{ etc.}$$

diuidamus, quotus erit

$$2 = \frac{4}{2} \cdot \frac{4}{6} \cdot \frac{8}{6} \cdot \frac{12}{10} \cdot \frac{12}{14} \text{ etc.}$$

vbi omnes numeri primi tam vnitate aucti quam minuti occurrunt, et numeri pariter pares in numeratore, impariter pares vero in denominatore conſtituuntur.

§. 16. Poſtrema haec expreſſio igitur hoc modo exhiberi poterit:

$$2 = \frac{3+1}{3-1} \cdot \frac{5-1}{5+1} \cdot \frac{7+1}{7-1} \cdot \frac{11+1}{11-1} \cdot \frac{13-1}{13+1} \text{ etc.}$$

hinc ergo logarithmis hyperbolicis ſumendis habebimus

$$l\,2 = l\frac{3+1}{3-1} + l\frac{5-1}{5+1} + l\frac{7+1}{7-1} + l\frac{11+1}{11-1} + l\frac{13-1}{13+1} + \text{etc.}$$

Conſtat autem per ſeries infinitas eſſe in genere

$$\tfrac{1}{2}\,l\frac{a+1}{a-1} = \frac{1}{a} + \frac{1}{3a^3} + \frac{1}{5a^5} + \frac{1}{7a^7} + \frac{1}{9a^9} + \text{etc.}$$

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$$\tfrac{1}{2}\,l\frac{a-1}{a+1} = -\frac{1}{a} - \frac{1}{3a^3} - \frac{1}{5a^5} - \frac{1}{7a^7} - \frac{1}{9a^9} - \text{etc.}$$

Quodſi igitur harum formularum ope omnes illos logarithmos in ſeries infinitas conuertamus, nanciſcemur quidem innumeras ſeries infinitas, quas autem ad ſeries facilius tractabiles reducere licebit.

§. 17. Primo igitur omnium illorum logarithmorum ſemiſſes accipi oportet, et quia hic de logarithmis hyperbolicis agitur, ob

$$l\,2 = 0,6931471805, \text{ erit}$$
$$\tfrac{1}{2}\,l\,2 = 0,3465735902,$$

ex altera autem parte logarithmi ita ordinentur:

$$\tfrac{1}{2}\,l\frac{3+1}{3-1} = \frac{1}{3} + \frac{1}{3.3^3} + \frac{1}{5.3^5} + \frac{1}{7.3^7} + \frac{1}{9.3^9} + \text{etc.}$$
$$\tfrac{1}{2}\,l\frac{5-1}{5+1} = -\frac{1}{5} - \frac{1}{3.5^3} - \frac{1}{5.5^5} - \frac{1}{7.5^7} - \frac{1}{9.5^9} - \text{etc.}$$
$$\tfrac{1}{2}\,l\frac{7+1}{7-1} = \frac{1}{7} + \frac{1}{3.7^3} + \frac{1}{5.7^5} + \frac{1}{7.7^7} + \frac{1}{9.7^9} + \text{etc.}$$
$$\tfrac{1}{2}\,l\frac{11+1}{11-1} = \frac{1}{11} + \frac{1}{3.11^3} + \frac{1}{5.11^5} + \frac{1}{7.11^7} + \frac{1}{9.11^9} + \text{etc.}$$
$$\tfrac{1}{2}\,l\frac{13-1}{13+1} = -\frac{1}{13} - \frac{1}{3.13^3} - \frac{1}{5.13^5} - \frac{1}{7.13^7} - \frac{1}{9.13^9} - \text{etc.}$$

etc. etc.

§. 18. Hinc iam verticaliter deſcendendo conſideremus ſequentes ſeries pariter infinitas:

$$O = \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{1}{11} - \frac{1}{13} - \frac{1}{17} + \frac{1}{19} + \text{etc.}$$
$$P = \frac{1}{3^3} - \frac{1}{5^3} + \frac{1}{7^3} + \frac{1}{11^3} - \frac{1}{13^3} - \frac{1}{17^3} + \frac{1}{19^3} + \text{etc.}$$
$$Q = \frac{1}{3^5} - \frac{1}{5^5} + \frac{1}{7^5} + \frac{1}{11^5} - \frac{1}{13^5} - \frac{1}{17^5} + \frac{1}{19^5} + \text{etc.}$$

R =

$$R = \frac{1}{3^7} - \frac{1}{5^7} + \frac{1}{7^7} + \frac{1}{11^7} - \frac{1}{13^7} - \frac{1}{17^7} + \frac{1}{19^7} + \text{etc.}$$
$$S = \frac{1}{3^9} - \frac{1}{5^9} + \frac{1}{7^9} + \frac{1}{11^9} - \frac{1}{13^9} - \frac{1}{17^9} + \frac{1}{19^9} + \text{etc.}$$
etc. etc.

Quarum ſerierum prima O eſt ea ipſa, cuius ſummam hic inueſtigare nobis eſt propoſitum.

§. 19. His igitur ſeriebus ita per litteras maiuſculas deſignatis habebimus iſtam aequationem:

$$\tfrac{1}{2}\, l\, 2 = O + \tfrac{1}{3} P + \tfrac{1}{5} Q + \tfrac{1}{7} R + \tfrac{1}{9} S + \tfrac{1}{11} T + \text{etc.}$$

vnde ſi ſummae ſerierum P, Q, R, S eſſent cognitae, inde impetraremus facile ſummam ſeriei O quaeſitam; foret enim

$$O = \tfrac{1}{2}\, l\, 2 - \tfrac{1}{3} P - \tfrac{1}{5} Q - \tfrac{1}{7} R - \tfrac{1}{9} S - \text{etc.}$$

§. 20. At vero ſummas ſerierum P, Q, R etc. ex ſeriebus ordinis, vbi omnes numeri impares occurrunt, concludere poterimus eodem modo quo ſupra ſeriem ipſam O ex ſerie Leibnitziana

$$1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \tfrac{1}{9} - \tfrac{1}{11} \text{ etc.}$$

elicuimus. Hunc in finem euolui hac methodo oportebit ſequentes ſeries ordinatas:

$$\mathfrak{P} = 1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \frac{1}{9^3} - \frac{1}{11^3} + \frac{1}{13^3} - \frac{1}{15^3} + \text{etc.}$$
$$\mathfrak{Q} = 1 - \frac{1}{3^5} + \frac{1}{5^5} - \frac{1}{7^5} + \frac{1}{9^5} - \frac{1}{11^5} + \frac{1}{13^5} - \frac{1}{15^5} + \text{etc.}$$
$$\mathfrak{R} = 1 - \frac{1}{3^7} + \frac{1}{5^7} - \frac{1}{7^7} + \frac{1}{9^7} - \frac{1}{11^7} + \frac{1}{13^7} - \frac{1}{15^7} + \text{etc.}$$
$$\mathfrak{S} = 1 - \frac{1}{3^9} + \frac{1}{5^9} - \frac{1}{7^9} + \frac{1}{9^9} - \frac{1}{11^9} + \frac{1}{13^9} - \frac{1}{15^9} + \text{etc.}$$
$$\mathfrak{T} = 1 - \frac{1}{3^{11}} + \frac{1}{5^{11}} - \frac{1}{7^{11}} + \frac{1}{9^{11}} - \frac{1}{11^{11}} + \frac{1}{13^{11}} - \frac{1}{15^{11}} + \text{etc.}$$
etc. etc.

harum

etc.
etc.
is ſummam hic
itteras maiuſcu-
T + etc.
at cognitae, in-
quaeſitam; foret
etc.
P, Q, R etc.
impares occur-
uo ſupra ſeriem
thodo oportebit
$\frac{1}{13^3} - \frac{1}{15^3} + \text{etc.}$
$\frac{1}{13^5} - \frac{1}{15^5} + \text{etc.}$
$\frac{1}{13^7} - \frac{1}{15^7} + \text{etc.}$
$\frac{1}{11^9} - \frac{1}{13^9} + \text{etc.}$
$\frac{1}{13^{11}} - \frac{1}{15^{11}} + \text{etc.}$
harum

Harum autem omnium ſerierum ſummas iam pridem per quadraturam circuli, ſcilicet per ſimiles poteſtates ipſius π expreſſas dedi, ſequenti modo:

$$\mathfrak{P} = \frac{1}{1 \cdot 2} \cdot \frac{\pi^3}{2^4} \qquad \mathfrak{T} = \frac{50521}{1 \cdot \ldots \cdot 10} \cdot \frac{\pi^{11}}{2^{12}}$$
$$\mathfrak{Q} = \frac{5}{1 \cdot 2 \cdot 3 \cdot 4} \cdot \frac{\pi^5}{2^6} \qquad \mathfrak{U} = \frac{2702765}{1 \cdot \ldots \cdot 12} \cdot \frac{\pi^{13}}{2^{14}}$$
$$\mathfrak{R} = \frac{61}{1 \cdot \ldots \cdot 6} \cdot \frac{\pi^7}{2^8} \qquad \mathfrak{V} = \frac{199360981}{1 \cdot \ldots \cdot 14} \cdot \frac{\pi^{15}}{2^{16}}$$
$$\mathfrak{S} = \frac{1385}{1 \cdot \ldots \cdot 8} \cdot \frac{\pi^9}{2^{10}} \qquad \mathfrak{W} = \frac{19391512145}{1 \cdot \ldots \cdot 16} \cdot \frac{\pi^{17}}{2^{18}}$$
etc. etc.

§. 21. Hos igitur valores in fractionibus decimalibus vsque ad ſextam figuram euoluamus, eritque

	Differentiae
$\mathfrak{P}$ = 0, 9689462	0, 0272095
$\mathfrak{Q}$ = 0, 9961557	0, 0033990
$\mathfrak{R}$ = 0, 9995547	0, 0003952
$\mathfrak{S}$ = 0, 9999499	0, 0000448
$\mathfrak{T}$ = 0, 9999947	0, 0000050
$\mathfrak{U}$ = 0, 9999997	0, 0000005
etc.	etc.

§. 22. Vt nunc hinc valores litterarum P, Q, R, etc. eruamus, eadem methodo vtamur, qua ſupra ex ſerie

$$1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \tfrac{1}{9} - \tfrac{1}{11} + \text{etc.}$$

omnes terminos compoſitos exterminauimus, quandoquidem loco horum numerorum ſimplicium eorum poteſtates ſcribi conuenit. Hanc igitur operationem in genere pro his litteris doceamus. Conſideremus igitur hanc ſeriem:

$\mathfrak{Z} =$

$$\mathfrak{Z} = 1 - \frac{1}{3^n} + \frac{1}{5^n} - \frac{1}{7^n} + \frac{1}{9^n} - \frac{1}{11^n} + \text{etc.}$$

cuius summam, vt supra factum est, littera A designemus, vt sit $A = \mathfrak{Z}$, hincque sequentes litteras B, C, D etc. eliciamus per sequentes formulas:

$$B = A + \frac{1}{3^n}(A - a) \text{ existente } a = 1$$

$$C = B - \frac{1}{5^n}(B - b) \text{ - - - - } b = 1 - \frac{1}{3^n}$$

$$D = C + \frac{1}{7^n}(C - c) \text{ - - - - } c = 1 - \frac{1}{3^n} + \frac{1}{5^n}$$

$$E = D + \frac{1}{11^n}(D - d) \text{ - - - - } d = 1 - \frac{1}{3^n} + \frac{1}{5^n} - \frac{1}{7^n}$$

$$F = E - \frac{1}{13^n}(E - e) \text{ - - - - } e = 1 - \frac{1}{3^n} + \frac{1}{5^n} - \frac{1}{7^n} - \frac{1}{11^n}$$

etc. etc.

Quibus valoribus inuentis eorum complementa ad vnitatem, scilicet: $1 - A$, $1 - B$, $1 - C$, $1 - D$, etc. promptissime ad valorem quaesitum

$$Z = \frac{1}{3^n} - \frac{1}{5^n} + \frac{1}{7^n} + \frac{1}{11^n} - \frac{1}{13^n} - \text{etc.}$$

appropinquabunt.

§. 23. Haec igitur praecepta generalia applicemus primo ad valorem litterae P, vnde incipiendum erit a valore

$$\mathfrak{P} = 0,9689462 = A,$$

et quia hic est $n = 3$, habebimus

$a = 1$, $b = 0,9629630$, $c = 0,9709630$, $d = 0,9680476$; pluri-

– etc.
A designemus, C, D etc. eli-
$- \frac{1}{7^n}$
$- \frac{1}{7^n} - \frac{1}{11^n}$
e.
ta ad vnitatem, promptissime ad
etc.
ralia applicemus dum erit a valore
$d = 0,9680476$; pluri-

pluribus valoribus non erit opus. Hinc igitur colligemus sequentes valores:

$$B = A - \tfrac{1}{3^3}.\,0,0310538 = 0,9677961$$

$$C = B - \tfrac{1}{5^3}.\,0,0048331 = 0,9677574$$

$$D = C - \tfrac{1}{7^3}.\,0,0032056 = 0,9677481$$

$$E = D - \tfrac{1}{11^3}.\,0,0002995 = 0,9677479$$

Vlterius procedi non est opus; quamobrem hinc habebimus

$$P = 1 - E = 0,0322521,$$

vnde iam colligimus

$$\tfrac{1}{4} l 2 - \tfrac{1}{3}.\,0,0322521 = 0,3358229.$$

§. 24. Sumamus nunc $n = 5$ et habebimus

$$A = \mathfrak{Q} = 0,9961557,$$

tum vero erit

$a = 1$; $b = 0,9958847$; $c = 0,9962048$; $d = 0,9961753$, hinc igitur reperiemus

$$B = A - \tfrac{1}{3^5}.\,0,0038443 = 0,9961899$$

$$C = B - \tfrac{1}{5^5}.\,0,0002551 = 0,9961398.$$

Erit igitur

$$Q = 1 - C = 0,0038602, \text{ ideoque}$$

$$\tfrac{1}{4} l 2 - \tfrac{1}{3} P - \tfrac{1}{5} Q = 0,3350509.$$

§. 25. Sit nunc $n = 7$ et $A = \mathfrak{R} = 0,9995547$, tum vero $a = 1$; $b = 0,9995428$, hinc igitur fiet

$$B = A - \tfrac{1}{3^7}\,0,0004453 = 0,9995545,$$

 vnde

vnde iam habemus

$$R = 1 - \mathfrak{B} = 0,0004455 \text{ ideoque}$$
$$\tfrac{1}{2} l2 - \tfrac{1}{3}P - \tfrac{1}{5}Q - \tfrac{1}{7}R = 0,3349873.$$

§. 26. Cum in hoc calculo tantum non fuerit $\mathfrak{B} = A$, in fequentibus nequidem littera B erit opus, quamobrem habebimus

$$S = 1 - \mathfrak{S} = 0,0000501 \text{ hincque}$$
$$\tfrac{1}{9}S = 0,0000056.$$

Deinde vero erit

$$T = 1 - \mathfrak{T} = 0,0000053 \text{ hincque}$$
$$\tfrac{1}{11}T = 0,0000005, \text{ denique}$$
$$U = 1 - \mathfrak{U} = 0,0000003 \text{ et } \tfrac{1}{13}U = 0,000000.$$

Particulis igitur his a praecedente valore ablatis prodit

$$O = 0,3349812.$$

Vnde patet, hunc valorem adhuc aliquanto maiorem effe quam $\frac{1}{3}$.

§. 27. Nunc igitur certi effe poffumus fummam feriei infinitae huius

$$\tfrac{1}{3} - \tfrac{1}{5} + \tfrac{1}{7} + \tfrac{1}{11} - \tfrac{1}{13} - \tfrac{1}{17} + \tfrac{1}{19} + \text{ etc.}$$

effe fatis exacte $= 0,3349812$. Inueftigandum iam foret, num ifte valor non quampiam teneat rationem notabilem, fiue ad peripheriam circuli π, fiue ad eius logarithmum hyperbolicum, quandoquidem fupra obferuauimus, feriem reciprocam numerorum primorum

$$\tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{5} + \tfrac{1}{7} + \tfrac{1}{11} + \text{ etc.}$$

expri-

3.
tum non fuerit
erit opus, quam-
= 0, 000000.
ablatis prodit
nto maiorem effe
offumus fummam
tc.
gandum iam fo-
at rationem nota-
ue ad eius loga-
pra obferuauimus,
expri-

exprimere logarithmum hyperbolicum feriei harmonicae completae

$$1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{5} + \tfrac{1}{6} + \text{ etc.}$$

vnde videri poteft, iftam feriem numerorum primorum

$$\tfrac{1}{3} - \tfrac{1}{5} + \tfrac{1}{7} + \tfrac{1}{11} - \tfrac{1}{13} - \tfrac{1}{17} + \text{ etc.}$$

etiam continere logarithmum eiusdem feriei completae:

$$1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \tfrac{1}{9} - \tfrac{1}{11} + \text{ etc.}$$

cuius fumma eft $\frac{\pi}{4}$. Hunc in finem fubiungam logarithmum hyperbolicum ipfius π, quem olim reperi

$$1,14472,98858,49400,17414,34273,51353,05865.$$

Videndum igitur erit num forte fit fumma inuenta $O = l\pi - lN$, ita vt N fit numerus fatis fimplex. Verum huiusmodi inueftigationes plerumque fine vllo fucceffu inftituuntur.

§. 28. Ope pofterioris methodi autem non folum fummam feriei propofitae elicuimus, fed etiam eius poteftatum imparium, quas fummas hic confpectui exponamus.

$$\tfrac{1}{3} - \tfrac{1}{5} + \tfrac{1}{7} + \tfrac{1}{11} - \tfrac{1}{13} - \tfrac{1}{17} + \tfrac{1}{19} + \text{ etc.} = 0,3349812$$
$$\tfrac{1}{3^3} - \tfrac{1}{5^3} + \tfrac{1}{7^3} + \tfrac{1}{11^3} - \tfrac{1}{13^3} - \tfrac{1}{17^3} + \tfrac{1}{19^3} + \text{ etc.} = 0,0322521$$
$$\tfrac{1}{3^5} - \tfrac{1}{5^5} + \tfrac{1}{7^5} + \tfrac{1}{11^5} - \tfrac{1}{13^5} - \tfrac{1}{17^5} + \tfrac{1}{19^5} + \text{ etc.} = 0,0038602$$
$$\tfrac{1}{3^7} - \tfrac{1}{5^7} + \tfrac{1}{7^7} + \tfrac{1}{11^7} - \tfrac{1}{13^7} - \tfrac{1}{17^7} + \tfrac{1}{19^7} + \text{ etc.} = 0,0004455$$
$$\tfrac{1}{3^9} - \tfrac{1}{5^9} + \tfrac{1}{7^9} + \tfrac{1}{11^9} - \tfrac{1}{13^9} - \tfrac{1}{17^9} + \tfrac{1}{19^9} + \text{ etc.} = 0,0000501$$
$$\tfrac{1}{3^{11}} - \tfrac{1}{5^{11}} + \tfrac{1}{7^{11}} + \tfrac{1}{11^{11}} - \tfrac{1}{13^{11}} - \tfrac{1}{17^{11}} + \tfrac{1}{19^{11}} + \text{ etc.} = 0,0000056$$
$$\tfrac{1}{3^{13}} - \tfrac{1}{5^{13}} + \tfrac{1}{7^{13}} + \tfrac{1}{11^{13}} - \tfrac{1}{13^{13}} - \tfrac{1}{17^{13}} + \tfrac{1}{19^{13}} + \text{ etc.} = 0,0000005$$

etc. etc.

§. 29.

§. 29. Ipſae quidem hae ſummae ſine dubio parum attentionis merentur, niſi forte ad quantitates cognitas reduci potuerint. Verum quia in his ſeriebus neque ipſi termini ſecundum certam legem progrediuntur, neque etiam in ſignis plus vel minus certus ordo obſeruatur; iſta disquiſitio primo intuitu plane impoſſibilis videri potuiſſet, quamobrem ipſa methodus, qua ad earum ſummas pertigimus, vtique omni attentione digna eſt cenſenda, idque eo magis, quod ſatis abſtruſis ſerierum poteſtatum proprietatibus innititur. Niſi enim ſummae ſerierum

$$1 - \frac{1}{3^n} + \frac{1}{5^n} - \frac{1}{7^n} + \frac{1}{9^n} \text{ etc.}$$

pro caſibus quibus n eſt numerus impar, fuiſſent cognitae, ota haec inueſtigatio fruſtra fuiſſet ſuſcepta.

DE

dubio parum cognitas reneque ipſi terneque etiam ur; iſta diseri potuiſſet, ımmas pertiıſenda, idque teſtatum proum

ſent cognitae,

DE

DE SERIEBVS POTESTATVM RECIPROCIS

METHODO NOVA ET FACILLIMA SVMMANDIS.

§. 1.

Cum primum ſummas harum ſerierum docuiſſem, eas ex hoc principio deduxi, quod cuique ſinui et coſinui innumerabiles arcus circulares reſpondent, qui omnes ſint radices aequationum infinitarum, quibus arcus per ſinum vel coſinum exprimi ſolent. Hinc enim ex coefficientibus iſtarum aequationum non ſolum ſummas ipſarum radicum, ſed etiam earum poteſtatum quarumcunque aſſignaui. Poſtea vero easdem ſummas etiam ex aliis principiis deriuaui, quae autem omnia memorata circuli proprietate innitebantur. Nunc vero obſeruaui, iſtas ſummas ex alio principio multo ſimpliciori, et ſolis operationibus analyticis innixo, deduci poſſe, quam methodum hic accuratius expoſuiſſe iuuabit.

§. 2. Hoc autem principium mihi ſuppeditauit integratio huius formulae: $\int \left(\frac{z^{m-1} + z^{n-m-1}}{1 + z^n} \right) dz$, pro caſu quo poſt integrationem ſtatuitur $z = 1$. Oſtendi enim in