E595

$$\cfrac{n+1}{3n+\cfrac{1}{5n+\cfrac{1}{7n+\cfrac{1}{9n+\text{etc.}}}}} = \frac{e^{\frac{2}{n}}}{e^{\frac{2}{n}}-1} \text{ et}$$

$$\cfrac{n-1}{3n-\cfrac{1}{5n-\cfrac{1}{7n-\cfrac{1}{9n-\text{etc.}}}}} = \cot.\tfrac{1}{n}$$

Harum fractionum prior cum formulis postremi exempli collata praebet $m-b=n$, $2m-b=3n$, ideoque $m=2n$ et $b=n$, vnde fit

$$A = \tfrac{1}{2n}\int \frac{dx}{x^{\frac{5}{2}}\, e^{\frac{1+xx}{2nx}}} \text{ et}$$

$$B = \tfrac{1}{2n}\int \frac{dx}{x^{\frac{3}{2}}\, e^{\frac{1+xx}{2n}}};$$

vnde iam discimus si hae duae formulae integrentur à termino $x=0$ vsque ad terminum $x=\infty$, tum fore

$$\frac{A}{B} = \frac{1+e^{\frac{2}{n}}}{1-e^{\frac{2}{n}}},$$

quanquam nulla adhuc via analytica patet, hanc conuenientiam demonstrandi.

SVM-

exempli
$m=2n$
ur à ter-
re
conueni-
SVM-

SVMMATIO
FRACTIONIS CONTINVAE,
CVIVS INDICES PROGRESSIONEM ARITHMETICAM CONSTITVVNT, DVM NVMERATORES OMNES SVNT VNITATES;
VBI SIMVL RESOLVTIO AEQVATIONIS RICCATIANAE PER HVIVSMODI FRACTIONES DOCETVR.

§. 1.

Cum in praecedente dissertatione methodum exposuissem; fractiones continuas ad duas formulas integrales reducendi, ea quidem infinitis casibus feliciter successit: at vero casus, qui simplicissimus videtur, vbi omnes numeratores inter se ponuntur aequales, ad eiusmodi formulas integrales perduxit, quas nullo adhuc modo euoluere et inter se comparare licuit, cum tamen ex hoc genere binae fractiones continuae habeantur, quarum valores satis commode exhiberi possunt:

$$\cfrac{n+1}{3n+\cfrac{1}{5n+\cfrac{1}{7n+\text{etc.}}}} = \frac{e^{\frac{2}{n}}+1}{e^{\frac{2}{n}}-1} \text{ et}$$

$$\cfrac{n-1}{3n-\cfrac{1}{5n-\cfrac{1}{7n-\text{etc.}}}} = \cot.\tfrac{1}{n}$$

quarum quidem altera ex altera facile deducitur, si loco n scribatur $n\sqrt{-1}$.

§. 2. Quando autem indices aliam quamcunque progressionem arithmeticam sequuntur, summationem talium fractionum continuarum iam olim in Tomo XI veterum Commentar. nostrae Academiae singulari prorsus modo ad aequationem Riccatianam reduxi. Methodus autem, qua hoc praestiti, ibi nimis succincte est exposita; quare, cum ea plurimum in recessu habere videatur, eam hic operae pretium erit vberius explicare; praecipue cum non solum nemo vim illius methodi animaduertisse videatur, sed etiam ipse eius penitus essem oblitus.

§. 3. Quo isthanc inuestigationem clarius ob oculos ponam, exordiar a fractione generali, cuius quidem omnes numeratores sint vnitates, indices autem in genere litteris a, b, c, d, e, f, etc. designentur, ita vt ipsa fractio continua hanc habeat formam:

$$a+\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{d+\cfrac{1}{e+\text{etc.}}}}}$$

cuius valor littera S indicetur, ad quem proxime saltem cognoscendum, ex indicibus a, b, c, d, e, etc. formetur more solito series sequentium fractionum:

$$\begin{array}{cccccc} a & b & c & d & e & \\ \frac{A}{\mathfrak{A}} & \frac{B}{\mathfrak{B}} & \frac{C}{\mathfrak{C}} & \frac{D}{\mathfrak{D}} & \frac{E}{\mathfrak{E}} & \text{etc.} \end{array}$$

quarum tam numeratores quam denominatores sequenti modo ex binis praecedentibus determinantur:

$$A = a,\ B = Ab+1,\ C = Bc+A,\ D = Cd+B, \text{ etc.}$$
$$\mathfrak{A} = 1,\ \mathfrak{B} = \mathfrak{A}b,\quad \mathfrak{C} = \mathfrak{B}c+\mathfrak{A},\ \mathfrak{D} = \mathfrak{C}d+\mathfrak{B}, \text{ etc.}$$

§. 4. Nunc autem loco litterarum A, B, C, D, et $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}$, alias in calculum introducamus, quae sint

$$A' = \tfrac{A}{a},\ B' = \tfrac{B}{ab},\ C' = \tfrac{C}{abc},\ D' = \tfrac{D}{abcd}, \text{ etc.}$$
$$\mathfrak{A}' = \tfrac{\mathfrak{A}}{a},\ \mathfrak{B}' = \tfrac{\mathfrak{B}}{ab},\ \mathfrak{C}' = \tfrac{\mathfrak{C}}{abc},\ \mathfrak{D}' = \tfrac{\mathfrak{D}}{abcd}, \text{ etc.}$$

atque hae nouae litterae sequenti modo per indices et binos antecedentes terminos determinabuntur:

$$A' = 1,\ B' = A' + \tfrac{1}{ab},\ C' = B' + \tfrac{A'}{bc},\ D' = C' + \tfrac{B'}{cd}, \text{ etc.}$$
$$\mathfrak{A}' = \tfrac{1}{a},\ \mathfrak{B}' = \mathfrak{A}',\qquad \mathfrak{C}' = \mathfrak{B}' + \tfrac{\mathfrak{A}'}{bc},\ \mathfrak{D}' = \mathfrak{C}' + \tfrac{\mathfrak{B}'}{cd}, \text{ etc.}$$

His igitur valoribus pro quouis casu euolutis, istae fractiones

$$\tfrac{A}{\mathfrak{A}},\ \tfrac{B}{\mathfrak{B}},\ \tfrac{C}{\mathfrak{C}},\ \tfrac{D}{\mathfrak{D}},\ \tfrac{E}{\mathfrak{E}} \text{ etc.}$$

continuo propius ad valorem S fractionis continuae propositae accedent, et in infinitum continuatae ei prorsus aequabuntur.

§. 5. Quo formae harum litterarum, melius perspiciantur, eos simpliciter per indices euoluamus, ac primo quidem pro numeratoribus reperiemus sequentes formulas:

$A' = 1$

$B' = 1 + \frac{1}{ab}$

$\mathfrak{C}' = 1 + \frac{1}{ab} + \frac{1}{bc}$

$\mathfrak{D}' = 1 + \frac{1}{ab} + \frac{1}{bc} + \frac{1}{cd} + \frac{1}{abcd}$

$\mathfrak{E}' = 1 + \frac{1}{ab} + \frac{1}{bc} + \frac{1}{cd} + \frac{1}{de} + \frac{1}{abcd} + \frac{1}{abde} + \frac{1}{bcde}$

$\mathfrak{F}' = 1 + \frac{1}{ab} + \frac{1}{bc} + \frac{1}{cd} + \frac{1}{de} + \frac{1}{ef} + \frac{1}{abcd} + \frac{1}{abde} + \frac{1}{bcde} + \frac{1}{abef} + \frac{1}{bcef} + \frac{1}{cdef} + \frac{1}{abcdef}$

etc. etc.

Pro denominatoribus vero prodibunt sequentes formulae:

$\mathfrak{A}' = \frac{1}{a}$

$\mathfrak{B}' = \frac{1}{a}$

$\mathfrak{C}' = \frac{1}{a} + \frac{1}{abc}$

$\mathfrak{D}' = \frac{1}{a} + \frac{1}{abc} + \frac{1}{acd}$

$\mathfrak{E}' = \frac{1}{a} + \frac{1}{abc} + \frac{1}{acd} + \frac{1}{ade} + \frac{1}{abcde}$

$\mathfrak{F}' = \frac{1}{a} + \frac{1}{abc} + \frac{1}{acd} + \frac{1}{ade} + \frac{1}{aef} + \frac{1}{abcde} + \frac{1}{abcef} + \frac{1}{acdef}$

etc. etc.

in quibus posterioribus formulis singuli termini factorem habent $\frac{1}{a}$, quo omisso reliqui factores eodem modo per indices b, c, d, e, f, etc. definientur, quo litterae latinae antecedentes per omnes indices a, b, c, d, e sunt determinati.

§. 6. Accommodemus nunc istas euolutiones ad casum qui hic nobis est propositus, vbi indices a, b, c, d, e, secundum progressionem arithmeticam procedere assumimus. Statuamus igitur differentiam, qua hi indices continuo crescunt $= \Delta$, eruntque indices post primum sequentes

$$b = a + \Delta,\ c = a + 2\Delta,\ d = a + 3\Delta,\ e = a + 4\Delta, \text{ etc.}$$

Hos quidem valores in denominatoribus nostrarum formularum non substituemus, sed iis praecique vtemur ad formulas contrahendas.

§. 7.

$\frac{1}{cdef} + \frac{1}{abcdef}$
formulae.
i factorem
odo per in-
latinae an-
determinati.
lutiones ad
a, b, c, d, e,
assumimus.
ontinuo cre-
entes
$a+4\Delta$, etc.
cum formu-
mur ad for-
§. 7.

§. 7. Hac igitur progressione stabilita euoluamus primo numeratores nostrarum fractionum sequenti modo:

$A' = 1$

$B' = 1 + \frac{1}{ab}$

$C' = 1 + \frac{1}{ab} + \frac{1}{bc} = 1 + \frac{2}{ac}$

$D' = 1 + \frac{2}{ac} + \frac{1}{cd} + \frac{1}{abcd} = 1 + \frac{3}{ad} + \frac{1}{abcd}$

$E' = 1 + \frac{3}{ad} + \frac{1}{de} + \frac{1}{abcd} + \frac{2}{acde} = 1 + \frac{4}{ae} + \frac{3}{abde}$

$F' = 1 + \frac{5}{af} + \frac{6}{abef} + \frac{1}{abcdef}$

$G' = 1 + \frac{6}{ag} + \frac{10}{abfg} + \frac{4}{abcefg}$

$H' = 1 + \frac{7}{ah} + \frac{15}{abgh} + \frac{10}{abcfgh} + \frac{1}{abcdefgh}$

$I' = 1 + \frac{8}{ai} + \frac{21}{abhi} + \frac{20}{abcghi} + \frac{5}{abcdfghi}$

$K' = 1 + \frac{9}{ak} + \frac{28}{abik} + \frac{35}{abchik} + \frac{15}{abcdghik} + \frac{1}{abcdefghik}$.

§. 8. Quemadmodum in his formis omnes termini primi sunt vnitates, ita numeratores secundorum secundum numeros naturales, tertiorum secundum trigonales, quartorum secundum pyramidales primos, tum secundos, tertios etc. progrediuntur. In denominatoribus ordo pariter satis est manifestus. Hinc igitur in genere eam formulam exhibere poterimus, quae indefinite respondeat indici i. Ita si ista expressio designetur littera Z', tum vero in ordine litterarum a, b, c, d, e, etc. fuerit $z = a + i\Delta$, antecedentes vero

$$y = a + (i-1)\Delta,\ x = a + (i-2)\Delta,\ v = a + (i-3)\Delta, \text{ etc.}$$

habebimus

$$Z' = 1 + \frac{i}{az} + \frac{(i-1)(i-2)}{1.2.ab.yz} + \frac{(i-2)(i-3)(i-4)}{1.2.3.abcxyz} + \frac{(i-3)(i-4)(i-5)(i-6)}{1.2.3.4.abcdvxyz} + \text{etc.}$$

§. 9.

§. 9. Quod iam ad litteras germanicas $\mathfrak{A}'$, $\mathfrak{B}'$, $\mathfrak{C}'$, $\mathfrak{D}'$, etc. attinet, earum quaelibet ex antecedente latina formatur, dum litterae a, b, c, d, e vno gradu promouentur, tum vero finguli termini per $\frac{1}{a}$ multiplicentur, ita erit vt fequitur:

$$\mathfrak{A}' = 1$$
$$\mathfrak{B}' = \frac{1}{a}$$
$$\mathfrak{C}' = \frac{1}{a} + \frac{1}{abc}$$
$$\mathfrak{D}' = \frac{1}{a} + \frac{2}{abd}$$
$$\mathfrak{E}' = \frac{1}{a} + \frac{3}{abe} + \frac{1}{abcde}$$
$$\mathfrak{F}' = \frac{1}{a} + \frac{4}{abf} + \frac{3}{abcef}$$
$$\mathfrak{G}' = \frac{1}{a} + \frac{5}{abg} + \frac{6}{abcfg} + \frac{1}{abcdefg}$$
$$\mathfrak{H}' = \frac{1}{a} + \frac{6}{abh} + \frac{10}{abchi} + \frac{4}{abcdfgh}$$
$$\mathfrak{I}' = \frac{1}{a} + \frac{7}{abi} + \frac{15}{abchi} + \frac{10}{abcdghi} + \frac{1}{abcdefghi}$$
etc. etc.

vnde in genere, fi indici i refpondeat $\mathfrak{Z}'$, erit

$$\mathfrak{Z}' = \frac{1}{a} + \frac{(i-1)}{abz} + \frac{(i-2)(i-3)}{1.2.abcyz} + \frac{(i-3)(i-4)(i-5)}{1.2.3.abcdxyz} + \frac{(i-4)(i-5)(i-6)(i-7)}{1.2.3.4.abcdevxyz} + \text{etc.}$$

§. 10. Augeamus nunc indicem i in infinitum, vt fractio $\frac{Z'}{\mathfrak{Z}'}$ exprimat ipfum valorem fractionis continuae, quem vocauimus S, ita vt futurum fit $S = \frac{Z'}{\mathfrak{Z}'}$, atque reductio formularum inuentarum fequenti modo peragetur. Primo fcilicet pro Z' erit $\frac{i}{z} = \frac{i}{a+i\Delta} = \frac{1}{\Delta}$, ob $i = \infty$; pro termino tertio erit

$$\frac{(i-1)(i-2)}{yz} = \frac{(i-1)(i-2)}{(a+(i-1)\Delta)(a+i\Delta)} = \frac{1}{\Delta^2};$$

porro

is $\mathfrak{A}'$, $\mathfrak{B}'$, $\mathfrak{C}'$, nte latina for- promouentur, , ita erit vt

rit

in infinitum, vt continuae, quem atque reductio eragetur. Primo ∞; pro termino

porro

porro autem fimili modo

$$\frac{(i-2)(i-3)(i-4)}{xyz} = \frac{(i-2)(i-3)(i-4)}{(a+(i-2)\Delta)(a+(i-1)\Delta)(a+i\Delta)} = \frac{1}{\Delta^3} \text{ et}$$
$$\frac{(i-3)(i-4)(i-5)(i-6)}{vxyz} = \frac{1}{\Delta^4}, \text{ etc.}$$

Pari modo pro formula $\mathfrak{Z}'$ erit etiam

$$\frac{i-1}{z} = \frac{1}{\Delta}, \quad \frac{(i-2)(i-3)}{yz} = \frac{1}{\Delta^2}$$

et ita porro; quamobrem pro cafu $i = \infty$ ambae noftrae formulae ita commode contrahuntur, vt fiat

$$Z' = 1 + \frac{1}{a\Delta} + \frac{1}{1.2ab\Delta^2} + \frac{1}{1.2.3abc\Delta^3} + \frac{1}{1.2.3.4abcd\Delta^4} + \text{etc.}$$

fimilique modo

$$\mathfrak{Z}' = \frac{1}{a} + \frac{1}{ab\Delta} + \frac{1}{1.2abc\Delta^2} + \frac{1}{1.2.3abcd\Delta^3} + \frac{1}{1.2.3.4abcde\Delta^4} + \text{etc.}$$

§. 11. Cum igitur fit valor quaefitus $S = \frac{Z'}{\mathfrak{Z}'}$, videamus quomodo ambas feries infinitas inuentas ad expreffiones finitas reuocare queamus. Hunc in finem ambas feries generaliores reddamus, dum loco numeratorum, qui omnes funt 1, progreffionem quandam geometricam fubftituimus. Statuamus igitur

$$p = 1 + \frac{x^{\Delta}}{a\Delta} + \frac{x^{2\Delta}}{1.2\,ab\Delta^2} + \frac{x^{3\Delta}}{1.2.3\,abc\Delta^3} \text{ etc. et}$$
$$q = \frac{1}{a} + \frac{x^{\Delta}}{ab\Delta} + \frac{x^{2\Delta}}{1.2\,abc\Delta^2} + \frac{x^{3\Delta}}{1.2.3\,abcd\Delta^3} \text{ etc.}$$

Quod fi autem hinc valores p et q eruerimus in genere, tum vero ponamus $x = 1$, vtique proueniet $S = \frac{p}{q}$. Hic autem manifeftum eft, ambas has feries egregiam inter fe tenere affinitatem, ac per differentiationem vnam in alteram conuerti poffe, quam inueftigationem fequenti modo inftituamus.

§. 12.

§. 12. Primo igitur feries prior fimpliciter differentiata dat

$$\frac{x\,dp}{dx}=\frac{x^{\Delta}}{a}+\frac{x^{2\Delta}}{ab\Delta}+\frac{x^{3\Delta}}{1.2\,abc\Delta^2}+\frac{x^{4\Delta}}{1.2.3\,abcd\Delta^3}\text{ etc.}$$

quae feries cum altera q comparata manifefto praebet

$$\frac{x\,dp}{dx}=x^{\Delta}q,$$

vnde patet, fi modo fumma alterius harum duarum feriērum effet cognita, alteram quoque affignari poffe, quandoquidem ex cognito valore p prodit $q=\frac{dp}{x^{\Delta-1}dx}$; contra vero ex cognito valore q fit $dp=x^{\Delta-1}q\,dx$, ideoque $p=\int x^{\Delta-1}q\,dx$, quod integrale ita fumi debet, vt pofito $x=0$ fiat $p=1$.

§. 13. Ante autem quam alteram feriem differentiemus, eam multiplicemus per x, atque ob

$$a+\Delta=b,\ a+2\Delta=c,\ a+3\Delta=d,\text{ etc. erit}$$

$$x^a q=\frac{x^a}{a}+\frac{x^b}{ab\Delta}+\frac{x^c}{1.2\,abc\Delta^2}+\frac{x^d}{1.2.3\,abcd\Delta^3}+\text{ etc.}$$

Nunc ifta aequatio differentiata, iterumque per x multiplicata dabit

$$x\,d.\frac{x^a q}{dx}=x^a+\frac{x^b}{a\Delta}+\frac{x^c}{1.2\,ab\Delta^2}+\frac{x^d}{1.2.3\,abc\Delta}+\text{ etc.}$$

at vero prior feries p itidem qer x multiplicata praebet

$$x^a p=x^a+\frac{x^b}{a\Delta}+\frac{x^c}{1.2\,ab\Delta^2}+\frac{x^d}{1.2.3\,abc\Delta^3}+\text{ etc.}$$

quae

quae feries cum fint perfecte aequales, erit $\frac{x}{dx}d.x^a q=x^a p$, ideoque $d.x^a q=p x^{a-1}dx$, ficque duas nacti fumus aequationes differentiales inter p et q, ex quibus valorem vtriusque eruere licebit.

§. 14. Cum ex priore aequatione fit

$$q=\frac{dp}{x^{\Delta-1}dx},\text{ erit } x^a q=\frac{x^{a-\Delta+1}dp}{dx},$$

vnde fumto elemento dx conftante fiet

$$d.x^a q=\frac{x^{a-\Delta+1}ddp+(a-\Delta+1)x^{a-\Delta}dp}{dx}$$

ficque elifa quantitate q pro altera p nancifcimur hanc aequationem differentialem fecundi gradus:

$$p x^{a-1}dx=\frac{x^{a-\Delta+1}ddp+(a-\Delta+1)x^{a-\Delta}dp}{dx}$$

quam fi refoluere licuerit, totum negotium erit confectum. Vbi probe notandum, cum fit

$$p=1+\frac{x^{\Delta}}{a\Delta}+\frac{x^{2\Delta}}{1.2\,ab\Delta^2}+\text{ etc.}$$

integrationem ita inftitui debere, vt pofito $x=0$ fiat $p=1$; cum vero quia eft

$$\frac{dp}{dx}=\frac{x^{\Delta-1}}{a}+\frac{x^{2\Delta-1}}{ab\Delta}+\text{ etc.}$$

altera conditio integrationis poftulat vt pofito $x=0$ etiam fiat $\frac{dp}{dx}=0$, fiquidem fuerit $\Delta>1$, fi enim effet $=1$, cafu $x=0$ fieri debet $\frac{dp}{dx}=\frac{1}{a}$. At fi $\Delta<1$, fieri debebit $\frac{dp}{dx}=\infty$.

§. 15.

§. 15. Cum fractio $\frac{p}{q}$ poſito $x = 1$ praebeat valorem noſtrae fractionis continuae S, ponamus in genere eſſe $\frac{p}{q} = z$, ita vt poſito $x = 1$ fiat $z = S$, vnde patet, caſu $x = 0$ fieri debere $z = a$. Cum igitur ſit $p = qz$, erit $dp = q\,dz + z\,dq$; erat autem ex prima aequatione

$$q = \frac{dp}{x^{\Delta-1}\,dx},$$

quamobrem habebimus

$$q = \frac{q\,dz + z\,dq}{x^{\Delta-1}\,dx},$$

ſiue $x^{\Delta-1}\,q\,dx = q\,dz + z\,dq$, vnde fit

$$dq = \frac{x^{\Delta-1}\,q\,dx - q\,dz}{z};$$

erat autem $dp = x^{\Delta-1}\,q\,dx$.

§. 16. Haec igitur ſequuntur ex priori aequatione differentiali inuenta $\frac{dp}{x^{\Delta-1}\,dx} = q$. Altera vero, quae eſt

$$d.\,x^a q = p\,x^{a-1}\,dx, \text{ quia}$$

$$d.\,x^a\,q = a\,x^{a-1}\,q\,dx + x^a\,dq$$

loco dq poſito valore modo inuento prodit

$$d.\,x^a q = a x^{a-1}\,q\,dx + \frac{x^{a+\Delta-1}\,q\,dx - x^a\,q\,dz}{z},$$

habebimus

$$p x^{a-1}\,dx = a x^{a-1}\,q\,dx + \frac{x^{a+\Delta-1}\,q\,dx - x^a\,q\,dz}{z},$$

quae

quae aequatio, ob $p = qz$, per q diuiſa ſuppeditat iſtam aequationem differentialem primi gradus:

$$x^{a-1}\,z\,dx = a\,x^{a-1}\,dx + \frac{x^{a+\Delta-1}\,dx - x^a\,dz}{z},$$

quae per z multiplicata et per x^{a-1} diuiſa praebet

$$z\,z\,dx = a\,z\,dx + x^{\Delta}\,dx - x\,dz,$$

cuius ergo reſolutio ſi ita inſtituatur, vt poſito $x = 0$ fiat $z = a$, tum vero fiat $x = 1$, valor ipſius z dabit ipſum valorem fractionis continuae quem quaerimus.

§. 17. Totum igitur negotium perductum eſt ad reſolutionem aequationis differentialis primi gradus

$$z\,z\,dx = a\,z\,dx + x^{\Delta}\,dx - x\,dz,$$

quae manifeſto in celeberrima illa aequatione Riccatiana continetur. Vt enim ad tres tantum terminos reducatur, ponatur $z = x^a y$, ita vt ſit $y = \frac{z}{x^a}$, vnde fit

$$dz = a\,x^{a-1}\,y\,dx + x^a\,dy,$$

quibus valoribus ſubſtitutis noſtra aequatio hanc induet formam:

$$x^{a+1}\,dy + x^{2a}\,y\,y\,dx = x^{\Delta}\,dx,$$

quam ergo ita integrari oportet, vt poſito $x = 0$, ſiue infinite paruo, fiat $y = \frac{a}{x^a}$, hoc eſt $y = \infty$, quo facto, ſi poſt integrationem ſtatuatur $x = 1$, valor ipſius y dabit ſummam fractionis continuae propoſitae.

 §. 18,

§. 18. Quo hanc expreſſionem ſimpliciorem reddamus, diuidamus per x^{a+1}, vt habeamus

$$dy + x^{a-1} yy\,dx = x^{\Delta-a-1}\,dx;$$

nunc vero ſtatuamus $x^a = t$, ita vt caſu $x=0$ fiat quoque $t=0$, et caſu $x=1$ etiam $t=1$, vnde ſi t euaneſcat, fieri debet $y=\frac{a}{t}$, ſiue $y=\infty$. Hoc autem valore introducto, ob $x=t^{\frac{1}{a}}$ et

$$dx = \frac{t^{\frac{1-a}{a}}\,dt}{a},$$

aequatio noſtra fiet

$$a\,dy + yy\,dt = t^{\frac{\Delta-2a}{a}}\,dt;$$

quae eſt forma maxime vſitata aequationis Riccatianae.

§. 19. Quando ergo propoſita fuerit talis fractio continua:

$$a+\cfrac{1}{a+\Delta+\cfrac{1}{a+2\Delta+\cfrac{1}{a+3\Delta+\cfrac{1}{a+4\Delta+\text{etc.}}}}}$$

pro eius valore inueſtigando reſolui debet iſta aequatio Riccatiana:

$$a\,dy + yy\,dt = t^{\frac{\Delta-2a}{a}}\,dt;$$

vbi integrationem ita inſtitui oportet, vt ſumta t infinite parua fiat $y=\frac{a}{t}$, quo facto ſtatuatur $t=1$, et valor pro y reſultans erit valor huius fractionis continuae.

§. 20

n redda-
: quoque
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ıtroducto,
anae.
ılis fractio
-4Δ+ etc.
ſta aequatio
ımta t infinite
et valor pro
:.
§. 20

§. 20. Euoluamus caſum ſimpliciſſimum, quo ad dextram partem exponens ipſius t fit nihilo aequalis, quod ergo euenit, ſi $\Delta = 2a$, ideoqne ipſa fractio continua

$$a+\cfrac{1}{3a+\cfrac{1}{5a+\cfrac{1}{7a+\text{etc.}}}}$$

pro cuius ſumma habebimus hanc aequationem differentialem:

$$a\,dy + yy\,dt = dt, \text{ vnde } dt = \frac{a\,dy}{1-yy},$$

et integrando

$$t = \frac{a}{2}\,l\frac{1+y}{1-y} + C,$$

quae conſtans C ita eſt capienda, vt poſito $t=0$ fiat $y=\frac{a}{t}$, ideoque $t=\frac{a}{y}$; vnde patet hoc caſu fieri $y=\infty$, ex quo ſtatim intelligitur, aequationem integralem ita inſtrui debere:

$$t = \frac{a}{2}\,l\frac{y+1}{y-1} + C.$$

ſiue integrationem ita inſtitui, vt facto $t=0$ fiat y infinitum. Nunc vero ſi y infinitum, erit $l\frac{y+1}{y-1}=\frac{2}{y}$, quare cum fieri debeat $t=\frac{a}{y}$ fit $C=0$, ita vt iuſta aequatio integralis ſit $t=\frac{a}{2}\,l\frac{y+1}{y-1}$, vnde, denotante e numerum cuius logarithmus hyperbolicus $=1$, fiet $e^{\frac{2t}{a}}=\frac{y+1}{y-1}$, hincque porro $y=\frac{e^{\frac{2t}{a}}+1}{e^{\frac{2t}{a}}-1}$, vnde poſito $t=1$ ſumma noſtrae fractionis continuae erit $\frac{e^{\frac{2}{a}}+1}{e^{\frac{2}{a}}-1}$, qui eſt idem valor, quem iam dudum inueneram.

§. 21.

§. 21. Contemplemur nunc etiam reliquos casus integrabilitatis aequationis Riccatianae, quibus exponens ipsius t ad partem dextram est vel -4, vel $-\frac{4}{3}$, vel $-\frac{8}{3}$, vel $-\frac{8}{5}$, vel $-\frac{12}{5}$, vel $-\frac{12}{7}$ etc. Sit igitur primo $\frac{\Delta-2a}{a}=-4$, vel $\Delta=-2a$, vnde nascitur haec fractio continua:

$$a+\cfrac{1}{-a+\cfrac{1}{-3a+\cfrac{1}{-5a+\cfrac{1}{-7a+\text{etc.}}}}}$$

quae manifesto a praecedente pendet. Si enim ponamus

$$-a+\cfrac{1}{-3a+\cfrac{1}{-5a+\text{etc.}}}=s,$$

mutatis signis erit

$$-s=a+\cfrac{1}{3a+\cfrac{1}{5a+\cfrac{1}{7a+\text{etc.}}}}$$

cuius valor cum iam sit inuentus, iste casus nihil noui nobis offert.

§. 22. Si sumatur $\frac{\Delta-2a}{a}=-\frac{4}{3}$, siue $\Delta=\frac{2}{3}a$, inde sequens nascitur fractio continua:

$$a+\cfrac{1}{\frac{5}{3}a+\cfrac{1}{\frac{7}{3}a+\cfrac{1}{\frac{9}{3}a+\text{etc.}}}}$$

siue

liquos casus ponens ipsius $-\frac{8}{5}$, vel $-\frac{8}{3}$, $=-4$, vel a: ponamus nihil noui no- $\Delta=\frac{2}{3}a$, inde siue

siue statuendo $a=3\alpha$, erit fractio

$$3\alpha+\cfrac{1}{5\alpha+\cfrac{1}{7\alpha+\cfrac{1}{9\alpha+\text{etc.}}}}$$

quae est ipsa ipsa forma superior primo membro truncata. Idem vsu venit, si sumatur $\frac{\Delta-2a}{a}=-\frac{8}{3}$, siue $\Delta=-\frac{2}{3}a$, vnde oritur haec fractio continua, ponendo scil. $a=3\alpha$

$$3\alpha+\cfrac{1}{\alpha+\cfrac{1}{-\alpha+\cfrac{1}{-3\alpha+\cfrac{1}{-5\alpha+\text{etc.}}}}}$$

sicque semper ad principalem formam reducimur.

§. 23. Casus autem integrabilitatis in genere hoc continentur exponente: $-\frac{4i}{2i\pm1}$. Posito igitur

$$\frac{\Delta-2a}{a}=-\frac{4i}{2i\pm1}, \text{ fiet } \Delta=\frac{\pm2a}{2i\pm1},$$

vnde posito $\frac{a}{2i\pm1}=\alpha$, erit $\Delta=\pm2\alpha$, ergo ob $a=(2i\pm1)\alpha$ fractio continua erit

$$(2i\pm1)\alpha+\cfrac{1}{a\pm2\alpha+\cfrac{1}{a\pm4\alpha+\cfrac{1}{a\pm6\alpha+\text{etc.}}}}$$

vbi manifesto iterum omnes numeri impares occurrunt, ita vt fractio continua inde nata semper formari possit ex nostra principali

$a+1$

$$a+\cfrac{1}{3a+\cfrac{1}{5a+\cfrac{1}{7a+\text{etc.}}}}$$

ſi vel aliquot membris truncetur, vel retro ad aliquot membra ſuperne continuetur.

§. 24. Cum igitur omnes caſus integrabilitatis aequationis Riccatianae ad eandem fractionem continuam perducant; praeterea vero nulli adhuc alii caſus euolui potuerint, hinc manifeſto ſequitur, ſi indices fractionis continuae a, b, c, d, e, f etc. quamcunque aliam progreſſionem arithmeticam conſtituant, tum ſummam nullo plane modo aſſignari poſſe; quandoquidem ea pendet a caſu irreſolubili aequationis Riccatianae. Ita ſi proponatur haec fractio continua:

$$a+\cfrac{1}{2a+\cfrac{1}{3a+\cfrac{1}{4a+\text{etc.}}}}$$

vbi eſt $\Delta = a$, ſummatio pendebit ab iſta aequatione differentiali: $a\,dy + yy\,dt = \frac{dt}{t}$, cuius reſolutio cum per nullas quantitates tranſcendentes etiamnunc vſu receptas expediri poſſit, valorem huius fractionis continuae fruſtra inter quantitates a circulo vel a logarithmis pendentes, vel adeo inter omnes quadraturas curuarum algebraicarum quaerimus; vnde mirum non eſt, quod methodus in ſuperiori diſſertatione vſitata pro talibus caſibus omni ſucceſſu caruerit.

§. 25. Quoniam tamen in diſſertatione ſuperiore ſummam talium fractionum continuarum per binas formulas inte-

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integralas expreſſam dedimus, illa ipſa expreſſio etiam ad aequationem Riccatianam pro iisdem caſibus accommodari poterit, id quod vtique maximam attentionem meretur, cum nullo adhuc modo iſta aequatio praeter caſus integrabiles tractari potuerit. Quamobrem maxime operae erit pretium ſolutiones inter ſe comparare, quandoquidem hinc haud contemnendum ſubſidium, aequationem Riccatianam feliciori ſucceſſu tractandi, expectari poterit.

§. 26. In ſuperiori autem diſſertatione oſtendi, ſi haec propoſita fuerit fractio continua:

$$m-b+\cfrac{1}{2m-b+\cfrac{1}{3m-b+\cfrac{1}{4m-b+\text{etc.}}}}$$

eius valorem exprimi per hanc fractionem: $-\frac{A}{B}$, exiſtente

$$A = \int \frac{dx}{x^{2+\frac{b}{m}} \cdot e^{\frac{1+xx}{mx}}} \text{ et}$$

$$B = \int \frac{dx}{x^{1+\frac{b}{m}} \cdot e^{\frac{1+xx}{mx}}},$$

ſiquidem haec duo integralia a termino $x = 0$ vsque ad terminum $x = \infty$ extendantur.

§. 27. Comparemus nunc hanc fractionem continuam cum generali, quam hic tractauimus:

 $a+1$

$$a+\cfrac{1}{a+\Delta+\cfrac{1}{a+2\Delta+\cfrac{1}{a+3\Delta+\text{etc.}}}}$$

cuius valor continetur in hac aequatione:

$$a\,dy+yy\,dt=t^{\frac{\Delta-2a}{a}}\,dt,$$

integratione scilicet ita instituta, vt casu $t=0$ fiat $y=\infty$; tum vero statuatur $t=1$, vnde valor ipsius y summam istius fractionis continuae exprimet.

§. 28. Comparatione igitur instituta fiet $m=\Delta$ et $b=\Delta-a$, quibus valoribus introductis formulae illae integrales erunt

$$A=\int\frac{dx}{x^{3-\frac{a}{\Delta}}\cdot e^{\frac{1+xx}{\Delta x}}}\ \text{et}$$

$$B=\int\frac{dx}{x^{2-\frac{a}{\Delta}}\cdot e^{\frac{1+xx}{\Delta x}}},$$

integralibus iterum sumtis ab $x=0$ ad $x=\infty$; quamobrem valor ipsius y, qui ex aequatione

$$a\,dy+yy\,dt=t^{\frac{\Delta-2a}{a}}\,dt$$

pro casu $t=1$ resultat, aequalis erit isti fractioni: $-\frac{\Delta}{a}$, siue erit

$$y=\frac{\int dx:x^{3-\frac{a}{\Delta}}\cdot e^{\frac{1+xx}{\Delta x}}}{\int dx:x^{2-\frac{a}{\Delta}}\cdot e^{\frac{1+xx}{\Delta x}}}.$$

Quanquam autem haec aequalitas tantum casus speciales, quibus ibi fit $t=1$, hic vero $x=\infty$, spectat, tamen fortasse

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tasse eiusmodi relatio inter binas variabiles t et x inueniri poterit, vt in genere quantitas y illi formulae aequetur.

§. 29. Quemadmodum consideratio fractionis nostrae continuae nos ad resolutionem aequationis Riccatianae perduxit, ita vicissim datur methodus directa, qua ista aequatio per fractiones continuas resolui potest. Quod quo facilius ostendatur, aequatio Riccatiana, quae vulgo hac forma proponi solet:

$$dy+ayy\,dx=acx^{2m}\,dx,$$

in aliam formam ad praesens institutum magis accommodatam transfundatur, ponendo $y=x^{m}z$ et $x^{m+1}=t$; tum enim si loco $\frac{a}{m+1}$ scribatur b, prodit ista aequatio:

$$dz+\frac{mz\,dt}{(m+1)t}+bzz\,dt=bc\,dt.$$

§. 30. Ponamus porro $\frac{m}{m+1}=-n$, vt sit $m=-\frac{n}{1+n}$ et nobis proposita sit ista aequatio:

$$dz-n\frac{z\,dt}{t}+bz\,dt=bc\,dt,$$

quae, si adhibeamus hanc substitutionem: $z=\frac{n+1}{bt}+\frac{c}{v}$, transmutatur in hanc formam:

$$dv-(n+2)\frac{v\,dt}{t}+bvv\,dt=bc\,dt,$$

quae a priore hoc tantum differt, vt hic numerus n binario maior sit factus; quare si porro faciamus $v=\frac{n+3}{bt}+\frac{c}{u}$, prodibit haec aequatio:

$$du-(n+4)u\,dt+buu\,dt=bc\,dt,$$

vnde patet, si prima aequatio resolutionem admittat casu $n=k$, tum etiam eius resolutionem in potestate fore casibus

$$n=k+2,\ n=k+4,\ n=k+6,\ \text{etc.}$$

et

et in genere casu $n = k + 2i$, denotante i numerum integrum quemcunque.

§. 31. Quod si iam successiue loco v et u et sequentium litterarum valores istos debitos substituamus, pro quantitate z sequens prodibit fractio continua:

$$z = n + \cfrac{1}{bt + \cfrac{c}{\frac{n+1}{bt} + \cfrac{c}{\frac{n+3}{bt} + \text{etc.}}}}$$

quae ergo expressio exhibet valorem quantitatis z, qui ipsi conuenit vi huius aequationis:

$$dz - n\frac{z\,dt}{t} + bzz\,dt = bc\,dt,$$

si scilicet ita integretur, vt posito $t = 0$ fiat $z = \infty$.

§. 32. Liberemus hanc formam a fractionibus partialibus, et obtinebimus hanc formam:

$$\frac{1}{z} = \cfrac{bt}{n+1+\cfrac{bbctt}{n+3+\cfrac{bbctt}{n+5+\cfrac{bbctt}{n+7+\text{etc.}}}}}$$

vnde statim patet, posito $t = 0$ fore $\frac{1}{z} = 0$ ideoque $z = \infty$. Simili modo resolutionem ab aequatione transformata exordiri possumus, quae erat

$$dv - (n+2)\frac{v\,dt}{t} + bvv\,dt = bc\,dt,$$

quae in primam transformatur ponendo $v = \frac{bct}{-n-1+btz}$; prodibit enim

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lo $v = \frac{bct}{-n-1+btz}$;
dz

$$dz - \frac{nz\,dt}{t} + bzz\,dt = bc\,dt,$$

in qua aequatione numerus n binario redditus est minor; vnde patet, si aequatio resolutionem admittat casu $n = k$, tum etiam resolutionem esse successuram casibus

$$n = k - 2,\ n = k - 4,\ n = k - 6$$

et in genere $n = k - 2i$. Quare cum resolutio nulla labores difficultate casu $n = 0$, sumto $k = 0$ omnes casus resolutionem admittentes continebuntur in hac formula: $n = \pm 2i$ hincque pro forma consueta fiet

$$m = \frac{\mp 2i}{\pm 2i + 1} = -\frac{2i}{2i \pm 1},$$

quae continet casus notissimos integrabilitatis.

§. 33. Ponamus $n + 2 = -\nu$, vt aequatio, a qua hic inchoamus, fit

$$dv = \frac{\nu v\,dt}{t} + bvv\,dt = bc\,dt,$$

quae ergo, posito $v = \frac{bct}{\nu+1+btz}$ transmutatur in hanc formam:

$$dz + (\nu+2)\frac{z\,dt}{t} + bzz\,dt = bc\,dt,$$

in qua nunc numerus ν binario augetur. Quare si vlterius ponamus $z = \frac{bct}{\nu+3+bty}$, orietur haec aequatio:

$$dy = +(\nu+4)\frac{y\,dt}{t} + byy\,dt = bc\,dt,$$

sicque vlterius progrediendo peruenietur ad $\nu+5, \nu+7$ etc.

§. 34. Substituamus ergo istos valores in superiore aequatione, quae est

$$dv + \frac{\nu v\,dt}{t} + bvv\,dt = bc\,dt,$$

ac pro v prodibit sequens fractio continua:

$v =$

$$v = \cfrac{bct}{v+1+\cfrac{bbctt}{v+3+\cfrac{bbctt}{v+5+\cfrac{bbctt}{v+7+\text{etc.}}}}}$$

Haec igitur expreſſio locum habet, ſi aequatio differentialis ita integretur, vt poſito $t=0$ fiat $v=0$.

§. 35. Geminas igitur ex aequatione Ricciatiana elicuimus fractiones continuas, quas quo facilius inter ſe comparare queamus, loco v ſcribamus iterum n, vt habeamus has duas aequationes differentiales:

I. $dz - n\frac{z\,dt}{t} + bzz\,dt = bc\,dt$,

II. $dv + n\frac{v\,dt}{t} + bvv\,dt = bc\,dt$,

atque ex priore naſcetur iſta fractio continua:

$$\frac{1}{z} = \cfrac{bt}{n+1+\cfrac{bbctt}{n+3+\cfrac{bbctt}{n+5+\text{etc.}}}}$$

ex altera vera oritur

$$v = \cfrac{bct}{n+1+\cfrac{bbctt}{n+3+\cfrac{bbctt}{n+5+\text{etc.}}}}$$

quae ergo fractiones prorſus inter ſe conueniunt, cum hinc fiat $v=\frac{c}{z}$, quippe quo modo altera aequatio in alteram actu conuertitur, ita vt hae duae formae pro vnica ſint habendae.

§. 36.

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§. 36.

§. 36. Iſta reſolutio aequationis Riccatianae in fractionem continuam eo magis eſt attentione digna, quod haec aequatio nullo adhuc modo in ſeriem infinitam regularem reſolui potuerit. Quae enim ſeries olim pro eius reſolutione exhibui, ita ſunt comparatae vt vna ſeries infinita per aliam diuiſa reſolutionem aequationis Riccatianae ſuppeditet; hic autem de vnica ſerie ſimplici ſermo inſtituitur. Hinc igitur naſcitur quaeſtio, num forte non etiam aliae aequationes differentiales dentur, quarum reſolutionem pariter per fractiones continuas expedire liceat.

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